Yan Dong (yd372)

1. Compute the first six terms of the sequence satisfying the recurrence equation

$$x_n = x_{n-1} + 2x_{n-2} - x_{n-3} \quad , \quad n \geq 3$$

subject to the initial conditions

$$x_0 = 1, x_1 = 2, x_2 = -1 \quad .$$

Solution:

$$\begin{aligned} x_3 &= x_2 + 2x_1 - x_0 \\ &= -1 + 2(2) - 1 \\ &= -1 + 4 - 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} x_4 &= x_3 + 2x_2 - x_1 \\ &= 2 + 2(-1) - 2 \\ &= 2 - 2 - 2 \\ &= -2 \end{aligned}$$

$$\begin{aligned} x_5 &= x_4 + 2x_3 - x_2 \\ &= -2 + 2(2) - (-1) \\ &= -2 + 4 + 1 \\ &= 3 \end{aligned}$$

so, the first six term is $x_0 = 1$, $x_1 = 2$, $x_2 = -1$,
$x_3 = 2$, $x_4 = -2$, $x_5 = 3$. ∎

2. Solve explicitly the recurrence equation

$$x_n = 5x_{n-1} - 6x_{n-2} \quad ,$$

with initial conditions

$$x_0 = 0, x_1 = 1 \quad .$$

<u>Sol</u>. $x_n = 5x_{n-1} - 6x_{n-2}$, $n \geq 2$

$x_0 = 0$, $x_1 = 1$

Assume $x_n = r^n$, $r \neq 0$

so, $r^n = 5r^{n-1} - 6r^{n-2}$

$\Rightarrow r^2 = 5r - 6$

$\Rightarrow r^2 - 5r + 6 = 0$

$(r-2)(r-3) = 0$

$r = 2$ or $r = 3$

so, general solution is $X_n = A \cdot 2^n + B \cdot 3^n$

initial condition $x_0 = 0$, $x_1 = 1$

$$\Rightarrow \begin{cases} x_0 = A \cdot 2^0 + B \cdot 3^0 = 0 \\ x_1 = A \cdot 2^1 + B \cdot 3^1 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} A + B = 0 \\ 2A + 3B = 1 \end{cases} \Rightarrow \begin{cases} A = -1 \\ B = 1 \end{cases}$$

Thus, the solution is $X_n = -2^n + 3^n$ ∎

3. (Corrected Sept. 6, 2025, thanks to Caroline Hill [who won a dollar].)

In a certain species of animals, only one-year-old, two-year-old are fertile.

The probabilities of a one-year-old, two-year-old, female to give birth to a new female are p_1, p_2, respectivley.

Assuming that there were c_0 females born at $n = 0$, c_1 females born at $n = 1$ Set up a recurrence that will enable you to find the **expected** number of females born at time n.

In terms of c_0, c_1, p_1, p_2, how many females were born at $n = 4$?

Sol: one-year-old gives birth to a new female with probability p_1

two-year-old gives birth to a new female with probability p_2,

Initial: C_0 females born at $n=0$,

C_1 females born at $n=1$.

Let f_n be the expected number of females born at time n

At time n,

- Thoes born at $n-1$ (one-year-old): expected number is f_{n-1}, expected offspring is $p_1 f_{n-1}$
- Thoes born at $n-2$ (two-year-old): expected number is f_{n-2}, expected offspring is $p_2 f_{n-2}$

Therefore, $f_n = p_1 f_{n-1} + p_2 f_{n-2}$ $\quad n \geq 2$

with initical condition $f_0 = C_0$, $f_1 = C_1$

want to find f_4 where $n=4$.

$f_2 = p_1 f_1 + p_2 f_0 = p_1 C_1 + p_2 C_0$

$$f_3 = p_1 f_2 + p_2 f_1 = p_1 (p_1 c_1 + p_2 c_0) + p_2 c_1$$

$$= p_1^2 c_1 + p_1 p_2 c_0 + p_2 c_1$$

$$f_4 = p_1 f_3 + p_2 f_2$$

$$= p_1 (p_1^2 c_1 + p_1 p_2 c_0 + p_2 c_1) + p_2 (p_1 c_1 + p_2 c_0)$$

$$= p_1^3 c_1 + p_1^2 p_2 c_0 + p_1 p_2 c_1 + p_1 p_2 c_1 + p_2^2 c_0$$

$$= p_1^3 c_1 + p_1^2 p_2 c_0 + 2 p_1 p_2 c_1 + p_2^2 c_0 \quad \blacksquare$$