MATH435: GEOMETRY NOTES ON THE PARALLEL POSTULATE

We now examine the issue whether Euclid's Postulate 5 can actually be deduced from the rest of his Postulates. If this could be done, there would be no need to make Postulate 5 an axiom: in a geometry which upholds the first four Postulates, all the propositions given in the "Elements" would be true. It was the attempts to understand this postulate that led to the invention(discovery?) of non-Euclidean geometry, which then stimulated a lot study on the foundation of Euclidean geometry.

In one line of attempts to prove Euclid's Postulate 5, some authors tried to build up properties (of triangles, etc), using only Euclid's first four Postulates and their consequences, which, they hope, would lead to a proof of Postulate 5. A geometry is called a neutral geometry, if in it all of Euclid's Postulates except Postulate 5 is assumed. This is a tentative definition for now, as we saw, even to have a satisfactory theory to prove Euclid's first 28 propositions, more axioms need to be assumed. A complete remedy was first proposed by D. Hilbert, a prominent mathematician around the turn of the 20th century. But we will not have time to go through his theory in detail.

A great number of Euclidean geometry theorems are valid in a neutral geometry. For, as long as a proof of an Euclidean theorem does not depend on Postulate 5 or its equivalent version in either way, but depends only on the rest of the axioms, that theorem should be valid in a neutral geometry. For instance, the first 28 propositions in Euclid are valid in neutral geometry. In particular, Prop. I.27, I.28 and I.31 hold in neutral geometry.

In the following we will describe a few more properties in neutral geometry (Lambert quadrilaterals and the **Saccheri-Legendre Theorem**; properties of Saccheri quadrilaterals have appeared in the last set of homework). Then we will study what could happen if we impose some additional properties in neutral geometry, see **Propositions N1** and **N2**. Finally we will study properties of a geometry in which the negation of Euclid's Postulate 5 holds, namely, **hyberbolic geometry**. Our approach is synthetic. Chapter 6 of the text by Brannan et al contains an analytic approach to hyperbolic geometry. In particular, it establishes an interpretation and a model for hyberbolic geometry (so that it is not just a possible existence in our mental exercise). Chapter 6 also gives more detailed properties of hyperbolic geometry.

A *parallelogram* is a quadrilateral such that the opposite sides are parallel. It is easy to show in neutral geometry that parallelograms exist. But the familiar properties of parallelograms (as in Prop. 33, 34) depend on the Euclidean parallel postulate. For instance, starting with a straight line segment AB, construct line segments $AC \perp AB$, and $BD \perp AB$, and C, D stay on the same side of AB such that $AC \perp CD$. All these steps can be justified by I.11. Such a quadrilateral is called a Lambert quadrilateral, because Lambert tried to prove Postulate 5 by studying properties of such a parallelogram. Note that $ABDC$ is a parallelogram by I.27. But it can not be proved in neutral geometry that $\angle D$ is a right angle, or $|AC| = |BD|$. It can be proved, however, that $\angle D$ is not an obtuse angle. This is based on the following peculiar theorem in neutral geometry: (**Saccheri-Legendre**) *The sum of the three interior angles of a triangle in a neutral geometry is not more than two right angles.* This is how much we can say if we only assume the axioms in neutral geometry.

If we identify two right angels as having measure π, then we can define the *defect* of any triangle (in a neutral geometry) to be π minus the angle sum. It is very easy to convince yourself that for any point D between A and B of a triangle ΔABC, defect(ΔABC) = defect(ΔADC) + defect(ΔDBC).

Exercise 1. Prove the above statement.

With this property, we can easily prove

Fact 1. *If the angle sum of a triangle is π, then any triangle contained in this triangle has angle sum equal to π too.*

In fact, a stronger result holds.

Proposition N1. *In a neutral geometry, if the angle sum of a triangle is π, then the angle sum of any triangle is equal to π.*

Note that Fact 1 and Proposition N1, and the following Proposition, are properties of a neutral geometry, when some additional information (such as information on a particular triangle or on a particular line) is assumed. We postpone the proof of Proposition N1 and first state

Proposition N2. In a neutral geometry, if there is one triangle with angle sum equal to π, then Playfair's parallel axiom holds in this neutral geometry.

Proof. Let l be a straight line and P a point not on l. By I.12 we can drop perpendicular PQ to l with Q being the foot. Construct line m through P parallel to l by I.31. Let n be any other line through P. We need to prove that n meets l. Let X be a point on n on the same side of m as Q. Then $\angle QPX$ is less than $\frac{\pi}{2}$. If we can prove the existence of a point S on l which is on the same side of PQ as X and $\angle QPS > \angle QPX$, then X is contained in the interior of $\angle QPS$ and n has to meet l, if produced along PX. Here is how one proves the existence of such an S: Pick a point S_0 on l on the same side of PQ as X. Extend QS_0 along l so that $S_0S_1 = S_0P$. Then triangle PS_0S_1 is isosceles, and by I.5, $\angle S_0S_1P = \angle S_0PS_1$. Under our assumption (that there is one triangle with angle sum equal to π), it follows from Proposition N1 that the angle sum of triangle PS_0S_1 is π, thus $\angle QS_0P = \angle QS_1P + \angle S_0PS_1 = 2\angle QS_1P$, and $\angle QS_1P = \frac{1}{2}\angle QS_0P$. Repeating this procedure, if necessary, we arrive at a point S_k on l such that $\angle QS_kP = \frac{1}{2^k}\angle QS_0P$. However, using Proposition N1 again, $\angle QS_kP + \angle QPS_k = \frac{\pi}{2}$. If k is large enough to make $\frac{1}{2^k}\angle QS_0P < \frac{\pi}{2} - \angle QPX$, then $\angle QPS_k > \angle QPX$, making X in the interior of triangle PQS_k, therefore, n, along the extension of PX, will have to meet that part of l between Q and S_k somewhere.

We now come to discuss some details of a proof of Proposition N1. It can be best understood as consisting of several steps.

Lemma 1. In a neutral geometry, if there is a triangle with angle sum equal to π, then there is a *right* triangle with angle sum equal to π.

Lemma 2. In a neutral geometry, if there is a *right* triangle with angle sum equal to π, then there is a rectangle (a quadirlateral with all its interior angles equal to right angles).

Lemma 3. In a neutral geometry, if there is a rectangle, then any triangle can be contained in a rectangle.

Lemma 4. In a neutral geometry, if a triangle is contained in a rectangle, then its angle sum is π.

Proof of Lemma 1. Let ABC be a triangle in a neutral geometry with angle sum equal to π. By Saccheri-Legendre theorem, at least two of the angles, $\angle A, \angle B, \angle C$ are acute angles, say, $\angle A$ and $\angle B$ are acute. Drop a perpendicular to AB from C, with D being the foot.

Then D has to be between A and B, for otherwise, either CAD or CBD would be a right triangle with $\angle A$ or $\angle C$ as an exterior angle. But by I.16, that would make $\angle A$ or $\angle B$ an angle greater than a right angle, which contradicts our assumption on $\angle A$ and $\angle B$. Now that D is between A and B, it follows from Exercise 1 and Saccheri-Legendre theorem that both defect(ΔADC) and defect(ΔDBC) have to be zero, making either of the right triangles ΔADC and ΔDBC have angle sum equal to π.

Proof of Lemma 2. Let ΔADC be a right triangle in a neutral geometry with angle sum equal to π and with $\angle D$ being a right angle. Construct AB perpendicular to AD with B on the same side of AD as C and with $AB = DC$. Join BC and consider ΔADC and ΔCBA. $\angle ACD + \angle CAD = \frac{\pi}{2}$, and $\angle CAD + \angle CAB = \frac{\pi}{2}$. Thus $\angle ACD = \angle CAB$. By construction $AB = DC$. Now AC is a shared side. So by I.4, ΔADC and ΔCBA are congruent, making $\angle B = \angle D =$ right angle, and $\angle BCA = \angle DAC$, so that $\angle BCD = \angle BCA + \angle DCA = \angle DAC + \angle BAC =$ right angle. Therefore $ABCD$ is a rectangle.

Proof of Lemma 3. Let $ABCD$ be a rectangle. Extend AB to E so that $BE = AB$. Extend DC to F so that $CF = DC$. Join EF. Then ΔCBA and ΔCBE are congruent by I.4, making $CA = CE$, $\angle CEA = \angle CAB$, and $\angle BCA = \angle BCE$. Since $\angle BCA + \angle DCA = \frac{\pi}{2} = \angle BCE + \angle FCE$, so $\angle DCA = \angle FCE$. Now ΔDCA and ΔFCE are congruent by I.4. Therefore, $\angle F = \angle D = \frac{\pi}{2}$, and $\angle FEC = \angle DAC$, making $\angle FEB = \angle FEC + \angle CEB = \angle DAC + \angle CBA = \angle DBA = \frac{\pi}{2}$. We have now produced a rectangle $AEFD$ which has doubled the side AB and CD. The same argument can be used to show that a rectangle can be doubled in the other direction too. Applying this procedure a finite number of times, any given triangle will be contained in a rectangle.

The proof of Lemma 4 will be left as an exercise.

Up to the beginning of the nineteenth century, the prevailing attitude of mathematicians and philosophers was that Euclidean geometry was the only possible geometry that is conceivable to describe our physical world. The numerous attempts to prove Euclid's fifth Postulate on parallel lines were not because there were doubts on the validity of Euclidean geometry, but rather were efforts to clarify the the relations between this Postulate and other Postulates in Euclidean geometry. Before the work of J. Bolyai, C. Gauss, and N. Lobachevsky, there were approaches which assumed the negation of Euclid's fifth Postulate, and tried to arrive at a contradiction. G. Saccheri and J. Lambert's work in this direction was very noteworthy. Assuming a version of the negation of Euclid's fifth Postulate, Saccheri was able to derive many results which seemed strange, but not a contradiction. However, his aim was to try to deduce a contradiction from his argument. Although not being able to deduce any contradictions, at some point, he exclaimed in frustration: "The hypothesis of the acute angle is absolutely false, because [it is]repugnant to the nature of the straight line!" In other words, he had essentially discovered some theorems of non-Euclidean geometry, but unable to believe what he discovered, he announced them to be impossible. The new steps that J. Bolyai, C. Gauss, and N. Lobachevsky were able to make were that they boldly came to the conclusion that another geometry, in which Euclid's fifth Postulate does not hold, is possible! This geometry is now called *hyperbolic geometry.* It assumed all the axioms of Euclidean geometry except the fifth Postulate. Instead, that Populate is replaced by the following (hyperbolic) axiom: *there exist a line l and a point P not on l such that at least two distinct lines parallel to l pass through P* This is a negation of Playfair's Postulate(which is equivalent to Euclid's fifth Postulate): *For every line l and every point P not on l, there exists a unique line parallel to l which passes through P.* Another possible negation of this

Postulate is that *there exist a line l and a point P not on l such that no line through P is parallel to l.* Although the round sphere satisfies this property, but it does not satisfy Euclid's Postulate 1. There is a geometry, called *elliptic geometry*, in which the above negation and Euclid's first four Postulates hold.

In hyperbolic geometry, as proposed by J. Bolyai, C. Gauss, and N. Lobachevsky, a lot of geometric statements countering our conventional thinking hold true. But we can not claim them to be impossible, or logical contradictions, because the point of an axiomatic[deductive] geometric system is to investigate what possible conclusions can be drawn from the initial Postulates, and nowhere in the initial Postulates did it specify that "points" and "lines" are what our conventional thinking perceives them to be.

We next discuss how some of the "strange" new theorems of hyperbolic geometry are derived from its basic Postulates. First, we have the following

Theorem H1. In hyperbolic geometry, all triangles have angle sum less than π.

This is the contra-positive of Proposition N2, so holds automatically after Proposition N2 is proved. But to learn the ideas of proof, it is worthwhile to outline the steps one more time.

Proof. By Proposition N1, it suffices to prove the existence of one triangle with angle sum less than π. This, we indicate through several steps.

(1) Under the hyperbolic axiom, there exists a line l and a point P not on l such that at least two parallel to l passing P exist.
(2) By two facts we stated at the beginning (for neutral geometry), we can drop the perpendicular PQ to l(Q is the foot on l) and take m to be the perpendicular to PQ at P. Them m is a parallel to l.
(3) By Step 1, there exists at least another line, say n parallel to l through P.
(4) There exist point X on n and Y on m such that X is in the interior of $\angle QPY$.
(5) For any point R on l on the same side of PQ as X and Y, R lies in the interior of $\angle QPX$. Otherwise, X would be lying in the interior of $\angle QPR$, therefore n would have to intersect l somewhere between Q and R, contradicting our choice of n.
(6) Now pick any R as in Step 5 as R_1. Extend QR_1 to R_2 such that $PR_1 \cong R_1R_2$. Then $\triangle R_2PR_2$ is isosceles, and hence $\angle R_1PR_2 \cong \angle PR_2R_1$. As a corollary of Saccheri-Legendre Theorem that (*The sum of the degree measures of the three angles in any triangle is less than or equal to π.*) $\angle R_1PR_2 + \angle PR_2R_1 \leq \angle QR_1P$. Thus $\angle PR_2R_1 = \angle PR_2Q \leq \frac{1}{2}\angle PR_1Q$. Repeating this procedure enough times, we can reach an R_k such that $\angle PR_kQ < \angle XPY$. Now $\angle PR_kQ + \angle QPR_k < \angle XPY + \angle QPR_k < \frac{\pi}{2}$, thus the angle sum of $\triangle PQR_k$ is less than π.

Theorem H2 (Universal Hyperbolic Theorem). *In hyperbolic geometry, for every line l and every point P not on l there pass through P at least two distinct parallels to l.*

Proof. Drop perpendicular PQ to l, here Q is the foot on l. Let m be the perpendicular to PQ through P. Then m is a parallel to l. Pick another point R on l and draw perpendicular t to l through R. Finally drop perpendicular PS to t from P.

Claim. the line through PS is distinct from m.

Otherwise, PS would be perpendicular to PQ, and all interior angles of the quadrilateral $PQRS$ are $\frac{\pi}{2}$. Then using argument similar to the proof in Lemma 1 or Lemma 4, it is easy to see that the angle sum of $\triangle PQR$ would have to π. This contradicts Theorem H1.

The next theorem is very different from the corresponding one in Euclidean geometry.
Theorem H3. *In hyperbolic geometry, if the triangles ABC and $A'B'C'$ are similar, then they are congruent.*

The next conclusion may look more strange to you.

Theorem H4. *In hyperbolic geometry if l and l' are two distinct parallel lines, then there can't be more than two points on l which are equidistant to l'.*

Exercises.
1. Based on Theorem H1, the sum of interior angles of any triangle in hyperbolic triangle is less than π, prove that the sum of interior angles of any quadrilaterals in hyperbolic geometry is less than 2π. Furthermore, provide a complete proof of Lemma 4 in these notes.
2. Let ΔABC be a triangle in hyperbolic geometry. Let B' be a point on AB and C' be a point on AC. Prove that it is impossible for $\angle AB'C' = \angle B$ and $\angle AC'B' = \angle C$ to hold simultaneously. (Hint: Use Problem 1.)
3. True or false question (also give your reasons or examples)

   (1) In hyperbolic geometry if $l \parallel m$ and $m \parallel n$, then $l \parallel n$.
   (2) In hyperbolic geometry some triangles have angle sum less than π and some triangles have angle sum equal to π.
   (3) In hyperbolic geometry if $m \parallel l$, then points on m are equidistant to l.
   (4) In hyperbolic geometry opposite sides of any parallelogram are congruent to each other.

Implications of Non-Euclidean geometry.
The first implication of the creation of non-Euclidean geometry was the realization that it is impossible to prove Euclid's fifth Postulate based on the rest of his Postulates. However this is based on the assumption, that the new axiomatic system adopted in hyperbolic geometry is consistent, *i.e.*, no contradictions will ever be derived from this axiomatic system. This question was considered by some of the creators of this new geometry, but was not firmly established. In fact, consideration of questions of this type led to a series of efforts to firmly establish the foundation of, not just geometry, but the whole mathematics—the relation between mathematics and reality, the deep investigation of logic and set theory, etc. You may find relevant discussion in chapters 7 and 8 of M. Greenberg's book. If time permits, we will discuss the question of the possibility of conceiving hyperbolic geometry in the visual sense.

The following are a few guided extra exercises to help you get more familiar with properties of hyperbolic geometry. First we explain some properties of the so called Saccheri quadrilaterals(in a neutral geometry), that is, a quadrilateral $ABCD$, such that $\angle A = \angle B = \frac{\pi}{2}$ and $AD = BC$. It is a routine exercise in neutral geometry to show that a). $AD \parallel BC$(Prop. I.27 or 28), b). $\Delta ABC \equiv \Delta BAD$ (SAS), c). $AC = BD$, d). $\Delta ADC \equiv \Delta BCD$ (SSS), and e). $\angle C = \angle D$. We would run into difficulty to try to prove the familiar Euclidean property $\angle C = \angle D = \frac{\pi}{2}$, without assuming Euclid's fifth postulate or one of its equivalent forms. Although we know $AD \parallel BC$, we can not deduce that $\angle C + \angle D = \pi$, or $\angle ADB = \angle CBD$

without quoting Postulate 5 or one of its equivalent forms. In fact, as we saw that without assuming Euclid's fifth postulate or one of its equivalent forms, it is impossible to prove that $\angle C = \angle D = \frac{\pi}{2}$.

Using these elementary properties of Saccheri quadrilaterals, you can easily fill in the details for a proof of Theorem H4.

EX1. Complete a proof of Theorem H4. (Hint: Let l and l' be two distinct lines in hyperbolic geometry. Suppose that there are three distinct points A, B, and C on l such that their distances to l', AA', BB', and CC' are equal. Then $ABB'A'$, $BCC'B'$, and $ACC'A'$ are all Saccheri quadrilaterals. Thus $\angle A = \angle ABB'$, $\angle C = \angle CBB'$, and $\angle A = \angle C$. Using these, prove that the sum of interior angles of $ACC'A'$ is 2π, which is a contradiction in hyperbolic geometry.

Next we discuss further properties concerning parallel lines in hyperbolic geometry.

Question: Do all parallel lines in hyperbolic geometry share a common perpendicular segment as above?

EX2. Let l and m be two parallel lines in hyperbolic geometry, such that there are two points A and B on l which are equidistant from m. Let A' be the foot of the perpendicular segment from A to m, and B' be the foot of the perpendicular segment from B to m. Let M be the midpoint of AB on l, and M' the midpoint of $A'B'$ on l'. Then MM' is a common perpendicular segment to l and m (and the only one, by our homework from last week). (Hint: First show $\Delta AA'M \cong \Delta BB'M$, using $\angle A = \angle B$ from last time. Then show that $\Delta A'M'M \cong \Delta B'M'M$.)

This problem shows that when there are two points on one of a pair of parallel lines which are equidistant to the other line, then these two parallel lines share a unique common perpendicular segment. Furthermore

EX3. Let l and m be two parallel lines in hyperbolic geometry with a common perpendicular segment MM'. Let B and C be two points on l such that B is between M and C. Let B' and C' be the feet of perpendicular segments from B, C to m, respectively. Show that $MM' < BB' < CC'$. *This is very different from the Euclidean notion that given a pair of parallel lines, then as a point moves on one of them, its distance from the other line stays constant.* (Hint: I will provide a proof for the first inequality. If, on the contrary, $MM' \geq BB'$, then either $MM' = BB'$ or $MM' > BB'$. In the first case, since $\angle M' = \angle B' = \frac{\pi}{2}$ by construction, and $MM' = BB'$ by assumption, it would imply that $MM'B'B$ would be a Saccheri quadrilateral. As a consequence, $\angle M = \angle B$. But $\angle M = \frac{\pi}{2}$ by construction. Thus we have $\angle M = \angle B = \angle B' = \angle M' = \frac{\pi}{2}$, which is impossible in hyperbolid geometry. Thus $MM' = BB'$ can not occur, and we are left with ruling out the possibility that $MM' > BB'$. If $MM' > BB'$, then we can extend $B'B$ to $B'B''$ such that $B'B'' = M'M$. Join MB''. Then $MB''B'M'$ is a Saccheri quadrilateral. Therefore $\angle MB''B' = \angle B''MM'$. But $\angle B''MM' > \angle BMM' = \frac{\pi}{2}$. This makes, $\angle B''MM'$, $\angle MB''B'$, the two top angles of Saccheri quadrilateral $MB''B'M'$ obtuse angles, which is impossible in hyperbolic geometry. Model on this proof to complete the second inequality).

This problem shows that, in hyperbolic geometry, if two lines share a common perpendicular segment, then as a point recedes on one line away from the foot of the common perpendicular segment, its distance to the other line increases (It can be shown that this distance actually diverges to infinity).

But not all parallel lines in hyperbolic geometry share a common perpendicular segment. Given, in hyperbolic geometry, a line l and a point P not on l. From a perpendicular segment PQ to l. Next construct a line m through P and perpendicular to PQ. Then $m \parallel l$, as mentioned at the beginning of this note. Since we are in hyperbolic geometry, there must be at least another line n passing through P and parallel to l. Pick a point X on m which is in the region between l and m. One can draw infinitely many lines through P which are between PQ and PX. By a continuity argument, it can be proved that there is a limiting ray PZ such that any ray between PQ and PZ will intersect l and any ray between PZ and the part of m which is on the same side of PQ as X will be parallel to l. This limiting parallel ray PZ will have the property that it *does not* share a common perpendicular segment with l.

One can go on to deduce a rich collection of results in hyperbolic geometry. But so far we can only say that such a geometry is logical possibility. How do we assure ourselves that the set of axioms in hyperbolic geometry would never give rise to any contradictions? Once questions of this nature are raised, one could equally well ask: how do we assure ourselves that the set of axioms in Euclidean geometry would never give rise to any contradictions? These kind of questions are non trivial, and are ultimately questions in theory of logic. We would be happy to be convinced that hyperbolic geometry is as consistent as Euclidean geometry. By this, I mean, if Euclidean geometry is known to be consistent, then hyperbolic geometry would also be consistent. In axiomatic method, this is called relative consistency. The question of the above mentioned relative consistency will be settled by exhibiting a model of hyperbolic geometry using Euclidean objects, *i.e.*, an appropriate interpretation of certain Euclidean objects as points, lines, etc, and an appropriate interpretation of congruence of lines, angles, etc, so that all the axioms of hyperbolic geometry hold under this interpretation.