2. Find the potential steady-states of the third-order recurrence

$$a(n+3) = \frac{1 + 2a(n+2) + 3a(n+1) + 4a(n)}{2 + 3a(n+2) + 4a(n+1) + 5a(n)} \quad .$$

$$z = \frac{1 + 2z + 3z + 4z}{2 + 3z + 4z + 5z}$$

$$z = \frac{1+9z}{2+12z} \rightarrow 2z + 12z^2 = 1 + 9z$$

$$12z^2 - 7z - 1 = 0$$

$$= \frac{7 \pm \sqrt{48 - 4(12)(-1)}}{2(12)}$$

$$= \frac{7 \pm \sqrt{97}}{24} \Big\} \text{possible steady-states}$$

$$= -0.1187,\ 0.702$$

3. Convert the third-order recurrence

$$a(n+3) = \frac{1 + 2a(n+2) + 3a(n+1) + 4a(n)}{2 + 3a(n+2) + 4a(n+1) + 5a(n)} \quad .$$

$$X(n) = \begin{bmatrix} X_1(n) \\ X_2(n) \\ X_3(n) \end{bmatrix} := \begin{bmatrix} a(n+2) \\ a(n+1) \\ a(n) \end{bmatrix}$$

$$f(X_1, X_2, X_3) = \frac{1 + 2X_1 + 3X_2 + 4X_3}{2 + 3X_1 + 4X_2 + 5X_3}$$

$$X(n+1) = F(X(n))$$

$$F\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} \dfrac{1 + 2X_1 + 3X_2 + 4X_3}{2 + 3X_1 + 4X_2 + 5X_3} \\ X_1 \\ X_2 \end{pmatrix}$$

$$D = \{(X_1, X_2, X_3) \in \mathbb{R}^3 : 2 + 3X_1 + 4X_2 + 5X_3 \neq 0\}$$

4. By using SS in the Maple file

`http://sites.math.rutgers.edu/~zeilberg/Bio25/DMB8.pdf`

find the two steady-states, in terms of the parameter r of the non-linear recurrence

$$a(n+1) = \frac{(1 - 2\,a(n))(1 - 3\,a(n))}{r} \quad .$$

By using SSS and experimenting with different r (playing *high-low*), estimate the cut-off r_0 such for $r < r_0$ there is no stable steady-state, but for $r > r_0$ one them is a stable steady-state.

4'. Optional challenge (5 dollars): Find the **exact** value of this cut-off r_0.

$$f := \frac{((1 - 2\cdot x)\cdot(1 - 3\cdot x))}{r};$$

$$f := \frac{(1 - 2x)\,(1 - 3x)}{r} \tag{2}$$

$SS(f, x);$

$$\left[\frac{r}{12} + \frac{5}{12} + \frac{\sqrt{r^2 + 10r + 1}}{12}, \frac{r}{12} + \frac{5}{12} - \frac{\sqrt{r^2 + 10r + 1}}{12}\right] \tag{3}$$

$$f := \frac{((1 - 2\cdot x)\cdot(1 - 3\cdot x))}{3};$$

$$f := \frac{(1 - 2x)\,(1 - 3x)}{3} \tag{4}$$

$SS(f, x);$

$$\left[\frac{2}{3} - \frac{\sqrt{10}}{6}, \frac{2}{3} + \frac{\sqrt{10}}{6}\right] \tag{5}$$

$SSS(f, x);$

$$\left[\frac{3}{4} - \frac{\sqrt{57}}{12}\right] \tag{6}$$

#zero stable steady states

$$f := \frac{((1 - 2\cdot x)\cdot(1 - 3\cdot x))}{4};$$

$$f := \frac{(1 - 2x)\,(1 - 3x)}{4} \tag{7}$$

$SS(f, x);$

$$\left[\frac{3}{4} - \frac{\sqrt{57}}{12}, \frac{3}{4} + \frac{\sqrt{57}}{12}\right] \tag{8}$$

$SSS(f, x);$

$$\left[\frac{3}{4} - \frac{\sqrt{57}}{12}\right] \tag{9}$$

#one stable steady state

$$f := \frac{((1 - 2\cdot x)\cdot(1 - 3\cdot x))}{3.4};$$

$$f := 0.2941176471\,(1 - 2x)\,(1 - 3x) \tag{10}$$

$SS(f, x);$

$$[0.1313759297, 1.268624070] \tag{11}$$

$SSS(f, x);$

$$[\,] \tag{12}$$

← no SS

$$f := \frac{((1 - 2\cdot x)\cdot(1 - 3\cdot x))}{3.5};$$

$$f := 0.2857142857\,(1 - 2x)\,(1 - 3x) \tag{13}$$

$SS(f, x);$

$$[0.1294815004, 1.287185166] \tag{14}$$

$SSS(f, x);$

$$[0.1294815004] \tag{15}$$

5. (i) By playing 'high-low' estimate the number r_1 such that for $r < r_1$ the orbit tends to period 2 ultimate orbit until it starts having period 4.

(ii) By playing 'high-low' estimate the number r_2 such that for $r < r_2$ the orbit tends to period 4 ultimate orbit until it starts having period 8.

(iii) By playing 'high-low' estimate the number r_3 such that for $r < r_3$ the orbit tends to period 8 ultimate orbit until it starts having period 16.

#Question 5

$$f(x, r) := \frac{((1 - 2\cdot x)\cdot(1 - 3\cdot x))}{r}$$

$$f := (x, r) \mapsto \frac{(1 - 2\cdot x)\cdot(1 - 3\cdot x)}{r} \tag{16}$$

$x0 := 0.2;$

$$x0 := 0.2 \tag{17}$$

$Orb(f(x, 1.0), x, x0, 200, 210);$

$$[\text{Float}(\infty), \text{Float}(\infty), \text{Float}(\infty), \text{Float}(\infty), \text{Float}(\infty), \text{Float}(\infty), \text{Float}(\infty), \text{Float}(\infty), \text{Float}(\infty), \text{Float}(\infty), \text{Float}(\infty)] \tag{18}$$

$Orb(f(x, 1.5), x, x0, 200, 210);$

$$[0.7466253960, 0.4077132745, -0.02745712491, 0.7612059914, 0.4470516075, -0.02408479923, 0.7492696407, 0.4147211752, -0.02776263807, 0.7622918494, 0.4500492901] \tag{19}$$

$Orb(f(x, 2.0), x, x0, 200, 210);$

$$[0.4156350870, -0.02083014086, 0.5533770370, 0.03523584274, 0.4156350870, -0.02083014086, 0.5533770370, 0.03523584274, 0.4156350870, -0.02083014086, 0.5533770370] \tag{20}$$

#period 4 above

$Orb(f(x, 2.3), x, x0, 200, 210);$

[0.4626096244, −0.01260962368, 0.4626096235, −0.01260962388, 0.4626096244, −0.01260962368, 0.4626096235, −0.01260962388, 0.4626096244, −0.01260962368, 0.4626096235] **(21)**

$Orb(f(x, 2.9), x, x0, 200, 220);$

[0.3570027476, −0.007002747341, 0.3570027476, −0.007002747341, 0.3570027476, −0.007002747341, 0.3570027476, −0.007002747341, 0.3570027476, −0.007002747341, 0.3570027476, −0.007002747341, 0.3570027476, −0.007002747341, 0.3570027476, −0.007002747341, 0.3570027476, −0.007002747341, 0.3570027476, −0.007002747341, 0.3570027476] **(22)**

$Orb(f(x, 3.38), x, x0, 200, 220);$

[0.1955204140, 0.07448737388, 0.1955187408, 0.07448868758, 0.1955171449, 0.07448994060, 0.1955156227, 0.07449113575, 0.1955141708, 0.07449227572, 0.1955127859, 0.07449336309, 0.1955114649, 0.07449440027, 0.1955102050, 0.07449538954, 0.1955090032, 0.07449633314, 0.1955078569, 0.07449723320, 0.1955067635] **(23)**

[0.1955204140, 0.07448737388, 0.1955187408, 0.07448868758, 0.1955171449, 0.07448994060, 0.1955156227, 0.07449113575, 0.1955141708, 0.07449227572, 0.1955127859, 0.07449336309, 0.1955114649, 0.07449440027, 0.1955102050, 0.07449538954, 0.1955090032, 0.07449633314, 0.1955078569, 0.07449723320, 0.1955067635] **(24)**

$Orb(f(x, 3.39), x, x0, 200, 220);$

[0.1886161542, 0.07975679964, 0.1886085204, 0.07976296217, 0.1886011711, 0.07976889524, 0.1885940955, 0.07977460754, 0.1885872833, 0.07978010736, 0.1885807246, 0.07978540267, 0.1885744099, 0.07979050114, 0.1885683300, 0.07979541017, 0.1885624761, 0.07980013683, 0.1885568397, 0.07980468798, 0.1885514128] **(25)**

$Orb(f(x, 3.42), x, x0, 200, 230);$

[0.1654530462, 0.09853305029, 0.1653764095, 0.09860061210, 0.1653010011, 0.09866711119, 0.1652267944, 0.09873257002, 0.1651537638, 0.09879701029, 0.1650818842, 0.09886045318, 0.1650111314, 0.09892291926, 0.1649414817, 0.09898442859, 0.1648729122, 0.09904500061, 0.1648054005, 0.09910465429, 0.1647389250, 0.09916340803, 0.1646734646, 0.09922127973, 0.1646089986, 0.09927828698, 0.1645455072, 0.09933444666, 0.1644829709, 0.09938977537, 0.1644213707] **(26)**

$Orb(f(x, 3.45), x, x0, 200, 230);$

[0.1451809021, 0.1161044799, 0.1450316238, 0.1162454819, 0.1448842501, 0.1163847609, 0.1447387452, 0.1165223479, 0.1445950742, 0.1166582730, 0.1444532032, 0.1167925657, 0.1443130991, 0.1169252545, 0.1441747300, 0.1170563671, 0.1440380647, 0.1171859307, 0.1439030727, 0.1173139718, 0.1437697243, 0.1174405160, 0.1436379910, 0.1175655884, 0.1435078445, 0.1176892134, 0.1433792576, 0.1178114148, 0.1432522037, 0.1179322159, 0.1431266568] **(27)**

#complete chaos

#period 4 above

$Orb(f(x, 2.3), x, x0, 200, 210);$

[0.4626096244, −0.01260962368, 0.4626096235, −0.01260962388, 0.4626096244, −0.01260962368, 0.4626096235, −0.01260962388, 0.4626096244, −0.01260962368, 0.4626096235] **(21)**

$Orb(f(x, 2.9), x, x0, 200, 220);$

[0.3570027476, −0.007002747341, 0.3570027476, −0.007002747341, 0.3570027476, −0.007002747341, 0.3570027476, −0.007002747341, 0.3570027476, −0.007002747341, 0.3570027476, −0.007002747341, 0.3570027476, −0.007002747341, 0.3570027476, −0.007002747341, 0.3570027476, −0.007002747341, 0.3570027476, −0.007002747341, 0.3570027476] **(22)**

$Orb(f(x, 3.38), x, x0, 200, 220);$

[0.1955204140, 0.07448737388, 0.1955187408, 0.07448868758, 0.1955171449, 0.07448994060, 0.1955156227, 0.07449113575, 0.1955141708, 0.07449227572, 0.1955127859, 0.07449336309, 0.1955114649, 0.07449440027, 0.1955102050, 0.07449538954, 0.1955090032, 0.07449633314, 0.1955078569, 0.07449723320, 0.1955067635] **(23)**

[0.1955204140, 0.07448737388, 0.1955187408, 0.07448868758, 0.1955171449, 0.07448994060, 0.1955156227, 0.07449113575, 0.1955141708, 0.07449227572, 0.1955127859, 0.07449336309, 0.1955114649, 0.07449440027, 0.1955102050, 0.07449538954, 0.1955090032, 0.07449633314, 0.1955078569, 0.07449723320, 0.1955067635] **(24)**

$Orb(f(x, 3.39), x, x0, 200, 220);$

[0.1886161542, 0.07975679964, 0.1886085204, 0.07976296217, 0.1886011711, 0.07976889524, 0.1885940955, 0.07977460754, 0.1885872833, 0.07978010736, 0.1885807246, 0.07978540267, 0.1885744099, 0.07979050114, 0.1885683300, 0.07979541017, 0.1885624761, 0.07980013683, 0.1885568397, 0.07980468798, 0.1885514128] **(25)**

$Orb(f(x, 3.42), x, x0, 200, 230);$

[0.1654530462, 0.09853305029, 0.1653764095, 0.09860061210, 0.1653010011, 0.09866711119, 0.1652267944, 0.09873257002, 0.1651537638, 0.09879701029, 0.1650818842, 0.09886045318, 0.1650111314, 0.09892291926, 0.1649414817, 0.09898442859, 0.1648729122, 0.09904500061, 0.1648054005, 0.09910465429, 0.1647389250, 0.09916340803, 0.1646734646, 0.09922127973, 0.1646089986, 0.09927828698, 0.1645455072, 0.09933444666, 0.1644829709, 0.09938977537, 0.1644213707] **(26)**

$Orb(f(x, 3.45), x, x0, 200, 230);$

[0.1451809021, 0.1161044799, 0.1450316238, 0.1162454819, 0.1448842501, 0.1163847609, **(27)**