

From our earlier discussions we have seen the power of the maximum principle in establishing the uniqueness, estimation, convergence theorems, and existence of solutions. It turns out that the maximum principle has extensions to second order variable coefficient elliptic and parabolic operators, and the proofs can be given by fairly elementary means. So we will present some of these generalizations, together with some applications.

Definition. $L[u] = \sum_{i,j=1}^n a_{ij}(x) \partial_{ij}^2 u(x) + \sum_{i=1}^n b_i(x) \partial_i u(x) + c(x)u(x)$ is called *elliptic at* $x \in U$ if the symmetric matrix $(a_{ij}(x))$ is positive definite at x ; L is called *elliptic in* U if it is elliptic at every $x \in U$. L is called *uniformly elliptic in* U if

$$\sup_U [\Lambda(x) \lambda^{-1}(x)] < \infty, \quad \text{in } U,$$

where $\Lambda(x)$ and $\lambda(x)$ are the largest and smallest eigenvalue of $(a_{ij}(x))$, respectively.

We will often need to assume

$$|b_i(x)| \lambda^{-1}(x) \quad \text{to be bounded in } U \text{ or in any compact subdomain of } U, \quad (1)$$

where $\lambda(x)$ is the smallest eigenvalue of $(a_{ij}(x))$.

We will often employ the summation convention, and will denote $a_{ij}(x) \partial_{ij}^2 u(x) + b_i(x) \partial_i u(x)$ by $M[u]$. Unless otherwise noted, U always stands for a bounded domain.

Theorem 1 (Weak Maximum Principle). (i) Suppose M is elliptic in U and (1) holds on any compact subset of U . Assume $u \in C^2(U) \cap C(\bar{U})$ satisfies $M[u] \geq 0$ in U . Then $\max_{\bar{U}} u = \max_{\partial U} u$.

(ii) Suppose L is elliptic in U , (1) holds on any compact subset of U and $c(x) \leq 0$. If $u \in C^2(U) \cap C(\bar{U})$ satisfies $L[u] \geq 0$ in U , then $\max_{\bar{U}} u \leq \max(\max_{\partial U} u, 0) := \max_{\partial U} u^+$.

Proof. (i) First, we assume that $M[u] > 0$ in U . Then if $\max_{\bar{U}} u > \max_{\partial U} u$, $\max_{\bar{U}} u$ must be attained at some interior point $x_0 \in U$. This implies that $\nabla u(x_0) = 0$ and $(\nabla_{ij}^2 u(x_0))$ is a non-positive definite matrix. Since $(a_{ij}(x_0))$ is assumed to be positive definite, we see that $\sum_{i,j} a_{ij}(x_0) \nabla_{ij}^2 u(x_0) \leq 0$, which implies that $M[u](x_0) \leq 0$, contradicting our assumption that $M[u] > 0$ in U . Thus we have proved $\max_{\bar{U}} u = \max_{\partial U} u$ under the assumption $M[u] > 0$ in U .

For the general case, for any compact domain $V \subset\subset U$, we will construct a function v on V such that $M[v] > 0$ in V ; and then apply the above argument to $u + \epsilon v$ on V for any $\epsilon > 0$ to conclude that

$$\max_{\bar{V}} (u + \epsilon v) = \max_{\partial V} (u + \epsilon v).$$

Since this equality holds for any $\epsilon > 0$, by sending $\epsilon \rightarrow 0$, we obtain

$$\max_{\bar{V}} u = \max_{\partial V} u.$$

Finally, if $\max_{\overline{U}} u > \max_{\partial U} u$, then we can easily construct a compact domain $V \subset\subset U$ such that $\max_{\overline{V}} u = \max_{\overline{U}} u > \max_{\partial V} u$, contradicting our argument in the paragraph above. This would conclude that $\max_{\overline{U}} u = \max_{\partial U} u$.

The construction of v can be made in the simple form of $v(x) = e^{\gamma x_1}$ for some $\gamma > 0$ large, as $M[e^{\gamma x_1}] = (a_{11}(x)\gamma^2 + b_1(x)\gamma) e^{\gamma x_1}$, and $a_{11}(x) \geq \lambda(x)$, thus

$$M[e^{\gamma x_1}] \geq \gamma \lambda(x) \left(\gamma + \frac{b_1(x)}{\lambda(x)} \right) e^{\gamma x_1} > 0$$

if γ is chosen to be larger than the bound of $|\frac{b_1(x)}{\lambda(x)}|$ on V .

For (ii), since $L[u] \geq 0$ in U , it follows that $M[u] = L[u] - c(x)u(x) \geq -c(x)u(x) \geq 0$ in the subdomain $U_+ := \{x \in U : u(x) > 0\}$. Applying the argument in (i) to u on U_+ , we have $\max_{\overline{U}_+} u = \max_{\partial U_+} u$. But $\max_{\overline{U}} u \leq \max_{\overline{U}_+} u$, and $\max_{\partial U_+} u = \max_{\partial U} u^+$, thus we have proved (ii). $\square$

Theorem 2 (Uniqueness). *Suppose L is elliptic in U , (1) holds on any compact subset of U and $c(x) \leq 0$. If $u \in C^2(U) \cap C(\overline{U})$ satisfies $L[u] = 0$ in U , $u = 0$ on ∂U , then $u \equiv 0$ in U .*

Remark 1. *The conclusion $\max_{\overline{U}} u \leq \max(\max_{\partial U} u, 0)$ in (ii) of Theorem 1 can not be improved, as one can see from the case of $U = (-1, 1)$, $u(x) = -\cosh(x)$ satisfying $u'' - u = 0$, $u(-1) = u(1) = -\cosh(1) < u(0) = \max_{\overline{U}} u$. The condition $c(x) \leq 0$ in both (ii) of Theorem 1 and Theorem 2 can not be dropped. For $U = (0, \pi)$, $u(x) = \sin x$ is a nonzero solution to $u'' + u = 0$ in U and $u(0) = u(\pi) = 0$. However, we have the following*

Theorem 3. *Suppose L is uniformly elliptic in U , (1) holds on any compact subset of U and there exists $w \in C^2(U) \cap C(\overline{U})$ satisfies $L[w] \leq 0$ in U , $w > 0$ in $\overline{U}$. Let $u \in C^2(U) \cap C(\overline{U})$ satisfy $L[u] = 0$ in U , $u = 0$ on ∂U . Then $u \equiv 0$ in U .*

Proof. Set $u(x) = w(x)v(x)$. Then $L[u] = w(x)\widetilde{M}[v] + v(x)L[w]$, where

$$\widetilde{M}[v] = a_{ij}(x)\partial_{ij}^2 v + (b_i(x) + 2a_{ij}(x)\partial_j w(x)/w(x))\partial_i v(x).$$

So $\widetilde{M}[v] + v(x)L[w]/w = 0$, with $L[w]/w \leq 0$ in U , and $v = 0$ on ∂U . We can apply (ii) of the Weak Maximum Principle to conclude $v \equiv 0$ in U . Therefore $u \equiv 0$ in U . $\square$

The maximum principle does not hold on unbounded domains without requiring some conditions on the solution's behavior at infinity. For example, if $U = \mathbb{R}_+^n$, then $u(x) = x_n$ is a harmonic function in U such that $u = 0$ on ∂U , yet u is not $\equiv 0$ in U . Some extension of the maximum principle to unbounded domains appear in the problems.

Maximum principle can also be used to estimate the solution.

Theorem 4 (Estimation). *Suppose L is elliptic in U , (1) is satisfied on U , and $c(x) \leq 0$ in U . Suppose $u \in C^2(U) \cap C(\overline{U})$ satisfies*

$$\begin{cases} L[u] = f(x), & \text{in } U, \\ u = g(x), & \text{on } \partial U. \end{cases}$$

Then

$$\max_{\bar{U}} |u| \leq \max_{\partial U} |g| + C \max_{\bar{U}} [|f(x)|/\lambda(x)],$$

where $C > 0$ depends only the diameter of U and the bound $\max_U [|b_i(x)|\lambda^{-1}(x)]$, where $\lambda(x)$ is the smallest eigenvalue of $(a_{ij}(x))$.

Proof. The key is to construct a function $v > 0$ in U satisfying $M[v] \leq -\lambda(x)$ in U . Then $w = \left(\sup_U \frac{|f(x)|}{\lambda(x)}\right) v(x) + \sup_{\partial U} |g|$ satisfies $L[w] \leq M[w] \leq -|f(x)|$ in U . So $L[w \pm u] \leq 0$ in U , and $w \pm u \geq 0$ on ∂U . By (ii) of the Weak Maximum Principle, $w \pm u \geq 0$ in U . Thus

$$|u| \leq w \leq \left(\sup_U \frac{|f(x)|}{\lambda(x)}\right) \max_U v(x) + \sup_{\partial U} |g|,$$

in U . A required v can be found in the form of $v(x) = e^{\gamma d} - e^{\gamma x_1}$ for some $\gamma > 0$ depending on $\max_U [|b_i(x)|\lambda^{-1}(x)]$, where we assume U lie in the slab $0 < x_1 < d$. $\square$

Notice that the weak maximum principle and the uniqueness statements do not require any quantitative bound on the coefficients of L , but the estimation does require quantitative bound on the coefficients of L . For some purposes the following Strong Maximum Principle is very useful.

Theorem 5 (Strong Maximum Principle). *(i) Suppose M is uniformly elliptic on any compact subset of U , (1) holds, and $u \in C^2(U) \cap C(\bar{U})$ satisfies $M[u] \geq 0$ in U . Suppose U is connected and u attains its maximum $\max_{\bar{U}} u$ at a point in U , then $u \equiv$ a constant in U .*

(ii) Suppose L is uniformly elliptic on any compact subset of U , $c(x) \leq 0$ in U and

$$|b_i(x)|/\lambda(x), \quad |c(x)|/\lambda(x) \quad \text{are bounded on any any compact subset of } U. \quad (2)$$

Suppose U is connected and $u \in C^2(U) \cap C(\bar{U})$ satisfies $L[u] \geq 0$ in U , and has a nonnegative maximum in U , then $u \equiv$ a constant in U .

(iii) Suppose L is uniformly elliptic on any compact subset of U and satisfies (2). Suppose U is connected and $u \in C^2(U) \cap C(\bar{U})$ satisfies $L[u] \geq 0$ in U . If $u \leq 0$ in U with $u(\bar{x}) = 0$ for some $\bar{x} \in U$. Then $u \equiv 0$ in U —this is no sign condition on $c(x)$ in this situation.

Remark 2. *The condition in (i) that u has a nonnegative maximum in U can not be dropped, as one can see through the example $U = (-1, 1)$, $u(x) = -\cosh(x)$ satisfying $u'' - u = 0$, $u(-1) = u(1) = -\cosh(1) < u(0) = \max_{\bar{U}} u$.*

Corollary. *Suppose $u \leq v$ in a connected domain U and*

$$F(x, u, u_i, u_{jk}) \geq F(x, v, v_i, v_{jk}) \quad \text{in } U,$$

where F is of class C^1 in its argument and is elliptic everywhere, i.e.,

$$\frac{\partial F}{\partial u_{jk}}(x, z, z_i, z_{jk}) \quad \text{is positive definite for any } C^2 \text{ function } z \text{ in } U.$$

Then either $u < v$ in U or $u \equiv v$ in U .

As an application of this corollary, two minimal surfaces can never touch each other, unless they are identical. The proof of the Strong Maximum Principle depends on the Hopf boundary Lemma as given below.

Hopf Lemma. *Suppose L is uniformly elliptic in a closed ball $\bar{B}$ and satisfies (1) in B . Let $x_0 \in \partial B$ and $u \in C^2(B) \cap C(\bar{B})$ satisfies $u(x) < u(x_0)$ for all $x \in B$.*

(i) If $M[u] \geq 0$ in B , then

$$\frac{\partial u}{\partial \nu}(x_0) > 0, \quad \text{in the sense} \quad \liminf_{\epsilon \rightarrow 0+} \frac{u(x_0) - u(x_0 - \epsilon \nu)}{\epsilon} > 0. \quad (3)$$

(ii) If $L[u] \geq 0$ in B , $c(x) \leq 0$ in B and satisfies (2), and $u(x_0) \geq 0$, then (3) also holds; furthermore, if $u(x_0) = 0$, then (3) continues to hold regardless of the sign on $c(x)$.

The Strong Maximum Principle and the Hopf Lemma give the uniqueness to the Neumann boundary value problem.

Theorem 6. *Assume L is uniformly elliptic in U and satisfies (2), and $c(x) \leq 0$ in U . Assume U is connected and ∂U is C^2 , and $u \in C^2(U) \cap C^1(\bar{U})$ satisfies*

$$\begin{cases} L[u] = 0, & \text{in } U, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial U. \end{cases}$$

Then $u \equiv a$ constant in U (in fact $u \equiv 0$ unless $c(x) \equiv 0$).

Proof. Suppose u is not identically a constant, then by considering $-u$ if necessary, we may assume that $\max_{\bar{U}} u > 0$. By (ii) of Strong Maximum Principle, $\max_{\bar{U}} u$ can not be attained in the interior of U , thus must be attained at a boundary point x_0 , and $u(x) < u(x_0)$ for all $x \in U$. But by Hopf Lemma, this would imply $\frac{\partial u}{\partial \nu}(x_0) > 0$, contradicting the boundary condition at x_0 . Therefore, u must be a constant in U . $\square$

Remark 3. *In the simple case $L[u] = \Delta u + c(x)u$, one can prove the uniqueness (up to a constant) to the Neumann boundary value problem by the energy method. But that method is not very suitable for general variable coefficient case.*

Proof of Strong Maximum Principle. The set $V = \{x \in U : u(x) = \max_{\bar{U}} u\}$ is (relatively) closed in U . We will prove that, under the assumptions for the Strong Maximum Principle, it is also open, therefore conclude that $V = U$, and $u \equiv a$ constant in U . Suppose V is not open, then there exists a point $\bar{x} \in V$, and a sequence of points $x_i \in U$ such that $x_i \rightarrow \bar{x}$ as $i \rightarrow \infty$, and $u(x_i) < u(\bar{x})$. For i sufficiently large, the distance from x_i to ∂U is obviously greater than its distance to V , thus we can construct a ball B centered at x_i such that $\bar{B} \subset U$ and $\bar{B} \cap V$ is a non empty subset of ∂B . We can now apply the appropriate form of the Hopf lemma on a perhaps smaller ball B' tangent to B and $\bar{B}' \cap V = \{x'\}$ to conclude that $\frac{\partial u}{\partial \nu}(x') \neq 0$. But x' is an interior maximum point of u , so we are supposed to have $\nabla u(x') = 0$. This contradiction shows that the Strong Maximum Principle holds. $\square$

Proof of the Hopf Lemma. The key idea is to construct a function v on the annulus region $A := B \setminus B'$, where B' is a strictly smaller concentric ball to B , satisfying

$$\begin{cases} L[v] \geq 0, & \text{in } A, \\ v = 0, & \text{on } \partial B, \\ \frac{\partial v}{\partial \nu} < 0, & \text{on } \partial B, \end{cases}$$

Then for $\epsilon > 0$ small, $\max_{\partial B'}(u + \epsilon v) \leq u(x_0) = \max_{\partial B}(u + \epsilon v)$, and

$$\begin{cases} L[u + \epsilon v] \geq 0, & \text{in } A, \\ u + \epsilon v \leq u(x_0), & \text{on } \partial A. \end{cases}$$

By the weak maximum principle, the maximum of $u + \epsilon v$ in A must be attained on its boundary, and in fact, $x_0 \in \partial B$ must be a maximum point. Thus $\frac{\partial(u + \epsilon v)}{\partial \nu}(x_0) \geq 0$. It follows now $\frac{\partial u}{\partial \nu}(x_0) \geq -\epsilon \frac{\partial v}{\partial \nu}(x_0) > 0$. A choice of v satisfying all the requirements can be found in the form of $v = e^{-\alpha|x|^2} - e^{-\alpha R^2}$ for sufficiently large α , if $B = B_R(0)$. $\square$

All of these maximum principles have their counterparts for second order parabolic operators. Many of such extensions have less strict requirements on the coefficients of the operator.

Definition. When $L = a_{ij}(x, t)\partial_{x_i}\partial_{x_j} + b_i(x, t)\partial_{x_i} + c(x, t)$ is elliptic (uniformly elliptic), we say $\partial_t - L$ is parabolic (uniformly parabolic).

For considerations in parabolic problems, it is often convenient to consider domains of the form $U_T = U \times (0, T]$ in spacetime. The *parabolic boundary* of U_T is defined to be $\partial'U_T = \partial_t U_T \cup \partial_x U_T$, where $\partial_t U_T = \{(x, 0) : x \in \overline{U}\}$, and $\partial_x U_T = \{(x, t) : x \in \partial U, 0 < t \leq T\}$. Because solutions to parabolic equations have different degrees of differentiability in t and x , we define $C^{2,1}(U_T)$ to consist of those functions $u(x, t)$ that have continuous derivatives in x up to order 2 and continuous derivative u_t in U_T .

Theorem 7 (Weak Maximum Principle for Parabolic Operators). (i) Suppose $\partial_t - L$ is parabolic in U_T and satisfies

$$c(x, t) \leq \gamma, \quad \text{in } U_T. \quad (4)$$

Suppose $u \in C^{2,1}(U_T) \cap C(\overline{U}_T)$ satisfies

$$\begin{cases} (\partial_t - L)[u] \leq 0, & \text{in } U_T, \\ u \leq 0, & \text{on } \partial'U_T. \end{cases}$$

Then $u \leq 0$ in U_T .

(ii) Suppose $\partial_t - L$ is parabolic in U_T and $c(x) \leq 0$. If $u \in C^{2,1}(U_T) \cap C(\overline{U}_T)$ satisfies $(\partial_t - L)[u] \leq 0$ in U_T , then $\max_{\overline{U}_T} u \leq \max(\max_{\partial'U_T} u, 0) := \max_{\partial'U_T} u^+$.

Note that (i) above does not require the nonpositive sign condition on $c(x)$. As a consequence, neither does the uniqueness to the mixed Dirichlet-Cauchy problem require the sign condition on $c(x)$. The reason is because if we introduce a new variable $v(x, t) = e^{-\gamma t}u(x, t)$, then

$$\begin{cases} \partial_t v - (L - \gamma)v = 0, & \text{in } U_T, \\ v = 0, & \text{on } \partial' U_T. \end{cases}$$

$L - \gamma$ would have nonpositive coefficient in front of v , so we can apply maximum principle on v .

Proof of (i). Because of the above reduction, we may assume $\gamma = 0$ in (4). For any $\epsilon > 0$, we note that $(\partial_t - L)[u - \epsilon t] = -\epsilon + (\partial_t - L)[u] + \epsilon c(x, t)t < 0$ in U_T , so $u - \epsilon t$ can not take a positive maximum in U_T . Therefore $u - \epsilon t \leq 0$ in U_T . Since $\epsilon > 0$ is arbitrary, we conclude that $u \leq 0$ in U_T . $\square$

Proof of (ii). $(\partial_t - L)[u - \max_{\partial' U_T} u^+] \leq c(x, t) \max_{\partial' U_T} u^+ \leq 0$, and $u - \max_{\partial' U_T} u^+ \leq 0$ on $\partial' U_T$. So $u - \max_{\partial' U_T} u^+ \leq 0$ in U_T by (i). $\square$

Theorem 8 (Uniqueness). *Suppose $\partial_t - L$ is parabolic in U_T and satisfies (4). Suppose $u \in C^{2,1}(U_T) \cap C(\overline{U}_T)$ satisfies*

$$\begin{cases} (\partial_t - L)[u] = 0, & \text{in } U_T, \\ u = 0, & \text{on } \partial' U_T. \end{cases}$$

Then $u \equiv 0$ in U_T .

Proof. We can apply the weak maximum principle to u and $-u$ to conclude that $u \equiv 0$ in U_T . $\square$

Estimation on the solution to parabolic equation also follows routinely.

Theorem 9 (Estimation). *Suppose $\partial_t - L$ is parabolic in U_T and satisfies (4). Suppose $u \in C^{2,1}(U_T) \cap C(\overline{U}_T)$ satisfies*

$$\begin{cases} (\partial_t - L)[u] = f(x, t), & \text{in } U_T, \\ u = g(x, t), & \text{on } \partial' U_T. \end{cases}$$

Then

$$\max_{U_T} |u| \leq e^{\gamma T} \left[T \max_{U_T} |f| + \max_{\partial' U_T} |g| \right]. \quad (5)$$

Proof. By our trick above, we may work with $v(x, t) = e^{-\gamma t}u(x, t)$ to get $(\partial_t - [L - \gamma])v = e^{-\gamma t}f(x, t)$. Note that $(\partial_t - [L - \gamma])[v - t \max_{U_T} |f| - \max_{\partial' U_T} |g|] \leq 0$ in U_T . Thus by the maximum principle,

$$v \leq t \max_{U_T} |f| + \max_{\partial' U_T} |g|, \quad \text{in } U_T.$$

Similarly

$$-v \leq t \max_{U_T} |f| + \max_{\partial' U_T} |g|, \quad \text{in } U_T.$$

Thus (5) holds. $\square$

Uniqueness to the mixed Dirichlet-Cauchy problem for fully nonlinear parabolic equations follow in a similar way.

Theorem 10. *Suppose $F = F(x, t, u, u_{x_i}, u_{x_i x_j})$ is of class C^1 in its argument and is elliptic everywhere. Then there exists at most one solution u in the class $C^{2,1}(U_T) \cap C(\bar{U}_T)$ to*

$$\begin{cases} \partial_t u - F(x, t, u, u_{x_i}, u_{x_i x_j}) = 0, & \text{in } U_T, \\ u = g(x, t), & \text{on } \partial' U_T. \end{cases}$$

Proof. Suppose u_1 and u_2 are two solutions. Then $v = u_1 - u_2$ satisfies a linear parabolic equation in U_T with zero boundary data on $\partial' U_T$. By the uniqueness for the linear problem, $v \equiv 0$ in U_T . $\square$

There are also versions of the strong maximum principle and Hopf boundary point lemma. These are useful for proving the uniqueness to the mixed Neumann-Cauchy problems. They are formulated and proved in similar ways as for the elliptic versions. We will omit the details.

Problems.

- (1) Prove that under the assumption that U satisfies the interior sphere condition, and that $c(x) \leq 0$ for $x \in U$ and $\alpha(x) \geq 0$ for $x \in \partial U$, there exists at most one solution (up to a constant) u to

$$\begin{cases} \Delta u + c(x)u = f, & \text{in } U, \\ \frac{\partial u}{\partial \nu} + \alpha(x)u(x) = g(x), & \text{on } \partial U, \end{cases}$$

in the class $C^2(U) \cap C^1(\bar{U})$. Give an example of the failure of the uniqueness when the condition on $c(x)$ or α is not satisfied.

- (2) Suppose that 0 is an interior point of the domain U in $\mathbb{R}^n$, $n \geq 3$, and $u(x)$ is a nonnegative harmonic function on $U \setminus \{0\}$. Prove that there exists a constant $A \geq 0$ and a smooth harmonic function $h(x)$ in U such that

$$u(x) = A|x|^{2-n} + h(x), \quad \text{for all } x \in U.$$

(Hint: Let $\bar{u}(r)$ denote the average of u over the sphere $|x| = r$. First establish that $\bar{u}''(r) + \frac{n-1}{r}\bar{u}'(r) = 0$ for small $r > 0$. Thus $\bar{u}(r) = Ar^{2-n} + B$ for some $A \geq 0$ and B . Next try to use Harnack/Green's identity or Maximum principle.)

- (3) Suppose U is a bounded domain and $x_0 \in \partial U$. Let $u \in C(\bar{U} \setminus \{x_0\})$ be a bounded harmonic function in U such that $u \equiv 0$ on $\partial U \setminus \{x_0\}$. Prove that $u \equiv 0$ in U .

- (4) Suppose U is a bounded domain in $\mathbb{R}^2$ with C^1 boundary, g is a C^0 function on ∂U that is locally Hölder at $x_0 \in \partial U$: $|g(x) - g(x_0)| \leq A|x - x_0|^\alpha$ for $x \in \partial U$ in a neighborhood of x_0 and some $0 < \alpha < 1$, $A > 0$. Let u be the harmonic function in U with g as boundary value. Prove that u is locally Hölder at x_0 , i.e., for some $B > 0$, $|u(x) - u(x_0)| \leq B|x - x_0|^\alpha$ for $x \in U$ in a neighborhood of x_0 . (Hint: Try to modify the construction of the barrier function in the barrier argument in the form of $r^\beta f(\theta)$, where $r = |x - x_0|$, and θ is the polar angle with respect to x_0 .)
- (5) (a). Let u be a bounded harmonic function on $U = \{x = (x', x_n) : 0 < x_n < h\}$. Prove that

$$\sup_{\bar{U}} |u| = \sup_{\partial U} |u|.$$

- (b). Let u be a bounded harmonic function on $\mathbb{R}_+^n = \{x = (x', x_n) : x_n > 0\}$. Prove that

$$\sup_{\mathbb{R}_+^n} |u| = \sup_{\partial \mathbb{R}_+^n} |u|.$$

- (6) Let B^+ denote the half disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1, y > 0\}$. Suppose $u \in C^2(B^+) \cap C(\bar{B}^+)$ is a solution to

$$\begin{cases} \partial_x^2 u + y \partial_y^2 u + c(x, y)u = f(x, y), & \text{in } B^+, \\ u(x, y) = g(x, y), & \text{on } \partial B^+. \end{cases} \quad (*)$$

- (a) There is at most one solution to (*) under the assumption $c(x, y) \leq 0$.
(b) Assume $-c_0 \leq c(x, y) \leq 0$ in B^+ . Then there exists a constant $C > 0$ depending only on c_0 such that for any solution u to (*)

$$\max_{B^+} |u| \leq C \left[\max_{B^+} |f| + \max_{\partial B^+} |g| \right].$$

- (7) Suppose $\partial_t - L$ is parabolic in U_T and satisfies (4), and $u \in C^{2,1}(U_T) \cap C(\bar{U}_T)$ satisfies

$$\begin{cases} (\partial_t - L)[u] = f, & \text{in } U_T, \\ u = 0, & \text{on } \partial' U_T. \end{cases}$$

Suppose f and the coefficients of L are independent of t , and $f \geq 0$ in U_T . Prove that $u_t \geq 0$ in U_T .

- (8) Suppose f is a locally Lipschitz function and $u, v \in C^{2,1}(U_T) \cap C(\bar{U}_T)$ satisfy

$$\begin{cases} u_t - \Delta u - f(u) \geq v_t - \Delta v - f(v), & \text{in } U_T, \\ u(x, 0) \geq v(x, 0), & \text{for } x \in U, \\ u(x, t) \geq v(x, t), & \text{for } x \in \partial U \text{ and } 0 < t < T. \end{cases}$$

Prove that $u(x, t) \geq v(x, t)$ in U_T .

- (9) (a) (Maximum principle for boundary value problem of the heat equation with Neumann or Robin type boundary condition.) Suppose $u \in C^{2,1}(U_T) \cap C(\overline{U}_T)$ satisfies

$$\begin{cases} u_t(x, t) - \Delta u(x, t) \geq 0, & \text{for } (x, t) \in U_T, \\ u(x, 0) \geq 0, & \text{for } x \in U, \\ \frac{\partial u(x, t)}{\partial \nu(x)} + h(x, t)u(x, t) \geq 0, & \text{for } (x, t) \in \partial U \times (0, T], \end{cases}$$

where U is a bounded convex domain with C^1 boundary (you may take U to be a bounded interval in $\mathbb{R}^1$) and $h(x, t) \geq 0$ for $(x, t) \in \partial U \times (0, T]$. Then $u(x, t) \geq 0$ in U_T .

- (b) Under the same assumptions on U and $h(x, t)$, prove that if $u \in C^{2,1}(U_T) \cap C(\overline{U}_T)$ satisfies

$$\begin{cases} u_t(x, t) - \Delta u(x, t) = f(x, t), & \text{for } (x, t) \in U_T, \\ u(x, 0) = g(x), & \text{for } x \in U, \\ \frac{\partial u(x, t)}{\partial \nu(x)} + h(x, t)u(x, t) = b(x, t), & \text{for } (x, t) \in \partial U \times (0, T]. \end{cases}$$

Then

$$\max_{U_T} |u| \leq C \left[\max_{U_T} |f| + \max_{\overline{U}} |g| + \max_{\partial U \times [0, T]} |b| \right].$$

where C depends only on U and T .

- (10) Consider the parabolic operator $L[u] = u_t - \sum_{i,j=1}^n a_{ij}(x, t) \partial_{x_i x_j}^2 u(x, t)$ in $Q_r := \{(x, t) : |x| < r, 0 < t < r^2\}$, where we assume that for some $0 < \lambda \leq \Lambda$, $\lambda I \leq (a_{ij}(x, t)) \leq \Lambda I$ for all $(x, t) \in Q_r$. Assume, in addition, that $a_{ij}(x, t) \in C_x^1(Q_r)$, and there exists $M > 0$ such that

$$\frac{r |\partial_x a_{ij}(x, t)|}{\lambda} \leq M$$

for all $(x, t) \in Q_r$. Suppose that $u(x, t)$ is a solution to $L[u] = 0$ in Q_r and $\partial_x^3 u, \partial_{xt}^2 u \in C(Q_r)$. Modify Bernstein's method to prove that there exists some $A > 0$ depending on M and Λ/λ , such that

$$\max\{|\partial_x u(x, t)| : |x| \leq r/2, \frac{3}{4}r^2 \leq t \leq r^2\} \leq \frac{A}{r} \max_{\overline{Q_r}} |u(x, t)|.$$