# Lecture notes on vertex operator algebras and tensor categories

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# 1 Vertex operator algebras, modules and intertwining operators

## 1.1 Vertex operator algebras

For a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$, let $V' = \coprod_{n\in\mathbb{Z}} V_{(n)}^*$ its graded dual space and $\overline{V} = \prod_{n\in\mathbb{Z}} V_{(n)}$ be its algebraic completion. For a $\mathbb{C}$-graded vector space, we use the same notations.

**Definition 1.1.** A *grading-restricted vertex algebra* is a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$, equipped with a linear map

$$\begin{aligned} Y_V : V \otimes V &\to V[[x, x^{-1}]], \\ u \otimes v &\mapsto Y_V(u, x)v, \end{aligned}$$

or equivalently, an analytic map

$$\begin{aligned} Y_V : \mathbb{C}^\times &\to \mathrm{Hom}(V \otimes V, \overline{V}), \\ z &\mapsto Y_V(\cdot, z)\cdot : u \otimes v \mapsto Y_V(u, z)v \end{aligned}$$

called the *vertex operator map* and a *vacuum* $\mathbf{1} \in V_{(0)}$ satisfying the following axioms:

1. Axioms for the grading: (a) *Grading-restriction condition*: When $n$ is sufficiently negative, $V_{(n)} = 0$ and $\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$. (b) *$L(0)$-commutator formula*: Let $d_V : V \to V$ be defined by $d_V v = nv$ for $v \in V_{(n)}$. Then

$$[d_V, Y_V(v, x)] = x\frac{d}{dx}Y_V(v, x) + Y_V(d_V v, x)$$

   for $v \in V$.

2. Axioms for the vacuum: (a) *Identity property*: Let $1_V$ be the identity operator on $V$. Then $Y_V(\mathbf{1}, x) = 1_V$. (b) *Creation property*: For $u \in V$, $\lim_{x\to 0} Y_V(u, x)\mathbf{1}$ exists and is equal to $u$.

3. *D-derivative property* and *D-commutator formula*: Let $D : V \to V$ be the operator given by

$$Dv = \lim_{x\to 0} \frac{d}{dx} Y_V(v, x)\mathbf{1}$$

for $v \in V$. Then for $v \in V$,

$$\frac{d}{dx} Y_V(v, x) = Y_V(Dv, x) = [D, Y_V(v, x)].$$

4. *Duality*: For $u_1, u_2, v \in V$ and $v' \in V'$, the series

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle, \tag{1.1}$$
$$\langle v', Y_V(u_2, z_2)Y_V(u_1, z_1)v\rangle, \tag{1.2}$$
$$\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle, \tag{1.3}$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in $z_1$ and $z_2$ with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.

We shall also use $L_V(0)$ and $L_V(-1)$ to denote $d_V$ and $D_V$.

**Remark 1.2.** A rational function in $z_1$ and $z_2$ with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$ must be of the form

$$\frac{g(z_1, z_2)}{z_1^m z_2^n (z_1 - z_2)^l},$$

where $g(z_1, z_2)$ is a polynomial in $z_1, z_2$ and $m, n, l \in \mathbb{N}$.

**Definition 1.3.** A *quasi-vertex operator algebra* or a *Möbius vertex algebra* is a grading-restricted vertex algebra $(V, Y_V, \mathbf{1})$ together with an operator $L_V(1)$ of weight 1 on $V$ satisfying

$$\begin{aligned}[L_V(-1), L_V(1)] &= -2L_V(0),\\ [L_V(1), Y_V(v, x)] &= Y_V(L_V(1)v, x) + 2xY_V(L_V(0)v, x) + x^2 Y_V(L_V(-1)v, x)\end{aligned}$$

for $v \in V$. (Note that here we have used $L_V(0)$ and $L_V(-1)$ to denote $d_V$ and $D_V$.)

Note that in the definition above, we have used $L_V(0)$ and $L_V(-1)$ to denote $d_V$ and $D_V$.

**Definition 1.4.** Let $V_1$ and $V_2$ be grading-restricted vertex algebras. A homomorphism from $V_1$ to $V_2$ is a grading-preserving linear map $g : V_1 \to V_2$ such that $gY_{V_1}(u, x)v = Y_{V_2}(gu, x)gv$. An isomorphism from $V_1$ to $V_2$ is an invertible homomorphism from $V_1$ to $V_2$. When $V_1 = V_2 = V$, an isomorphism from $V$ to $V$ is called an automorphism of $V$.

**Definition 1.5.** Let $(V, Y_V, \mathbf{1})$ be a grading-restricted vertex algebra. A *conformal element* of $V$ is an element $\omega \in V$ satisfying the following axioms:

1. $D_V = \mathrm{Res}_x Y_V(\omega, x)$ and $d_V = \mathrm{Res}_x x Y_V(\omega, x)$ ($\mathrm{Res}_x$ being the operation of taking the coefficient of $x^{-1}$ of a Laurent series).

2. Let
$$L_V(n) = \mathrm{Res}_x x^{-n-1} Y_V(\omega, x)$$
for $n \in \mathbb{Z}$. Then there exists $c \in \mathbb{C}$ such that
$$[L_V(m), L_V(n)] = (m-n)L_V(m+n) + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$
for $m, n \in \mathbb{N}$.

A grading-restricted vertex algebra equipped with a conformal element is called a *vertex operator algebra* (or *grading-restricted conformal vertex algebra*).

**Remark 1.6.** Condition 1 in Definition 1.5 can be replaced by the following condition: There exists $c \in \mathbb{C}$ such that $Y_V(\omega, x)\omega$ is equal to $L_V(-1)\omega x^{-1} + 2\omega x^{-2} + \frac{c}{2}\mathbf{1}x^{-4}$ plus a $V$-valued power series in $x$.

We now briefly describe the examples of vertex (operator) algebras constructed from affine Lie algebras. For details, see [FZ], [LL] and [H8].

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with a symmetric invariant bilinear form $(\cdot, \cdot)$. We define the affine Lie algebra $\hat{\mathfrak{g}}$ by
$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k},$$
where $\mathbf{k}$ is central and
$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + \delta_{m+n,0} m(a, b)\mathbf{k}.$$
We write $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_- \oplus \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+$, where $\hat{\mathfrak{g}}_+$ is the span of all $a \otimes t^n$ with $a \in \mathfrak{g}$ and $n > 0$, $\hat{\mathfrak{g}}_-$ is the span of all $a \otimes t^n$ with $a \in \mathfrak{g}$ and $n < 0$, and $\hat{\mathfrak{g}}_0$ is the span of $\mathbf{k}$ and all $a \otimes t^0$ with $a \in \mathfrak{g}$.

Fix $\ell \in \mathbb{C}$. Let $\mathbb{C}_\ell$ be a copy of $\mathbb{C}$, with the structure of a module for $\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+$ by defining $a(n)1 = 0$ for all $a \in \mathfrak{g}$ and $n \geq 0$, and $\mathbf{k}1 = \ell$, where we use $a(n)$ to denote the action of $a \otimes t^n$ (and we shall use the same notation below). Now define
$$V(\ell, 0) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+)} \mathbb{C}_\ell,$$
where $U(\cdot)$ is the universal enveloping algebra (see [Hum, Section 17]). In other words, $V(\ell, 0)$ is the induced $U(\hat{g})$-module constructed from the $U(\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+)$-module $\mathbb{C}_\ell$.

Let $\mathbf{1} := 1 \otimes 1 \in V$. Then $\mathbf{k}\mathbf{1} = \ell\mathbf{1}$ and if $n \geq 0$, $a(n)\mathbf{1} = 0$. Moreover, $V(\ell, 0)$ is spanned by elements of the form $a_1(n_1) \cdots a_k(n_k)\mathbf{1}$ for $a_1, \ldots, a_k \in \mathfrak{g}$ and $n_1, \ldots, n_k \in \mathbb{Z}$. Using the Poincare-Birkhoff-Witt (PBW) Theorem (see [Hum, Section 17]), we can show that $V(\ell, 0)$ is canonically linearly isomorphic to $U(\hat{g}_-)$. In particular, $V(\ell, 0)$ is spannedby elements of the form $a_1(-n_1) \cdots a_k(-n_k)\mathbf{1}$ for $a_1, \ldots, a_k \in \mathfrak{g}$ and $n_1, \ldots, n_k \in \mathbb{Z}_+$.

For $n \in \mathbb{Z}$, we define $V_{(n)}(\ell, 0)$ to be the span of $a_1(-n_1) \cdots a_k(-n_k)\mathbf{1}$ for $n_1, \dots, n_k \in \mathbb{Z}_+$ such that $n = n_1 + \cdots + n_l$. Then $V_{(n)}(\ell, 0) = 0$ for $n < 0$, $V_{(0)}(\ell, 0) = \mathbb{C}\mathbf{1}$ and $V_{(n)}(\ell, 0)$ is finite dimensional for $n \in \mathbb{Z}$. We also have

$$V(\ell, 0) = \coprod_{n \in \mathbb{Z}} V_{(n)}(\ell, 0).$$

For $a \in \mathfrak{g}$, let

$$a(x) = \sum_{n \in \mathbb{Z}} a(n) x^{-n-1} \in (\mathrm{End}\, V(\ell, 0))[[x, x^{-1}]].$$

For $z \in \mathbb{C}^\times$, we have a linear map

$$a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} : V(\ell, 0) \to \overline{V(\ell, 0)}.$$

We can prove that for $a_1, \dots, a_k \in \mathfrak{g}$, $v \in V(\ell, 0)$ and $v' \in V(\ell, 0)'$, $\langle v', a_1(z_1) \cdots a_k(z_k) v\rangle$ is absolutely convergent to a rational function $R(\langle v', a_1(z_1) \cdots a_k(z_k) v\rangle)$ with the only possible poles at $z_1 = 0$, $z_2 = 0$ and $z_1 - z_2 = 0$.

We now define the vertex operator map. For $v' \in V(\ell, 0)'$, $v \in V(\ell, 0)$, $a_1, \dots, a_k \in \mathfrak{g}$, $n_1, \dots, n_k \in \mathbb{Z}_+$, we define

$$Y_{V(\ell,0)} : \mathbb{C}^\times \to \mathrm{Hom}(V(\ell, 0) \otimes V(\ell, 0), \overline{V(\ell, 0)})$$

by

$$\begin{aligned} &\langle v', Y_{V(\ell,0)}(a_1(-n_1) \cdots a_k(-n_k)\mathbf{1}, z) v\rangle \\ &\quad = \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{-n_1} \cdots \xi_k^{-n_k} R(\langle v', a_1(\xi_1 + z) \cdots a_k(\xi_k + z) v\rangle). \end{aligned} \tag{1.4}$$

**Theorem 1.7** (see [FZ], [LL], [H8])**.** *The triple $(V(\ell, 0), Y_{V(\ell,0)}, \mathbf{1})$ is a grading-restricted vertex algebra.*

Next we discuss the conformal element of $V(\ell, 0)$. We now assume that the invariant bilinear form on $\hat{\mathfrak{g}}$ is positive definite (for example, in the case that $\hat{\mathfrak{g}}$ is semisimple and the form is obtained from the Killing form). Let

$$\Omega = \sum_{i=1}^{\dim \mathfrak{g}} u^i u^i \in U(\mathfrak{g})$$

be the Casimir element of $\mathfrak{g}$, where $\{u^i \mid 1 \le i \le \dim \mathfrak{g}\}$ is an orthonormal basis for $\mathfrak{g}$ with respect to the form $(\cdot, \cdot)$. We also assume that that $\Omega$ acts on $\mathfrak{g}$ by a scalar $2h^\vee$, where $h^\vee \in \mathbb{C}$ is called the dual Coxeter number of $\mathfrak{g}$. This assumption is satisfied if $\mathfrak{g}$ is a simple Lie algebra.

For $\ell \neq -h^\vee$, we define $\omega_{V(\ell,0)} \in V_{(2)}(\ell, 0)$ by

$$\omega_{V(\ell,0)} = \frac{1}{2(\ell + h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}} u^i(-1) u^i(-1)\mathbf{1} = \frac{1}{2(\ell + h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}} u^i(-1)^2 \mathbf{1}.$$

**Theorem 1.8** (see [FZ], [LL], [H8]). *The quadruple* $(V(\ell,0), Y_{V(\ell,0)}, \mathbf{1}, \omega_{V(\ell,0)})$ *is a Vvertex operator algebra of the central charge* $\frac{\ell \dim \mathfrak{g}}{\ell + h^\vee}$.

The vertex operator algebra $V(\ell,0)$ is still not the vertex algebra for the Wess-Zumino-Witten model. We need to take an irreducible quotient of $V(\ell,0)$.

Consider all $\hat{\mathfrak{g}}$-submodules of $V(\ell,0)$ that do not contain $\mathbf{1}$. Let $I(\ell,0)$ to be the sum of all such $\hat{\mathfrak{g}}$-submodules. Then $I(\ell,0)$ is the maximal proper submodule of the $\hat{\mathfrak{g}}$-module $V(\ell,0)$. Let $L(\ell,0) = V(\ell,0)/I(\ell,0)$. Then as a $\hat{\mathfrak{g}}$-module, $L(\ell,0)$ is irreducible, that is, there is no $\hat{\mathfrak{g}}$-submodule of $L(\ell,0)$ that is not 0 or $L(\ell,0)$ itself.

We take the vacuum of $L(\ell,0)$ to be the coset containing the vacuum of $V(\ell,0)$. We define the vertex operator map $Y_{L(\ell,0)} : L(\ell,0) \otimes L(\ell,0) \to L(\ell,0)((x))$ by

$$Y_{L(\ell,0)}(u + I(\ell,0), x)(v + I(\ell,0)) = Y_{V(\ell,0)}(u,x)v + I(\ell,0).$$

The vacuum $\mathbf{1}_{L(\ell,0)}$ is defined to $\mathbf{1}_{V(\ell,0)} + I(\ell,0)$ and in the case $\ell + h^\vee \neq 0$, the conformal element $\omega_{L(\ell,0)}$ is defined to be $\omega_{V(\ell,0)} + I(\ell,0)$.

**Theorem 1.9** (see [FZ], [LL], [H8]). *The triple* $(L(\ell,0), Y_{L(\ell,0)}, \mathbf{1}_{L(\ell,0)}$ *is a grading-restricted vertex algebra. When* $\ell + h^\vee \neq 0$, $(L(\ell,0), Y_{L(\ell,0)}, \mathbf{1}_{L(\ell,0)}, \omega_{L(\ell,0)})$ *is a vertex operator algebra.*

The vertex operator algebra underlying the Wess-Zumino-Witten model associated to a finite-dimensional simple Lie algebra $\mathfrak{g}$ and a level $\ell \in \mathbb{Z}_+$ is exactly $L(\ell,0)$. In this case, there is an explicit formula $I(\ell,0) = U(\hat{\mathfrak{g}})e_\theta(-1)^{\ell+1}\mathbf{1}$, where $\theta$ is the highest root of $\mathfrak{g}$ and $e_\theta$ is a root vector in $\mathfrak{g}_\theta$ (see [K] and [LL]).

## 1.2 Modules

In this subsection, we introduce various notions of (generalized) $V$-modules for a grading-restricted vertex algebra, a Möbius vertex algebra or a vertex operator algebra $V$.

**Definition 1.10.** Let $V$ be a grading-restricted vertex superalgebra. A *generalized $V$-module* is a $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ equipped with a *vertex operator map*

$$\begin{aligned} Y_W : V \otimes W &\to W((x)), \\ u \otimes w &\mapsto Y_W(u,x)w \end{aligned}$$

satisfying the following axioms:

1. Axioms for the gradings: There are operators $d_W$ (also denoted by $L_W(0)$), $(d_W)_S$ (also denoted by $L_W(0)_S$) and $(d_W)_N$ (also denoted by $L_W(0)_N$) on $W$ such that $d_W = (d_W)_S + (d_W)_N$, $(d_W)_S v = nv$ for $v \in W_{[n]}$ , $(d_W)_N$ is nilpotent (for $w \in W$, there exists $K \in \mathbb{N}$ such that $((d_W)_N)^K w = 0$), and

$$[d_W, Y_W(v,x)] = x\frac{d}{dx}Y_W(v,x) + Y_W(d_V v, x)$$

for $v \in V$.

2. *Identity property*: Let $1_W$ be the identity operator on $W$. Then $Y_W(\mathbf{1}, z) = 1_W$.

3. *$L(-1)$-derivative property*: There exists $D_W : W \to W$ (also denoted by $L_W(-1)$) such that for $u \in V$,
$$\frac{d}{dz} Y_W(u, z) = Y_W(D_V u, z) = [D_W, Y_W(u, z)].$$

4. *Duality*: For $u_1, u_2 \in V$, $w \in W$ and $w' \in W'$, the series
$$\langle w', Y_W(u_1, z_1) Y_W(u_2, z_2) w\rangle,$$
$$\langle w', Y_W(u_2, z_2) Y_W(u_1, z_1) w\rangle,$$
$$\langle w', Y_W(Y_V(u_1, z_1 - z_2)u_2, z_2) w\rangle$$
are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in $z_1$ and $z_2$ with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.

A *lower-bounded generalized $V$-module* is a generalized $V$-module $(W, Y_W, d_W, D_W)$ such that $W_{[n]} = 0$ when $\Re(n)$ is sufficiently negative. A *grading-restricted generalized $V$-module* is a lower-bounded generalized $V$-module $(W, Y_W, d_W, D_W)$ such that $\dim W_{[n]} < \infty$. An *ordinary $V$-module* or simply a *$V$-module* is a grading-restricted generalized $V$-module $(W, Y_W, d_W, D_W)$ such that $(D_W)_N = 0$. When $V$ is a Möbius vertex algebra, a *generalized $V$-module* or a *lower-bounded generalized $V$-module* or *grading-restricted generalized $V$-module* or an *ordinary $V$-module* is such a $V$-module when $V$ is viewed as a grading-restricted vertex algebra with an operator $L_W(1)$ of weight 1 on $W$ such that

$$\begin{aligned}
[L_W(-1), L_W(1)] &= -2L_W(0), \\
[L_W(1), Y_W(v, x)] &= Y_W(L_V(1)v, x) + 2xY_W(L_V(0)v, x) + x^2 Y_W(L_V(-1)v, x)
\end{aligned}$$

for $v \in V$. (Note that here we have used $L_V(0)$, $L_V(-1)$, $L_W(0)$ and $L_W(-1)$ to denote $d_V$, $D_V$, $d_W$ and $D_W$.) When $V$ is a vertex operator algebra, a *generalized $V$-module* or a *lower-bounded generalized $V$-module* or *grading-restricted generalized $V$-module* or an *ordinary $V$-module* is such a $V$-module when $V$ is viewed as a grading-restricted vertex algebra such that $d_W = \mathrm{Res}_x xY_W(\omega, x)$ and $D_W = \mathrm{Res}_x Y_W(\omega, x)$.

For a finite-dimensional Lie algebra $\mathfrak{g}$ with an invariant nondegenerate bilinear form $(\cdot, \cdot)$ and complex number $\ell \neq -h^\vee$, we can construct lower-bounded generalized $V(\ell, 0)$-modules as follows:

Let $M$ be a $\mathfrak{g}$-module. Defining $a(0)m = am$, $a(n)m = 0$ and $\mathbf{k}m = \ell m$ for $a \in \mathfrak{g}$, $m \in M$ and $n > 0$. Then $M$ is a module for $\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+$. We then consider the induced $\hat{\mathfrak{g}}$-module

$$V(\ell, M) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+)} M.$$

We will often omit the tensor product symbol when writing elements of $V(\ell, M)$. For example, for $m \in M$, we write $1 \otimes m$ as $m$. Using the PBW theorem, we see that $V(\ell, M)$

is linearly isomorphic to $U(\hat{\mathfrak{g}}_-) \otimes M$. Then $V(\ell, M)$ is spanned by elements of the form $a_1(-n_1) \cdots a_k(-n_k)m$ for $a_1, \dots, a_k \in \mathfrak{g}$, $n_1, \dots, n_k \in \mathbb{Z}_+$ and $m \in M$.

Let $a_{V(\ell,M)}(x) : V(\ell, M) \to V(\ell, M)((x))$ for $a \in \mathfrak{g}$ be defined by

$$a_{V(\ell,M)}(x)w := \sum_{n \in \mathbb{Z}} a(n)wx^{-n-1}$$

for $w \in V(\ell, M)$ where $a(n)$ is action of $a \otimes t^n \in \hat{\mathfrak{g}}$ on $V(\ell, M)$. For simplicity, we shall denote $a_{V(\ell,M)}(x)$ simply by $a(z)$. We can prove that for $a_1, \dots, a_k \in \mathfrak{g}$, $w \in V(\ell, M)$ and $w' \in V(\ell, M)'$, $\langle w', a_1(z_1) \cdots a_k(z_k)w\rangle$ is absolutely convergent to a rational function $R(\langle v', a_1(z_1) \cdots a_k(z_k)v\rangle)$ with the only possible poles at $z_1 = 0$, $z_2 = 0$ and $z_1 - z_2 = 0$.

For $w' \in V(\ell, M)'$, $w \in V(\ell, M)$, $a_1, \dots, a_k \in \mathfrak{g}$, $n_1, \dots, n_k \in \mathbb{Z}_+$, we define

$$Y_{W(\ell,M)} : \mathbb{C}^\times \to \mathrm{Hom}(V(\ell, 0) \otimes W(\ell, M), \overline{W(\ell, M)})$$

by

$$\begin{aligned} &\langle w', Y_{V(\ell,M)}(a_1(-n_1) \cdots a_k(-n_k)\mathbf{1}, z)w\rangle \\ &\quad = \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{-n_1} \cdots \xi_k^{-n_k} R(\langle v', a_1(\xi_1 + z) \cdots a_k(\xi_k + z)v\rangle). \end{aligned} \tag{1.5}$$

**Theorem 1.11** (see [FZ], [LL], [H8]). *The pair $(V(\ell, M), Y_{V(\ell,M)})$ is a $V(\ell, 0)$-module.*

## 1.3 Intertwining operators

**Definition 1.12.** Let $V$ be a grading-restricted vertex algebra and $W_1, W_2, W_3$ lower-bounded generalized $V$-modules (grading-restricted generalized $V$-modules and ordinary $V$-modules are special cases). An *intertwining operator of type* $\binom{W_3}{W_1 W_2}$ is a linear map

$$\begin{aligned} \mathcal{Y} : W_1 \otimes W_2 &\to W_3\{x\}[\log x] \\ w_1 \otimes w_2 &\mapsto \mathcal{Y}(w_1, x)w_2 \end{aligned}$$

(where $W_3\{x\}[\log x]$ is the space of formal series of the form $\sum_{k=0}^{K} \sum_{n \in \mathbb{C}} a_{n,k} x^n (\log x)^k$ for $a_{n,k} \in W_3$ and $x$ and $\log x$ is formal variables such that $\frac{d}{dx} \log x = x^{-1}$) satisfying the following axioms:

1. *$L(0)$-bracket formula*: For $w_1 \in W_1$,

$$L_{W_3}(0)\mathcal{Y}(w_1, x) - \mathcal{Y}(w_1, x)L_{W_2}(0) = x\frac{d}{dx}\mathcal{Y}(w_1, x) + \mathcal{Y}(L_{W_1}(0)w_1, x).$$

2. *$L(-1)$-derivative property*: For $w_1 \in W_1$,

$$\frac{d}{dx}\mathcal{Y}(w_1, x) = \mathcal{Y}(L_{W_1}(-1)w_1, x) = L_{W_3}(-1)\mathcal{Y}(w_1, x) - \mathcal{Y}(w_1, x)L_{W_2}(-1).$$

3. *Duality with vertex operators*: For $u \in V$, $w_1 \in W_1$, and $w_2 \in W_2$, $w_3' \in W_3'$, for any single-valued branch $l(z_2)$ of the logarithm of $z_2$ in the region $z_2 \neq 0$, $0 \leq \arg z_2 \leq 2\pi$, the series

$$\begin{aligned}
&\langle w_3', Y_{W_3}(u, z_1)\mathcal{Y}(w_1, x_2)w_2\rangle\Big|_{x_2^n = e^{nl(z_2)}, \log x_2 = l(z_2)} \\
&\quad = \sum_{n\in\mathbb{C}} \langle w_3', Y_{W_3}(u, z_1)\pi_n\mathcal{Y}(w_1, x_2)w_2\rangle\Big|_{x_2^n = e^{nl(z_2)}, n\in\mathbb{C}, \log x_2 = l(z_2)}, \qquad (1.6)\\
&\langle w_3', \mathcal{Y}(w_1, x_2)Y_{W_2}(u, z_1)w_2\rangle\Big|_{x_2^n = e^{nl(z_2)}, n\in\mathbb{C}, \log x_2 = l(z_2)} \\
&\quad = \sum_{n\in\mathbb{C}} \langle w_3', \mathcal{Y}(w_1, x_2)\pi_n Y_{W_2}(u, z_1)w_2\rangle\Big|_{x_2^n = e^{nl(z_2)}, n\in\mathbb{C}, \log x_2 = l(z_2)}, \qquad (1.7)\\
&\langle w_3', \mathcal{Y}(Y_{W_1}(u, z_1 - z_2)w_1, x_2)w_2\rangle\Big|_{x_2^n = e^{nl(z_2)}, n\in\mathbb{C}, \log x_2 = l(z_2)} \\
&\quad = \sum_{n\in\mathbb{C}} \langle w_3', \mathcal{Y}(\pi_n Y_{W_1}(u, z_1 - z_2)w_1, x_2)w_2\rangle\Big|_{x_2^n = e^{nl(z_2)}, n\in\mathbb{C}, \log x_2 = l(z_2)} \qquad (1.8)
\end{aligned}$$

(where $\pi_n$ for $n \in \mathbb{C}$ is the projection from a generalized $V$-module $W$ to its homogeneous subspace $W_{[n]}$ of weight $n$) are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, and the sums can be analytically extended to a common multivalued analytic functions with the only possible poles $z_1 = 0$, $z_1 = z_2$, $z_1 = \infty$ and the only possible branch point $z_2 = 0$, $z_2 = \infty$..

Let

$$\begin{gathered}
F(\langle w_3', Y_{W_3}(u, z_1)\mathcal{Y}(w_1, z_2)w_2\rangle), \\
F(\langle w_3', \mathcal{Y}(w_1, z_2)Y_{W_2}(u, z_1)w_2\rangle), \\
F(\langle w_3', \mathcal{Y}(Y_{W_1}(u, z_1 - z_2)w_1, z_2)w_2\rangle)
\end{gathered}$$

be the multivalued function obtained by analytically extending the sums of the series (1.6), (1.7) and (1.8). Then they are of the form

$$\sum_{k=0}^{K}\sum_{i=1}^{N} \frac{g_{i,k}(z_1, z_2)}{z_1^{m_{i,k}}(z_1 - z_2)^{n_{i,k}} z_2^{p_{i,k}}} z_2^{r_{i,k}} (\log z_2)^k \qquad (1.9)$$

for polynomials $g_{i,k}(z_1, z_2)$ of $z_1$ and $z_2$, $m_{i,k}, n_{i,k}, p_{i,k} \in \mathbb{N}$, $r_i \in \mathbb{C}$ satisfying $0 \leq \Re(r_i) < 1$ for $i = 1, \ldots, N$. In the case that $w_1$, $w_2$ and $w_3'$ are homogeneous, $N$ can be taken to be 1 and $r_{1,k}$ can be taken to be $-\mathrm{wt}\, w_3' + \mathrm{wt}\, w_1 + \mathrm{wt}\, w_2$.

It is clear from the definition that the set of all intertwining operators of type $\binom{W_3}{W_1 W_2}$ is a vector space and is denoted by $\mathcal{V}_{W_1 W_2}^{W_3}$. The dimension of $\mathcal{V}_{W_1 W_2}^{W_3}$ is called fusion rules of type $\binom{W_3}{W_1 W_2}$.

# 2 Tensor categories

We review the basic concepts and properties in the theory of tensor categories in this section. The main references for this section are [J], [M], [T] and [EGNO].

## 2.1 Basic concepts in category theory

**Definition 2.1.** A *category* consists of the following data:

1. A collection of *objects*.
2. For two objects $A$ and $B$, a set $\mathrm{Hom}(A, B)$ of *morphisms from* $A$ *to* $B$.
3. For an object $A$, an *identity* $1_A \in \mathrm{Hom}(A, A)$.
4. For three objects $A, B, C$, a map

$$\begin{array}{rcl} \circ : \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) & \to & \mathrm{Hom}(A, C) \\ (f, g) & \mapsto & f \circ g \end{array}$$

   called *composition* or *multiplication*.

These data must satisfy the following axioms:

1. The composition is associative, that is, for objects $A, B, C, D$ and $f \in \mathrm{Hom}(C, D)$, $g \in \mathrm{Hom}(B, C)$, $h \in \mathrm{Hom}(A, B)$, we have $f \circ (g \circ h) = (f \circ g) \circ h$.
2. For an object $A$, the identity $1_A$ is the identity for the composition of morphisms when the morphisms involving $A$, that is, for an object $B$, $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, A)$, we have $1_A \circ g = g$ and $f \circ 1_A = f$.

We shall use $\mathcal{C}$, $\mathcal{D}$ and so on to denote categories. For a category $\mathcal{C}$, we use $\mathrm{Ob}\,\mathcal{C}$ to denote the collection of objects of $\mathcal{C}$.

**Definition 2.2.** Let $\mathcal{C}$ be a category. For any $A, B \in \mathrm{Ob}\,\mathcal{C}$, an element $f \in \mathrm{Hom}(A, B)$ is called an isomorphism if there exists $f^{-1} \in \mathrm{Hom}(B, A)$ such that $f \circ f^{-1} = 1_B$ and $f^{-1} \circ f = 1_A$.

**Definition 2.3.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant functor* (or a *contravariant functor*) from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data:

1. A map $\mathcal{F}$ from the collection $\mathrm{Ob}\,\mathcal{C}$ of objects of $\mathcal{C}$ to the collection $\mathrm{Ob}\,\mathcal{D}$ of objects of $\mathcal{D}$.
2. Given objects $A$ and $B$ of $\mathcal{C}$, a map, still denoted by $\mathcal{F}$, from $\mathrm{Hom}(A, B)$ to $\mathrm{Hom}(\mathcal{F}(A), \mathcal{F}(B))$ (or from $\mathrm{Hom}(A, B)$ to $\mathrm{Hom}(\mathcal{F}(B), \mathcal{F}(A))$ for a contravariant functor).

These data must satisfy the following axioms:

1. For objects $A, B, C$ of $\mathcal{C}$ and morphisms $f \in \operatorname{Hom}(B, c)$, $g \in \operatorname{Hom}(A, B)$, we have
$$\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$$
(or
$$\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f)$$
for a contravariant functor).

2. For an object $A$ of $\mathcal{C}$, $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$.

We shall denote the functor defined above by $\mathcal{F}$.

**Definition 2.4.** Let $\mathcal{F}$ and $\mathcal{G}$ be functors from $\mathcal{C}$ to $\mathcal{D}$. A *natural transformation $\eta$ from $\mathcal{F}$ to $\mathcal{G}$* consists of an element $\eta_A \in \operatorname{Hom}(\mathcal{F}(A), \mathcal{G}(A))$ for each object $A \in \operatorname{Ob} \mathcal{C}$ such that the following diagram is commutative for $A, B \in \operatorname{Ob} \mathcal{C}$ and $f, g \in \operatorname{Hom}(A, B)$:

$$\begin{array}{ccc}
\mathcal{F}(A) & \xrightarrow{\eta_A} & \mathcal{G}(A) \\
{\scriptstyle \mathcal{F}(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(g)} \\
\mathcal{F}(B) & \xrightarrow[\eta_B]{} & \mathcal{G}(B).
\end{array}$$

A *natural isomorphism* from $\mathcal{C}$ to $\mathcal{D}$ is a natural transformation $\eta$ from $\mathcal{C}$ to $\mathcal{D}$ such that $\eta_A \in \operatorname{Hom}(\mathcal{F}(A), \mathcal{G}(A))$ for each object $A \in \operatorname{Ob} \mathcal{C}$ is an isomorphism.

**Definition 2.5.** Let $\mathcal{F}$ be a functor from a category $\mathcal{C}$ to a category $\mathcal{D}$ and $\mathcal{G}$ a functor from a category $\mathcal{D}$ to a category $\mathcal{E}$. The *composition* $\mathcal{G} \circ \mathcal{F}$ of $\mathcal{G}$ and $\mathcal{F}$ is a functor from $\mathcal{C}$ to $\mathcal{E}$ given by $(\mathcal{G} \circ \mathcal{F})(A) = \mathcal{G}(\mathcal{F}(A))$ for $A \in \operatorname{Ob} \mathcal{C}$ and $(\mathcal{G} \circ \mathcal{F})(f) = \mathcal{G}(\mathcal{F}(f))$ for $f \in \operatorname{Hom}(A, B)$ and $A, B \in \operatorname{Ob} \mathcal{C}$. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We say that $\mathcal{C}$ *is isomorphic to* $\mathcal{D}$ if there is a functor $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$ and a functor $\mathcal{F}^{-1}$ such that $\mathcal{F} \circ \mathcal{F}^{-1} = 1_{\mathcal{D}}$ and $\mathcal{F}^{-1} \circ \mathcal{F} = 1_{\mathcal{C}}$. We say that $\mathcal{C}$ *is equivalent to* $\mathcal{D}$ if there is a functor $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$ and a functor $\mathcal{G}$ such that $\mathcal{F} \circ \mathcal{G}$ is naturally isomorphic to $1_{\mathcal{D}}$ and $\mathcal{G} \circ \mathcal{F}$ is naturally isomorphic to $1_{\mathcal{C}}$.

**Definition 2.6.** Let $A_j$ for $j \in I$ be objects of a category $\mathcal{C}$. A *product* of $A_j$ for $j \in I$ is an object $\prod_{j \in I} A_j$ together with morphisms $p_j : \prod_{j \in I} A_j \to A_j$ satisfying the following universal property: For any object $A$ of $\mathcal{C}$ and any morphism $f_j : A \to A_i$, there exists a unique morphism $f : A \to \prod_{j \in I} A_j$ such that such that $f_j = p_j \circ f$ for $i \in I$. A *coproduct* of $A_j$ for $j \in J$ is an object $\coprod_{j \in I} A_j$ together with morphisms $i_j : A_j \to \prod_{j \in I} A_j$ satisfying the following universal property: For any object $A$ of $\mathcal{C}$ and any morphism $f_j : A_j \to A$, there exists a unique morphism $f : \prod_{j \in I} A_i \to A$ such that $f_j = f \circ i_j$ for $i \in I$.

**Exercise 2.7.** Prove that products and coproducts of objects $A_j$ for $j \in I$ in a category $\mathcal{C}$ are unique up to isomorphisms.

**Definition 2.8.** An *initial object* in a category $\mathcal{C}$ is an object $I$ in $\mathcal{C}$ such that for any object $X$ in $\mathcal{C}$, $\operatorname{Hom}(I, X)$ has one and only one element. An *terminal object* in a category $\mathcal{C}$ is an object $T$ in $\mathcal{C}$ such that for any object $X$ in $\mathcal{C}$, $\operatorname{Hom}(X, T)$ has one and only one element. A *zero object* in a category $\mathcal{C}$ is both an initial object and a terminal object.

**Definition 2.9.** Let $\mathcal{C}$ be a category containing a zero object 0. Let $A$ and $B$ be objects of $\mathcal{C}$ and let $f \in \mathrm{Hom}(A, B)$. A *kernel* of $f$ is an object $K$ and a morphism $k \in \mathrm{Hom}(K, A)$ satisfying $f \circ k = 0$ and the following universal property: For any object $K'$ and morphism $k' \in \mathrm{Hom}(K', A)$ satisfying $f \circ k' = 0$, there exists a unique $g \in \mathrm{Hom}(K', K)$ such that $k' = k \circ g$. A *cokernel* of $f$ is an object $Q$ and a morphism $q \in \mathrm{Hom}(B, Q)$ satisfying $q \circ f = 0$ and the following universal property: For any object $Q'$ and morphism $q' \in \mathrm{Hom}(B, Q')$ satisfying $q' \circ f = 0$, there exists a unique $u \in \mathrm{Hom}(Q, Q')$ such that $q' = u \circ q$.

**Exercise 2.10.** Prove that kernels and cokernels of of a morphism are unique up to isomorphisms.

**Definition 2.11.** Let $\mathcal{C}$ be a category containing a zero object 0. Let $A_1, \ldots, A_n$ be objects of $\mathcal{C}$. A *biproduct* of $A_1, \ldots, A_n$ is an object $A_1 \oplus \cdots \oplus A_n$ of $\mathcal{C}$ and $p_k : A_1 \oplus \cdots \oplus A_n \to A_k$ and $i_k : A_k \to A_1 \oplus \cdots \oplus A_n$ for $k = 1, \ldots, n$ such that $p_k \circ i_k = 1_{A_k}$ for $k = 1, \ldots, n$, $p_l \circ i_k = 0$ for $l \neq k$, $A_1 \oplus \cdots \oplus A_n$ equipped with $p_k$ for $k = 1, \ldots, n$ is a product of $A_1, \ldots, A_n$ and $A_1 \oplus \cdots \oplus A_n$ equipped with $i_k$ for $k = 1, \ldots, n$ is a coproduct of $A_1, \ldots, A_n$.

**Definition 2.12.** Let $\mathcal{C}$ be a category. Let $A$ and $B$ be objects of $\mathcal{C}$. A morphism $f \in \mathrm{Hom}(A, B)$ is said to be a *monomorphism* if for any object $C$ and any $g_1, g_2 \in \mathrm{Hom}(C, A)$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$. A morphism $f \in \mathrm{Hom}(A, B)$ is said to be aN *epimorphism* if for any object $C$ and any $g_1, g_2 \in \mathrm{Hom}(B, C)$, $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$.

**Definition 2.13.** An *abelian category* is a category $\mathcal{C}$ satisfying the following conditions:

1. For any objects $A$ and $B$, $\mathrm{Hom}(A, B)$ is an abelian group and for any objects $A$, $B$ and $C$, the map from $\mathrm{Hom}(B, A) \times \mathrm{Hom}(C, B)$ to $\mathrm{Hom}(C, A)$ given by the composition of morphisms is bilinear.

2. Every finite set of objects has a biproduct.

3. Every morphism has a kernel and cokernel.

4. Every monomorphism is a kernel of some morphism and every epimorphism is a cokernel of some morphism.

## 2.2 Monoidal categories and tensor categories

**Definition 2.14.** An *monoidal category* consists of the following data:

1. A category $\mathcal{C}$.

2. A bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *tensor product bifunctor*.

3. A natural isomorphism $\mathcal{A}$ from $\otimes \circ (1_{\mathcal{C}} \times \otimes)$ to $\otimes \circ (\otimes \times 1_{\mathcal{C}})$ called the *associativity isomorphism*.

4. An object $\mathbf{1}$ called the *unit object*.

5. A natural isomorphism $l$ from $\mathbf{1}\otimes\cdot$ to $1_{\mathcal{C}}$ called the *left unit isomorphism* and a natural isomorphism $r$ from $\cdot\otimes\mathbf{1}$ to $1_{\mathcal{C}}$ called the *right unit isomorphism.*

These data satisfy the following axioms:

1. The following *pentagon diagram* is commutative for objects $A_1, A_2, A_3, A_4$:

$$\begin{array}{ccc} & A_1 \otimes (A_2 \otimes (A_3 \otimes A_4)) & \\ \swarrow & & \searrow \\ (A_1 \otimes A_2) \otimes (A_3 \otimes A_4) & & A_1 \otimes ((A_2 \otimes A_3) \otimes A_4) \\ \downarrow & & \downarrow \\ ((A_1 \otimes A_2) \otimes A_3) \otimes A_4 & \longleftarrow & (A_1 \otimes (A_2 \otimes A_3)) \otimes A_4 \end{array}$$

2. The following *triangle diagram* is commutative for objects $A_1, A_2$:

$$\begin{array}{ccc} (A_1 \otimes \mathbf{1}) \otimes A_2 & \longrightarrow & A_1 \otimes (\mathbf{1} \otimes A_2) \\ \downarrow & & \downarrow \\ A_1 \otimes A_2 & \underset{=}{\longrightarrow} & A_1 \otimes A_2. \end{array}$$

**Definition 2.15.** A *tensor category* is an abelian category equipped with a monoidal category structure such that the abelain category structure and the monoidal category structure are compatible in the sense that for objects $A$, $B$, $C$ and $D$, the map $\otimes : \mathrm{Hom}(A, B) \times \mathrm{Hom}(C, D) \to \mathrm{Hom}(A \times C, B \otimes D)$ is bilinear.

**Definition 2.16.** Let $\mathcal{C}$ be a monoidal category. A *graph diagram in $\mathcal{C}$* is a graph whose vertices are functors obtained from the tensor product bifunctor and the unit objects and the edges are natural isomorphisms obtained from the associativity isomorphisms, the left and the right unit isomorphisms. A graph diagram is *commutative* if the compositions of the isorphisms in any two paths with the same starting and ending vertices must be equal.

**Theorem 2.17** (Mac Lane)**.** *Let $\mathcal{C}$ be a monoidal category. Any graph diagram in $\mathcal{C}$ is commutative*

We omit the proof here; see [M] and [EGNO].

**Definition 2.18.** A *monoidal functor* from a monoidal categoory $\mathcal{C}$ to a monoidal category $\mathcal{D}$ is a triple $(\mathcal{F}, J, \varphi)$ where $\mathcal{F}$ is a functor from $\mathcal{C}$ to $\mathcal{D}$, $J$ a natural transformation from the functor $\mathcal{F}(\cdot) \otimes_{\mathcal{D}} \mathcal{F}(\cdot)$ to the functor $\mathcal{F}(\cdot \otimes_{\mathcal{C}} \cdot)$ and $\varphi$ an isomorphism from $\mathbf{1}_{\mathcal{D}}$ to $\mathcal{F}(\mathbf{1}_{\mathcal{C}})$

such that the diagram

$$\begin{array}{ccc}
(\mathcal{F}(A_1) \otimes_{\mathcal{D}} \mathcal{F}(A_2)) \otimes_{\mathcal{D}} \mathcal{F}(A_3) & \longrightarrow & \mathcal{F}(A_1) \otimes_{\mathcal{D}} (\mathcal{F}(A_2) \otimes_{\mathcal{D}} \mathcal{F}(A_3)) \\
\downarrow & & \downarrow \\
(\mathcal{F}(A_1 \otimes_{\mathcal{C}} A_2) \otimes_{\mathcal{D}} \mathcal{F}(A_3) & & \mathcal{F}(A_1) \otimes_{\mathcal{D}} \mathcal{F}(A_2 \otimes_{\mathcal{C}} A_3) \\
\downarrow & & \downarrow \\
(\mathcal{F}((A_1 \otimes_{\mathcal{C}} A_2) \otimes_{\mathcal{C}} A_3) & \longrightarrow & \mathcal{F}(A_1 \otimes_{\mathcal{C}} (A_2 \otimes_{\mathcal{C}} A_3))
\end{array}$$

for objects $A_1$, $A_2$ and $A_3$ in $\mathcal{C}$ and the diagram

$$\begin{array}{ccc}
\mathbf{1}_{\mathcal{D}} \otimes_{\mathcal{D}} \mathcal{F}(A) & \longrightarrow & \mathcal{F}(A) \\
\downarrow & & \uparrow \\
\mathcal{F}(\mathbf{1}_{\mathcal{C}} \otimes_{\mathcal{D}} \mathcal{F}(A) & \longrightarrow & \mathcal{F}(\mathbf{1}_{\mathcal{C}} \otimes_{\mathcal{C}} A)
\end{array}$$

for an object $A$ in $\mathcal{C}$ are commutative. A *monoidal eqivalence* from a monoidal categoory $\mathcal{C}$ to a monoidal category $\mathcal{D}$ is a monoidal functor $(\mathcal{F}, J, \varphi)$ from $\mathcal{C}$ to $\mathcal{D}$ such that $\mathcal{F}$ is an equivalence of categories and $J$ is a natural isomorphism.

**Definition 2.19.** A monoidal category is *strict* if

$$\begin{array}{rcl}
\otimes \circ (1_{\mathcal{C}} \times \otimes) & = & \otimes \circ (\otimes \times 1_{\mathcal{C}}), \\
\mathbf{1} \otimes \cdot & = & 1_{\mathcal{C}} \\
\cdot \otimes \mathbf{1} & = & 1_{\mathcal{C}}
\end{array}$$

and the associativity, the left and the right unit isomorphisms are identities.

**Theorem 2.20** (Mac Lane)**.** *Any monoidal category is monoidal equivalent to a strict monoidal category.*

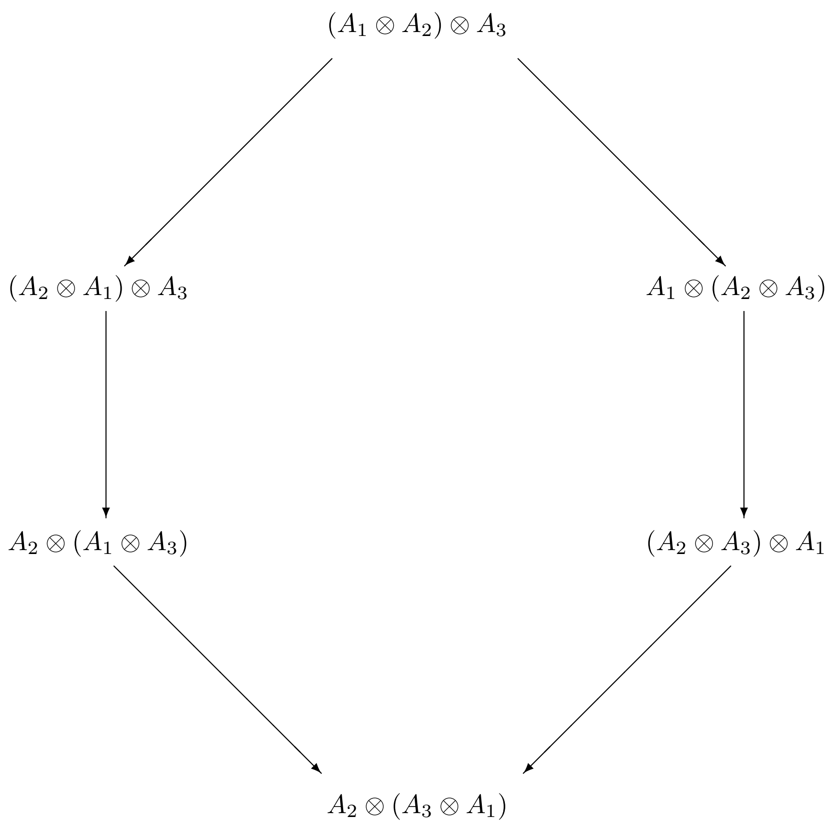
**Exercise 2.21.** Consider the category of bimodules for an associative algebra and the tensor product bifunctor we defined in the section on associative algebras. Show that there exists an associaitivity isomorphism such that the pentagon diagram is commutative.

## 2.3 Symmetries and braidings

**Definition 2.22.** Let $\mathcal{C}$ be a monoidal category. A *symmetry* of $\mathcal{C}$ is a natural isomorphism $C$ from $\otimes$ to $\otimes \circ \sigma_{12}$ ($\sigma_{12}$ being the functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C} \times \mathcal{C}$ induced from the nontrivial element of $S_2$) such that for objects $A_1, A_2$, the morphism

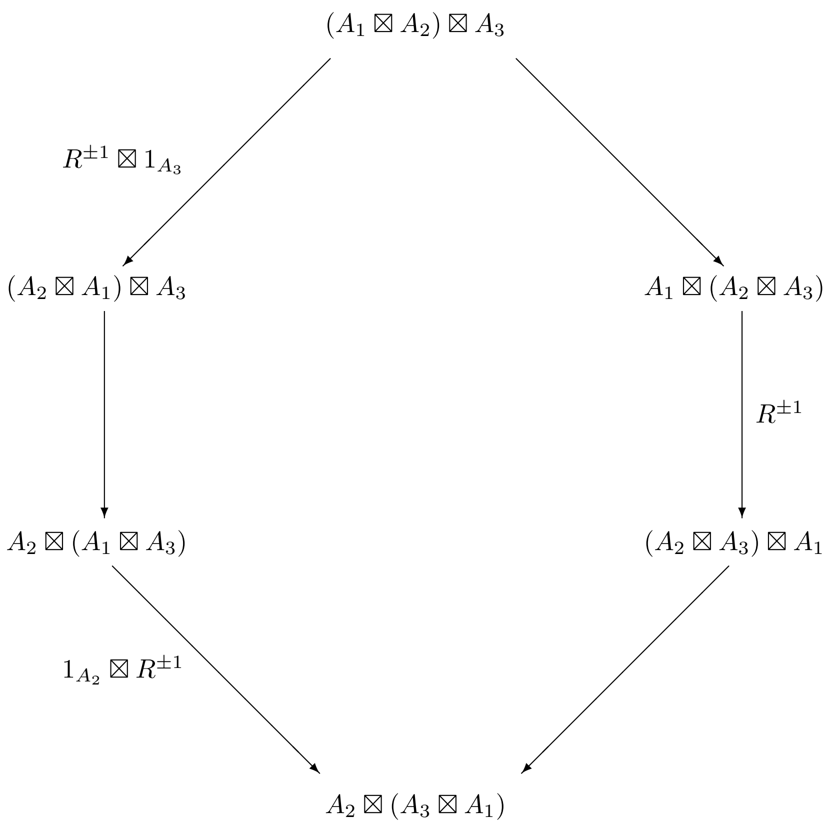
$$C_{A_2,A_1} \circ C_{A_1,A_2} : A_1 \otimes A_2 \to A_2 \otimes A_1 \to A_1 \otimes A_2$$

is equal to the identity $1_{A_1 \otimes A_2}$ and for objects $A_1$, $A_2$ and $A_2$, the *hexagon diagram*




is commutative. A *symmetric monoidal category* is a monoidal category with a symmetry. A *symmetric tensor category* is a tensor category with a symmetry.

**Definition 2.23.** Let $\mathcal{C}$ be a monoidal category. A *braiding* of $\mathcal{C}$ is a natural isomorphism $R$ from $\otimes$ to $\otimes \circ \sigma_{12}$ such that for objects $A_1$, $A_2$ and $A_2$, the *hexagon diagrams*




is commutative. A *braided monoidal category* is a monoidal category with a braiding. A *braided tensor category* is a tensor category with a braiding.

## 2.4 Rigidity

**Definition 2.24.** Let $\mathcal{C}$ be a monoidal category. For an object $A$, a *right dual* of $A$ is an object $A^*$ and morphisms $\mathrm{ev}_A : A^* \otimes A \to \mathbf{1}$ and $\mathrm{coev}_A : \mathbf{1} \to A \otimes A^*$ such that the morphism obtained by composing the morphisms in

$$A \to \mathbf{1} \otimes A \to (A \otimes A^*) \otimes A \to A \otimes (A^* \otimes A) \to A \otimes \mathbf{1} \to A$$

is equal to the identity $1_A$ and the morphism obtained by composing the morphisms in

$$A^* \to A^* \otimes \mathbf{1} \to A^* \otimes (A \otimes A^*) \to (A^* \otimes A) \otimes A^* \to \mathbf{1} \otimes A^* \to A^*$$

is equal to the identity $1_{A^*}$. A *left dual* of $A$ is an object ${}^*A$ and morphisms $\mathrm{ev}'_A : A \otimes^* A \to \mathbf{1}$ and $\mathrm{coev}'_A : \mathbf{1} \to^* A \otimes A$ such that the morphism obtained by composing the morphisms in

$$A \to A \otimes \mathbf{1} \to A \otimes ({}^*A \otimes A) \to (A \otimes^* A) \otimes A \to \mathbf{1} \otimes A \to A$$

is equal to the identity $1_A$ and the morphism obtained by composing the morphisms in

$${}^*A \to \mathbf{1} \otimes^* A \to ({}^*A \otimes A) \otimes^* A \to^* A \otimes (A \otimes^* A) \to^* A \otimes \mathbf{1} \to^* A$$

is equal to the identity $1_{^*A}$.

**Definition 2.25.** A monoidal category $\mathcal{C}$ is said to be *rigid* if there are contravariant functors ${}^*\cdot : \mathcal{C} \to \mathcal{C}$ and $\cdot^* : \mathcal{C} \to \mathcal{C}$ such that for an object $A$, ${}^*A$ and $A^*$ are left and right duals of $A$.

**Exercise 2.26.** Show that the category of finite-dimensional representations for a finite group and the category of finite-dimensional modules for a finite-dimensional Lie algebra are rigid symmetric tensor categories.

## 2.5 Ribbon categories and modular tensor categories

**Definition 2.27.** Let $\mathcal{C}$ be a braided onoidal category. A *twist* of $\mathcal{C}$ is a natural isomorphism $\theta : 1_{\mathcal{C}} \to 1_{\mathcal{C}}$ such that for objects $A_1$ and $A_2$,

$$\theta_{A_1 \otimes A_2} = R_{A_2,A_1} \circ R_{A_1,A_2} \circ (\theta_{A_1} \otimes \theta_{A_2}).$$

**Definition 2.28.** A *ribbon category* is a rigid braided monoidal category equipped with a twist.

**Lemma 2.29.** *In a ribbon category, the left dual and right dual can be taken to be the same.*

We omit the proof of this lemma.

Let $\mathcal{C}$ be a ribbon category and let $K = \mathrm{Hom}(\mathbf{1}, \mathbf{1})$. Then $K$ is a monoid (a set with an associative product and an identity).

**Lemma 2.30.** *$K$ is in fact commutative.*

In a ribbon category, we can define the "trace" of a morphism and the "dimension" of an object as follows:

**Definition 2.31.** Let $f \in \mathrm{Hom}(A, A)$ be a morphism in a ribbon category. The *trace* of $f$ is defined to be

$$\mathrm{Tr}\, f = \mathrm{ev}_A \circ R_{A,A^*} \circ ((\theta_A \circ f) \otimes 1_{A^*}) \circ \mathrm{coev}_A \in K.$$

The dimension $\dim A$ of an object $A$ is defined to be $\mathrm{Tr}\ 1_A$.

The trace of a morphism satisfies the properties that a trace should have.

**Proposition 2.32.** *Let $\mathcal{C}$ be a ribbon category. Then we have:*

1. *For $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, A)$, $\mathrm{Tr}\, fg = \mathrm{Tr}\, gf$.*

2. *For $f \in \mathrm{Hom}(A_1, A_2)$ and $g \in \mathrm{Hom}(A_3, A_4)$, $\mathrm{Tr}\,(f \otimes g) = (\mathrm{Tr}\, f)(\mathrm{Tr}\, g)$.*

3. *For $k \in K$, $\mathrm{Tr}\, k = k$.*

**Example 2.33.** The category of finite-dimensional representations of a finite group and the category of finite-dimensional modules for a finite-dimensional Lie algebra are ribbon categories whose braidings and twists are trivial.

**Example 2.34.** Let $G$ be an mulplicative abelian group (an abelian group whose operation is written as a multiplication instead of an addition), $K$ a commutative ring with identity and $c : G \times G \to K^*$ a bilinear form ($K^*$ being the set of invertible elements of $K$), that is, for $g, g', h, h' \in G$, we have

$$\begin{aligned} c(gg', h) &= c(g, h)c(g', h), \\ c(g, hh') &= c(g, h)c(g, h'). \end{aligned}$$

We construct a ribbon category as follows: The objects of the category are elements of $G$. For any $g, h \in G$, $\mathrm{Hom}(g, h)$ is $K$ if $g = h$ and 0 if $g \neq h$. The composition of two morphisms $g \to h \to\to f$ is the product of the two elements of $K$ is $g = h = f$ and 0 otherwise. The tensor product of two objects $g, h \in G$ is their product $gh$. The tensor product $gg' \to hh'$ of two morphisms $g \to g'$ and $h \to h'$ is the product of the two elements in $K$ if $g = h$ and $g' = h'$ and is 0 otherwise. The unit object is the identity of $G$. The associativity and left and right unit isomorphisms are the identity natural isomorphisms. For $g, h \in G$, the briading $gh \to hg = gh$ is defined to be $c(g, h)$. For $g \in G$, the twist $g \to g$ is defined to be $c(g, g)$. For $g \in G$, the (left and right) dual of $g$ is $g^{-1}$. The morphisms $\mathrm{ev}_g$, $\mathrm{coev}_g$, $\mathrm{ev}'_g$ and $\mathrm{coev}'_g$ are the indentity of $K$. Then we have a ribbon category.

**Exercise 2.35.** Verify that the example above is indeed a ribbon category.

We now consider ribbon tensor categories, that is, rigid braided tensor categories with twists.

Let $\mathcal{C}$ be a ribbon tensor category. Then $K = \mathrm{Hom}(\mathbf{1}, \mathbf{1})$ acts on $\mathrm{Hom}(A, B)$ for any objects $A$ and $B$ by $kf = l_B \circ (k \otimes f) \circ l_A^{-1}$ for $k \in K$ and $f \in \mathrm{Hom}(A, B)$. This action gives $\mathrm{Hom}(A, B)$ a $K$-module structure.

**Definition 2.36.** An object $A$ of a ribbon tensor category is said to be *irreducible* if $\mathrm{Hom}(A, A)$ is a free $K$-module of rank 1. A ribbon tensor category is said to be *semisimple* if the following conditions are satisfied:

1. For any simple objects $A$ and $B$, $\mathrm{Hom}(A, B) = 0$ if $A$ is not isomorphic to $B$.

2. Every object is a direct sum of finitely many irreducible objects.

**Example 2.37.** The unit object is an irreducible object.

**Example 2.38.** The ribbon tensor category of finite-dimensional representations over a field of a finite group such that the characteristic of the field does not divide the order of the group and the ribbon tensor category of finite-dimensional modules for a finite-dimensional semisimple Lie algebra are semisimple.

**Definition 2.39.** A *modular tensor category* is a semisimple ribbon tensor category $\mathcal{C}$, with finitely many equivalence classes of irreducible objects satisfying the following *nondegeneracy property*: Let $\{A_i\}_{i=1}^n$ be a set of representatives of the equivalence classes of irreducible objects of $\mathcal{C}$. Then the matrix $(S_{ij})$ where

$$S_{ij} = \mathrm{Tr}\ R_{A_j, A_i} \circ R_{A_i, A_j}$$

for $i, j = 1, \ldots, n$ is invertible.

Let $I$ be the set of equivalence classes of irreduible objects in a modular tensor category. We shall use 0 to denote the equivalence class in $I$ containing the unit object.

**Proposition 2.40.** *The dual object of an irreducible object is also irreducible.*

We omit the proof.

From this proposition, we see that there is a map $*: I \to I$ such that for any $i \in I$, $i^*$ is the equivalence class in $I$ such that objects in $i^*$ are duals of objects in $i$.

We now choose one object $A_i$ for each equivalence class $i \in I$. Then by definition, we have

$$S_{0,i} = S_{i,0} = \dim A_i$$

for $i \in I$.

**Definition 2.41.** Let $\mathcal{C}$ be a modular tensor category. Assuming that there exists $\mathcal{D} \in K$ such that

$$\mathcal{D}^2 = \sum_{i \in I} (\dim A_i)^2.$$

We call $\mathcal{D}$ the *rank* of $\mathcal{C}$.

If there is no such $\mathcal{D}$ in $K$, we can always enlarge $K$ and the sets of morphisms such that in the new category, there exists such a $\mathcal{D}$.

For $i \in I$, the twist $\theta_{A_i}$ as an element of $\mathrm{Hom}(A_i, A_i)$ must be proportional to $1_{A_i}$, that is, there exists $A_i \in K$ such that $\theta_i = A_i 1_{A_i}$. Since $\theta_{A_i}$ is an isomorphism, $A_i$ is invertible. Let $\Delta = \sum_{i \in I} v_i^{-1} (\dim A_i)^2$, $T = (\delta_i^j v_i)$ and $J = (\delta_{i^*}^j)$. Then we have

$$\begin{aligned} (\mathcal{D}^{-1} S)^4 &= I, \\ (\mathcal{D}^{-1} T^{-1} S)^3 &= \Delta \mathcal{D}^{-1} (\mathcal{D}^{-1} S)^2. \end{aligned}$$

Let

$$\begin{aligned} s &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ t &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Then $s$ and $t$ are the generators of the modular group

$$SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}$$

satisfying the relations

$$s^4 = I,\ (ts)^3 = s^2.$$

Thus we see that $s \mapsto \mathcal{D}^{-1} S$ and $t \mapsto T^{-1}$ give a projective matrix representation of $SL(2, \mathbb{Z})$.

Since $\mathcal{C}$ is semisimple and $I$ is the set of equivalence classes of irreducible objects in $\mathcal{C}$, we see that $A_i \otimes A_j$ for $i, j \in I$ must be isomorphic to $\oplus_{k \in I} N_{ij}^k A_k$, where $N_{ij}^k$ are nonnegative integers giving the numbers of copies of $A_k$. These numbers $N_{ij}^k$ afre called *fusion rules.*

**Theorem 2.42.** *For $i, l, m \in I$, we have*

$$\sum_{j,k\in I} S^{-1}_{mj} N^k_{ij} S_{kl} = (\dim A_m)^{-1} S_{il} \delta_{lm}.$$

In fact, if we let

$$N_i = (N^k_{ij})$$

for $i \in I$, then the theorem above says that the matrix $S$ diagonalizes $N_i$ for $i \in I$ simultaneously.

**Corollary 2.43.** *For $i, j, k \in I$, we have*

$$N^k_{ij} = \mathcal{D}^{-2} \sum_{l\in I} (\dim A_l)^{-1} S_{il} S_{jl} S_{k^* l}.$$

We omit the proofs of these results.

# 3 Tensor product modules and their construction

## 3.1 Definition of tensor product module

We have mentioned above that for two $V$-modules $W_1$ and $W_2$, $W_1 \otimes W_2$ is not a $V$-module. But tensor products for $V$-modules are important. They describe interations of the quantum objects whose state spaces are $W_1$ and $W_2$. Mathematically tensor products also give us new $V$-modules. Using intertwining operators, we can introduce a notion of tensor product of two $V$-modules. Such a tensor product does not always exist. In order to prove the existence, $V$ must satisfies certain conditions. In this subsection we give the definition of tensor product $V$-module of two $V$-modules. But we will not discuss the existence of the tensor product $V$-modules.

Our definition of tensor product $V$-module is given in terms of intertwining operators. To motivate our definition of tensor product $V$-module, we first give a definition of tensor product of two vector spaces using analogues of intertwining operators. Let $W_1$, $W_2$ and $W_3$ be vector spaces. A bilinear map $I : W_1 \times W_2 \to W_3$ is called an intertwining operator of type $\binom{W_3}{W_1 W_2}$. We call a pair $W_3, I)$ consisting of a vector space $W_3$ and an intertwining operator $I$ of type $\binom{W_3}{W_1 W_2}$ a product of $W_1$ and $W_2$. We define a tensor product vector space of $W_1$ and $W_2$ to be a product $(W_1 \otimes W_2, \otimes)$ such that the following universal property holds: Given any product $(W_3, I)$ of $W_1$ and $W_2$, there exists a unique linear map $f : W_1 \otimes W_2 \to W_3$ such that $I = f \circ \otimes$.

Here is a construction of a tensor product vector space: Let $\mathbb{C}(W_1 \times W_2)$ be the free vector space generated by the direct product $W_1 \times W_2$. Let $W_1 \otimes W_2$ be the quotient vector space $\mathbb{C}(W_1 \times W_2)/J$, where $J$ is the subspace of $W_1 \times W_2$ spanned by elemenets of the form $(\lambda w_1, w_2) - (w_1, \lambda w_2)$, $\lambda(w_1, w_2) - (\lambda w_1, w_2)$, $(w_1 + \tilde{w}_1, w_2) - (w_1, w_2) - (\tilde{w}_1, w_2)$ and $(w_1, w_2 + \tilde{w}_2) - (w_1, w_2) - (w_1, \tilde{w}_2)$ for $w_1, \tilde{w}_1 \in W_1$, $w_2, \tilde{w} \in W_2$ and $\lambda \in \mathbb{C}$. We use $w_1 \otimes w_2$

to denote the coset $(w_1, w_2) + J$. Let $\otimes : W_1 \times W_2 \to W_1 \otimes W_2$ be the projection map. Then $\otimes$ is an intertwining operator of type $\binom{W_1 \otimes W_2}{W_1 W_2}$ and $(W_1 \otimes W_2, \otimes)$ is a tensor product vector space of $W_1$ and $W_2$.

We now give the definition of tensor product $V$-module of two $V$-modules. For simplicity, we work with the category of lower bounded generalized $V$-modules. For other categories of $V$-modules, the definition is the same except that we replace the words "lower bounded generalized $V$-module" by the names for the types of $V$-modules in the other categories.

One crucial new feature for the tensor product $V$-module is that it involves $z \in \mathbb{C}^\times$. For such $z \in \mathbb{C}^\times$, we use $\log z$ to denote the value $\log |z| + i \arg z$, where $0 \leq \arg z < 2\pi$ of the logarithm of $z$. For an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$, we use $\mathcal{Y}(w_1, z)w_2$ to denote $\mathcal{Y}(w_1, x)w_2\Big|_{x^n = e^{n \log z}, n \in \mathbb{C}, \log x = \log z}$.

**Definition 3.1.** Let $z \in \mathbb{C}^\times$ and $W_1$ and $W_2$ lower-bounded generalized $V$-modules. A *$P(z)$-product of $W_1$ and $W_2$* is a pair $(W_3, I)$ consisiting of a lower-bounded generalized $V$-module $W_3$ and the value $I = \mathcal{Y}(\cdot, z)\cdot : W_1 \otimes W_2 \to \overline{W}_3$ (called *a $P(z)$-intertwining map of type $\binom{W_3}{W_1 W_2}$*) of an intertwining operator $\mathcal{Y}(\cdot, x)\cdot : W_1 \otimes W_2 \to W_3\{x\}[\log x]$ at $z$ (with $x^n$ for $n \in \mathbb{C}$ and $\log x$ be substituted by $e^{n \log z}$ and $\log z$, respectively). A *$P(z)$-tensor product of $W_1$ and $W_2$* is a $P(z)$-product $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$ such that the following universal property holds: Given any $P(z)$-product $(W_3, I)$ of $W_1$ and $W_2$, there exists a unique module map $f : W_1 \boxtimes_{P(z)} W_2 \to W_3$ such that $I = \bar{f} \circ \boxtimes_{P(z)}$, where $\bar{f} : \overline{W_1 \boxtimes_{P(z)} W_2} \to \overline{W}_3$ is the unique extension of $f$ to $\overline{W_1 \boxtimes_{P(z)} W_2}$ (note that $f$ as a module map must preserve wegihts).

The value $I$ of an intertwining operator $\mathcal{Y}(\cdot, x)\cdot$ at $z$ is called a $P(z)$-intertwining map.

The first question about the $P(z)$-tensor product is its existence. For vector spaces, the existence is trivial (see above). But for $V$-modules, it is not trivial in general. As we mentioned above, in general the $P(z)$-tensor product might not exist. Under certain conditions, the existence of $P(z)$-tensor product was proved in [HL4] and [H7].

The category of $V$-mdules form a braided tensor category under certain conditions on $V$ or a modular tensor category under stronger conditions. The two difficult part of the construction is the construction of the associativity isomorphism and the proof of the rigidity. These two difficult parts corresponding to the associativity of intertwining operators and the modular invariance of intertwining operators, respectively. See [H1], [H3], [H4], [H6], [H7] and [HLZ2].

## 3.2 A construction of tensor product modules

We now give a construction of the $P(z)$-tensor product $W_1 \boxtimes_{P(z)} W_2$ in the category of grading-restricted generalized $V$-modules. For a grading-restricted generalized $V$-module $W_3$, $w_3' \in W_3'$ and an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$, we have an element $\lambda^z_{\mathcal{Y}}(w_3') \in (W_1 \otimes W_2)^*$ given by

$$(\lambda^z_{\mathcal{Y}}(w_3'))(w_1 \otimes w_2) = \langle w_3', \mathcal{Y}(w_1, z)w_2 \rangle,$$

where for simplicity we use $\mathcal{Y}(w_1, z)$ to denote $\mathcal{Y}(w_1, x)w_2\Big|_{x^n=e^{n\log z}, n\in\mathbb{C}, \log x=\log z} \in \mathrm{Hom}(W_2, \overline{W}_3$. Then we obtain a linear map $\lambda^z_{\mathcal{Y}} : W_3' \to (W_1 \otimes W_2)^*$. Let $W_1 \boxslash_{P(z)} W_2$ be the subspace of $(W_1 \otimes W_2)^*$ spanned by all elements of the form $\lambda^z_{\mathcal{Y}}(w_3')$ for a grading-restricted generalized $V$-module $W_3$, $w_3' \in W_3'$ and an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1W_2}$. Then $\lambda^z_{\mathcal{Y}}$ is in fact a linear map from $W_3'$ to $W_1 \boxslash_{P(z)} W_2$.

We define a vertex operator map

$$\begin{aligned} Y_{W_1\boxslash_{P(z)}W_2} : V \otimes (W_1 \boxslash_{P(z)} W_2) &\to (W_1 \boxslash_{P(z)} W_2)[[x, x^{-1}]] \\ v \otimes \lambda &\mapsto Y_{W_1\boxslash_{P(z)}W_2}(v, x)\lambda \end{aligned}$$

by

$$(Y_{W_1\boxslash_{P(z)}W_2}(v, x)\lambda^z_{\mathcal{Y}}(w_3'))(w_1 \otimes w_2) = \langle Y_{W_3'}(v, x)w_3', \mathcal{Y}(w_1, z)w_2\rangle$$

for $w_1 \in W_1$, $w_2 \in W_2$, $w_3' \in W_3'$ and $v \in V$.

**Proposition 3.2.** *The space $W_1 \boxslash_{P(z)} W_2$ equipped with $Y_{W_1\boxslash_{P(z)}W_2}$ is a generalized $V$-module such that for a grading-restricted generalized $V$-module $W_3$, $w_3' \in W_3'$ and an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1W_2}$, $\lambda^z_{\mathcal{Y}}$ is a $V$-module map from $W_3'$ to $W_1 \boxslash_{P(z)} W_2$.*

**Exercise 3.3.** *Prove Proposition 3.2.*

Let $W_3$ be a grading-restricted generalized $V$-module and $J : W_3' \to W_1 \boxslash_{P(z)} W_2$ a $V$-module map. Let $\mathcal{Y}_J(\cdot, z)\cdot : W_1 \otimes W_2 \to \overline{W}_3$ be defined by

$$\langle w_3', \mathcal{Y}_J(w_1, z)w_2\rangle = (J(w_3'))(w_1 \otimes w_2).$$

Then we define

$$\mathcal{Y}_J(w_1, x)w_2 = x^{L_{W_3}(0)}z^{-L_{W_3}(0)}\mathcal{Y}_J(x^{-L_{W_1}(0)}z^{L_{W_1}(0)}w_1, z)x^{-L_{W_2}(0)}z^{L_{W_2}(0)}w_2.$$

In this way, we obtain a linear map

$$\mathcal{Y}_J : W_1 \otimes W_2 \to W_3\{x\}[\log x].$$

**Proposition 3.4.** *The linear map $\mathcal{Y}_J$ is an intertwining operator of type $\binom{W_3}{W_1W_2}$ such that $\mathcal{Y}_{\lambda^z_{\mathcal{Y}}} = \mathcal{Y}$ and $\lambda^z_{\mathcal{Y}_J} = J$.*

**Exercise 3.5.** *Prove Proposition 3.4.*

One immediate consequence of this proposition is the following:

**Corollary 3.6.** *The map given by $\mathcal{Y} \mapsto \lambda^z_Y$ is a linear isomorphism from the space of intertwining operators of type $\binom{W_3}{W_1W_2}$ to the space of $V$-module maps from $W_3'$ to $W_1 \boxslash_{P(z)} W_2$. The inverse of this map is the linear map given by $J \mapsto \mathcal{Y}_J$.*

We shall now make the following assumption:

**Assumption 3.7.** *For any grading-restricted generalized $V$-modules $W_1$ and $W_2$, $W_1 \boxbackslash_{P(z)} W_2$ is a grading-restricted generalized $V$-module.*

Under this assumption, for grading-restricted generalized $V$-modules $W_1$ and $W_2$, the graded dual $W_1 \boxtimes_{P(z)} W_2 = (W_1 \boxbackslash_{P(z)} W_2)'$ of $W_1 \boxbackslash_{P(z)} W_2$ is also a grading-restricted generalized $V$-module. Consider the identity operator $1_{W_1 \boxbackslash_{P(z)} W_2}$ on $W_1 \boxbackslash_{P(z)} W_2$. This is certainly a $V$-module map from the graded dual $W_1 \boxbackslash_{P(z)} W_2$ of $W_1 \boxtimes_{P(z)} W_2$ to $W_1 \boxbackslash_{P(z)} W_2$. Then by Proposition 3.4, we have an intertwining operator $\mathcal{Y}_{1_{W_1 \boxbackslash_{P(z)} W_2}}$ of type $\binom{W_1 \boxtimes_{P(z)} W_2}{W_1 W_2}$. We denote the evaluation $\mathcal{Y}_{1_{W_1 \boxbackslash_{P(z)} W_2}}(\cdot, z)$ of $\mathcal{Y}_{1_{W_1 \boxbackslash_{P(z)} W_2}}$ at $z$ by $\boxtimes_{P(z)}$. Let

$$W_1 \boxtimes_{P(z)} W_2 = (W_1 \boxbackslash_{P(z)} W_2)'.$$

**Theorem 3.8.** *The pair $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$ is a $P(z)$-tensor product of $W_1$ and $W_2$.*

*Proof.* Let $(W_3, I)$ be a $P(z)$-tensor product. Then by the definition of $P(z)$-intertwining map, we have an intertwining operator $\mathcal{Y}^I$ of type $\binom{W_3}{W_1 W_2}$ such that $I = \mathcal{Y}^I(\cdot, z)\cdot$. Then by Propositions 3.2, we have a $V$-module map $\lambda^z_{\mathcal{Y}^I} : W_3' \to W_1 \boxbackslash_{P(z)} W_2$ given by

$$\langle w_3', \mathcal{Y}^I(w_1, z)w_2\rangle = (\lambda^z_{\mathcal{Y}^I}(w_3'))(w_1 \otimes w_2).$$

The adjoint of $\lambda^z_{\mathcal{Y}}$ is a $V$-module map $f : W_1 \boxtimes_{P(z)} W_2 \to W_3$. Then we have

$$\begin{aligned}
\langle w_3', I(w_1 \otimes w_2)\rangle &= \langle w_3', \mathcal{Y}(w_1, z)w_2\rangle \\
&= (\lambda^z_{\mathcal{Y}}(w_3'))(w_1 \otimes w_2) \\
&= (1_{W_1 \boxbackslash_{P(z)} W_2}(\lambda^z_{\mathcal{Y}}(w_3')))(w_1 \otimes w_2) \\
&= \langle \lambda^z_{\mathcal{Y}}(w_3'), \mathcal{Y}_{1_{W_1 \boxbackslash_{P(z)} W_2}}(w_1, z)w_2\rangle \\
&= \langle w_3', (\bar{f} \circ \boxtimes_{P(z)})(w_1 \otimes w_2)\rangle
\end{aligned}$$

for $w_1 \in W_1$, $w_2 \in W_2$ and $w_3' \in W_3'$. Thus we obtain $I = \bar{f} \circ \boxtimes_{P(z)}$. The uniqueness of $f$ follows from the uniqueness of $\mathcal{Y}^I$ and $\lambda^I_{\mathcal{Y}}$. ∎

Note that the tensor product elements of the tensor product of two vector spaces span the tensor product space. We also have a tensor product element of $w_1 \in W_1$ and $w_2 \in W_2$ defined by

$$w_1 \boxtimes_{P(z)} w_2 = \boxtimes_{P(z)}(w_1 \otimes w_2) = \mathcal{Y}_{1_{W_1 \boxbackslash_{P(z)} W_2}}(w_1, z)w_2.$$

But note that $w_1 \boxtimes_{P(z)} w_2$ is in $\overline{W_1 \boxtimes_{P(z)} W_2}$ instead of $W_1 \boxtimes_{P(z)} W_2$.

**Proposition 3.9.** *The homogeneous components $\pi_n(w_1 \boxtimes_{P(z)} w_2)$ ($\pi_n$ is the projection from $\overline{W_1 \boxtimes_{P(z)} W_2}$ to its homogeneous subspace of weight $n$) of $w_1 \boxtimes_{P(z)} w_2$ for $n \in \mathbb{C}$, $w_1 \in W_1$ and $w_2 \in W_2$ span $W_1 \boxtimes_{P(z)} W_2$.*

*Proof.* Let $W_3$ be the subspace of $W_1 \boxtimes_{P(z)} W_2$ spanned by homogeneous components $\pi_n(w_1 \boxtimes_{P(z)} w_2)$ of $w_1 \boxtimes_{P(z)} w_2$ for $n \in \mathbb{C}$, $w_1 \in W_1$ and $w_2 \in W_2$. Then we have $w_1 \boxtimes_{P(z)} w_2 \in \overline{W}_3$ for $w_1 \in W_1$ and $w_2 \in W_2$. Recall the intertwining operator $\mathcal{Y}_{1_{W_1 \boxbackslash_{P(z)} W_2}}$ of type $\binom{W_1 \boxtimes_{P(z)} W_2}{W_1 W_2}$ such that $w_1 \boxtimes_{P(z)} w_2 = \mathcal{Y}_{1_{W_1 \boxbackslash_{P(z)} W_2}}(w_1, z)w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$.

If $W_3$ is not equal to $W_1 \boxtimes_{P(z)} W_2$, then there exists a nonzero subspace $W_0$ of $W_1 \boxbackslash_{P(z)} W_2 = (W_1 \boxtimes_{P(z)} W_2)'$ such that for $\lambda \in W_0$, $\langle \lambda, \pi_n(w_1 \boxtimes_{P(z)} w_2)\rangle = 0$ for all $n \in \mathbb{C}$, $w_1 \in W_1$ and $w_2 \in W_2$. Then for $\lambda \in W_0$, $\langle \lambda, w_1 \boxtimes_{P(z)} w_2\rangle = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$. Since

$$\langle \lambda, w_1 \boxtimes_{P(z)} w_2\rangle = \langle \lambda, \mathcal{Y}_{1_{W_1 \boxbackslash_{P(z)} W_2}}(w_1, z)w_2\rangle = \lambda(w_1 \otimes w_2),$$

we obtain $\lambda(w_1 \otimes w_2) = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$. So $\lambda = 0$ and thus $W_0 = 0$. Contradiction. ∎

The construction above is based on Assumption 3.7. We have the the following results on this assumption:

**Theorem 3.10** ([HL2]). *Assume that $V$ satisfies the following condition:*

1. *There are only finitely many irreducible $V$-modules (up to equivalence).*
2. *Every grading-restricted generalized $V$-module is completely reducible (and is in particular a finite direct sum of irreducible $V$-modules).*
3. *All the fusion rules for $V$ are finite (for triples of irreducible $V$-modules and hence arbitrary $V$-modules).*

*Then $W_1 \boxbackslash_{P(z)} W_2$ is a (ordinary) $V$-module.*

*Proof.* Elements of $W_1 \boxbackslash_{P(z)} W_2$ are spanned by elements of the form $\lambda_{\mathcal{Y}}^z(w_3')$ for a grading-restricted generalized $V$-module $W_3$, an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$ and $w_3' \in W_3'$. Then $W_1 \boxbackslash_{P(z)} W_2$ is a sum of grading-restricted generalized $V$-modules. Since every grading-restricted generalized $V$-module is completely reducible, $W_1 \boxbackslash_{P(z)} W_2$ is a direct sum of irreducible $V$-modules. Since there are only finitely many irreducible $V$-modules, $W_1 \boxbackslash_{P(z)} W_2$ is a direct sum of finitely or infinitely many copies of these finitely many irreducible $V$-modules. To show that $W_1 \boxbackslash_{P(z)} W_2$ is a grading-restricted generalized $V$-module, we still need to show that there are only finitely many copies of these irreducible $V$-modules appearing in the decomposition of $W_1 \boxbackslash_{P(z)} W_2$. If there are infinitely many copies of an irreducible $V$-module $W$ appearing in the decomposition of $W_1 \boxbackslash_{P(z)} W_2$, then for each copy of this irreducible $V$-module in $W_1 \boxbackslash_{P(z)} W_2$ we have an embedding map from $W$ to $W_1 \boxbackslash_{P(z)} W_2$. This embedding map is clearly a $V$-module map. Also, these infinitely many embedding $V$-module maps are linearly independent. But by Corollary 3.6, these infinitely many linearly independent embedding $V$-module maps corresponds to infinitely many linearly independent intertwining operators of the type $\binom{W'}{W_1 W_2}$. In particular, the dimension of the space of intertwining operators of the type $\binom{W'}{W_1 W_2}$ is infinite. But all the fusion rules are finitely.

Contradiction. So $W_1 \boxtimes_{P(z)} W_2$ is a direct sum of finitely many irreducible $V$-modules and thus must be grading restricted. ∎

A vertex operator algebra $V$ is said to be of positive energy (or of CFT type) if $V_{(n)} = 0$ for $n < 0$ and $V_{(0)} = \mathbb{C}\mathbf{1}$. A vertex operator algebra is called $C_2$-cofinite if $\dim V/C_2(V) < \infty$, where $C_2(V)$ is the subspace of $V$ spanned by elements of the form $\mathrm{Res}_x x^{-2} Y_V(u, x)v$ for $u, v \in V$.

**Theorem 3.11** ([H7]). *Assume that $V$ is of positive energy and $C_2$-cofinite. Then $W_1 \boxtimes_{P(z)} W_2$ is a grading-restricted generalized $V$-module.*

We omit the proof of this theorem here. See [H7] for a proof.

# 4 Associativity of intertwining operators and associativity isomorphisms

## 4.1 Associativity of intertwining operators

We formulate the associativity of intertwining operators in the category of grading-restricted generalized $V$-modules as the main assumption in this section. We also state without proof a result on the associativity of intertwining operators.

**Assumption 4.1** (Associativity of intertwining operators in the category of grading-restricted generalized $V$-modules). *Let $W_1$, $W_2$, $W_3$, $W_4$, $W_5$ be grading-restricted generalized $V$-modules and $\mathcal{Y}_1$ and $\mathcal{Y}_2$ intertwining operators of types $\binom{W_4}{W_1\, W_5}$ and $\binom{W_5}{W_2\, W_3}$, respectively.*

1. *For $w_1 \in W_1, w_2 \in W_2$, $w_3 \in W_3$ and $w_4' \in W_4'$, the series*

$$\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle$$

*is absolutely convergent in the region $|z_1| > |z_2| > 0$ and its sum can be analytically continued to a multivalued analytic function*

$$F(\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle)$$

*on the region*

$$M^2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1, z_2 \neq 0, z_1 - z_2 \neq 0\} \subset \mathbb{C}^2$$

*with the only possible singular points $z_1 = 0$, $z_2 = 0$ and $z_1 = z_2$ being regular singular points.*

2. *There exist a grading-restricted generalized $V$-module $W_6$ and intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of types $\binom{W_4}{W_1\, W_5}$ and $\binom{W_5}{W_2\, W_3}$, respectively, such that for $w_1 \in W_1, w_2 \in W_2$, $w_3 \in W_3$ and $w_4' \in W_4'$,*

$$\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle = \langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3\rangle$$

*in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$. (The absolute convergence of*

$$\langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3\rangle$$

*in the region $|z_2| > |z_1 - z_2| > 0$ is a consequence of Part 1 above. See Proposition 4.2 below.)*

**Proposition 4.2.** *Let $W_1$, $W_2$, $W_3$, $W_4$, $W_6$ be grading-restricted generalized $V$-modules and $\mathcal{Y}_3$ and $\mathcal{Y}_4$ intertwining operators of types $\binom{W_4}{W_6W_3}$ and $\binom{W_6}{W_1W_2}$, respectively.*

1. *Suppose that Part 1 of Assumption 4.1 holds. Then for $w_1 \in W_1, w_2 \in W_2$, $w_3 \in W_3$ and $w_4' \in W_4'$, the series*

$$\langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3\rangle$$

*is absolutely convergent in the region $|z_2| > |z_1 - z_2| > 0$ and its sum can be analytically extended to a multivalued analytic function*

$$F(\langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3\rangle)$$

*on the region $M^2$ with the only possible singular points $z_1 = 0$, $z_2 = 0$ and $z_1 = z_2$ being regular singular points.*

2. *Suppose that Assumption 4.1 holds. Then there exist a grading-restricted generalized $V$-module $W_5$ and intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of the types $\binom{W_4}{W_6\ W_3}$ and $\binom{W_6}{W_1\ W_2}$, respectively, such that for $w_1 \in W_1, w_2 \in W_2$, $w_3 \in W_3$ and $w_4' \in W_4'$,*

$$\langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3\rangle = \langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle$$

*in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$.*

We need skew-symmetry isomorphism for intertwining operators to prove this result. The skew-symmetry isomorphism will be discussed in Section 4. We shall prove this result in that section.

The associativity of intertwining operators holds only when the vertex operator algebra satisfies certain conditions. Here is a result proved in [H7]:

**Theorem 4.3** ([H7]). *Assume that $V$ is of positive energy and $C_2$-cofinite. Then associativity of intertwining operators in the category of grading-restricted generalized $V$-modules hold.*

We omit the proof of this theorem here. See [H7] for a proof.

## 4.2 Associativity isomorphisms

Assuming that the associativity of intertwining operators hold, we construct associativity isomorphisms for the vertex tensor category structure and the braided tensor category structure.

Recall the tensor product element $w_1 \boxtimes_{P(z)} w_2$ of $\overline{W_1 \boxtimes_{P(z)} W_2}$. To construct and study associativity isomorphisms , we need tensor products of three elements of three grading-restricted generalized $V$-modules. Let $W_1$, $W_2$ and $W_3$ be grading-restricted generalized $V$-modules and $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$. By Assumption 4.1,

$$\langle w_4', w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)\rangle = \langle w_4', \mathcal{Y}_{1_{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}}(w_1, z_1)\mathcal{Y}_{1_{W_2 \boxtimes_{P(z_2)} W_3}}(w_2, z_2)w_3\rangle$$

is absolutely convergent for $w_4' \in (W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))'$, $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$. Thus we have a well-defined element

$$w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3) \in \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}.$$

Similarly, we have a well-defined element

$$(w_1 \boxtimes_{P(z_1 - z_2)} w_2) \boxtimes_{P(z_2)} w_3 \in \overline{(W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3}.$$

We now construct the associativity isomorphsim. We have the foloowing theorem:

**Theorem 4.4.** *Let $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$. Suppose that Assumptions 3.7 and 4.1 hold. Then there exist a unique natural isomorphism*

$$\mathcal{A}_{P(z_1),P(z_2)}^{P(z_2),P(z_1-z_2)} : \boxtimes_{P(z_1)} \circ (1_{W_1} \times \boxtimes_{P(z_2)}) \to \boxtimes_{P(z_2)} \circ (\boxtimes_{P(z_2)} \times 1_{W_3}),$$

*called associativity isomorphism, such that for grading-restricted generalized $V$-modules $W_1$, $W_2$ and $W_3$, the extension $\overline{\mathcal{A}_{P(z_1),P(z_2)}^{P(z_2),P(z_1-z_2)}}$ of the module map*

$$\mathcal{A}_{P(z_1),P(z_2)}^{P(z_2),P(z_1-z_2)} : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \to (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$$

*to the algebraic extension $\overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}$ of $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ satisfies*

$$\overline{\mathcal{A}_{P(z_1),P(z_2)}^{P(z_2),P(z_1-z_2)}}(w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)) = (w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3 \tag{4.1}$$

*for $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$.*

*Proof.* For simplicity, we denote $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$ and $W_1 \boxtimes_{P(z_1-z_2)} W_2$ by $W_4$ and $W_6$. We also denote the intertwining operators $\mathcal{Y}_{1_{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3}}$ and $\mathcal{Y}_{1_{W_1 \boxtimes_{P(z_1-z_2)} W_3}}$ by $\mathcal{Y}_3$ and $\mathcal{Y}_4$. For $w_4' \in W_4'$, $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$, by Proposition 4.2, there exist grading-restricted generalized $V$-module $W_5$ and intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of type $\binom{W_4}{W_1 W_5}$ and $\binom{W_5}{W_2 W_3}$, respectively, such that

$$\langle w_4', (w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3\rangle = \langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3\rangle$$

$$= \langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle.$$

Let $I_2 = \mathcal{Y}_2(\cdot, z_2)\cdot$. Then $I_2$ is a $P(z_2)$-intertwining map. So we have a $P(z_2)$-product $(W_5, I_2)$. By the universal property of the tensor product $(W_2 \boxtimes_{P(z_2)} W_3, \boxtimes_{P(z_2)})$, there exists a unique $V$-module map $f : W_2 \boxtimes_{P(z_2)} W_3 \to W_5$ such that $I_2 = \bar{f} \circ \boxtimes_{P(z_2)}$. We use $\mathcal{Y}^2$ to denote that intertwining operator $\mathcal{Y}_{1_{W_2 \boxtimes_{P(z_2)} W_3}}$. Then we have $w_2 \boxtimes_{P(z_2)} w_3 = \mathcal{Y}^2(w_2, z_2)w_3$ and $\mathcal{Y}_2(w_2, z_2)w_3 = \bar{f}(\mathcal{Y}^2(w_2, z_2)w_3)$.

Let $I_1 = \mathcal{Y}_1(\cdot, z_1)f(\cdot)$. Since $\mathcal{Y}_1 \circ (1_{W_1} \otimes f(\cdot))$ is an intertwining operator of type $\binom{W_4}{W_1 W_5}$, $I_1$ is a $P(z_1)$-intertwining map of type $\binom{W_4}{W_1 (W_2 \boxtimes_{P(z_2)} W_3)}$. In particular, we have a $P(z_1)$-product $(W_4, I_1)$ of $W_1$ and $W_2 \boxtimes_{P(z_2)} W_3$. By the universal property of the tensor product

$$(W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3), \boxtimes_{P(z_1)}).$$

there exists a $V$-module map

$$\mathcal{A}_{P(z_1),P(z_2)}^{P(z_2),P(z_1-z_2)} : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \to (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 (= W_4)$$

such that $I_1 = \overline{\mathcal{A}_{P(z_1),P(z_2)}^{P(z_2),P(z_1-z_2)}} \circ \boxtimes_{P(z_1)}$. We use $\mathcal{Y}^1$ to denote the intertwining operator $\mathcal{Y}_{1_{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}}$. Then

$$\overline{\mathcal{A}_{P(z_1),P(z_2)}^{P(z_2),P(z_1-z_2)}}(w_1 \boxtimes_{P(z_1)} w) = I_1(w_1 \otimes w) = \mathcal{Y}_1(w_1, z_1)f(w)$$

for $w_1 \in W_1$ and $w \in W_2 \boxtimes_{P(z_2)} W_3$.

Using all the calculations above, we obtain

$$\begin{aligned}
&\langle w_4', \overline{\mathcal{A}_{P(z_1),P(z_2)}^{P(z_2),P(z_1-z_2)}}(w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3))\rangle \\
&\quad = \sum_{n \in \mathbb{C}} \langle w_4', \overline{\mathcal{A}_{P(z_1),P(z_2)}^{P(z_2),P(z_1-z_2)}}(w_1 \boxtimes_{P(z_1)} (\pi_n(w_2 \boxtimes_{P(z_2)} w_3)))\rangle \\
&\quad = \sum_{n \in \mathbb{C}} \langle w_4', \mathcal{Y}_1(w_1, z_1) f(\pi_n(w_2 \boxtimes_{P(z_2)} w_3))\rangle \\
&\quad = \langle w_4', \mathcal{Y}_1(w_1, z_1)\bar{f}(w_2 \boxtimes_{P(z_2)} w_3)\rangle \\
&\quad = \langle w_4', \mathcal{Y}_1(w_1, z_1)\bar{f}(\mathcal{Y}^2(w_2, z_2)w_3)\rangle \\
&\quad = \langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle \\
&\quad = \langle w_4', (w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3\rangle
\end{aligned}$$

for $w_4' \in W_4'$, $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$. Thus we obatin (4.1).

The same construction also gives a $V$-module map from $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$ to $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ such that its extension to the algebraic completion of $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$ maps $(w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3$ to $w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)$. The last property of this $V$-module map shows that this $V$-module map must be the inverse of $\mathcal{A}_{P(z_1),P(z_2)}^{P(z_2),P(z_1-z_2)}$. So $\mathcal{A}_{P(z_1),P(z_2)}^{P(z_2),P(z_1-z_2)}$ is an isomorphism. ∎

Now we choose the tensor product $\boxtimes_{P(1)}$ to be the tensor product bifunctor of the category of grading-restricted generalized $V$-modules. We need to construct the associativity isomorphism for this tensor product bifunctor.

We construct this associativity isomorphism using the associativity isomorphism $\mathcal{A}_{P(z_1),P(z_2)}^{P(z_2),P(z_1-z_2)}$ constructed above. But we first have to introduce and construct what we call parallel transport isomorphisms.

Let $z_1, z_2 \in \mathbb{C}^\times$ and $\gamma$ a path in $\mathbb{C}^\times$ from $z_1$ to $z_2$. We denote the homotopy class of $\gamma$ by $[\gamma]$. The following theorem and its proof gives a construction of the parallel transport isomorphism:

**Theorem 4.5.** *Let $z_1, z_2 \in \mathbb{C}^\times$ and $\gamma$ a path in $\mathbb{C}^\times$ from $z_1$ to $z_2$. Then there exists a unique natural isomorphism*

$$\mathcal{T}_{[\gamma]} : \boxtimes_{P(z_1)} \to \boxtimes_{P(z_2)},$$

*called the parallel transport isomorphism associated to $[\gamma]$, such that for grading-rsetricted generalized $V$-modules $W_1$ and $W_2$, the extension $\overline{\mathcal{T}_{[\gamma]}}$ of the $V$-module map*

$$\mathcal{T}_{[\gamma]} : W_1 \boxtimes_{P(z_1)} W_2 \to W_1 \boxtimes_{P(z_2)} W_2$$

*to the algebraic extension $\overline{W_1 \boxtimes_{P(z_1)} W_2}$ of $W_1 \boxtimes_{P(z_1)} W_2$ satisfies the following property: For $w_1 \in W_1$ and $w_2 \in W_2$, the image $\overline{\mathcal{T}_{[\gamma]}}(w_1 \boxtimes_{P(z_1)} w_2)$ of the tensor product element $w_1 \boxtimes_{P(z_1)} w_2$ under $\overline{\mathcal{T}_{[\gamma]}}$ is the element $\mathcal{Y}(w_1, x)w_2\Big|_{x^n = e^{nl_p(z_1)}, \log x = l_p(z_1)}$ of $\overline{W_1 \boxtimes_{P(z_2)} W_2}$, where $\mathcal{Y}$ is the intertwining operator of type $\binom{W_1 \boxtimes_{P(z_2)} W_2}{W_1 W_2}$ and $l_p(z_1) = \log|z_1| + i \arg z_1 + i2\pi p$ is the value of logarithm of $z_1$ obtained by analytically extending the value $\log z_2$ along the path $\gamma$.*

*Proof.* Let $I = \mathcal{Y}(w_1, x)w_2\Big|_{x^n = e^{nl_p(z_1)}, \log x = l_p(z_1)}$. Then $I$ is a $P(z_1)$-intertwining map of type $\binom{W_1 \boxtimes_{P(z_2)} W_2}{W_1 W_2}$ and we have a $P(z_2)$-product $(W_1 \boxtimes_{P(z_2)} W_2, I)$ of $W_1$ and $W_2$. By the universal property of the $P(z_1)$-tensor product $W_1 \boxtimes_{P(z_1)} W_2$, there exists a unique $V$-module map

$$\mathcal{T}_{[\gamma]} : W_1 \boxtimes_{P(z_1)} W_2 \to W_1 \boxtimes_{P(z_2)} W_2$$

such that $\overline{\mathcal{T}_{[\gamma]}} \circ \boxtimes_{P(z_1)} = \overline{I}$. The property follows immediately. The $V$-module map $\mathcal{T}_{[\gamma]}$ is invertible since the same construction also gives $V$-module map

$$\mathcal{T}_{[\gamma^{-1}]} : W_1 \boxtimes_{P(z_2)} W_2 \to W_1 \boxtimes_{P(z_1)} W_2$$

which is clearly the inverse of $\mathcal{T}_{[\gamma]}$. Thus the natural transformation $\mathcal{T}_{[\gamma]}$ is a natural isomorphism. ∎

We have constructed the associativity isomorphisms $\mathcal{A}_{P(z_1),P(z_2)}^{P(z_4),P(z_3)}$ for $z_1, z_2 \in \mathbb{C}^\times$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$, $z_3 = z_1 - z_2$ and $z_4 = z_2$. Let $W_1$, $W_2$ and $W_3$ be grading-restricted generalized $V$-modules. We now constructe associativity isomorphisms

$$\mathcal{A}_{P(z_1),P(z_2)}^{P(z_4),P(z_3)} : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \to (W_1 \boxtimes_{P(z_4)} W_2) \boxtimes_{P(z_3)} W_3$$

for general $z_1, z_2, z_3, z_4 \in \mathbb{C}^\times$.

Let $\zeta_1$ and $\zeta_2$ be nonzero complex numbers satisfying $|\zeta_1| > |\zeta_2| > |\zeta_1 - \zeta_2| > 0$. Let $\gamma_1$ and $\gamma_2$, be paths from $z_1$ and $z_2$ to $\zeta_1$ and $\zeta_2$, respectively, in the complex plane with a cut along the positive real line, and let $\gamma_3$ and $\gamma_4$ be paths from $\zeta_2$ and $\zeta_1 - \zeta_2$ to $z_3$ and $z_4$, respectively, also in the complex plane with a cut along the positive real line. Then we define

$$\mathcal{A}_{P(z_1),P(z_2)}^{P(z_4),P(z_3)} = \mathcal{T}_{\gamma_3} \circ (\mathcal{T}_{\gamma_4} \boxtimes_{P(\zeta_2)} 1_{W_3}) \circ \mathcal{A}_{P(\zeta_1),P(\zeta_2)}^{P(\zeta_1-\zeta_2),P(\zeta_2)} \circ (1_{W_1} \boxtimes_{P(\zeta_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1},$$

that is, $\mathcal{A}_{P(z_1),P(z_2)}^{P(z_4),P(z_3)}$ is given by the commutative diagram

$$\begin{array}{ccc}
W_1 \boxtimes_{P(\zeta_1)} (W_2 \boxtimes_{P(\zeta_2)} W_3) & \xrightarrow{\mathcal{A}_{P(\zeta_1),P(\zeta_2)}^{P(\zeta_1-z_2),P(\zeta_2)}} & (W_1 \boxtimes_{P(\zeta_1-\zeta_2)} W_2) \boxtimes_{P(\zeta_2)} W_3 \\
{\scriptstyle (1_{W_1}\boxtimes_{P(\zeta_1)}\mathcal{T}_{\gamma_2})\circ\mathcal{T}_{\gamma_1}} \uparrow & & \downarrow {\scriptstyle \mathcal{T}_{\gamma_3}\circ(\mathcal{T}_{\gamma_4}\boxtimes_{P(\zeta_2)}1_{W_3})} \\
W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) & \xrightarrow{\mathcal{A}_{P(z_1),P(z_2)}^{P(z_4),P(z_3)}} & (W_1 \boxtimes_{P(z_4)} W_2) \boxtimes_{P(z_3)} W_3
\end{array}$$

The inverse of $\mathcal{A}_{P(z_1),P(z_2)}^{P(z_4),P(z_3)}$ is denoted $\alpha_{P(z_1),P(z_2)}^{P(z_4),P(z_3)}$. It is clear that $\mathcal{A}_{P(z_1),P(z_2)}^{P(z_4),P(z_3)}$ is independent of $\zeta_1, \zeta_2$ and $\gamma_1, \gamma_2, \gamma_3, \gamma_4$.

In particular, when $z_1 = z_2 = z_3 = z_4 = 1$, we have the corresponding natural associativity isomorphism

$$\mathcal{A}_{P(1),P(1)}^{P(1),P(1)} : W_1 \boxtimes (W_2 \boxtimes W_3) \to (W_1 \boxtimes W_2) \boxtimes W_3.$$

We shall simply denote this associativity isomorphism by $\mathcal{A}$.

In the case $|z_1| > |z_2| > |z_1 - z_2| > 0$, the associativity isomorphism $\mathcal{A}_{P(z_1),P(z_2)}^{P(z_2),P(z_1-z_2)}$ satisfies (4.1). In fact, (4.1) also holds when $|z_1| = |z_2| = |z_1 - z_2| > 0$.

**Proposition 4.6.** *For any $z_1, z_2 \in \mathbb{C}^\times$ such that $z_1 \neq z_2$ but $|z_1| = |z_2| = |z_1 - z_2|$, we have*

$$\overline{\mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}}(w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)) = (w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3 \tag{4.2}$$

*for $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$, where*

$$\overline{\mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}} : \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \to \overline{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3}$$

*is the natural extension of $\mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}$.*

*Proof.* To prove (4.2), we choose $\epsilon_1 \in \mathbb{C}$ such that $|z_1 + \epsilon_1| > |z_2| > |(z_1 + \epsilon_1) - z_2| > 0$. Then we know that (4.2) holds when $z_1$ is replaced by $z_1 + \epsilon_1$. But $\mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}$ is defined to be the composition of $\mathcal{A}_{P(z_1+\epsilon_1),P(z_2)}^{P((z_1+\epsilon_1)-z_2),P(z_2)}$ and the parallel transport isomorphism $\mathcal{T}_\gamma$ associated to a path $\gamma$ from $z_1 + \epsilon_1$ to $z_1$ in the complex plane with a cut along the nonnegative real line. We further choose $\epsilon_1$ and the path $\gamma$ such that $|z_1 + \epsilon_1| > |\epsilon_1|$ and the path $\gamma - z_2$ from $(z_1 + \epsilon_1) - z_2$ to $z_1 - z_2$ is also in the complex plane with a cut along the nonnegative real line.

Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be intertwining operators of types

$$\binom{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}{W_1 \;\; W_2 \boxtimes_{P(z_2)} W_3}$$

and

$$\binom{W_2 \boxtimes_{P(z_2)} W_3}{W_2 \;\; W_3},$$

respectively, corresponding to the intertwining maps $\boxtimes_{P(z_1)}$ and $\boxtimes_{P(z_2)}$, respectively. Then the series

$$\langle w', \mathcal{Y}_1(w_1, z_1 + \epsilon_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle$$

is absolutely convergent for $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$ and

$$w' \in (W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))'.$$

The sums of these series define elements

$$\mathcal{Y}_1(w_1, z_1 + \epsilon_1)\mathcal{Y}_2(w_2, z_2)w_3 \in \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}.$$

By the definition of the parallel transport isomorphism, for any path $\gamma$ from $z_1 + \epsilon_1$ to $z_1$ in the complex plane with a cut along the nonnegative real line, we have

$$\overline{\mathcal{T}_\gamma}(w_1 \boxtimes_{P(z_1+\epsilon_1)} (w_2 \boxtimes_{P(z_2)} w_3)) = \mathcal{Y}_1(w_1, z_1 + \epsilon_1)\mathcal{Y}_2(w_2, z_2)w_3. \tag{4.3}$$

By definition, we know that

$$\mathcal{T}_\gamma^{-1} = \mathcal{T}_{\gamma^{-1}},$$

so that (4.3) can be written as

$$\overline{\mathcal{T}_{\gamma_1^{-1}}}(\mathcal{Y}_1(w_1, z_1 + \epsilon_1)\mathcal{Y}_2(w_2, z_2)w_3) = w_1 \boxtimes_{P(z_1+\epsilon_1)} (w_2 \boxtimes_{P(z_2)} w_3). \tag{4.4}$$

Since $|z_1 + \epsilon_1| > |z_2| > |(z_1 + \epsilon_1) - z_2| > 0$, we have

$$\overline{\mathcal{A}_{P(z_1+\epsilon_1),P(z_2)}^{P((z_1+\epsilon_1)-z_1),P(z_2)}}(w_1 \boxtimes_{P(z_1+\epsilon_1)} (w_2 \boxtimes_{P(z_2)} w_3)) = (w_1 \boxtimes_{P((z_1+\epsilon_1)-z_2)} w_2) \boxtimes_{P(z_2)} w_3. \tag{4.5}$$

Let $\mathcal{Y}_3$ and $\mathcal{Y}_4$ be intertwining operators of types

$$\binom{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3}{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \;\; W_3}$$

and

$$\binom{W_1 \boxtimes_{P(z_1-z_2)} W_2}{W_1 \;\; W_2},$$

respectively, corresponding to the intertwining maps $\boxtimes_{P(z_2)}$ and $\boxtimes_{P(z_1-z_2)}$, respectively. Then the series

$$\langle \tilde{w}', \mathcal{Y}_3(\mathcal{Y}_4(w_1, (z_1 + \epsilon_1) - z_2)w_2, z_2)w_3\rangle$$

is absolutely convergent for $m, n \in \mathbb{R}$ and

$$\tilde{w}' \in ((W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3)',$$

and the sums of these series define elements

$$\mathcal{Y}_3(\mathcal{Y}_4(w_1, (z_1+\epsilon_1)-z_2)w_2), z_2)w_3 \in \overline{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3}.$$

Since the path $\gamma - z_2$ from $(z_1+\epsilon_1)-z_2$ to $z_1 - z_2$ is also in the complex plane with a cut along the nonnegative real line, by the definition of the parallel transport isomorphism, we have

$$\overline{\mathcal{T}_{\gamma-z_2}}((w_1 \boxtimes_{P((z_1+\epsilon_1)-z_2)} w_2) \boxtimes_{P(z_2)} w_3) = \mathcal{Y}_3(\mathcal{Y}_4(w_1, (z_1+\epsilon_1)-z_2)w_2, z_2)w_3. \tag{4.6}$$

Combining (4.4)–(4.6) and using the definition of the associativity isomorphism $\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}$, we obtain

$$\overline{\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}}(\mathcal{Y}_1(w_1, z_1+\epsilon_1)\mathcal{Y}_2(w_2, z_2)w_3) = \mathcal{Y}_3(\mathcal{Y}_4(w_1, (z_1+\epsilon_1)-z_2)w_2), z_2)w_3.$$

In particular, for $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$ and $m \in \mathbb{C}$,

$$\begin{aligned} &\overline{\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}}(\mathcal{Y}_1(\pi_m(e^{-\epsilon_1 L(-1)}w_1), z_1+\epsilon_1)\mathcal{Y}_2(w_2, z_2)w_3) \\ &\quad = \mathcal{Y}_3(\mathcal{Y}_4(\pi_m(e^{-\epsilon_1 L(-1)}w_1), (z_1+\epsilon_1)-z_2)w_2), z_2)w_3. \end{aligned} \tag{4.7}$$

For $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$, since $|z_1+\epsilon_1| > |\epsilon_1| > 0$, both the series

$$\begin{aligned} &\sum_{m\in\mathbb{R}} e^{-\epsilon_1 \frac{\partial}{\partial z_1}} \langle w', \mathcal{Y}_1(\pi_m(w_1), z_1+\epsilon_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle \\ &\quad = e^{-\epsilon_1 \frac{\partial}{\partial z_1}} \langle w', \mathcal{Y}_1(w_1, z_1+\epsilon_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle \end{aligned}$$

and

$$\begin{aligned} &\sum_{m\in\mathbb{R}} \langle \tilde{w}', \mathcal{Y}_3(\mathcal{Y}_4(\pi_m(e^{-\epsilon_1 L(-1)}w_1), (z_1+\epsilon_1)-z_2)w_2, z_2)w_3\rangle \\ &\quad = \langle \tilde{w}', \mathcal{Y}_3(\mathcal{Y}_4(e^{-\epsilon_1 L(-1)}w_1, (z_1+\epsilon_1)-z_2)w_2, z_2)w_3\rangle \end{aligned}$$

are absolutely convergent for

$$w' \in (W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))'$$

and

$$\tilde{w}' \in ((W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3)'.$$

We know that

$$\langle \tilde{w}', \mathcal{Y}_1(w_1, z_1+\epsilon_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle$$

and

$$\langle \tilde{w}', \mathcal{Y}_3(\mathcal{Y}_4(w_1, (z_1+\epsilon_1)-z_2)w_2, z_2)w_3\rangle$$

are the values on a neighborhood of the point $(\zeta_1, \zeta_2) = (z_1+\epsilon_1, z_2)$ containing the point $(\zeta_1, \zeta_2) = (z_1, z_2)$ of single-valued branches $F(w', w_1, w_2, w_3; \zeta_1, \zeta_2)$ and $G(\tilde{w}', w_1, w_2, w_3; \zeta_1, \zeta_2)$, respectively, of some multivalued functions of $\zeta_1$ and $\zeta_2$. Then by the definition of the tensor product elements $w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)$ and $(w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3$, we have

$$\langle w', w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)\rangle = F(w', w_1, w_2, w_3; z_1, z_2) \tag{4.8}$$

and

$$\langle \tilde{w}', (w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3\rangle = G(\tilde{w}', w_1, w_2, w_3; z_1, z_2). \tag{4.9}$$

On the other hand, since $F(w', w_1, w_2, w_3; \zeta_1, \zeta_2)$ and $G(\tilde{w}', w_1, w_2, w_3; \zeta_1, \zeta_2)$ are analytic extensions of matrix elements of products and iterates of intertwining maps, properties of these products and iterates also hold for these functions if they still make sense. In particular, they satisfy the $L(-1)$-derivative property:

$$\begin{aligned}\frac{\partial}{\partial \zeta_1} F(w', w_1, w_2, w_3; \zeta_1, \zeta_2) &= F(w', L(-1)w_1, w_2, w_3; \zeta_1, \zeta_2), & (4.10)\\ \frac{\partial}{\partial \zeta_1} G(\tilde{w}', w_1, w_2, w_3; \zeta_1, \zeta_2) &= G(\tilde{w}', L(-1)w_1, w_2, w_3; \zeta_1, \zeta_2), & (4.11)\end{aligned}$$

From the Taylor theorem (which applies since $|z_1+\epsilon_1| > |\epsilon_1|$) and (4.10)–(4.11), we have

$$\begin{aligned}F(w', w_1, w_2, w_3; z_1, z_2) &= \sum_{m\in\mathbb{R}} F(w', \pi_m(e^{-\epsilon_1 L(-1)})w_1, w_2, w_3; z_1+\epsilon_1, z_2), & (4.12)\\ G(\tilde{w}', w_1, w_2, w_3; z_1, z_2) &= \sum_{m\in\mathbb{R}} G(\tilde{w}', \pi_m(e^{-\epsilon_1 L(-1)})w_1, w_2, w_3; z_1+\epsilon_1, z_2). & (4.13)\end{aligned}$$

Thus by the definitions of

$$F(w', \pi_m(e^{-\epsilon_1 L(-1)})w_1, w_2, w_3; z_1+\epsilon_1, z_2),$$
$$G(\tilde{w}', \pi_m(e^{-\epsilon_1 L(-1)})w_1, w_2, w_3; z_1+\epsilon_1, z_2),$$

and by (4.12), (4.13), (4.8) and (4.9), we obtain

$$\begin{aligned}&\sum_{m\in\mathbb{R}} \langle w', \mathcal{Y}_1(\pi_m(e^{-\epsilon_1 L(-1)}w_1), z_1+\epsilon_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle\\ &\quad = \langle w', w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)\rangle & (4.14)\end{aligned}$$

and

$$\begin{aligned}&\sum_{m\in\mathbb{R}} \langle \tilde{w}', \mathcal{Y}_3(\mathcal{Y}_4(\pi_m(e^{-\epsilon_1 L(-1)}w_1), (z_1+\epsilon_1)-z_2)w_2, z_2)w_3\rangle\\ &\quad = \langle \tilde{w}', (w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3\rangle. & (4.15)\end{aligned}$$

Since $w'$ and $\tilde{w}'$ are arbitrary, (4.14) and (4.15) gives

$$\sum_{m\in\mathbb{R}} \mathcal{Y}_1(\pi_m(e^{-\epsilon_1 L(-1)}w_1), z_1+\epsilon_1)\mathcal{Y}_2(w_2, z_2)w_3 = w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3) \tag{4.16}$$

and

$$\sum_{m\in\mathbb{R}} \mathcal{Y}_3(\mathcal{Y}_4(\pi_m(e^{-\epsilon_1 L(-1)}w_1), (z_1+\epsilon_1)-z_2)w_2, z_2)w_3 = (w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3. \tag{4.17}$$

Taking the sum $\sum_{m\in\mathbb{R}}$ on both sides of (4.7) and then using (4.16) and (4.17), we obtain (4.2). ∎

# 5 Skew-symmetry and commutativity of intertwining operators and braiding isomorphisms

## 5.1 Skew-symmetry and commutativity of intertwining operators

For an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$ and an integer $p$, we introduce a linear map

$$\begin{aligned}\Omega_p(\mathcal{Y}) : W_2 \otimes W_1 &\to W_3\{x\}[\log x]\\ w_2 \otimes w_1 &\mapsto \Omega_p(\mathcal{Y})(w_2, x)w_1\end{aligned}$$

defined by

$$\Omega(\mathcal{Y})_p(w,x)w_1 = e^{xL_{W_2\boxtimes_{P(-z)}W_1}(-1)}\mathcal{Y}(w_1,y)w_2\Big|_{y^n=e^{n(\pi i+2p\pi)}x^n,\ \log y=\log x+\pi i+2p\pi i}.$$

We have a commutativity isomorphism. Let $z\in\mathbb{C}^\times$. For grading-restricted generalized $V$-modules $W_1$ and $W_2$, let $\mathcal{Y}$ be the intertwining operator associated to the $P(z)$-tensor product $W_2 \boxtimes_{P(-z)} W_1$. The we have an intertwining operator $\Omega(\mathcal{Y})$ of type $\binom{W_2\boxtimes_{P(-z)}W_1}{W_2 W_1}$, where $\Omega(\mathcal{Y})$ is defined by

$$\Omega(\mathcal{Y})(w_2,x)w_1 = e^{xL_{W_2\boxtimes_{P(-z)}W_1}(-1)}\mathcal{Y}(w_1,y)w_2\Big|_{y^n=e^{n\pi i}x^n,\ \log y=\log x+\pi i}.$$

## 5.2 Commutativity and braiding isomorphisms

The pair $(W_2 \boxtimes_{P(-z)} W_1, \Omega(\mathcal{Y})(\cdot, z)\cdot)$ is a $P(z)$-product of $W_1$ and $W_2$. By the universal property of the tensor product $W_1 \boxtimes_{P(z)} W_2$, there exists a unique $V$-module map

$$\mathcal{R}_{P(z)} : W_1 \boxtimes_{P(z)} W_2 \to W_2 \boxtimes_{P(-z)} W_1$$

such that

$$\Omega(\mathcal{Y})(\cdot, z)\cdot = \overline{\mathcal{R}}_{P(z)} \circ \boxtimes_{P(z)},$$

where $\overline{\mathcal{R}}_{P(z)}$ is the natrual extension of $\mathcal{R}_{P(z)}$ and $\boxtimes_{P(z)}$ is the value at $z$ of the intertwining operator associated to the tensor product $W_1 \boxtimes_{P(z)} W_2$.

Let $\gamma$ be a path from $-1$ to $1$ in the closed upper half plane with 0 deleted. Let $W_1$ and $W_2$ be grading-restricted generalized $V$-modules. We define the brading isomorphsim $\mathcal{R} : W_1 \boxtimes W_2 \to W_2 \boxtimes W_1$ by

$$\mathcal{R} = \mathcal{T}_\gamma \circ \mathcal{R}_{P(1)}.$$

**Proposition 5.1.** *Let $z_1, z_2$ be nonzero complex numbers such that $z_1 \neq z_2$ but $|z_1| = |z_2| = |z_1 - z_2|$. Let $\gamma$ be a path from $z_2$ to $z_1$ in the complex plane with a cut along the nonnegative real line. Then we have*

$$\overline{\mathcal{T}_\gamma \circ (\mathcal{R}_{P(z_1-z_2)} \boxtimes_{P(z_2)} 1_{W_3})}((w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3) = (w_2 \boxtimes_{P(z_2-z_1)} w_1) \boxtimes_{P(z_1)} w_3 \quad (5.1)$$

*for $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$.*

*Proof.* To pove (5.1), we choose $\epsilon$ such that $|z_2| > |\epsilon|$, $|z_2 + \epsilon| > |z_1 - z_2| > 0$. Let $\mathcal{Y}_1 = \mathcal{Y}_{\boxtimes_{P(z_2)},0}$, $\tilde{\mathcal{Y}}_1 = \mathcal{Y}_{\boxtimes_{P(z_2)},0}$ and $\mathcal{Y}_2 = \mathcal{Y}_{\boxtimes_{P(z_1)},0}$ be the intertwining operator of types

$$\binom{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3}{(W_1 \boxtimes_{P(z_1-z_2)} W_2)\ \ W_3},$$

$$\binom{(W_2 \boxtimes_{P(z_2-z_1)} W_1) \boxtimes_{P(z_2)} W_3}{(W_2 \boxtimes_{P(z_2-z_1)} W_1)\ \ W_3},$$

and

$$\binom{(W_2 \boxtimes_{P(z_2-z_1)} W_1) \boxtimes_{P(z_1)} W_3}{(W_2 \boxtimes_{P(z_2-z_1)} W_1)\ \ W_3},$$

respectively, corresponding to the intertwining maps $\boxtimes_{P(z_2)}$, $\boxtimes_{P(z_2)}$ and $\boxtimes_{P(z_1)}$, respectively.

Using the definition of $\mathcal{R}_{P(z_1-z_2)}$, we obtain

$$\begin{aligned}
&\overline{(\mathcal{R}_{P(z_1-z_2)} \boxtimes_{P(z_2)} 1_{W_3})}(\mathcal{Y}_1(w_1 \boxtimes_{P(z_1-z_2)} w_2, z_2 + \epsilon)w_3) \\
&\quad = \tilde{\mathcal{Y}}_1(e^{(z_1-z_2)L(-1)}(w_2 \boxtimes_{P(z_2-z_1)} w_1), z_2 + \epsilon)w_3. \qquad (5.2)
\end{aligned}$$

On the other hand, by the definition of the parallel isomorphism, we obtain

$$\begin{aligned}
&\overline{\mathcal{T}_\gamma}(\tilde{\mathcal{Y}}_1(e^{(z_1-z_2)L(-1)}(w_2 \boxtimes_{P(z_2-z_1)} w_1), z_2 + \epsilon)w_3) \\
&\quad = \mathcal{Y}_3((w_2 \boxtimes_{P(z_2-z_1)} w_1), z_1 + \epsilon)w_3. \qquad (5.3)
\end{aligned}$$

From (5.2) and (5.3), we obtain

$$\begin{aligned}
&\overline{\mathcal{T}_\gamma \circ (\mathcal{R}_{P(z_1-z_2)} \boxtimes_{P(z_2)} 1_{W_3})}(\mathcal{Y}_1(w_1 \boxtimes_{P(z_1-z_2)} w_2, z_2 + \epsilon)w_3) \\
&\quad = \mathcal{Y}_3((w_2 \boxtimes_{P(z_2-z_1)} w_1), z_1 + \epsilon)w_3.
\end{aligned}$$

Taking the limit $\epsilon \to 0$, we obtain

$$\begin{aligned}&\overline{\mathcal{T}_\gamma \circ (\mathcal{R}_{P(z_1-z_2)} \boxtimes_{P(z_2)} 1_{W_3})}(\mathcal{Y}_1(w_1 \boxtimes_{P(z_1-z_2)} w_2, z_2)w_3)\\&\quad = \mathcal{Y}_3((w_2 \boxtimes_{P(z_2-z_1)} w_1), z_1)w_3,\end{aligned}$$

which is the same as (5.1). ∎

**Proposition 5.2.** *Let $z_1, z_2$ be nonzero complex numbers such that $z_1 \neq z_2$ but $|z_1| = |z_2| = |z_1 - z_2|$. Let $\gamma$ be a path from $z_2$ to $z_2 - z_1$ in the complex plane with a cut along the nonnegative real line. Then we have*

$$\begin{aligned}&\overline{\mathcal{T}_\gamma \circ (1_{W_2} \boxtimes_{P(z_2)} \mathcal{R}_{P(z_1)})}(w_2 \boxtimes_{P(z_2)} (w_1 \boxtimes_{P(z_1)} w_3))\\&\quad = e^{z_1 L(-1)}(w_2 \boxtimes_{P(z_2-z_1)} (w_3 \boxtimes_{P(-z_1)} w_1))\end{aligned} \tag{5.4}$$

*for $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$.*

*Proof.* To prove (5.4), we choose $\epsilon \in \mathbb{C}$ such that $|z_2| > |\epsilon|$, $|z_2 + \epsilon|, |z_2 - z_1 + \epsilon| > |z_1| > 0$. Let $\mathcal{Y}_1 = \mathcal{Y}_{\boxtimes_{P(z_2)},0}$, $\tilde{\mathcal{Y}}_1 = \mathcal{Y}_{\boxtimes_{P(z_2)},0}$ and $\mathcal{Y}_2 = \mathcal{Y}_{\boxtimes_{P(z_2-z_1)},0}$ be intertwining operators of types

$$\binom{W_2 \boxtimes_{P(z_2)} (W_1 \boxtimes_{P(z_1)} W_3)}{W_2 \;\; (W_1 \boxtimes_{P(z_1)} W_3)},$$

$$\binom{W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(-z_1)} W_1)}{W_2 \;\; (W_3 \boxtimes_{P(-z_1)} W_1)}$$

and

$$\binom{W_2 \boxtimes_{P(z_2-z_1)} (W_1 \boxtimes_{P(-z_1)} W_3)}{W_2 \;\; W_1 \boxtimes_{(-z_1)} W_3},$$

respectively, corresponding to the intertwining maps $\boxtimes_{P(z_2)}$, $\boxtimes_{P(z_2)}$ and $\boxtimes_{P(z_2-z_1)}$, respectively.

Using the definition of $\mathcal{R}_{P(z_1)}$ and and the $L(-1)$-conjugation property, we obtain

$$\begin{aligned}&\overline{(1_{W_2} \boxtimes_{P(z_2)} \mathcal{R}_{P(z_1)})}(\mathcal{Y}_1(w_2, z_2 + \epsilon)(w_1 \boxtimes_{P(z_1)} w_3))\\&\quad = \tilde{\mathcal{Y}}_1(w_2, z_2 - z_1 + \epsilon)(e^{z_1 L(-1)}(w_3 \boxtimes_{P(-z_1)} w_1))\\&\quad = e^{z_1 L(-1)} \tilde{\mathcal{Y}}_1(w_2, z_2 - z_1 + \epsilon)(w_3 \boxtimes_{P(-z_1)} w_1).\end{aligned} \tag{5.5}$$

Using the definition of the parallel transport isomorphism, we obtain

$$\begin{aligned}&\overline{\mathcal{T}_\gamma}(e^{z_1 L(-1)} \mathcal{Y}_1(w_2, z_2 - z_1 + \epsilon)(w_3 \boxtimes_{P(-z_1)} w_1))\\&\quad = e^{z_1 L(-1)} \mathcal{Y}_3(w_2, z_2 - z_1 + \epsilon)(w_3 \boxtimes_{P(-z_1)} w_1).\end{aligned} \tag{5.6}$$

From (5.5) and (5.6), we obtain

$$\overline{\mathcal{T}_\gamma \circ (1_{W_2} \boxtimes_{P(z_2)} \mathcal{R}_{P(z_1)})}(\mathcal{Y}_1(w_2, z_2 + \epsilon)(w_1 \boxtimes_{P(z_1)} w_3))$$

$$= e^{z_1 L(-1)} \mathcal{Y}_3(w_2, z_2 - z_1 + \epsilon)(w_3 \boxtimes_{P(-z_1)} w_1). \tag{5.7}$$

Taking the limit $\epsilon \to 0$, we obtain from (5.8)

$$\begin{aligned} &\overline{\mathcal{T}_\gamma \circ (1_{W_2} \boxtimes_{P(z_2)} \mathcal{R}_{P(z_1)})}(\mathcal{Y}_1(w_2, z_2)(w_1 \boxtimes_{P(z_1)} w_3)) \\ &\quad = e^{z_1 L(-1)} \mathcal{Y}_3(w_2, z_2 - z_1)(w_3 \boxtimes_{P(-z_1)} w_1), \end{aligned} \tag{5.8}$$

which is the same as (5.4).

# 6 Vertex tensor categories and braided tensor categories

## 6.1 Vertex tensor categories

We need the left and right unit isomorphisms. Given a grading-restricted generalized $V$-module $W$, let $\mathcal{Y}$ be the intertwining operator of type $\binom{V \boxtimes_{P(z)} W}{VW}$ given in the construction of the tensor product $V \boxtimes_{P(z)} W$. Then $V \boxtimes_{P(z)}$ is spanned by the homogeneous components of $\mathcal{Y}(v, z)w$ for $v \in V$ and $w \in W$. Using $v = \mathrm{Res}_x x^{-1} Y_W(v, x)\mathbf{1}$ and the associator formula for the intertwining operator $\mathcal{Y}$, we see that homogeneous components of $\mathcal{Y}(v, z)w$ for $v \in V$ and $w \in W$ are in fact spanned by elements of the form $\mathcal{Y}(\mathbf{1}, z)w$ for $w \in W$. But by the $L(-1)$-derivative property, $\mathcal{Y}(\mathbf{1}, z)w$ is independent of $z$ and by the $L(0)$-commutator formulas, it is homogeneous of wieght $\mathrm{wt}\, w$ if $w$ is homogeneous. In particular, it is a well defined element of $V \boxtimes_{P(z)} W$. Then $\mathcal{Y}(\mathbf{1}, z)$ is a linear map from $W$ to $V \boxtimes_{P(z)} W$. We denote this map by $\psi$. For $v \in V$ and $w \in W$, using the commutator formula for $\mathcal{Y}$ and $Y_V(v, x)\mathbf{1} \in V[[x_0]]$, we obtain

$$\begin{aligned} Y_{V \boxtimes_{P(z)} W}(v, x)\psi(w) &= Y_{V \boxtimes_{P(z)} W}(v, x)\mathcal{Y}(\mathbf{1}, z)w \\ &= \mathcal{Y}(\mathbf{1}, z)Y_W(v, x)w + \mathrm{Res}_x x^{-1} \delta\left(\frac{z + x}{x}\right) \mathcal{Y}(Y_V(v, x)\mathbf{1}, z)w \\ &= \mathcal{Y}(\mathbf{1}, z)Y_W(v, x)w \\ &= \psi(Y_W(v, x)w). \end{aligned}$$

So $\psi$ is a $V$-module map. Since $Y_W$ is an intertwining operator of type $\binom{W}{VW}$, $(W, Y_W(\cdot, z)\cdot)$ is a $P(z)$-product $(W, Y_W(\cdot, z)\cdot)$ of $V$ and $W$. By the universal property of the tensor product $(V \boxtimes_{P(z)} W, \mathcal{Y}(\cdot, z)\cdot)$, there exists a unique $V$-module map $\phi : V \boxtimes_{P(z)} W \to W$ such that $Y_W(v, z)w = \overline{\phi}(\mathcal{Y}(v, z)w)$ for $v \in V$ and $w \in W$. In particular,

$$w = Y_W(\mathbf{1}, z)w = \overline{\phi}(\mathcal{Y}(\mathbf{1}, z)w) = \phi(\psi(w)).$$

So $\phi$ and $\psi$ are inverse to each other and thus are equivalences. We define the left $P(z)$-unit isomorphism $l_{W;z} : V \boxtimes_{P(z)} W \to W$ to be $\phi$.

We can also define the right $P(z)$-unit isomorphism $r_{W;z} : W \boxtimes_{P(z)} V \to W$ similarly. We omit the details here.

Before we prove that our category equipped with these data is a vertex tensor category, we need to add another assumption on the covengence of products of intertwining operators.

**Assumption 6.1** (Convergence of products of intertwining operators)**.** Let $W_0, W_1, \dots W_{n+1}$, $\widetilde{W}_1, \dots, \widetilde{W}_{n-1}$ be grading-restricted generalized $V$-modules and $\mathcal{Y}_1, \dots, \mathcal{Y}_i, \dots, \mathcal{Y}_n$ intertwining operators of types $\binom{W_0}{W_1\, \widetilde{W}_1}, \dots, \binom{\widetilde{W}_{i-1}}{W_i\, \widetilde{W}_i}, \dots, \binom{\widetilde{W}_{n-1}}{W_n\, W_{n+1}}$, respectively. For $w_1 \in W_1, \dots, w_{n+1} \in W_{n+1}$ and $w_0' \in W_0'$, the series

$$\langle w_0', \mathcal{Y}_1(w_1, z_1) \cdots \mathcal{Y}_n(w_n, z_n) w_{n+1}\rangle \tag{6.1}$$

is absolutely convergent in the region $|z_1| > \cdots > |z_n| > 0$ and its sum can be analytically continued to a multivalued analytic function

$$F(\langle u_1', \mathcal{Y}_1(w_1, z_1) \cdots \mathcal{Y}_n(w_n, z_n) u_{n+1}\rangle)$$

on the region

$$\{(z_1, \dots, z_n) \mid z_i \neq 0, z_i - z_j \neq 0 \text{ for } i \neq j\} \subset \mathbb{C}^n$$

with the only possible singular points $z_i = 0, \infty$ and $z_i = z_j$ being regular singular points.

This assumption holds when all the grading-restricted generalized $V$-modules involved are $C_1$-cofinite.

**Theorem 6.2.** *In the setting of Assumption 6.1, if $W_0, W_1, \dots W_{n+1}$ are $C_1$-cofinite, then* (6.1) *is absolutely convergent in the region $|z_1| > \cdots > |z_n| > 0$ and its sum can be analytically continued to a multivalued analytic function with the only possible singular points $z_i = 0, \infty$ and $z_i = z_j$ being regular singular points.*

See [H3] for a proof of this result.

We need tensor product elements of four elements in four grading-restricted generalized $V$-modules. Let $W_1, W_2, W_3$ and $W_4$ be grading-restricted generalized $V$-modules. For $z_1, z_2, z_3 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_3| > 0$, we have the tensor product $V$-modules $W_3 \boxtimes_{P(z_3)} W_4$, $W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)$ and $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4))$. let $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3$ be the intertwining operators of the corresponding types such that $\boxtimes_{P(z_1)} = \mathcal{Y}_1(\cdot, z_1)\cdot$, $\boxtimes_{P(z_2)} = \mathcal{Y}_2(\cdot, z_2)\cdot$ and $\boxtimes_{P(z_3)} = \mathcal{Y}_3(\cdot, z_3)\cdot$. Then by Assumption 6.1, for $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$ and $w_4 \in W_4$,

$$\mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)\mathcal{Y}_3(w_3, z_3)w_4$$

is absolutely convergent to an element of $\overline{((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4}$. We define the tensor product element $w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} (w_3 \boxtimes_{P(z_3)} w_4))$ to be this element of $\overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4))}$. Similarly, we have other tensor product elements

$$(w_1 \boxtimes_{P(z_1 - z_2)} w_2) \boxtimes_{P(z_2)} (w_3 \boxtimes_{P(z_3)} w_4) \in \overline{(W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)},$$
$$w_1 \boxtimes_{P(z_1)} ((w_2 \boxtimes_{P(z_2 - z_3)} w_3) \boxtimes_{P(z_3)} w_4) \in \overline{W_1 \boxtimes_{P(z_1)} ((W_2 \boxtimes_{P(z_2 - z_3)} W_3) \boxtimes_{P(z_3)} W_4)},$$
$$(w_1 \boxtimes_{P(z_1 - z_3)} (w_2 \boxtimes_{P(z_2 - z_3)} w_3)) \boxtimes_{P(z_3)} w_4 \in \overline{(W_1 \boxtimes_{P(z_1 - z_3)} (W_2 \boxtimes_{P(z_2 - z_3)} W_3)) \boxtimes_{P(z_3)} W_4},$$

$$((w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2-z_3)} w_3) \boxtimes_{P(z_3)} w_4 \in \overline{((W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2-z_3)} W_3) \boxtimes_{P(z_3)} W_4}$$

for suitable $z_1, z_2 \in \mathbb{C}^\times$. The natural extensions of the associativity isomorphisms to the algebriac completions of the corresponding modules in our category send such an element to another such element. Also the homogeneous components of these elements span the tensor product modules.

## 6.2 Braided tensor categories

**Theorem 6.3.** *Under Assumptions 3.7, 4.1 and 6.1, the category of grading-restricted generalized $V$-modules equipped with the tensor product bifunctor $\boxtimes = \boxtimes_{P(1)}$, the associativity isomorphism $\mathcal{A}$, the braiding isomorpism $\mathcal{R}$, the unit object $V$ and the left and right unit isomorphisms $l = l_1$ and $r = r_1$ is a braided tensor category.*

*Proof.* We first prove the commtativity of pentagon diagrams for $\mathcal{A}_{P(z_1),P(z_2)}^{P(z_2),P(z_1-z_2)}$ and $\mathcal{A}$.

We first prove the commutativity of the pentagon diagram involving these $z$'s. This is in fact the prntagon diagram for vertex tensor categories. Let $W_1$, $W_2$, $W_3$ and $W_4$ be $V$-modules and let $z_1, z_2, z_3 \in \mathbb{C}$ satisfying

$$\begin{gathered}
|z_1| > |z_2| > |z_3| > |z_1 - z_3| > |z_2 - z_3| > |z_1 - z_2| > 0,\\
|z_1| > |z_2 - z_3| + |z_3|,\\
|z_2| > |z_1 - z_2| + |z_3|,\\
|z_2| > |z_1 - z_2| + |z_2 - z_3|.
\end{gathered}$$

For example, we can take $z_1 = 7$, $z_2 = 6$ and $z_3 = 4$. We want to prove the commutativity of the diagram:

$$\begin{array}{ccc}
 & W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)) & \\
\swarrow & & \searrow \\
(W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4) & & W_1 \boxtimes_{P(z_1)} ((W_2 \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4)) \\
\downarrow & & \downarrow \\
((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4 & \longleftarrow & (W_1 \boxtimes_{P(z_{13})} (W_2 \boxtimes_{P(z_{23})} W_3)) \boxtimes_{P(z_3)} W_4
\end{array} \tag{6.2}$$

where $z_{12} = z_1 - z_2$ and $z_{23} = z_2 - z_3$. For $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$ and $w_4 \in W_4$, we consider

$$w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} (w_3 \boxtimes_{P(z_3)} w_4)) \in \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4))}.$$

Since the natural extensions of the associativity isomorphisms send tensor products of elements to tensor products of elements, we see that the compositions of the natural extensions of the $V$-module maps in the two routes in (6.2) applying to this element both give

$$((w_1 \boxtimes_{P(z_{12})} w_2) \boxtimes_{P(z_{23})} w_3) \boxtimes_{P(z_3)} w_4 \in \overline{((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4}.$$

Since the homogeneous components of

$$w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} (w_3 \boxtimes_{P(z_3)} w_4))$$

for $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$ and $w_4 \in W_4$ span

$$W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)),$$

the diagram (6.2) above is commutative.

By the definition of $\mathcal{A}$, the diagrams

$$\begin{array}{ccc} W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)) & \longrightarrow & (W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4) \\ \downarrow & & \downarrow \\ W_1 \boxtimes (W_2 \boxtimes (W_3 \boxtimes W_4)) & \longrightarrow & (W_1 \boxtimes W_2) \boxtimes (W_3 \boxtimes W_4) \end{array} \tag{6.3}$$

$$\begin{array}{ccc} (W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4) & \longrightarrow & ((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4 \\ \downarrow & & \downarrow \\ (W_1 \boxtimes W_2) \boxtimes (W_3 \boxtimes W_4) & \longrightarrow & ((W_1 \boxtimes W_2) \boxtimes W_3) \boxtimes W_4 \end{array} \tag{6.4}$$

$$\begin{array}{ccc} W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)) & \longrightarrow & W_1 \boxtimes_{P(z_1)} ((W_2 \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4)) \\ \downarrow & & \downarrow \\ W_1 \boxtimes (W_2 \boxtimes (W_3 \boxtimes W_4)) & \longrightarrow & W_1 \boxtimes ((W_2 \boxtimes W_3) \boxtimes W_4) \end{array} \tag{6.5}$$

$$\begin{array}{ccc} W_1 \boxtimes_{P(z_1)} ((W_2 \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4)) & \longrightarrow & (W_1 \boxtimes_{P(z_{13})} (W_2 \boxtimes_{P(z_{23})} W_3)) \boxtimes_{P(z_3)} W_4 \\ \downarrow & & \downarrow \\ W_1 \boxtimes ((W_2 \boxtimes W_3) \boxtimes W_4) & \longrightarrow & (W_1 \boxtimes (W_2 \boxtimes W_3)) \boxtimes W_4 \end{array} \tag{6.6}$$

$$
\begin{array}{ccc}
(W_1 \boxtimes_{P(z_{13})} (W_2 \boxtimes_{P(z_{23})} W_3)) \boxtimes_{P(z_3)} W_4 & \longrightarrow & ((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4 \\
\downarrow & & \downarrow \\
(W_1 \boxtimes (W_2 \boxtimes W_3)) \boxtimes W_4 & \longrightarrow & ((W_1 \boxtimes W_2) \boxtimes W_3) \boxtimes W_4
\end{array}
\tag{6.7}
$$

are all commutative. Combining all the diagrams (6.2)–(6.7) above, we see that the pentagon diagram

$$
\begin{array}{ccc}
 & W_1 \boxtimes (W_2 \boxtimes (W_3 \boxtimes W_4)) & \\
\swarrow & & \searrow \\
(W_1 \boxtimes W_2) \boxtimes (W_3 \boxtimes W_4) & & W_1 \boxtimes ((W_2 \boxtimes W_3) \boxtimes W_4) \\
\downarrow & & \downarrow \\
((W_1 \boxtimes W_2) \boxtimes W_3) \boxtimes W_4 & \longleftarrow & (W_1 \boxtimes (W_2 \boxtimes W_3)) \boxtimes W_4
\end{array}
$$

is also commutative.

Next we prove the commutativity of the hexagon diagrams for the braiding isomorphisms. We prove only the commutativity of the hexagon diagram involving $\mathcal{R}$; the proof of the commutativity of the other hexagon diagram is the same. Let $W_1$, $W_2$ and $W_3$ be objects of $\mathcal{C}$ and let $z_1, z_2 \in \mathbb{C}^\times$ satisfying $|z_1| = |z_2| = |z_1 - z_2|$ and let $z_{12} = z_1 - z_2$. We first prove the commutativity of the following diagram:

$$
\begin{array}{ccc}
 & (W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} W_3 & \\
\mathcal{R}_{P(z_{12})} \boxtimes_{P(z_2)} 1_{W_3} \swarrow & & \searrow \left(\mathcal{A}^{P(z_{12}),P(z_2)}_{P(z_1),P(z_2)}\right)^{-1} \\
(W_2 \boxtimes_{P(-z_{12})} W_1) \boxtimes_{P(z_2)} W_3 & & W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \\
\mathcal{T}_{\gamma_1} \downarrow & & \\
(W_2 \boxtimes_{P(-z_{12})} W_1) \boxtimes_{P(z_1)} W_3 & & \downarrow \mathcal{R}_{P(z_1)} \\
\left(\mathcal{A}^{P(-z_{12}),P(z_1)}_{P(z_2),P(z_1)}\right)^{-1} \downarrow & & \\
W_2 \boxtimes_{P(z_2)} (W_1 \boxtimes_{P(z_1)} W_3) & & (W_2 \boxtimes_{P(z_2)} W_3) \boxtimes_{P(-z_1)} W_1 \\
1_{W_2} \boxtimes_{P(z_2)} \mathcal{R}_{P(z_1)} \searrow & & \\
W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(-z_1)} W_1) & & \swarrow \left(\mathcal{A}^{P(z_2),P(-z_1)}_{P(-z_{12}),P(-z_1)}\right)^{-1} \\
\mathcal{T}_{\gamma_2} \searrow & & \\
 & W_2 \boxtimes_{P(-z_{12})} (W_3 \boxtimes_{P(-z_1)} W_1) &
\end{array}
$$

$$(6.8)$$

where $\gamma_1$ and $\gamma_2$ are paths from $z_2$ to $z_1$ and from $z_2$ to $-z_{12}$, respectively, in $\mathbb{C}$ with a cut along the nonnegative real line.

Let $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$. By (4.2), (5.1), (5.1) and the definition of $\mathcal{R}_{P(z_1)}$, the images of the element

$$(w_1 \boxtimes_{P(z_{12})} w_2) \boxtimes_{P(z_2)} w_3$$

under the natural extension to

$$\overline{(W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} W_3}$$

of the compositions of the maps in both the left and right routes in (6.8) from

$$(W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} W_3$$

to

$$W_2 \boxtimes_{P(-z_{12})} (W_3 \boxtimes_{P(-z_1)} W_1)$$

are

$$e^{z_1 L(-1)}(w_2 \boxtimes_{P(-z_{12})} (w_3 \boxtimes_{P(-z_1)} w_1)).$$

Since the homogeneous components of

$$(w_1 \boxtimes_{P(z_{12})} w_2) \boxtimes_{P(z_2)} w_3$$

for $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$ span

$$(W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} W_3,$$

the diagram (6.8) commutes.

Now we consider the following diagrams:

$$\begin{array}{ccc} (W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} W_3 & \longrightarrow & (W_1 \boxtimes W_2) \boxtimes W_3 \\ \downarrow & & \downarrow \\ (W_2 \boxtimes_{P(-z_{12})} W_1) \boxtimes_{P(z_2)} W_3 & \longrightarrow & (W_2 \boxtimes W_1) \boxtimes W_3 \end{array} \tag{6.9}$$

$$\begin{array}{ccc} (W_2 \boxtimes_{P(-z_{12})} W_1) \boxtimes_{P(z_2)} W_3 & \longrightarrow & (W_2 \boxtimes W_1) \boxtimes W_3 \\ \downarrow & & \downarrow \\ (W_2 \boxtimes_{P(-z_{12})} W_1) \boxtimes_{P(z_1)} W_3 & \longrightarrow & (W_2 \boxtimes W_1) \boxtimes W_3 \end{array} \tag{6.10}$$

$$\begin{array}{ccc} (W_2 \boxtimes_{P(-z_{12})} W_1) \boxtimes_{P(z_1)} W_3 & \longrightarrow & (W_2 \boxtimes W_1) \boxtimes W_3 \\ \downarrow & & \downarrow \\ W_2 \boxtimes_{P(z_2)} (W_1 \boxtimes_{P(z_1)} W_3) & \longrightarrow & W_2 \boxtimes (W_1 \boxtimes W_3) \end{array} \tag{6.11}$$

$$\begin{array}{ccc} W_2 \boxtimes_{P(z_2)} (W_1 \boxtimes_{P(z_1)} W_3) & \longrightarrow & W_2 \boxtimes (W_1 \boxtimes W_3) \\ \downarrow & & \downarrow \\ W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(-z_1)} W_1) & \longrightarrow & W_2 \boxtimes (W_3 \boxtimes W_1) \end{array} \tag{6.12}$$

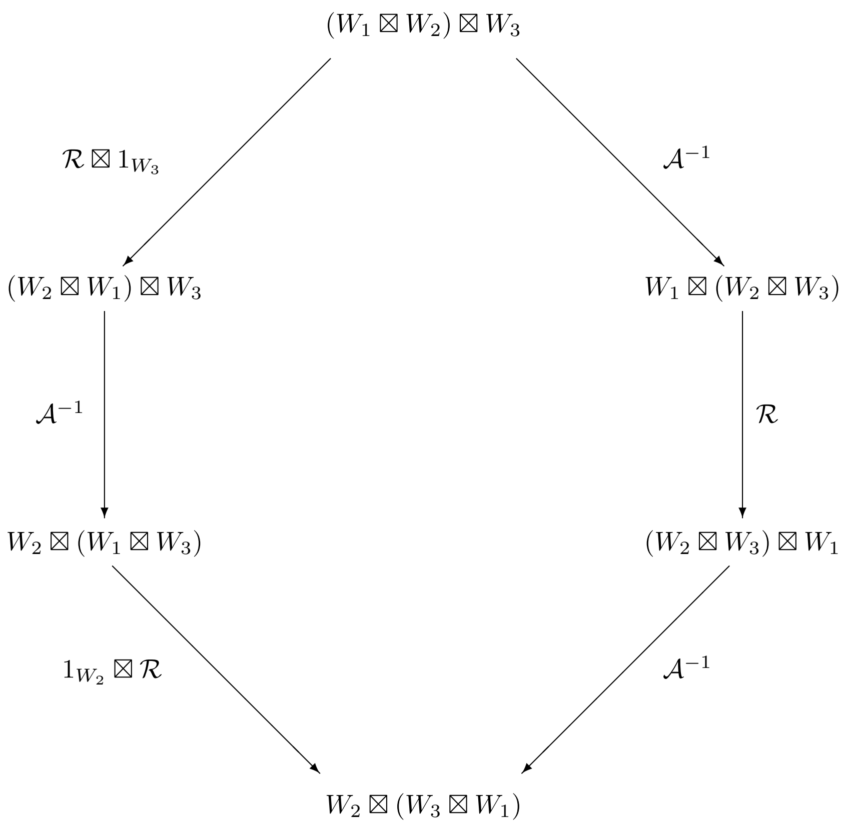
$$\begin{array}{ccc} (W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} W_3 & \longrightarrow & (W_1 \boxtimes W_2) \boxtimes W_3 \\ \downarrow & & \downarrow \\ W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) & \longrightarrow & W_1 \boxtimes (W_1 \boxtimes W_3) \end{array} \tag{6.13}$$

$$\begin{array}{ccc} W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) & \longrightarrow & W_1 \boxtimes (W_1 \boxtimes W_3) \\ \downarrow & & \downarrow \\ (W_2 \boxtimes_{P(z_2)} W_3) \boxtimes_{P(-z_1)} W_1 & \longrightarrow & (W_2 \boxtimes W_3) \boxtimes W_1 \end{array} \tag{6.14}$$

$$\begin{array}{ccc} (W_2 \boxtimes_{P(z_2)} W_3) \boxtimes_{P(-z_1)} W_1 & \longrightarrow & (W_2 \boxtimes W_3) \boxtimes W_1 \\ \downarrow & & \downarrow \\ W_2 \boxtimes_{P(-z_{12})} (W_3 \boxtimes_{P(-z_1)} W_1) & \longrightarrow & W_2 \boxtimes (W_3 \boxtimes W_1) \end{array} \tag{6.15}$$

$$\begin{array}{ccc} W_2 \boxtimes_{P(-z_{12})} (W_3 \boxtimes_{P(-z_1)} W_1) & \longrightarrow & W_2 \boxtimes (W_3 \boxtimes W_1) \\ \downarrow & & \downarrow \\ W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(-z_1)} W_1) & \longrightarrow & W_2 \boxtimes (W_3 \boxtimes W_1) \end{array} \tag{6.16}$$

The commutativity of the diagrams (6.9), (6.12) and (6.14) follows from the definition of the commutativity isomorphism for the braided tensor category structure and the naturality of the parallel transport isomorphisms. The commutativity of (6.11), (6.13) and (6.15) follows from the definition of the associativity isomorphism for the braided tensor product structure. The commutativity of (6.10) and (6.16) follows from the facts that compositions of parallel transport isomorphisms are equal to the parallel transport isomorphisms associated to the products of the paths and that parallel transport isomorphisms associated to homotopically equivalent paths are equal. The commutativity of the hexagon diagram (6.17) follows from (6.8)–(6.16).




(6.17)

We now prove the commutativity of the triangle diagram for the unit isomorphisms. Let $z_1$ and $z_2$ be complex numbers such that $|z_1| > |z_2| > |z_1 - z_2| > 0$ and let $z_{12} = z_1 - z_2$. Also let $\gamma$ be a path from $z_2$ to $z_1$ in $\mathbb{C}$ with a cut along the nonnegative real line. We first prove the commutativity of the following diagram:

$$\begin{array}{ccc} (W_1 \boxtimes_{P(z_{12})} V) \boxtimes_{P(z_2)} W_2 & \xrightarrow{(\mathcal{A}^{P(z_{12}),P(z_2)}_{P(z_1),P(z_2)})^{-1}} & W_1 \boxtimes_{P(z_1)} (V \boxtimes_{P(z_2)} W_2) \\ {\scriptstyle r_{z_{12};W_1} \boxtimes_{P(z_2)} 1_{W_2}} \downarrow & & \downarrow {\scriptstyle 1_{W_1} \boxtimes_{P(z_1)} l_{W_2}} \\ W_1 \boxtimes_{P(z_2)} W_2 & \xrightarrow[\mathcal{T}_\gamma]{} & W_1 \boxtimes_{P(z_1)} W_2. \end{array} \tag{6.18}$$

Let $w_1 \in W_1$ and $w_2 \in W_2$. Then we have

$$\begin{aligned} &\overline{(1_{W_1} \boxtimes_{P(z_1)} l_{z_2;W_2}) \circ (\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)})^{-1}}((w_1 \boxtimes_{P(z_{12})} \mathbf{1}) \boxtimes_{P(z_2)} w_2) \\ &\quad = \overline{(1_{W_1} \boxtimes_{P(z_1)} l_{z_2;W_2})}(w_1 \boxtimes_{P(z_1)} (\mathbf{1} \boxtimes_{P(z_2)} w_2) \\ &\quad = w_1 \boxtimes_{P(z_1)} w_2. \end{aligned} \tag{6.19}$$

But

$$\begin{aligned} &\overline{r_{z_{12};W_1} \boxtimes_{P(z_2)} 1_{W_2}}((w_1 \boxtimes_{P(z_{12})} \mathbf{1}) \boxtimes_{P(z_2)} w_2) \\ &\quad = (e^{z_{12}L(-1)} w_1) \boxtimes_{P(z_2)} w_2. \end{aligned} \tag{6.20}$$

Let $\mathcal{Y} = \mathcal{Y}_{\boxtimes_{P(z_1)},0}$ be the intertwining operator of type $\binom{W_1\boxtimes_{P(z_1)}W_2}{W_1\ W_2}$ corresponding to the $P(z_1)$-intertwining map $\boxtimes_{P(z_1)}$. Then by the definition of the parallel transport isomorphism and the $L(-1)$-derivative property for intertwining operators, we have

$$\begin{aligned}\overline{\mathcal{T}_\gamma}((e^{z_{12}L(-1)}w_1)\boxtimes_{P(z_2)} w_2) &= \mathcal{Y}(e^{z_{12}L(-1)}w_1, z_2)w_2\\ &= \mathcal{Y}(w_1, z_1)w_2\\ &= w_1\boxtimes_{P(z_1)} w_2. \end{aligned}\tag{6.21}$$

Since the elements $(w_1\boxtimes_{P(z_{12})}\mathbf{1})\boxtimes_{P(z_2)} w_3$ for $w_1\in W_1$ and $w_2\in W_2$ span $(W_1\boxtimes_{P(z_{12})} V)\boxtimes_{P(z_2)} W_3$, (6.19)–(6.21) give the commutativity of (6.18).

Let $\gamma_1$ be a path from $z_1$ to 1 in $\mathbb{C}$ with a cut along the nonnegative real line. Let $\gamma_2$ be the product of $\gamma$ and $\gamma_1$. In particular, $\gamma_2$ is a path from $z_2$ to 1 in $\mathbb{C}$ with a cut along the nonnegative real line. Also let $\gamma_{12}$ be a path from $z_{12} = z_1 - z_2$ to 1 in $\mathbb{C}$ with a cut along the nonnegative real line. Then we have the following commutative diagrams:

$$\begin{array}{ccc} (W_1\boxtimes V)\boxtimes W_2 & \xrightarrow{\mathcal{A}^{-1}} & W_1\boxtimes (V\boxtimes W_2)\\ {\scriptstyle \mathcal{T}_{\gamma_2}^{-1}\circ(\mathcal{T}_{\gamma_{12}}^{-1}\boxtimes_{P(z_2)}1_{W_2})}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{T}_{\gamma_1}^{-1}\circ(1_{W_1}\boxtimes_{P(z_1)}\mathcal{T}_{\gamma_2}^{-1})}\\ (W_1\boxtimes_{P(z_{12})} V)\boxtimes_{P(z_2)} W_2 & \xrightarrow[(\mathcal{A}_{P(z_1),P(z_2)}^{P(z_{12}),P(z_2)})^{-1}]{} & W_1\boxtimes_{P(z_1)} (V\boxtimes_{P(z_2)} W_2). \end{array}\tag{6.22}$$

$$\begin{array}{ccc} (W_1\boxtimes V)\boxtimes W_2 & \xrightarrow{\mathcal{T}_{\gamma_2}^{-1}\circ(\mathcal{T}_{\gamma_{12}}^{-1}\boxtimes 1_{W_2})} & (W_1\boxtimes_{P(z_{12})} V)\boxtimes_{P(z_2)} W_2\\ {\scriptstyle r_{W_1}\boxtimes 1_{W_2}}\Big\downarrow & & \Big\downarrow{\scriptstyle r_{W_1}\boxtimes_{P(z_2)}1_{W_2}}\\ W_1\boxtimes W_2 & \xrightarrow[\mathcal{T}_{\gamma_2}^{-1}]{} & W_1\boxtimes_{P(z_2)} W_2. \end{array}\tag{6.23}$$

$$\begin{array}{ccc} W_1\boxtimes_{P(z_1)} (W_2\boxtimes_{P(z_2)} W_2) & \xrightarrow{\mathcal{T}_{\gamma_1}\circ(1_{W_1}\boxtimes_{P(z_1)}\mathcal{T}_{\gamma_2})} & W_1\boxtimes (V\boxtimes W_2)\\ {\scriptstyle 1_{W_1}\boxtimes_{P(z_1)}l_{W_2}}\Big\downarrow & & \Big\downarrow{\scriptstyle 1_{W_1}\boxtimes l_{W_2}}\\ W_1\boxtimes_{P(z_1)} W_2 & \xrightarrow[\mathcal{T}_{\gamma_1}]{} & W_1\boxtimes W_2. \end{array}\tag{6.24}$$

$$\begin{array}{ccc} W_1\boxtimes_{P(z_2)} W_2 & \xrightarrow{\mathcal{T}_{\gamma}} & W_1\boxtimes_{P(z_1)} W_2\\ {\scriptstyle \mathcal{T}_{\gamma_2}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{T}_{\gamma_1}}\\ W_1\boxtimes W_2 & = & W_1\boxtimes W_2. \end{array}\tag{6.25}$$

The commutativity of (6.22) follows from the definition of $\mathcal{A}$. The commutativity of (6.23) and (6.24) follows from the definition of the left and right unit isomorphisms and the parallel transport isomorphisms. The commutativity of (6.25) follows from the fact that $\gamma_2$ is the product of $\gamma$ and $\gamma_1$. Combining (6.18) and (6.22)–(6.25), we obtain the commutativity of the triangle diagram for the unit isomorphisms.

Finally, it is clear from the definition that $l_V = r_V$.

Thus we have proved that the category $\mathcal{C}$ equipped with the data in the theorem is a braided monoidal category. ∎

# 7 Modular invariance of intertwining operators and the Verlide formula

In this section, we assume that every grading-restricted generalized $V$-module is completely reducible. In particular, every grading-restricted generalized $V$-module is an ordinary $V$-module (grading-restricted and $L(0)$-semisimple).

## 7.1 Modular invariance of intertwining operators

We first recall geometrically-modified intertwining operators from [H4] (see also [H8]). Given an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1\,W_2}$ and $w_1 \in W_1$, we have an operator (actually a series with linear maps from $W_2$ to $W_3$ as coefficients) $\mathcal{Y}_1(w_1, z)$. The corresponding geometrically-modified operator is

$$\mathcal{Y}_1(\mathcal{U}(q_z)w_1, q_z),$$

where $q^z = e^{2\pi i z}$, $\mathcal{U}(q_z) = (2\pi i q_z)^{L(0)}e^{-L^+(A)}$ and $A_j \in \mathbb{C}$ for $j \in \mathbb{Z}_+$ are defined by

$$\frac{1}{2\pi i}\log(1 + 2\pi i y) = \left(\exp\left(\sum_{j\in\mathbb{Z}_+} A_j y^{j+1}\frac{\partial}{\partial y}\right)\right) y.$$

See [H4] for details.

We assume the convergence and extension property of $q$-traces of products of geometrically-modified intertwining operators and the modular invariance of the spaces of the analytic extensions of such $q$-traces:

**Assumption 7.1.** *Let $W_1, \dots, W_n$ be irreducible ordinary $V$-modules.*

1. *Let $\widetilde{W}_1, \dots, \widetilde{W}_n$ be ordinary $V$-modules and $\mathcal{Y}_1, \dots, \mathcal{Y}_n$ intertwining operators of types $\binom{\widetilde{W}_0}{W_1\widetilde{W}_1}, \dots, \binom{\widetilde{W}_{n-1}}{W_n\tilde{W}_n}$, respectively, where we use the convention $\tilde{W}_0 = \tilde{W}_n$. For $w_1 \in W_1, \dots, w_n \in W_n$,*

$$\mathrm{Tr}_{\widetilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q_\tau^{L(0)-\frac{c}{24}}$$

*is absolutely convergent in the region $1 > |q_{z_1}| > \dots > |q_{z_n}| > |q_\tau| > 0$ and can be extended to a multivalued analytic function*

$$\overline{F}_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1, \dots, w_n; z_1, \dots, z_n; \tau).$$

*in the region $\Im(\tau) > 0$, $z_i \neq z_j + l + m\tau$ for $i \neq j$, $l, m \in \mathbb{Z}$.*

2. *For $w_1 \in W_1, \dots, w_n \in W_n$, let $\mathcal{F}_{w_1,\dots,w_n}$ be the vector space spanned by functions of the form*
$$\overline{F}^{\phi}_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1,\dots,w_n; z_1,\dots,z_n;\tau)$$
*for all ordinary $V$-modules $\widetilde{W}_1,\dots,\widetilde{W}_n$, all intertwining operators $\mathcal{Y}_1,\dots,\mathcal{Y}_n$ of types $\binom{\widetilde{W}_0}{W_1\widetilde{W}_1},\dots,\binom{\widetilde{W}_{n-1}}{W_n\tilde{W}_n}$, respectively. Then for*
$$\left(\begin{array}{cc}\alpha & \beta\\ \gamma & \delta\end{array}\right)\in SL(2,\mathbb{Z}),$$
$$\overline{F}_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}\left(\left(\frac{1}{\gamma\tau+\delta}\right)^{L(0)}w_1,\dots,\left(\frac{1}{\gamma\tau+\delta}\right)^{L(0)}w_n;\frac{z_1}{\gamma\tau+\delta},\dots,\frac{z_n}{\gamma\tau+\delta};\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)$$
*is in $\mathcal{F}_{w_1,\dots,w_n}$.*

**Theorem 7.2** (H. 2003[H4])**.** *Assume that $V$ is of positive energy, $C_2$-cofinite and assume in addition that every grading-restricted generalized $V$-module is completely reducible. Let $W_1,\dots,W_n$ be a complete set of representatives of irreducible $V$-modules. Then Assumption 7.1 holds.*

## 7.2 Verlinde formula

To prove the rigidity and nondegeracy property in the next section, we need the Verlinde formula, or more precisely, some formulas obtained in [MS1] and [MS2] and proved in [H5]. In this subsection, we give a proofs of these formulas under suitable assumptions.

We assume that every low-bounded generalized $V$-module is a direct sum of irreducible (grading-restricted) $V$-mdoules. Then one can prove that there are only finitely many irreducible ordinary $V$-mdoules. Let $\{W^a\}_{a\in A}$ be a complete set of representatives of equivalence classes of irreducible (grading-restricted) $V$-modules. Then $A$ is a finite set. We also assume that $V$ is irreducible. Then there exists $e\in A$ such that $W^e$ is equivalent to $V$. We shall take $W^e$ to be $V$.

We assume in addition that $V'$ as a $V$-module is equivalent to $V$. This assmption is equivalent to the existence of a nondegenerate invariant bilinear form (see [FHL]).

Since the contragredient module of an irreducible $V$-module is also irreducible, we have a map $' : A\to A$, $a\mapsto a'$ such that for $a\in A$, $(W^a)'$ is equivalent to $W^{a'}$.

For $a_1,a_2,a_3\in A$, let $\mathcal{V}^{a_3}_{a_1a_2}$ be the space of intertwining operators of type $\binom{W^{a_3}}{W^{a_1}W^{a_2}}$. For $a_1,a_2,a_3\in\mathcal{A}$, we have isomorphisms $\Omega_{-r}:\mathcal{V}^{a_3}_{a_1a_2}\to\mathcal{V}^{a_3}_{a_2a_1}$ and $A_{-r}:\mathcal{V}^{a_3}_{a_1a_2}\to\mathcal{V}^{a_2'}_{a_1a_3'}$ for $r\in\mathbb{Z}$. Using these isomorphisms, we define a left action of the symmetric group $S_3$ on
$$\mathcal{V}=\coprod_{a_1,a_2,a_3\in\mathcal{A}}\mathcal{V}^{a_3}_{a_1a_2}$$
as follows: For $a_1,a_2,a_3\in\mathcal{A}$, $\mathcal{Y}\in\mathcal{V}^{a_3}_{a_1a_2}$, we define
$$\sigma_{12}(\mathcal{Y}) = e^{\pi i\Delta(\mathcal{Y})}\Omega_{-1}(\mathcal{Y})$$

$$
\begin{aligned}
&= e^{-\pi i\Delta(\mathcal{Y})}\Omega_0(\mathcal{Y}),\\
\sigma_{23}(\mathcal{Y}) &= e^{\pi i h_{a_1}}A_{-1}(\mathcal{Y})\\
&= e^{-\pi i h_{a_1}}A_0(\mathcal{Y}).
\end{aligned}
$$

**Proposition 7.3.** *The actions $\sigma_{12}$ and $\sigma_{23}$ of* (12) *and* (23) *on $\mathcal{V}$ defined above generate an action of $S_3$ on $\mathcal{V}$.*

We also assume that the fusion rules among irreducible $V$-modules are finite. In particular, for $a_1, a_2, a_3 \in A$, we can find a finite basis of the space $\mathcal{V}_{a_1a_2}^{a_3}$ of intertwining operators of type $\binom{W^{a_3}}{W^{a_1}W^{a_2}}$. For $a_1, a_2, a_3 \neq e$, we choose an arbitrary basis $\{\mathcal{Y}_{a_1a_2;k}^{a_3}\}_{k=1}N_{a_1a_2}^{a_3}$, where $N_{a_1a_2}^{a_3} = \dim \mathcal{V}_{a_1a_2}^{a_3}$ is the fusion rule. For $a \in \mathcal{A}$, we choose $\mathcal{Y}_{ea;1}^{a}$ to be the the vertex operator $Y_{W^a}$ defining the module structure on $W^a$ and we choose $\mathcal{Y}_{ae;1}^{a}$ to be the intertwining operator defined using the action of $\sigma_{12}$, or equivalently the skew-symmetry in this case,

$$
\begin{aligned}
\mathcal{Y}_{ae;1}^{a}(w_a, x)u &= \sigma_{12}(\mathcal{Y}_{ea;1}^{a})(w_a, x)u\\
&= e^{xL(-1)}\mathcal{Y}_{ea;1}^{a}(u, -x)w_a\\
&= e^{xL(-1)}Y_{W^a}(u, -x)w_a
\end{aligned}
$$

for $u \in V$ and $w_a \in W^a$. Since $V'$ as a $V$-module is isomorphic to $V$, we have $e' = e$. From [FHL], we know that there is a nondegerate invariant bilinear form $(\cdot,\cdot)$ on $V$ such that $(\mathbf{1},\mathbf{1}) = 1$. We choose $\mathcal{Y}_{aa';1}^{e} = \mathcal{Y}_{aa';1}^{e'}$ to be the intertwining operator defined using the action of $\sigma_{23}$ by

$$\mathcal{Y}_{aa';1}^{e'} = \sigma_{23}(\mathcal{Y}_{ae;1}^{a}),$$

that is,

$$(u, \mathcal{Y}_{aa';1}^{e'}(w_a, x)w_{a'}) = e^{\pi i h_a}\langle \mathcal{Y}_{ae;1}^{a}(e^{xL(1)}(e^{-\pi i}x^{-2})^{L(0)}w_a, x^{-1})u, w_{a'}\rangle$$

for $u \in V$, $w_a \in W^a$ and $w_{a'} \in W^{a'}$. Since the actions of $\sigma_{12}$ and $\sigma_{23}$ generate the action of $S_3$ on $\mathcal{V}$, we have

$$\mathcal{Y}_{a'a;1}^{e} = \sigma_{12}(\mathcal{Y}_{aa';1}^{e})$$

for any $a \in \mathcal{A}$.

# 8 Rigidity, twists and modularity

In this section, we give proofs of the rigidity, the twists and the proof of nondegeneracy property.

### 8.1 Rigidity

### 8.2 Twisted

### 8.3 Nondegeneracy property

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