Dynamical Models in Biology — Homework 1

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1. **Recurrence with three terms.**

$$x_n = x_{n-1} + 2x_{n-2} - x_{n-3}, \quad n \geq 3, \qquad x_0 = 1,\ x_1 = 2,\ x_2 = -1.$$

Compute step by step (I'm just plugging straight into the rule):

$$\begin{aligned} x_3 &= x_2 + 2x_1 - x_0 = (-1) + 2(2) - 1 = 2, \\ x_4 &= x_3 + 2x_2 - x_1 = 2 + 2(-1) - 2 = -2, \\ x_5 &= x_4 + 2x_3 - x_2 = (-2) + 2(2) - (-1) = 3. \end{aligned}$$

So the first six terms are

$$\boxed{(x_0, x_1, x_2, x_3, x_4, x_5) = (1,\ 2,\ -1,\ 2,\ -2,\ 3)}.$$

2. **Solve the linear homogeneous recurrence.**

$$x_n = 5x_{n-1} - 6x_{n-2}, \qquad x_0 = 0,\ x_1 = 1.$$

Characteristic equation:

$$r^2 - 5r + 6 = 0 \quad \Longrightarrow \quad (r-2)(r-3) = 0, \ \ r = 2, 3.$$

General solution (distinct roots):

$$x_n = A \cdot 2^n + B \cdot 3^n.$$

Use the initials:

$$\begin{cases} x_0 = 0: \quad A + B = 0 \ \Rightarrow \ B = -A, \\ x_1 = 1: \quad 2A + 3B = 1 \ \Rightarrow \ 2A + 3(-A) = 1 \ \Rightarrow \ -A = 1 \ \Rightarrow \ A = -1,\ B = 1. \end{cases}$$

Hence

$$\boxed{x_n = 3^n - 2^n.}$$

(Quick check: $x_0 = 1 - 1 = 0$, $x_1 = 3 - 2 = 1$, and x_n satisfies the recurrence by linearity.)

3. **Births with age-structured fertility (one-year-olds and two-year-olds fertile).**
Let c_n be the expected number of females *born at time* n. One-year-old females at time n are exactly those born at time $n-1$ (i.e., c_{n-1}), and two-year-old females at time n are those born at time $n-2$ (i.e., c_{n-2}). If p_1 is the expected number of female births per one-year-old female and p_2 per two-year-old female, then the births at time n add linearly:

$$\boxed{c_n = p_1\, c_{n-1} + p_2\, c_{n-2} \quad (n \geq 2),}$$

with given c_0, c_1.

They asked specifically for $n = 4$ in terms of c_0, c_1, p_1, p_2. Unfold a couple of steps:

$$\begin{aligned}
c_2 &= p_1 c_1 + p_2 c_0, \\
c_3 &= p_1 c_2 + p_2 c_1 = (p_1^2 + p_2) c_1 + p_1 p_2 c_0, \\
c_4 &= p_1 c_3 + p_2 c_2 = \big(p_1^3 + 2p_1 p_2\big) c_1 + \big(p_1^2 p_2 + p_2^2\big) c_0.
\end{aligned}$$

Therefore

$$\boxed{c_4 = \big(p_1^3 + 2p_1 p_2\big) c_1 \;+\; \big(p_1^2 p_2 + p_2^2\big) c_0.}$$

(That matches the usual two-age Leslie-type linear recurrence when survivals to ages 1 and 2 are taken as certain.)