Lecture notes on vertex algebras and quantum vertex algebras

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Contents

1 Heisenberg vertex algebras (chiral algebras for free bosons)

Let $\mathfrak{h}$ be a finite-dimensional inner product vector space over $\mathbb{R}$ with the inner product $(\cdot, \cdot)$. Let $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$ be the Heisenberg algebra associated to $\mathfrak{h}$ with the commutator formula given by

$$[a \otimes t^m, b \otimes t^n] = m(a, b)\delta_{m+n,0}\mathbf{k},$$
$$[a \otimes t^m, \mathbf{k}] = 0$$

for $a, b \in \mathfrak{h}$ and $m, n \in \mathbb{Z}$. Note that $\mathfrak{h}$ is vector space over $\mathbb{R}$ while $\hat{\mathfrak{h}}$ is a vector space over $\mathbb{C}$. Let $\hat{\mathfrak{h}}_+ = \mathfrak{h} \otimes t\mathbb{C}[t]$, $\hat{\mathfrak{h}}_- = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]$ and $\hat{\mathfrak{h}}_0 = \mathfrak{h} \otimes t^0 \oplus \mathbb{C}\mathbf{k}$. These are all Lie subalgebras of $\hat{\mathfrak{h}}$.

Let $\hat{\mathfrak{h}}_+$ act on the one-dimensional space $\mathbb{C}$ as 0 and $\mathbf{k}$ acts on $\mathbb{C}$ as 1. Then $\mathbb{C}$ becomes an $\hat{\mathfrak{h}}_+ \oplus \hat{\mathfrak{h}}_0$-module. By the Poincaré-Birkhoff-Witt theorem, the induced module $U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}_+ \oplus \hat{\mathfrak{h}}_0)} \mathbb{C}$ is linearly isomorphic to $U(\hat{\mathfrak{h}}_-) \otimes \mathbb{C} \simeq S(\hat{\mathfrak{h}}_-)$. In particular, $S(\hat{\mathfrak{h}}_-)$ is equipped with an $\hat{\mathfrak{h}}$-module structure under this linear isomorphism. The grading on $\hat{\mathfrak{h}}_-$ gives a grading on $S(\hat{\mathfrak{h}}_-)$ called weight. It is easy to verify that this grading on $S(\hat{\mathfrak{h}}_-)$ is grading-restricted; in fact, it is easy to verify that $S(\hat{\mathfrak{h}}_-)_{(n)} = 0$ when $n < 0$ and $\dim S(\hat{\mathfrak{h}}_-)_{(n)} < \infty$.

The $\hat{\mathfrak{h}}$-module structure on $S(\hat{\mathfrak{h}}_-)$ can also be obtained explicitly as follows: For $a \in \mathfrak{h}$ and $n \in \mathbb{Z}$, we define the action of $a(n)$ on $S(\hat{\mathfrak{h}}_-)$ by

$$a(n)(a_1(-n_1) \cdots a_k(-n_k)) = a(n)a_1(-n_1) \cdots a_k(-n_k)$$

when $n < 0$ for $a_1, \ldots, a_k \in \mathfrak{h}$ and $n_1, \ldots, n_k \in \mathbb{Z}_+$,

$$a(n)(a_1(-n_1) \cdots a_k(-n_k)) = \sum_{i=1}^{k} a_1(-n_1) \cdots a_{i-1}(-n_{i-1})[a(n), a_i(-n_i)]a_{i+1} \cdots a_k(-n_k)$$

when $n \leq 0$ and

$$\mathbf{k}(a_1(-n_1)\cdots a_k(-n_k)) = (a_1(-n_1)\cdots a_k(-n_k)).$$

Then it is easy to verify that $S(\hat{\mathfrak{h}}_-)$ with this action of $\hat{\mathfrak{h}}$ is an $\hat{\mathfrak{h}}$-module.

For $a \in \mathfrak{h}$, let $a(x) = \sum_{n\in\mathbb{Z}} a(n)x^{-n-1}$.

Proposition 1.1. *For $a, b \in \mathfrak{h}$, we have*

$$[a(x_1), b(x_2)] = (a, b)\left((x_1 - x_2)^{-2} - (-x_2 + x_1)^{-2}\right). \tag{1.1}$$

Proof. For $a, b \in \mathfrak{h}$,

$$\begin{aligned}
[a(x_1), b(x_2)] &= \sum_{m,n\in\mathbb{Z}} [a(m), b(n)]x_1^{-m-1}x_2^{-n-1} \\
&= \sum_{m,n\in\mathbb{Z}} (a, b)m\delta_{m+n,0}x_1^{-m-1}x_2^{-n-1} \\
&= -(a, b)\sum_{n\in\mathbb{Z}} nx_1^{n-1}x_2^{-n-1} \\
&= -(a, b)\frac{\partial}{\partial x_1}\sum_{n\in\mathbb{Z}} x_1^{n}x_2^{-n-1} \\
&= -(a, b)\frac{\partial}{\partial x_1}\left((x_1 - x_2)^{-1} - (-x_2 + x_1)^{-1}\right) \\
&= (a, b)\left((x_1 - x_2)^{-2} - (-x_2 + x_1)^{-2}\right),
\end{aligned}$$

proving (1.1). ∎

Let $L_{S(\hat{\mathfrak{h}}_-)}(0)$ be the operator on $S(\hat{\mathfrak{h}}_-)$ giving the grading on $S(\hat{\mathfrak{h}}_-)$, that is, $L_{S(\hat{\mathfrak{h}}_-)}(0)v = nv$ for $v \in S(\hat{\mathfrak{h}}_-)_{(n)}$. We denote $1 \in S(\hat{\mathfrak{h}}_-)$ by $\mathbf{1}_{S(\hat{\mathfrak{h}}_-)}$. Then $S(\hat{\mathfrak{h}}_-)$ is spanned by elements of the form

$$\alpha_1(-n_1)\cdots\alpha_k(-n_k)\mathbf{1}_{S(\hat{\mathfrak{h}}_-)}$$

for $a_1, \ldots, a_k \in \mathfrak{h}$ and $n_1, \ldots, n_k \in \mathbb{Z}_+$. We define an operator $L_{S(\hat{\mathfrak{h}}_-)}(-1)$ on $S(\hat{\mathfrak{h}}_-)$ by

$$\begin{aligned}
&L_{S(\hat{\mathfrak{h}}_-)}(-1)a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}_{S(\hat{\mathfrak{h}}_-)} \\
&\qquad = \sum_{i=1}^{k} n_i a_1(-n_1)\cdots a_{i-1}(-n_{i-1})a_i(-n_i-1)a_{i+1}(-n_{i+1})\cdots a_k(-n_k)\mathbf{1}_{S(\hat{\mathfrak{h}}_-)}.
\end{aligned}$$

Proposition 1.2. *The series $a(x)$ for $a \in \mathfrak{h}$ and the operators $L_{S(\hat{\mathfrak{h}}_-)}(0)$ and $L_{S(\hat{\mathfrak{h}}_-)}(-1)$ have the following properties:*

1. *For $a \in \mathfrak{h}$, $[L_{S(\hat{\mathfrak{h}}_-)}(0), a(x)] = x\frac{d}{dx}a(x) + a(x)$.*

2. *$L_{S(\hat{\mathfrak{h}}_-)}(-1)\mathbf{1}_{S(\hat{\mathfrak{h}}_-)} = 0$, $[L_{S(\hat{\mathfrak{h}}_-)}(-1), a(x)] = \frac{d}{dz}a(x)$ for $a \in \mathfrak{h}$.*

3. *For $a \in \mathfrak{h}$, $a(x)\mathbf{1}_{S(\hat{\mathfrak{h}}_-)} \in S(\hat{\mathfrak{h}}_-)[[x]]$. Moreover, $\lim_{x\to 0} a(x)\mathbf{1}_{S(\hat{\mathfrak{h}}_-)} = a(-1)\mathbf{1}_{S(\hat{\mathfrak{h}}_-)}$.*

4. *The vector space $S(\hat{\mathfrak{h}}_-)$ is spanned by elements of the form*
$$a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}_{S(\hat{\mathfrak{h}}_-)}$$
for $a_1, \dots, a_k \in \mathfrak{h}$ and $n_1, \dots, n_k \in \mathbb{Z}_+$.

5. *For $a, b \in \mathfrak{h}$,*
$$(x_1 - x_2)^2 a(x_1)b(x_2) = (x_1 - x_2)^2 b(x_2)a(x_1).$$

Proof. Properties 1–4 are easily verified using the definitions. Property 5 follows from Proposition 1.1. ∎

We shall give the definition of grading-restricted vertex algebra later. Here we first state the main result of this section assuming the reader knows this definition. This theorem follows easily from a general theorem we shall discuss later. So the proof will be given after the general theorem is proved. Using (1.1), we can prove that for $a_1, \dots, a_k \in \mathfrak{h}$, $v \in S(\hat{\mathfrak{h}}_-)$, and $v' \in S(\hat{\mathfrak{h}}_-)'$,
$$\langle v', a_1(z_1)\cdots a_k(z_k)v\rangle$$
is the expansion in the region $|z_1| > \cdots > |z_k| > 0$ of a rational function denoted by
$$R(\langle v', a_1(z_1)\cdots a_k(z_k)v\rangle)$$
in $z_1, \dots, z_k$ with the only possible poles at $z_i = 0$ and $z_i - z_j = 0$ for $i, j = 1, \dots, k$. See Section 4 for more details.

Theorem 1.3. *The vector space $S(\hat{\mathfrak{h}}_-)$ equipped with the the vertex operator map $Y_{S(\hat{\mathfrak{h}}_-)}$ defined by*
$$\begin{aligned}\langle v', &Y_{S(\hat{\mathfrak{h}}_-)}(\alpha_1(-n_1)\cdots\alpha_k(-n_k)\mathbf{1}_{S(\hat{\mathfrak{h}}_-)}, z)v\rangle\\ &= \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{-n_1}\cdots\xi_k^{-n_k}\xi_{k+1}^{-1}R(\langle v', a_1(\xi_1+z)\cdots a_k(\xi_k+z)v\rangle)\end{aligned} \tag{1.2}$$
for $a_1, \dots, a_k \in \mathfrak{h}$, $v \in S(\hat{\mathfrak{h}}_-)$ and $v' \in S(\hat{\mathfrak{h}}_-)'$ and the vacuum $\mathbf{1}_{S(\hat{\mathfrak{h}}_-)}$ is a grading-restricted vertex algebra. Moreover, this is the unique grading-restricted vertex algebra structure on $S(\hat{\mathfrak{h}}_-)$ with the vacuum $\mathbf{1}_{S(\hat{\mathfrak{h}}_-)}$ such that $Y(a(-1)\mathbf{1}, z) = a(x)$ for $a \in \mathfrak{h}$.

In the study of representations of Heisenberg algebras, an operation on Heisenberg operators called normal ordering is very useful. For operators $a(n_1), \dots, a(n_k)$ on $S(\hat{\mathfrak{h}}_-)$, we define the normal ordered product
$$:a(n_1)\cdots a(n_k):$$
to be the operator obtained by taking the product these operators in an order such that $a_i(n_i)$ with $n_i \in \mathbb{N}$ are always to the right of those $a_i(n_i)$ with $n_i \in -Z_+$. Note that $a_i(n_i)$

with $n_i \in \mathbb{N}$ commute with themselves. So the normal ordered product can also be defined by taking the product these operators in an order such that $a_i(n_i)$ with $n_i \in \mathbb{N}$ are always to the right of all the other $a_i(n_i)$.

We now discuss what is called the stress-energy tensor in physics. Let $\{u^i\}_{i=1}^{\dim \mathfrak{h}}$ be an orthonormal basis of $\mathfrak{h}$. The stress-energy tensor is defined to be the series

$$T(x) = \frac{1}{2} {}^{\circ}_{\circ} u^i(x) u^i(x) {}^{\circ}_{\circ}.$$

Note that though $u^i(x)$ is a formal Laurent series and $u^i(x)u^i(x)$ is not well defined in general, the normal ordered product ${}^{\circ}_{\circ} u^i(x)u^i(x) {}^{\circ}_{\circ}$ is well defined. By definition

$$\begin{aligned} T(x) &= \frac{1}{2} {}^{\circ}_{\circ} u^i(x) u^i(x) {}^{\circ}_{\circ} \\ &= \frac{1}{2} \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} {}^{\circ}_{\circ} u^i(k) u^i(l) {}^{\circ}_{\circ} x^{-k-l-2} \\ &= \sum_{n \in \mathbb{Z}} \frac{1}{2} \sum_{k \in \mathbb{Z}} {}^{\circ}_{\circ} u^i(k) u^i(n-k) {}^{\circ}_{\circ} x^{-n-2}. \end{aligned}$$

We write

$$T(x) = \sum_{n \in \mathbb{Z}} L_{S(\hat{\mathfrak{h}}_-)}(n) x^{-n-2}.$$

Then for $n \in \mathbb{Z}$,

$$L_{S(\hat{\mathfrak{h}}_-)}(n) = \frac{1}{2} \sum_{k \in \mathbb{Z}} {}^{\circ}_{\circ} u^i(k) u^i(n-k) {}^{\circ}_{\circ} = \frac{1}{2} \sum_{k \in -\mathbb{Z}_+} u^i(k) u^i(n-k) + \frac{1}{2} \sum_{k \in \mathbb{N}} u^i(n-k) u^i(k). \quad (1.3)$$

Let

$$\omega = \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{h}} u^i(-1)^2 \mathbf{1}.$$

We now calculate $T(x)\omega$, which will be used to give the example of what we call the conformal element for the Heisenberg vertex operator algebra in Subsection 5.6.

Lemma 1.4. *We have*

$$T(x)\omega = L_{S(\hat{\mathfrak{h}}_-)}(-1)\omega x^{-1} + 2\omega x^{-1} + \frac{\dim \mathfrak{h}}{2} x^{-4} + G(x), \quad (1.4)$$

where $G(x) \in S(\hat{\mathfrak{h}}_-)[[x]]$.

Proof. By definition,

$$
\begin{aligned}
&{\circ\atop\circ}u^i(x)u^i(x){\circ\atop\circ}u^j(-1)^2\mathbf{1}\\
&\quad=\sum_{k,l\in\mathbb{Z}}{\circ\atop\circ}u^i(k)u^i(l){\circ\atop\circ}u^j(-1)^2\mathbf{1}x^{-k-l-2}\\
&\quad=\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{N}}u^i(k)u^i(l)u^j(-1)^2\mathbf{1}x^{-k-l-2}+\sum_{k\in\mathbb{Z}}\sum_{l\in-\mathbb{Z}_+}u^i(l)u^i(k)u^j(-1)^2\mathbf{1}x^{-k-l-2}\\
&\quad=\sum_{k\in\mathbb{Z}}\sum_{l=0}^{2}u^i(k)u^i(l)u^j(-1)^2\mathbf{1}x^{-k-l-2}+2\delta_{ij}\sum_{k<-2}u^i(k)u^j(-1)\mathbf{1}x^{-k-3}\\
&\qquad+\sum_{k=0}^{2}\sum_{l\in-\mathbb{Z}_+}u^i(l)u^i(k)u^j(-1)^2\mathbf{1}x^{-k-l-2}+\sum_{k\in-\mathbb{Z}_+}\sum_{l\in-\mathbb{Z}_+}u^i(l)u^i(k)u^j(-1)^2\mathbf{1}x^{-k-l-2}. \qquad (1.5)
\end{aligned}
$$

Note that the last term in the right-hand side of (1.5) is in $S(\hat{\mathfrak{h}}_-)[[x]]$. So we need only calculate the first two terms. By the commutator relations for the Heisenberg algebra, we have

$$
\begin{aligned}
&u^i(0)u^j(-1)^2\mathbf{1}=0,\\
&u^i(1)u^j(-1)^2\mathbf{1}=2(u^i,u^j)u^j(-1)\mathbf{1}=2\delta_{ij}u^j(-1)\mathbf{1},\\
&u^i(0)u^j(-1)^2\mathbf{1}=0.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}}\sum_{l=0}^{2}u^i(k)u^i(l)u^j(-1)^2\mathbf{1}x^{-k-l-2}\\
&\quad=2\delta_{ij}\sum_{k\in\mathbb{Z}}u^i(k)u^j(-1)\mathbf{1}x^{-k-3}\\
&\quad=2\delta_{ij}\sum_{k=-2}^{1}u^i(k)u^j(-1)\mathbf{1}x^{-k-3}+2\delta_{ij}\sum_{k<-2}u^i(k)u^j(-1)\mathbf{1}x^{-k-3}\\
&\quad=2\delta_{ij}\sum_{k=-2}^{1}u^i(k)u^j(-1)\mathbf{1}x^{-k-3}+2\delta_{ij}\sum_{k<-2}u^i(k)u^j(-1)\mathbf{1}x^{-k-3}\\
&\quad=2\delta_{ij}u^i(-2)u^j(-1)\mathbf{1}x^{-1}+2\delta_{ij}u^i(-1)u^j(-1)\mathbf{1}x^{-2}+2\delta_{ij}^2\mathbf{1}x^{-4}\\
&\qquad+2\delta_{ij}\sum_{k<-2}u^i(k)u^j(-1)\mathbf{1}x^{-k-3}
\end{aligned}
\qquad (1.6)
$$

and

$$
\begin{aligned}
&\sum_{k=0}^{2}\sum_{l\in-\mathbb{Z}_+} u^i(l)u^i(k)u^j(-1)^2\mathbf{1}x^{-k-l-2}\\
&\quad = 2\delta_{ij}\sum_{l\in-\mathbb{Z}_+} u^i(l)u^j(-1)\mathbf{1}x^{-l-3}\\
&\quad = 2\delta_{ij}u^i(-2)u^j(-1)\mathbf{1} + 2\delta_{ij}u^i(-1)u^j(-1)\mathbf{1} + 2\delta_{ij}\sum_{l<-2} u^i(l)u^j(-1)\mathbf{1}x^{-l-3} \qquad (1.7)
\end{aligned}
$$

Note that the last terms in both (1.6) and (1.7) are in $S(\hat{\mathfrak{h}}_-)[[x]]$. Sustituting (1.6) and (1.7) into (1.5), taking sum over $i, j = 1, \dots, \dim \mathfrak{h}$, dividing both sides by 4 and using the definition of ω and $L_{S(\hat{\mathfrak{h}}_-)}(-1)$, we obtain

$$
\begin{aligned}
T(x)\omega &= \frac{1}{2}\sum_{i,j=1}^{\dim\mathfrak{h}} \delta_{ij}u^i(-2)u^j(-1)\mathbf{1}x^{-1} + \frac{1}{2}\sum_{i,j=1}^{\dim\mathfrak{h}} \delta_{ij}u^i(-1)u^j(-1)\mathbf{1}x^{-2} + \frac{1}{2}\sum_{i,j=1}^{\dim\mathfrak{h}} \delta_{ij}^2\mathbf{1}x^{-4}\\
&\quad + \frac{1}{2}\sum_{i,j=1}^{\dim\mathfrak{h}} \delta_{ij}\sum_{k<-2} u^i(k)u^j(-1)\mathbf{1}x^{-k-3} + \frac{1}{2}\sum_{i,j=1}^{\dim\mathfrak{h}} \delta_{ij}u^i(-2)u^j(-1)\mathbf{1}\\
&\quad + \frac{1}{2}\sum_{i,j=1}^{\dim\mathfrak{h}} 2\delta_{ij}u^i(-1)u^j(-1)\mathbf{1} + \frac{1}{2}\sum_{i,j=1}^{\dim\mathfrak{h}} \delta_{ij}\sum_{l<-2} u^i(l)u^j(-1)\mathbf{1}x^{-l-3}\\
&\quad + \sum_{k\in-\mathbb{Z}_+}\sum_{l\in-\mathbb{Z}_+} u^i(l)u^i(k)u^j(-1)^2\mathbf{1}x^{-k-l-2}\\
&= \frac{1}{2}\sum_{i=1}^{\dim\mathfrak{h}} 2u^i(-2)u^i(-1)\mathbf{1}x^{-1} + 2\frac{1}{2}\sum_{i,j=1}^{\dim\mathfrak{h}} u^i(-1)u^i(-1)\mathbf{1}x^{-2} + \frac{\dim\mathfrak{h}}{2}\mathbf{1}x^{-4} + G(x)\\
&= L_{S(\hat{\mathfrak{h}}_-)}(-1)\omega x^{-1} + 2\omega x^{-1} + \frac{\dim\mathfrak{h}}{2}x^{-4} + G(x),
\end{aligned}
$$

where

$$
\begin{aligned}
G(x) &= \frac{1}{2}\sum_{i,j=1}^{\dim\mathfrak{h}} \delta_{ij}\sum_{k<-2} u^i(k)u^j(-1)\mathbf{1}x^{-k-3} + \frac{1}{2}\sum_{i,j=1}^{\dim\mathfrak{h}} \delta_{ij}\sum_{l<-2} u^i(l)u^j(-1)\mathbf{1}x^{-l-3}\\
&\quad + \sum_{k\in-\mathbb{Z}_+}\sum_{l\in-\mathbb{Z}_+} u^i(l)u^i(k)u^j(-1)^2\mathbf{1}x^{-k-l-2}\\
&\in S(\hat{\mathfrak{h}}_-)[[x]].
\end{aligned}
$$

∎

We now state the following result whose proof will be given in Subsection 5.6 using (1.4):

Theorem 1.5. *The element ω is a conformal element of the grading-restricted vertex algebra $S(\hat{\mathfrak{h}}_-)$. In particular, $S(\hat{\mathfrak{h}}_-)$ is a vertex operator algebra (see Definition 3.5).*

2 Lattice vertex algebras (chiral algebras for free bosons on tori)

Let L be a positive-definite even lattice of rank n with nondegenerate symmetric $\mathbb{Z}$-linear form $(\cdot,\cdot)$. Then $\mathfrak{h} = L\otimes_{\mathbb{Z}}\mathbb{R}$ is an n-dimensional vector space over $\mathbb{R}$ with a positive definite bilinear form still denoted by $(\cdot,\cdot)$. Then we have the Heisenberg algebra $\hat{\mathfrak{h}} = \mathfrak{h}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}\mathbf{k}$. As in the preceding section, we also have the subalgebras $\hat{\mathfrak{h}}_+ = \mathfrak{h}\otimes t\mathbb{C}[t]$, $\hat{\mathfrak{h}}_- = \mathfrak{h}\otimes t^{-1}\mathbb{C}[t^{-1}]$ and $\hat{\mathfrak{h}}_0 = \mathfrak{h}\otimes t^0\oplus\mathbb{C}\mathbf{k}$. Given a module M for the abelian Lie algebra $\mathfrak{h}\otimes t^0 = \mathfrak{h}$, let $\hat{\mathfrak{h}}_+$ act on M as 0 and $\mathbf{k}$ acts on M as 1. Then M becomes an $\hat{\mathfrak{h}}_+\oplus\hat{\mathfrak{h}}_0$-module. By the Poincaré-Birkhoff-Witt theorem, the induced module $U(\hat{\mathfrak{h}})\otimes_{U(\hat{\mathfrak{h}}_+\oplus\hat{\mathfrak{h}}_0)}M$ is linearly isomorphic to $U(\hat{\mathfrak{h}}_-)\otimes M = S(\hat{\mathfrak{h}}_-)\otimes M$. In particular, $S(\hat{\mathfrak{h}}_-)\otimes M$ is equipped with an $\hat{\mathfrak{h}}$-module structure under this linear isomorphism.

Fix a basis $\alpha_1,\dots,\alpha_r$ of L. We define a $\mathbb{Z}$-linear map $\epsilon: L\times L\to\mathbb{Z}$ by

$$\epsilon(\alpha_i,\alpha_j) = \begin{cases} (\alpha_i,\alpha_j) & i>j \\ 0 & i\le j \end{cases}$$

for $i,j=1,\dots,r$. Let $\hat{L} = \{1,-1\}\times L$. Define a multiplication on $\hat{L}$ by

$$(\theta,\alpha)\cdot(\tau,\beta) = (\theta\tau(-1)^{\epsilon(\alpha,\beta)},\alpha+\beta)$$

for $\theta,\tau\in\{1,-1\}$ and $\alpha,\beta\in L$. We have a surjective map $^-:\hat{L}\to L$ defined by $\overline{(\theta,\alpha)}=\alpha$ for $\theta\in\{1,-1\}$ and $\alpha\in L$. We also have an injective map from $\{1,-1\}$ to $\hat{L}$ defined by $\theta\mapsto(\theta,0)$ for $\theta\in\{1,-1\}$. It is clear that these maps are homomorphisms of groups and we have the exact sequence

$$1\to\{1,-1\}\to\hat{L}\xrightarrow{-}L\to 1,$$

that is, $\hat{L}$ is a central extension of L by the group $\{1,-1\}$. The commutator map $c: L\times L\to\mathbb{Z}/2\mathbb{Z}$ of this central extension is given by $c(\alpha,\beta) = (\alpha,\beta)+2\mathbb{Z}$. We shall denote $(1,\alpha)\in\hat{L}$ by e_α for $\alpha\in L$ and $(\theta,0)$ by θ for $m\in\mathbb{Z}/2\mathbb{Z}$. Then

$$(\theta,\alpha) = \theta e_\alpha = e_\alpha\theta$$

and

$$e_\alpha e_\beta = (-1)^{\epsilon(\alpha,\beta)}e_{\alpha+\beta}.$$

Let $\mathbb{C}[L]$ be the group algebra of L. We shall use e^α to denote the element $\alpha\in L$ in $\mathbb{C}[L]$. Then in $\mathbb{C}[L]$, we have $e^\alpha e^\beta = e^{\alpha+\beta}$ for $\alpha,\beta\in L$. We have an action of the abelian Lie algebra $\mathfrak{h}\otimes t^0=\mathfrak{h}$ on $\mathbb{C}[L]$ by $(a\otimes t^0)\cdot e^\alpha = (a,\alpha)e^\alpha$. Then $\mathbb{C}[L]$ is a $\mathfrak{h}\otimes t^0$-module. We also have an action of $\hat{L}$ on $\mathbb{C}[L]$ defined by $(\theta,\alpha)\cdot e^\beta = (\theta(-1)^{\epsilon(\alpha,\beta)})e^{\alpha+\beta}$ for $\theta\in\{1,-1\}$ and $\alpha,\beta\in L$. In particular, $e_\alpha\cdot e^\beta = (-1)^{\epsilon(\alpha,\beta)}e^{\alpha+\beta}$ for $\alpha\in L$ and $\theta\cdot e^\beta = \theta e^\beta$. It is clear that this action gives $\mathbb{C}[L]$ an $\hat{L}$-module structure.

Let $V_L = S(\hat{\mathfrak{h}}_-)\otimes\mathbb{C}[L]$. The grading on $\hat{\mathfrak{h}}_-$ gives a grading on $S(\hat{\mathfrak{h}}_-)$ called weight and we give a grading on $\mathbb{C}[L]$ also called weight by defining the weight of e^α to be $\frac{1}{2}(\alpha,\alpha)$.

These gradings give a grading on V_L also called weight. Then we have $V_L = \coprod_{n\in\mathbb{Z}}(V_L)_{(n)}$ where $(V_L)_{(n)}$ is the homogeneous subspace of V_L of weight n. It is clear that V_L is grading-restricted since the gradings on $S(\hat{\mathfrak{h}}_-)$ and $\mathbb{C}[L]$ are both grading restricted. In fact since L is positive definite, we have $(V_L)_{(n)} = 0$ when $n < 0$.

As is discussed above, we have an $\hat{h}$-module structure on $V_L = S(\hat{\mathfrak{h}}_-)\otimes\mathbb{C}[L]$. Since $\mathbb{C}[L]$ is an $\hat{L}$-module, $V_L = S(\hat{\mathfrak{h}}_-)\otimes\mathbb{C}[L]$ is also an $\hat{L}$-module with $\hat{L}$ acts only on $\mathbb{C}[L]$. We denote the action of $a\otimes t^n$ on V_L by $a(n)$ for $a \in \mathfrak{h}$ and $n\in\mathbb{Z}$. For $a\in\mathfrak{h}$, let $a(x) = \sum_{n\in\mathbb{Z}} a(n)x^{-n-1}$. For $\alpha\in L$, for simplicity, we shall also use e^α to denote $1\otimes e^\alpha \in V_L$. For $\alpha\in L$ and a formal variable x, we define $x^\alpha\cdot(u\otimes e^\beta) = x^{(\alpha,\beta)}(u\otimes e^\beta)$ for $u\in S(\hat{\mathfrak{h}}_-)$ and $\beta\in L$. For $\alpha \in L$, let

$$Y_{V_L}(e^\alpha, x) = \exp\left(-\sum_{n<0}\frac{\alpha(n)}{n}x^{-n}\right)\exp\left(-\sum_{n>0}\frac{\alpha(n)}{n}x^{-n}\right)e_\alpha x^\alpha \in (\mathrm{End}\, V_L)[[x, x^{-1}]].$$

For a vector space M, let $\int\cdot\, dx : M[[x]]\oplus x^{-2}M[[x^{-1}]] \to xV[[x]]\oplus x^{-1}V[[x^{-1}]]$ be the linear map given by the integrating the formal series in $M[[x]]\oplus x^{-2}M[[x^{-1}]]$ with constant terms being 0. Then the images of $M[[x]]$ and $x^{-2}M[[x^{-1}]]$ are $xM[[x]]$ and $x^{-1}M[[x^{-1}]]$, respectively. For $a\in\mathfrak{h}$, let $a(x)^+ = \sum_{n\in\mathbb{Z}_+} a(n)x^{-n-1}$ and $a(x)^- = \sum_{n\in\mathbb{Z}_-} a(n)x^{-n-1}$. Then $a(x) = a(x)^+ + a(x)^- + a(0)x^{-1}$. Also

$$\int a(x)^+ dx = -\sum_{n>0}\frac{\alpha(n)}{n}x^{-n},$$
$$\int a(x)^- dx = -\sum_{n<0}\frac{\alpha(n)}{n}x^{-n}.$$

Thus we have

$$Y_{V_L}(e^\alpha, x) = e^{\int\alpha(x)^- dx}e^{\int \alpha(x)^+ dx}e_\alpha x^\alpha$$

for $\alpha\in L$.

We need the following commutator formulas:

Proposition 2.1. *For $a, b\in\mathfrak{h}$, $\alpha,\beta\in L$, we have*

$$[a(x_1), b(x_2)] = (a,b)\left((x_1-x_2)^{-2} - (-x_2+x_1)^{-2}\right), \tag{2.1}$$
$$[a(x_1), Y_{V_L}(e^\alpha, x_2)] = (a,\alpha)\left((x_1-x_2)^{-1} - (-x_2+x_1)^{-1}\right)Y_{V_L}(e^\alpha, x_2), \tag{2.2}$$
$$\begin{aligned}[Y_{V_L}(e^\alpha, x_1), Y_{V_L}(e^\beta, x_2)] &= \left((x_1-x_2)^{(\alpha,\beta)} - (-x_2+x_1)^{(\alpha,\beta)}\right)\cdot\\ &\quad\cdot e^{\int\alpha(x_1)^- dx_1}e^{\int\beta(x_2)^- dx_2}e^{\int\alpha(x_1)^+ dx_1}e^{\int\beta(x_2)^+ dx_2}e_\alpha e_\beta x_1^\alpha x_2^\beta,\end{aligned} \tag{2.3}$$

where for $n\in\mathbb{Z}$, $(x_1-x_2)^n$ and $(-x_2+x_1)^n$ means their binomial expansions in nonnegative powers of x_2 and x_1, respectively.

Proof. The proof of (2.1) is completely the same as that of (1.1) above.

For $a \in \mathfrak{h}$ and $\alpha \in L$, taking $b = \alpha$ in (2.1), we obtain

$$[a(x_1)^{\pm}, \alpha(x_2)^{\pm}] = 0, \tag{2.4}$$
$$[a(0), \alpha(x_2)^{\pm}] = 0, \tag{2.5}$$
$$[a(x_1)^{+}, \alpha(x_2)^{-}] = \frac{(a, \alpha)}{(x_1 - x_2)^2}, \tag{2.6}$$
$$[a(x_1)^{-}, \alpha(x_2)^{+}] = -\frac{(a, \alpha)}{(x_2 - x_1)^2}. \tag{2.7}$$

Applying the map $\int \cdot \, dx_2$ to both sides of (2.4)–(2.7) and then switch the order of the commutators, we obtain

$$\left[\int \alpha(x_2)^{\pm} dx_2, a(x_1)^{\pm}\right] = 0, \tag{2.8}$$
$$\left[\int \alpha(x_2)^{\pm} dx_2, a(0)\right] = 0, \tag{2.9}$$
$$\left[\int \alpha(x_2)^{-} dx_2, a(x_1)^{+}\right] = -\frac{(a, \alpha)}{x_1 - x_2} + \frac{(a, \alpha)}{x_1}, \tag{2.10}$$
$$\left[\int \alpha(x_2)^{+} dx_2, a(x_1)^{-}\right] = -\frac{(a, \alpha)}{x_2 - x_1}, \tag{2.11}$$

where in (2.14), the term $-\frac{(a,\alpha)}{x_1}$ appears because the constant term of $\int \cdot \, dx_2$ must be 0. From (2.8)–(2.15), we obtain

$$\begin{aligned}
e^{\int \alpha(x_2)^{\pm} dx_2} a(x_1)^{\pm} e^{-\int \alpha(x_2)^{\pm} dx_2} &= a(x_1)^{\pm}, \\
e^{\int \alpha(x_2)^{\pm} dx_2} a(0) e^{-\int \alpha(x_2)^{\pm} dx_2} &= a(0), \\
e^{\int \alpha(x_2)^{-} dx_2} a(x_1)^{+} e^{-\int \alpha(x_2)^{-} dx_2} &= a(x_1)^{+} - \frac{(a, \alpha)}{x_1 - x_2} + \frac{(a, \alpha)}{x_1}, \\
e^{\int \alpha(x_2)^{+} dx_2} a(x_1)^{-} e^{-\int \alpha(x_2)^{+} dx_2} &= a(x_1)^{-} - \frac{(a, \alpha)}{x_2 - x_1},
\end{aligned}$$

or equivalently,

$$a(x_1)^{\pm} e^{\int \alpha(x_2)^{\pm} dx_2} = e^{\int \alpha(x_2)^{\pm} dx_2} a(x_1)^{\pm}, \tag{2.12}$$
$$a(0) e^{\int \alpha(x_2)^{\pm} dx_2} = e^{\int \alpha(x_2)^{\pm} dx_2} a(0), \tag{2.13}$$
$$a(x_1)^{+} e^{\int \alpha(x_2)^{-} dx_2} = e^{\int \alpha(x_2)^{-} dx_2} a(x_1)^{+} + \left(\frac{(a, \alpha)}{x_1 - x_2} - \frac{(a, \alpha)}{x_1}\right) e^{\int \alpha(x_2)^{-} dx_2}, \tag{2.14}$$
$$a(x_1)^{-} e^{\int \alpha(x_2)^{+} dx_2} = e^{\int \alpha(x_2)^{+} dx_2} a(x_1)^{-} + \frac{(a, \alpha)}{x_2 - x_1} e^{\int \alpha(x_2)^{+} dx_2}. \tag{2.15}$$

On the other hand, for $u \in S(\hat{\mathfrak{h}}_-)$ and $\beta \in L$,

$$\begin{aligned}
&a(0)e_\alpha x_2^\alpha(u \otimes e^\beta)\\
&\quad= (-1)^{\epsilon(\alpha,\beta)} x_2^{(\alpha,\beta)}(a, \alpha+\beta)(u \otimes e^{\alpha+\beta})\\
&\quad= (-1)^{\epsilon(\alpha,\beta)} x_2^{(\alpha,\beta)}(a, \alpha)(u \otimes e^{\alpha+\beta}) + (-1)^{\epsilon(\alpha,\beta)} x_2^{(\alpha,\beta)}(a, \beta)(u \otimes e^{\alpha+\beta})\\
&\quad= (a,\alpha)e_\alpha x_2^\alpha(u \otimes e^\beta) + e_\alpha x_2^\alpha a(0)(u \otimes e^\beta).
\end{aligned}$$

Thus we obtain

$$a(0)e_\alpha x_2^\alpha = (a,\alpha)e_\alpha x_2^\alpha + e_\alpha x_2^\alpha a(0). \tag{2.16}$$

Using (2.12)–(2.16), we have

$$\begin{aligned}
&a(x_1)Y_{V_L}(e^\alpha, x_2)\\
&\quad= a(x_1)^+ e^{\int \alpha(x_2)^- dx_2} e^{\int \alpha(x_2)^+ dx_2} e_\alpha x_2^\alpha + a(x_1)^- e^{\int \alpha(x_2)^- dx_2} e^{\int \alpha(x_2)^+ dx_2} e_\alpha x_2^\alpha\\
&\qquad + a(0)x_1^{-1} e^{\int \alpha(x_2)^- dx_2} e^{\int \alpha(x_2)^+ dx_2} e_\alpha x_2^\alpha\\
&\quad= e^{\int \alpha(x_2)^- dx_2} e^{\int \alpha(x_2)^+ dx_2} e_\alpha x_2^\alpha a(x_1)^+ + \left(\frac{(a,\alpha)}{x_1 - x_2} - \frac{(a,\alpha)}{x_1}\right) e^{\int \alpha(x_2)^- dx_2} e^{\int \alpha(x_2)^+ dx_2} e_\alpha x_2^\alpha\\
&\qquad + e^{\int \alpha(x_2)^- dx_2} e^{\int \alpha(x_2)^+ dx_2} e_\alpha x_2^\alpha a(x_1)^- + \frac{(a,\alpha)}{x_2 - x_1} e^{\int \alpha(x_2)^- dx_2} e^{\int \alpha(x_2)^+ dx_2} e_\alpha x_2^\alpha\\
&\qquad + e^{\int \alpha(x_2)^- dx_2} e^{\int \alpha(x_2)^+ dx_2} e_\alpha x_2^\alpha a(0)x_1^{-1} + \frac{(a,\alpha)}{x_1} e^{\int \alpha(x_2)^- dx_2} e^{\int \alpha(x_2)^+ dx_2} e_\alpha x_2^\alpha\\
&\quad= Y_{V_L}(e^\alpha, x_2)a(x_1) + (a,\alpha)\left((x_1 - x_2)^{-1} - (-x_2 + x_1)^{-1}\right) Y_{V_L}(e^\alpha, x_2).
\end{aligned}$$

This is (2.2).

Taking $a \in \mathfrak{h}$ to be $\beta \in L \subset \mathfrak{h}$ in (2.15), we obtain

$$[\beta(x_2)^-, e^{\int \alpha(x_1)^+ dx_1}] = \frac{(\alpha,\beta)}{x_1 - x_2} e^{\int \alpha(x_1)^+ dx_1}. \tag{2.17}$$

Applying $-\int \cdot\, dx_2$ to both sides of (2.17), we obtain

$$\begin{aligned}
&\left[-\int \beta(x_2)^- dx_2, e^{\int \alpha(x_1)^+ dx_1}\right]\\
&\quad= \left((\alpha,\beta)\log(x_1 - x_2) - (\alpha,\beta)\log x_1\right) e^{\int \alpha(x_1)^+ dx_1}\\
&\quad= \log\left(1 - \frac{x_2}{x_1}\right)^{(\alpha,\beta)} e^{\int \alpha(x_1)^+ dx_1}.
\end{aligned} \tag{2.18}$$

From (2.18), we obtain

$$e^{-\int \beta(x_2)^- dx_2} e^{\int \alpha(x_1)^+ dx_1} e^{\int \beta(x_2)^- dx_2} = \left(1 - \frac{x_2}{x_1}\right)^{(\alpha,\beta)} e^{\int \alpha(x_1)^+ dx_1},$$

or equivalently,

$$e^{\int \alpha(x_1)^+ dx_1} e^{\int \beta(x_2)^- dx_2} = \left(1 - \frac{x_2}{x_1}\right)^{(\alpha,\beta)} e^{\int \beta(x_2)^- dx_2} e^{\int \alpha(x_1)^+ dx_1}. \tag{2.19}$$

For $u \in S(\hat{\mathfrak{h}}_-)$ and $\gamma \in L$, we have

$$\begin{aligned} x_1^\alpha e_\beta(u \otimes e^\gamma) &= (-1)^{\epsilon(\beta,\gamma)} x_1^{(\alpha,\beta)+(\alpha,\gamma)}(u \otimes e^{\beta+\gamma}), \\ e_\beta x_1^\alpha(u \otimes e^\gamma) &= (-1)^{\epsilon(\beta,\gamma)} x_1^{(\alpha,\gamma)}(u \otimes e^{\beta+\gamma}). \end{aligned}$$

Therefore we obtain

$$x_1^\alpha e_\beta = x_1^{(\alpha,\beta)} e_\beta x_1^\alpha. \tag{2.20}$$

Using (2.19) and (2.20), we obtain

$$\begin{aligned} &Y_{V_L}(e^\alpha, x_1) Y_{V_L}(e^\beta, x_2) \\ &\quad = e^{\int \alpha(x_1)^- dx_1} e^{\int \alpha(x_1)^+ dx_1} e_\alpha x_1^\alpha e^{\int \beta(x_2)^- dx_2} e^{\int \beta(x_2)^+ dx_2} e_\beta x_2^\beta \\ &\quad = (x_1 - x_2)^{(\alpha,\beta)} e^{\int \alpha(x_1)^- dx_1} e^{\int \beta(x_2)^- dx_2} e^{\int \alpha(x_1)^+ dx_1} e^{\int \beta(x_2)^+ dx_2} e_\alpha e_\beta x_1^\alpha x_2^\beta. \end{aligned} \tag{2.21}$$

From (2.21), we also obtain

$$\begin{aligned} &Y_{V_L}(e^\beta, x_2) Y_{V_L}(e^\alpha, x_1) \\ &\quad = (x_2 - x_1)^{(\alpha,\beta)} e^{\int \beta(x_2)^- dx_2} e^{\int \alpha(x_1)^- dx_1} e^{\int \beta(x_2)^+ dx_2} e^{\int \alpha(x_1)^+ dx_1} e_\beta e_\alpha x_1^\alpha x_2^\beta. \end{aligned} \tag{2.22}$$

Since the commutator map of the central extension $\hat{L}$ is $c(\alpha, \beta) = (\alpha, \beta) + 2\mathbb{Z}$, we have

$$e_\beta e_\alpha = (-1)^{(\alpha,\beta)} e_\alpha e_\beta.$$

Thus the right-hand side of (2.22) is equal to

$$(-x_2 + x_1)^{(\alpha,\beta)} e^{\int \alpha(x_1)^- dx_1} e^{\int \beta(x_2)^- dx_2} e^{\int \alpha(x_1)^+ dx_1} e^{\int \beta(x_2)^+ dx_2} e_\alpha e_\beta x_1^\alpha x_2^\beta. \tag{2.23}$$

From (2.21)–(2.23), we obtain (2.3). ∎

Let $L_{V_L}(0)$ be the operator on V_L giving the grading on V_L, that is, $L_{V_L}(0)v = nv$ for $v \in (V_L)_{(n)}$. Note that V_L is spanned by elements of the form

$$a_1(-n_1) \cdots a_k(-n_k) e^\beta$$

for $a_1, \ldots, a_k \in \mathfrak{h}$, $n_1, \ldots, n_k \in \mathbb{Z}_+$ and $\beta \in L$. We define an operator $L_{V_L}(-1)$ on V_L by

$$\begin{aligned} &L_{V_L}(-1) a_1(-n_1) \cdots a_k(-n_k) e^\beta \\ &\qquad = \sum_{i=1}^k n_i a_1(-n_1) \cdots a_{i-1}(-n_{i-1}) \alpha_i(-n_i - 1) a_{i+1}(-n_{i+1}) \cdots a_k(-n_k) e^\beta \\ &\qquad\quad + a_1(-n_1) \cdots a_k(-n_k) \beta(-1) e^\beta. \end{aligned}$$

We denote $e^0 \in V_L$ by $\mathbf{1}_{V_L}$.

Proposition 2.2. *The series $a(x)$ for $a \in \mathfrak{h}$, $Y_{V_L}(e^{\alpha}, x)$ for $\alpha \in L$ and the operators $L_{V_L}(0)$ and $L_{V_L}(-1)$ have the following properties:*

1. *For $a \in \mathfrak{h}$, $[L_{V_L}(0), a(x)] = x\frac{d}{dx}a(x) + a(x)$ and for $\alpha \in L$, $[L_{V_L}(0), Y_{V_L}(e^{\alpha}, x)] = x\frac{d}{dx}Y_{V_L}(e^{\alpha}, x) + \frac{1}{2}(\alpha, \alpha)Y_{V_L}(e^{\alpha}, x)$.*

2. *$L_V(-1)\mathbf{1}_{V_L} = 0$, $[L_V(-1), a(x)] = \frac{d}{dz}a(x)$ for $a \in \mathfrak{h}$ and $[L_V(-1), Y_{V_L}(e^{\alpha}, x)] = \frac{d}{dz}Y_{V_L}(e^{\alpha}, x)$ for $\alpha \in L$.*

3. *For $a \in \mathfrak{h}$ and $\alpha \in L$, $a(x)\mathbf{1}_{V_L}, Y_{V_L}(e^{\alpha}, x)\mathbf{1}_{V_L} \in V_L[[x]]$. Moreover, $\lim_{x\to 0} a(x)\mathbf{1}_{V_L} = a(-1)\mathbf{1}_{V_L}$ and $\lim_{x\to 0} Y_{V_L}(e^{\alpha}, x)\mathbf{1}_{V_L} = e^{\alpha}$.*

4. *The vector space V is spanned by elements of the form*

$$\begin{aligned} &a_1(-n_1)\cdots a_k(-n_k)e^{\alpha} \\ &\quad = a_1(-n_1)\cdots a_k(-n_k)e_{\alpha}\mathbf{1}_{V_L} \\ &\quad = \mathrm{Res}_{x_1}\cdots\mathrm{Res}_{x_k} x_1^{-n_1}\cdots x_k^{-n_k}x_{k+1}^{-1}a_1(x_1)\cdots a_k(x_k)Y_{V_L}(e^{\alpha}, x_{k+1})\mathbf{1}_{V_L} \end{aligned} \tag{2.24}$$

   for $a_1, \ldots, a_k \in \mathfrak{h}$, $n_1, \ldots, n_k \in \mathbb{Z}_+$ and $\alpha \in L$.

5. *For $a, b \in \mathfrak{h}$,*

$$(x_1 - x_2)^2 a(x_1)b(x_2) = (x_1 - x_2)^2 b(x_2)a(x_1).$$

   For $a \in \mathfrak{h}$ and $\alpha \in L$,

$$(x_1 - x_2)a(x_1)Y_{V_L}(e^{\alpha}, x_2) = (x_1 - x_2)Y_{V_L}(e^{\alpha}, x_2)a(x_1).$$

   For $\alpha, \beta \in L$,

$$(x_1 - x_2)^{-(\alpha,\beta)}Y_{V_L}(e^{\alpha}, x_1)Y_{V_L}(e^{\beta}, x_2) = (x_1 - x_2)^{-(\alpha,\beta)}Y_{V_L}(e^{\beta}, x_2)Y_{V_L}(e^{\alpha}, x_1)$$

   when $(\alpha, \beta) < 0$ and

$$Y_{V_L}(e^{\alpha}, x_1)Y_{V_L}(e^{\beta}, x_2) = Y_{V_L}(e^{\beta}, x_2)Y_{V_L}(e^{\alpha}, x_1)$$

   when $(\alpha, \beta) \geq 0$.

Proof. Property 1 can be verified by the definition of $L_{V_L}(0)$ and straightforward calculations.

The first two formulas in Property 2 can also be verified by the definition of $L_{V_L}(-1)$ and straightforward calculations. Here we prove the third equality. We first need several commutator formulas. For $\alpha \in L$, from $[L_V(-1), \alpha(x)] = \frac{d}{dz}\alpha(x)$ whose proof we omitted, we obtain

$$\begin{aligned} [L_{V_L}(-1), \alpha(x)^-] &= \frac{d}{dx}\int a(x)^-, \\ [L_{V_L}(-1), \alpha(x)^+] &= \frac{d}{dx}a(x)^+ - \alpha(0)x^{-2}. \end{aligned}$$

Applying $\int \cdot\, dx$ to both sides, we obtain

$$\left[L_{V_L}(-1), \int \alpha(x)^- dx\right] = \frac{d}{dx}\left(\int a(x)^- dx\right) - \alpha(-1), \tag{2.25}$$

$$\left[L_{V_L}(-1), \int \alpha(x)^+ dx\right] = \frac{d}{dx}\int a(x)^+ dx + \alpha(0)x^{-1}. \tag{2.26}$$

By the definition of $L_{V_L}(-1)$, for a product A of operators of the form $a(-m)$ for $a \in \mathfrak{h}$ and $m \in \mathbb{Z}_+$, $[L_{V_L}, A]$ is a linear combinations of products of the operators of the same form. In particular, $[L_{V_L}(-1), A]$ commutes with $e_\alpha x^\alpha$. For such a product A and $\beta \in L$,

$$\begin{aligned}
&L_{V_L}(-1)e_\alpha x^\alpha A e^\beta \\
&\quad = L_{V_L}(-1)Ae_\alpha x^\alpha e^\beta \\
&\quad = [L_{V_L}(-1), A]e_\alpha x^\alpha e^\beta + AL_{V_L}(-1)e_\alpha x^\alpha e^\beta \\
&\quad = e_\alpha x^\alpha [L_{V_L}(-1), A]e^\beta + x^{(\alpha,\beta)}(-1)^{\epsilon(\alpha,\beta)} AL_{V_L}(-1)e^{\alpha+\beta} \\
&\quad = e_\alpha x^\alpha [L_{V_L}(-1), A]e^\beta + x^{(\alpha,\beta)}(-1)^{\epsilon(\alpha,\beta)} A(\alpha+\beta)(-1)e^{\alpha+\beta} \\
&\quad = e_\alpha x^\alpha [L_{V_L}(-1), A]e^\beta + x^{(\alpha,\beta)}(-1)^{\epsilon(\alpha,\beta)} A\alpha(-1)e^{\alpha+\beta} + x^{(\alpha,\beta)}(-1)^{\epsilon(\alpha,\beta)} A\beta(-1)\ e^{\alpha+\beta} \\
&\quad = e_\alpha x^\alpha [L_{V_L}(-1), A]e^\beta + \alpha(-1)e_\alpha x^\alpha A e^\beta + Ae_\alpha x^\alpha \beta(-1)e^\beta \\
&\quad = e_\alpha x^\alpha [L_{V_L}(-1), A]e^\beta + \alpha(-1)e_\alpha x^\alpha A e^\beta + e_\alpha x^\alpha A L_{V_L}(-1)(u \otimes e^\beta) \\
&\quad = e_\alpha x^\alpha L_{V_L}(-1)Ae^\beta + \alpha(-1)e_\alpha x^\alpha A e^\beta,
\end{aligned}$$

where we have used the fact that $S(\hat{\mathfrak{h}}_-)$ is a commutative algebra and e_α and x^α commute with A. So we obtain the commutator formula

$$[L_{V_L}(-1), e_\alpha x^\alpha] = \alpha(-1)e_\alpha x^\alpha. \tag{2.27}$$

For $u \in S(\hat{\mathfrak{h}}_-)$ and $\beta \in L$,

$$\begin{aligned}
\alpha(0)e_\alpha(u \otimes e^\beta) &= (-1)^{\epsilon(\alpha,\beta)}\alpha(0)(u \otimes e^{\alpha+\beta}) \\
&= (\alpha,\alpha)(-1)^{\epsilon(\alpha,\beta)}(u \otimes e^\beta) + (\alpha,\beta)(-1)^{\epsilon(\alpha,\beta)}(u \otimes e^\beta) \\
&= (\alpha,\alpha)e_\alpha(u \otimes e^\beta) + (\alpha,\beta)e_\alpha(u \otimes e^\beta) \\
&= (\alpha,\alpha)e_\alpha(u \otimes e^\beta) + e_\alpha\alpha(0)(u \otimes e^\beta),
\end{aligned}$$

which gives us the commutator formula

$$[\alpha(0), e_\alpha] = (\alpha,\alpha)e_\alpha. \tag{2.28}$$

Using the fact that $[\alpha(-1), \cdot]$ is a derivation on the algebra of operators on V_L as coefficients, we have

$$\begin{aligned}
[\alpha(-1), e^{\int \alpha(x)^+ dx}] &= e^{\int \alpha(x)^+ dx}\left[\alpha(-1), \int \alpha(x)^+ dx\right] \\
&= e^{\int \alpha(x)^+ dx}\left[\alpha(-1), -\alpha(1)\right] x^{-1} \\
&= e^{\int \alpha(x)^+ dx}(\alpha,\alpha)x^{-1}.
\end{aligned}$$

Note that both $[L_{V_L}(-1), \cdot]$ and $\frac{d}{dx}$ are derivations on the algebra of series in x with operators on V_L as coefficients. Using these properties, (2.25), (2.26) and (2.27) and the formula

$$\frac{d}{dx}x^{\alpha} = \alpha(0)x^{\alpha}x^{-1},$$

we have

$$\begin{aligned}
&[L_{V_L}(-1), Y(_{V_L}(e^{\alpha}, x)]\\
&\quad = [L_{V_L}(-1), e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} e_{\alpha} x^{\alpha}]\\
&\quad = [L_{V_L}(-1), e^{\int \alpha(x)^- dx}] e^{\int \alpha(x)^+ dx} e_{\alpha} x^{\alpha} + e^{\int \alpha(x)^- dx} [L_{V_L}(-1), e^{\int \alpha(x)^+ dx}] e_{\alpha} x^{\alpha}\\
&\qquad + e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} [L_{V_L}(-1), e_{\alpha} x^{\alpha}]\\
&\quad = e^{\int \alpha(x)^- dx} \left[L_{V_L}(-1), \int \alpha(x)^- dx\right] e^{\int \alpha(x)^+ dx} e_{\alpha} x^{\alpha}\\
&\qquad + e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} \left[L_{V_L}(-1), \int \alpha(x)^+ dx\right] e_{\alpha} x^{\alpha}\\
&\qquad + e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} \alpha(-1) e_{\alpha} x^{\alpha}\\
&\quad = e^{\int \alpha(x)^- dx} \left(\frac{d}{dx}\left(\int a(x)^- dx\right) - \alpha(-1)\right) e^{\int \alpha(x)^+ dx} e_{\alpha} x^{\alpha}\\
&\qquad + e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} \left(\frac{d}{dx}\int a(x)^+ dx + \alpha(0)x^{-1}\right) e_{\alpha} x^{\alpha}\\
&\qquad + e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} \alpha(-1) e_{\alpha} x^{\alpha}\\
&\quad = e^{\int \alpha(x)^- dx} \left(\frac{d}{dx}\int a(x)^- dx\right) e^{\int \alpha(x)^+ dx} e_{\alpha} x^{\alpha} - e^{\int \alpha(x)^- dx} \alpha(-1) e^{\int \alpha(x)^+ dx} e_{\alpha} x^{\alpha}\\
&\qquad + e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} \left(\frac{d}{dx}\int a(x)^+ dx\right) e_{\alpha} x^{\alpha} + e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} \alpha(0) x^{-1} e_{\alpha} x^{\alpha}\\
&\qquad + e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} \alpha(-1) e_{\alpha} x^{\alpha}\\
&\quad = \left(\frac{d}{dx} e^{\int \alpha(x)^- dx}\right) e^{\int \alpha(x)^+ dx} e_{\alpha} x^{\alpha} + e^{\int \alpha(x)^- dx} \left(\frac{d}{dx} e^{\int \alpha(x)^+ dx}\right) e_{\alpha} x^{\alpha}\\
&\qquad + e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} e_{\alpha} \alpha(0) x^{\alpha} x^{-1}\\
&\quad = \frac{d}{dx} e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} e_{\alpha} x^{\alpha}\\
&\quad = \frac{d}{dx} Y_{V_L}(e^{\alpha}, x). \qquad (2.29)
\end{aligned}$$

Properties 3 and 4 are clear. Property 5 follows immediately from Proposition 2.1. ∎

Just as in the Heisenberg case in the preceding section, we first state the following main result of this section assuming the reader knows grading-restricted vertex algebra:

Theorem 2.3. *The vector space V_L equipped with the the vertex operator map Y_{V_L} defined by*

$$\begin{aligned}\langle v', &Y_{V_L}(\alpha_1(-n_1)\cdots\alpha_k(-n_k)e^{\alpha}, z)v\rangle \\ &= \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{-n_1}\cdots\xi_k^{-n_k}\xi_{k+1}^{-1}R(\langle v', a_1(\xi_1+z)\cdots a_k(\xi_k+z)Y_{V_L}(e^{\alpha}, \xi_{k+1}+z)v\rangle).\end{aligned} \tag{2.30}$$

and the vacuum $\mathbf{1}_{V_L}$ is a grading-restricted vertex algebra. Moreover, this is the unique grading-restricted vertex algebra structure on V_L with the vacuum $\mathbf{1}_{V_L}$ such that $Y(a(-1)\mathbf{1}, z) = a(x)$ for $a \in \mathfrak{h}$ and $Y(e^{\alpha}, x) = Y_{V_L}(e^{\alpha}, x)$ for $\alpha \in L$.

3 Grading-restricted vertex algebras

For a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$, let $V' = \coprod_{n\in\mathbb{Z}} V_{(n)}^*$ be its graded dual space and $\overline{V} = \prod_{n\in\mathbb{Z}} V_{(n)}$ be its algebraic completion. On V and V', we use the topology given by the dual pair (V, V'). For $n \in \mathbb{N}$, a sequence (or more generally a net) $\{f_n\}$ in $\mathrm{Hom}(V\otimes\cdots\otimes V, \overline{V})$ is convergent to $f \in \mathrm{Hom}(V\otimes\cdots\otimes V, \overline{V})$ if for $v_1, \dots, v_n \in V$ and $v' \in V'$, $\langle v', f_n(v_1\otimes\cdots\otimes v_n)\rangle$ is convergent to $\langle v', f(v_1 \otimes \cdots \otimes v_n)\rangle$. In particular, analytic maps from a region in $\mathbb{C}$ to $\mathrm{Hom}(V^{\otimes n}, \overline{V})$ make sense. For a $\mathbb{C}$-graded vector space, we use the same notations and definition of convergence.

We give the definition of grading-restricted vertex algebra first.

Definition 3.1. A *grading-restricted vertex algebra* is a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$, equipped with a linear map

$$\begin{aligned} Y_V : V \otimes V &\to V[[x, x^{-1}]], \\ u \otimes v &\mapsto Y_V(u, x)v, \end{aligned}$$

or equivalently, an analytic map

$$\begin{aligned} Y_V : \mathbb{C}^{\times} &\to \mathrm{Hom}(V \otimes V, \overline{V}), \\ z &\mapsto Y_V(\cdot, z)\cdot : u \otimes v \mapsto Y_V(u, z)v \end{aligned}$$

called the *vertex operator map* and a *vacuum* $\mathbf{1} \in V_{(0)}$ satisfying the following axioms:

1. Axioms for the grading: (a) *Grading-restriction condition*: When n is sufficiently negative, $V_{(n)} = 0$ and $\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$. (b) $L(0)$-*commutator formula*: Let $L_V(0) : V \to V$ be defined by $L_V(0)v = nv$ for $v \in V_{(n)}$. Then

$$[L_V(0), Y_V(v, x)] = x\frac{d}{dx}Y_V(v, x) + Y_V(L_V(0)v, x)$$

for $v \in V$.

2. Axioms for the vacuum: (a) *Identity property*: Let 1_V be the identity operator on V. Then $Y_V(\mathbf{1}, x) = 1_V$. (b) *Creation property*: For $u \in V$, $\lim_{x\to 0} Y_V(u, x)\mathbf{1}$ exists and is equal to u.

3. $L(-1)$-*derivative property* and $L(-1)$-*commutator formula*: Let $L_V(-1) : V \to V$ be the operator given by

$$L_V(-1)v = \lim_{x\to 0} \frac{d}{dx} Y_V(v, x)\mathbf{1}$$

for $v \in V$. Then for $v \in V$,

$$\frac{d}{dx} Y_V(v, x) = Y_V(L_V(-1)v, x) = [L_V(-1), Y_V(v, x)].$$

4. *Duality*: For $u_1, u_2, v \in V$ and $v' \in V'$, the series

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle, \tag{3.1}$$
$$\langle v', Y_V(u_2, z_2)Y_V(u_1, z_1)v\rangle, \tag{3.2}$$
$$\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle, \tag{3.3}$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1-z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.

Remark 3.2. In Definition 3.1, the duality property can be stated separately as three axioms, that is, the *rationality* (the convergence of (3.1), (3.2) and (3.3) to rational functions in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively), the *commutativity* (the statement that the rational functions to which (3.1) and (3.2) converge are equal) and the *associativity* (the statement that the (3.1) and (3.3) are equal in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$). These axioms are not independent. In fact, the associativity follows from the rationality and commutativity (see [FHL]) and the commutativity also follows from the rationality and associativity (see [Hua2]).

Definition 3.3. A *quasi-vertex operator algebra* or a *Möbius vertex algebra* is a grading-restricted vertex algebra $(V, Y_V, \mathbf{1})$ together with an operator $L_V(1)$ of weight 1 on V satisfying

$$\begin{aligned}
[L_V(-1), L_V(1)] &= -2L_V(0),\\
[L_V(1), Y_V(v, x)] &= Y_V(L_V(1)v, x) + 2xY_V(L_V(0)v, x) + x^2Y_V(L_V(-1)v, x)
\end{aligned}$$

for $v \in V$.

Definition 3.4. Let V_1 and V_2 be grading-restricted vertex algebras. A homomorphism from V_1 to V_2 is a grading-preserving linear map $g : V_1 \to V_2$ such that $gY_{V_1}(u, x)v = Y_{V_2}(gu, x)gv$. An isomorphism from V_1 to V_2 is an invertible homomorphism from V_1 to V_2. When $V_1 = V_2 = V$, an isomorphism from V to V is called an automorphism of V.

Definition 3.5. Let $(V, Y_V, \mathbf{1})$ be a grading-restricted vertex algebra. A *conformal element* of V is an element $\omega \in V$ satisfying the following axioms:

1. There exists $c \in \mathbb{C}$ such that $Y_V(\omega, x)\omega$ is equal to $L_V(-1)\omega x^{-1} + 2\omega x^{-2} + \frac{c}{2}\mathbf{1}x^{-4}$ plus a V-valued power series in x.

2. $L_V(-1) = \mathrm{Res}_x Y_V(\omega, x)$ and $L_V(0) = \mathrm{Res}_x x Y_V(\omega, z)$ (Res_x being the operation of taking the coefficient of x^{-1} of a Laurent series).

A grading-restricted vertex algebra equipped with a conformal element is called a *vertex operator algebra* (or, more consistently, *grading-restricted conformal vertex algebra*).

4 A construction theorem

Let $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ be a $\mathbb{Z}$-graded vector space such that $V_{(n)} = 0$ for n sufficiently negative and $\dim V_{(n)} < \infty$ for $n \in \frac{\mathbb{Z}}{2}$. Since $\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$, we have $\overline{V} = (V')^*$. Elements of $V_{(n)}$ is said to have *weight* n. Let $L_V(0) : V \to V$ be the operator defined by the grading on V, that is, by $L_V(0)v = nv$ for $v \in V_{(n)}$. Then for $a \in \mathbb{C}$, the operator $e^{aL_V(0)}$ on V defined by $e^{aL_V(0)}v = e^{an}v$ for $v \in V_{(n)}$ has a natural extension to $\overline{V}$. For $n \in \mathbb{Z}$, we use π_n to denote the projection from V or $\overline{V}$ to $V_{(n)}$.

An operator O on V satisfying $[L_V(0), O] = nO$ is said to have *weight* n. Similarly for operators on the graded dual V' of V.

Lemma 4.1. *Let* $\phi(x) = \sum_{n\in\mathbb{Z}} \phi_n x^{-n-1} \in (\mathrm{End}\ V)[[x.x^{-1}]]$. *If there exists* $\mathrm{wt}\,\phi \in \mathbb{Z}$ *such that*

$$[L_V(0), \phi(x)] = x\frac{d}{dx}\phi(x) + (\mathrm{wt}\,\phi)\phi(x),$$

then $\phi_n \in \mathrm{Hom}(V, V)$ *is homogeneous of weight* $\mathrm{wt}\,\phi - n - 1$. *In particular, for* $v \in V$, $\phi(x)v$ *as a Laurent series in* x *has only finitely many negative power terms and for* $v' \in V'$, $\langle v', \phi(x)\cdot\rangle$ *as a Laurent series with coefficients in* V' *has only finitely many positive powers of* x.

Proof. Taking the coefficients of the bracket formula for $L_V(0)$ and $\phi(z)$, we obtain that ϕ_n is of weight $\mathrm{wt}\,\phi - n - 1$. Since $V_{(n)} = 0$ for n sufficiently negative and the weight of ϕ_n is $\mathrm{wt}\,\phi - n - 1$, for $v \in V$, $\phi(x)v$ has only finitely many negative power terms and for $v' \in V'$, $\langle v', \phi(x)\cdot\rangle$ as a Laurent series with coefficients in V' has only finitely many positive powers of x. ∎

Let $\phi^i(x) \in (\mathrm{End}\ V)[[x, x^{-1}]]$ for $i \in I$ and $\mathbf{1} \in V_{(0)}$. Write $\phi^i(x) = \sum_{n\in\mathbb{Z}} \phi^i_n x^{-n-1}$ for $i \in I$. Assume that $\phi^i(x) \in (\mathrm{End}\ V)[[x, x^{-1}]]$ for $i \in I$ and $\mathbf{1} \in V_{(0)}$ satisfy the following conditions:

1. For $i \in I$, there exists $\mathrm{wt}\,\phi^i \in \mathbb{Z}$ such that $[L_V(0), \phi^i(x)] = x\frac{d}{dx}\phi^i(x) + (\mathrm{wt}\,\phi^i)\phi^i(x)$.

2. There exists an operator $L_V(-1)$ on V such that $L_V(-1)\mathbf{1} = 0$ and $[L_V(-1), \phi^i(x)] = \frac{d}{dx}\phi^i(x)$ for $i \in I$.

3. For $i \in I$, $\phi^i(x)\mathbf{1} \in V[[x]]$.

4. The vector space V is spanned by elements of the form $\phi^{i_1}_{n_1} \cdots \phi^{i_k}_{n_k}\mathbf{1}$ for $i_1, \dots, i_k \in I$ and $n_1, \dots, n_k \in \mathbb{Z}$.

5. For $i, j \in I$, there exists $N_{ij} \in \mathbb{Z}_+$ such that

$$(x_1 - x_2)^{N_{ij}}\phi^i(x_1)\phi^j(x_2) = (x_1 - x_2)^{N_{ij}}\phi^j(x_2)\phi^i(x_1). \tag{4.1}$$

Proposition 4.2. *Let $V = \coprod_{n \in \frac{\mathbb{Z}}{2}} V_{(n)}$ be a $\frac{\mathbb{Z}}{2}$-graded vector space, ϕ^i for $i \in I$ linear maps from V to $V[[x, x^{-1}]]$, or equivalently, analytic maps from $\mathbb{C}^\times$ to $\mathrm{Hom}(V, \overline{V})$, $L_V(-1)$ an operator on V and $\mathbf{1} \in V_{(0)}$. Assume that they satisfy Conditions 1–4. Then Condition 5 is equivalent to the following two conditions:*

6. For $v' \in V'$, $v \in V$ and $i_1, \dots, i_k \in I$, the series $\langle v', \phi^{i_1}(z_1) \cdots \phi^{i_k}(z_k)v\rangle$ (a Laurent series in $z_1, \dots, z_k$ with complex coefficients) is absolutely convergent in the region $|z_1| > \cdots > |z_k| > 0$ to a rational function $R(\langle v', \phi^{i_1}(z_1) \cdots \phi^{i_k}(z_k)v\rangle)$ in $z_1, \dots, z_k$ with the only possible poles at $z_i = 0$ for $i = 1, \dots, k$ and $z_j = z_l$ for $j \neq l$. In addition, the order of the pole $z_j = z_l$ is independent of ϕ^{i_n} for $n \neq j, l$, v and v' and the order of the pole $z_j = 0$ is independent of ϕ^{i_n} for $n \neq j$ and v'.

7. For $v \in V$, $v' \in V'$, $i_1, i_2 \in I$,

$$R(\langle v', \phi^{i_1}(z_1)\phi^{i_2}(z_2)v\rangle) = R(\langle v', \phi^{i_2}(z_2)\phi^{i_1}(z_1)v\rangle).$$

Proof. Exercise: Prove that Conditions 6 and 7 imply Property 5.

Now we assume that Property 5 holds. Consider the Laurent series

$$\prod_{1 \le p < q \le k} (x_p - x_q)^{N_{i_p i_q}} \langle v', \phi^{i_1}(x_1) \cdots \phi^{i_k}(x_k)v\rangle. \tag{4.2}$$

For $1 \le l \le k$, using (4.1), the Laurent series (4.2) is equal to

$$\prod_{1 \le p < q \le k} (x_p - x_q)^{N_{i_p i_q}} \langle v', \phi^{i_1}(x_1) \cdots \phi^{i_{l-1}}(x_{l-1})\phi^{i_{l+1}}(x_{l+1}) \cdots \phi^{i_k}(x_k)\phi^{i_l}(x_l)v\rangle. \tag{4.3}$$

By Lemma 4.1, (4.3) has only finitely many negative power terms in x_l. So the same is true for (4.2). On the other hand, using (4.1) again, (4.2) is equal to

$$\prod_{1 \le p < q \le k} (x_p - x_q)^{N_{i_p i_q}} \langle v', \phi^{i_l}(x_l)\phi^{i_1}(x_1) \cdots \phi^{i_{l-1}}(x_{l-1})\phi^{i_{l+1}}(x_{l+1}) \cdots \phi^{i_k}(x_k)v\rangle. \tag{4.4}$$

By Lemma 4.1 again, (4.4) has only finitely many positive power terms in x_l. So the same is true for (4.2). Thus (4.2) must be a Laurent polynomial in x_l. Since this is true for $1 \leq l \leq k$, (4.2) is a Laurent polynomial in $x_1, \dots, x_k$.

For fixed $1 \leq p < q \leq k$, the expansion coefficients of

$$\langle v', \phi(x_1) \cdots \phi(x_k) v\rangle \tag{4.5}$$

as Laurent series in x_l for $l \neq p, q$ are of the form

$$\langle v', \phi^{i_1}_{n_1} \cdots \phi^{i_{p-1}}_{n_{p-1}} \phi^{i_p}(x_p) \phi^{i_{p+1}}_{n_{p+1}} \cdots \phi^{i_{q-1}}_{n_{q-1}} \phi^{i_q}(x_q) \phi^{i_{q+1}}_{n_{q+1}} \cdots \phi^{i_k}_{n_k} v\rangle \tag{4.6}$$

for $n_l \in \mathbb{Z}$, $l \neq p, q$. Clearly (4.6) contains only finitely many negative powers in x_q and finitely many positive powers in x_p. But we have shown that when multiplied by $(x_p - x_q)^{N_{pq}}$, it becomes a Laurent polynomial. Thus (4.6) must be the product of a Laurent polynomial in x_p and x_q and the expansion of $(x_p - x_q)^{-N_{pq}}$ as a Laurent series in nonnegative powers of x_q. Since p and q are arbitrary, we see that (4.5) is equal to the product of a Laurent polynomial and the expansion of $\prod_{1 \leq p < q \leq k} (z_p - z_q)^{-N_{pq}}$ in the region $|z_1| > \cdots > |z_k| > 0$. This is Condition 6. Condition 7 follows immediately from Condition 6 in the case $k = 2$ and (4.1). ∎

Proposition 4.3. *The space V, the fields ϕ^i for $i \in I$, $L_V(-1)$ and $\mathbf{1}$ have the following properties:*

8. *For $a \in \mathbb{C}$ and $i \in I$, $e^{aL_V(0)} \phi^i(x) e^{-aL_V(0)} = e^{a(\mathrm{wt}\, \phi^i)} \phi^i(e^a x)$.*

9. $L_V(-1) \phi^{i_1}_{n_1} \cdots \phi^{i_k}_{n_k} \mathbf{1} = \sum_{j=1}^{k} \phi^{i_1}_{n_1} \cdots \phi^{i_{j-1}}_{n_{j-1}} (-n_j \phi^{i_j}_{n_j - 1}) \phi^{i_{j+1}}_{n_{j+1}} \cdots \phi^{i_k}_{n_k} \mathbf{1}$.

10. *For $a \in \mathbb{C}$, $z \in \mathbb{C}^\times$ satisfying $|z| > |a|$ and $i \in I$, $e^{aL_V(-1)} \phi^i(z) e^{-aL_V(-1)} = \phi^i(z + a)$.*

11. *The operator $L_V(-1)$ has weight 1 and its adjoint $L_V(-1)'$ as an operator on V' has weight -1 (the weight of an operator on V' is defined in the same way as that of an operator on V). In particular, $e^{zL_V(-1)'} v' \in V'$ for $z \in \mathbb{C}$ and $v' \in V'$.*

12. *For $v \in V$, $v' \in V'$ and $\sigma \in S_k$,*

$$R(\langle v', \phi^{i_1}(z_1) \cdots \phi^{i_k}(z_k) v\rangle) = R(\langle v', \phi^{i_{\sigma(1)}}(z_{\sigma(1)}) \cdots \phi^{i_{\sigma(k)}}(z_{\sigma(k)}) v\rangle).$$

Proof. These properties follow immediately from Conditions 1–7. ∎

We now define a vertex operator map. We first give the motivation of this definition. The vertex operator map we want to define is a map

$$\begin{aligned} Y_V : \mathbb{C}^\times &\to \mathrm{Hom}(V \otimes V, \overline{V}), \\ z &\mapsto Y_V(\cdot, z) \cdot : u \otimes v \mapsto Y_V(u, z) v. \end{aligned}$$

We define $Y_V(\phi^i_{-1}\mathbf{1}, z)v = \phi^i(z)v$ for $i \in I$ and $v \in V$. The vertex operator map should satisfy the rationality and associativity property. In particular, we should have

$$R(\langle v', Y_V(\phi^{i_1}(\xi_1)\cdots\phi^{i_k}(\xi_k)\mathbf{1}, z)v\rangle) = R(\langle v', \phi^{i_1}(\xi_1+z)\cdots\phi^{i_k}(\xi_k+z)v\rangle)$$

for $i_1, \ldots, i_k \in I$, $v \in V$ and $v' \in V'$.

Motivated by this associativity formula, we define the vertex operator map as follows: For $v' \in V'$, $v \in V$, $i_1, \ldots, i_k \in I$, $m_1, \ldots, m_k \in \mathbb{Z}$, we define Y_V by

$$\begin{aligned}&\langle v', Y_V(\phi^{i_1}_{m_1}\cdots\phi^{i_k}_{m_k}\mathbf{1}, z)v\rangle\\ &\quad= \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{m_1}\cdots\xi_k^{m_k}R(\langle v', \phi^{i_1}(\xi_1+z)\cdots\phi^{i_k}(\xi_k+z)v\rangle).\end{aligned} \tag{4.7}$$

Note that for a meromorphic function $f(\xi)$, $\mathrm{Res}_{\xi=0}f(\xi)$ means expanding $f(\xi)$ as a Laurent series in $0 < |\xi| < r$ for r sufficiently small so that no other poles are in this disk and then taking the coefficient of ξ^{-1}. We can also expand $f(\xi)$ as a Laurent series in a different region. In general, the coefficient of ξ^{-1} in this Laurent series might be different from $\mathrm{Res}_{\xi=0}f(\xi)$. Also note that the order to take these residues is important. Different orders in general give vertex operators for different elements.

Since $\overline{V} = (V')^*$, for fixed $\phi^{i_1}_{m_1}\cdots\phi^{i_k}_{m_k}\mathbf{1}, v \in V$, the formula above indeed gives an element

$$Y_V(\phi^{i_1}_{m_1}\cdots\phi^{i_k}_{m_k}\mathbf{1}, z)v \in \overline{V},$$

which in turn gives

$$Y_V(\phi^{i_1}_{m_1}\cdots\phi^{i_k}_{m_k}\mathbf{1}, x)v \in V[[x, x^{-1}]].$$

Since there might be relations among elements of the form $\phi^{i_1}_{m_1}\cdots\phi^{i_k}_{m_k}\mathbf{1}$, we first have to show that the definition above indeed gives a well-defined map from $\mathbb{C}^\times$ to $\mathrm{Hom}(V\otimes V, \overline{V})$. Let ϕ^0 be the map from $\mathbb{C}^\times$ to $\mathrm{Hom}(V, \overline{V})$ given by $\phi^0(z) = 1_V$. Let $\mathrm{wt}\,\phi^0 = 0$. Then Conditions 1 to 5 and Properties 6 to 12 above still hold for ϕ^i, $i \in \tilde{I} = I \cup \{0\}$. Then any relation among such elements can always be written as

$$\sum_{p=1}^{q}\lambda_p\phi^{i_1^p}_{m_1^p}\cdots\phi^{i_k^p}_{m_k^p}\mathbf{1} = 0$$

for some $i_j^p \in \tilde{I}$ and $m_j^p \in \mathbb{Z}$, $p = 1, \ldots, q$, $j = 1, \ldots, k$.

Lemma 4.4. *If*

$$\sum_{p=1}^{q}\lambda_p\phi^{i_1^p}_{m_1^p}\cdots\phi^{i_k^p}_{m_k^p}\mathbf{1} = 0,$$

then

$$\sum_{p=1}^{q}\lambda_p\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{m_1^p}\cdots\xi_k^{m_k^p}R(\langle v', \phi^{i_1^p}(\xi_1+z)\cdots\phi^{i_k^p}(\xi_k+z)v\rangle) = 0$$

for $v \in V$ and $v' \in V'$.

Proof. By Condition 4, we can take v to be of the form $\phi^{j_1}_{n_1}\cdots\phi^{j_l}_{n_l}\mathbf{1}$. Moreover, in this case,

$$\begin{aligned}&R(\langle v', \phi^{i_1^p}(z_1)\cdots\phi^{i_k^p}(z_k)\phi^{j_1}_{n_1}\cdots\phi^{j_l}_{n_l}\mathbf{1}\rangle)\\&\quad=\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_l=0}\zeta_1^{n_1}\cdots\zeta_k^{n_l}R(\langle v', \phi^{i_1^p}(z_1)\cdots\phi^{i_k^p}(z_k)\phi^{j_1}(\zeta_1)\cdots\phi^{j_l}(\zeta_l)\mathbf{1}\rangle).\end{aligned}$$

Then

$$\begin{aligned}&\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{m_1^p}\cdots\xi_k^{m_k^p}R(\langle v', \phi^{i_1^p}(\xi_1+z)\cdots\phi^{i_k^p}(\xi_k+z)v\rangle)\\
&\quad=\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{m_1^p}\cdots\xi_k^{m_k^p}\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_l=0}\zeta_1^{n_1}\cdots\zeta_l^{n_l}\cdot\\
&\qquad\qquad\cdot R(\langle v', \phi^{i_1^p}(\xi_1+z)\cdots\phi^{i_k^p}(\xi_k+z)\phi^{j_1}(\zeta_1)\cdots\phi^{j_l}(\zeta_l)\mathbf{1}\rangle)\\
&\quad=\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{m_1^p}\cdots\xi_k^{m_k^p}\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_l=0}\zeta_1^{n_1}\cdots\zeta_l^{n_l}\cdot\\
&\qquad\qquad\cdot R(\langle v', \phi^{j_1}(\zeta_1)\cdots\phi^{j_l}(\zeta_l)\phi^{i_1^p}(\xi_1+z)\cdots\phi^{i_k^p}(\xi_k+z)\mathbf{1}\rangle)\\
&\quad=\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{m_1^p}\cdots\xi_k^{m_k^p}\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_l=0}\zeta_1^{n_1}\cdots\zeta_l^{n_l}\cdot\\
&\qquad\qquad\cdot R(\langle e^{zL_V(-1)'}v', \phi^{j_1}(\zeta_1-z)\cdots\phi^{j_l}(\zeta_l-z)\phi^{i_1^p}(\xi_1)\cdots\phi^{i_k^p}(\xi_k)\mathbf{1}\rangle)\\
&\quad=\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_l=0}\zeta_1^{n_1}\cdots\zeta_l^{n_l}\cdot\\
&\qquad\qquad\cdot R(\langle e^{zL_V(-1)'}v', \phi^{j_1}(\zeta_1-z)\cdots\phi^{j_l}(\zeta_l-z)\phi^{i_1^p}_{m_1^p}\cdots\phi^{i_k^p}_{m_k^p}\mathbf{1}\rangle).\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{p=1}^{q}\lambda_p\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{m_1^p}\cdots\xi_k^{m_k^p}R(\langle v', \phi^{i_1^p}(\xi_1+z)\cdots\phi^{i_k^p}(\xi_k+z)v\rangle)\\
&\quad=\sum_{p=1}^{q}\lambda_p\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_l=0}\zeta_1^{n_1}\cdots\zeta_l^{n_l}\cdot\\
&\qquad\qquad\cdot R(\langle e^{zL_V(-1)'}v', \phi^{j_1}(\zeta_1-z)\cdots\phi^{j_l}(\zeta_l-z)\phi^{i_1^p}_{m_1^p}\cdots\phi^{i_k^p}_{m_k^p}\mathbf{1}\rangle)\\
&\quad=\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_l=0}\zeta_1^{n_1}\cdots\zeta_l^{n_l}\cdot\\
&\qquad\qquad\cdot R\left(\left\langle e^{zL_V(-1)'}v', \phi^{j_1}(\zeta_1-z)\cdots\phi^{j_l}(\zeta_l-z)\left(\sum_{p=1}^{q}\lambda_p\phi^{i_1^p}_{m_1^p}\cdots\phi^{i_k^p}_{m_k^p}\mathbf{1}\right)\right\rangle\right)\\
&\quad=0,\end{aligned}$$

proving the lemma. ∎

From this lemma, we see that the vertex operator map Y_V is well defined. We are now ready to formulate and prove the main result of this section.

Theorem 4.5. *Let $V=\coprod_{n\in\mathbb{Z}}V_{(n)}$ be a $\mathbb{Z}$-graded vector space, ϕ^i for $i\in I$ linear maps from V to $V[[x,x^{-1}]]$, or equivalently, maps from $\mathbb{C}^\times$ to $\mathrm{Hom}(V,\overline{V})$, $L_V(-1)$ an operator on V and $\mathbf{1}\in V_{(0)}$. Assume that they satisfy Conditions 1–5. Then the triple $(V, Y_V, \mathbf{1})$ is a grading-restricted vertex algebra generated by $\phi^i_{-1}\mathbf{1}$ for $i\in I$. Moreover, this is the unique grading-restricted vertex algebra structure on V with the vacuum $\mathbf{1}$ such that $Y(\phi^1_{-1}\mathbf{1},z)=\phi^i(z)$ for $i\in I$.*

Proof. The vertex operator map Y_V is clearly analytic. The grading-restriction axiom is by assumption satisfied. The $L(-1)$-bracket formula follows from Condition 1 and the definition of Y_V. The identity property and the creation property also follow from of the definition of Y_V.

Let $L_V(0)'$ be the adjoint operator of $L_V(0)$. For $v' \in V'$, $v \in V$, $i_1, \dots, i_k \in I$ and $n_1, \dots, n_k \in \mathbb{Z}$, $a \in \mathbb{C}^\times$

$$\begin{aligned}
&\langle v', a^{L_V(0)} Y_V(\phi^{i_1}_{n_1} \cdots \phi^{i_k}_{n_k}\mathbf{1}, z) a^{-L_V(0)} v\rangle\\
&\quad = \langle a^{L_V(0)'} v', Y_V(\phi^{i_1}_{n_1} \cdots \phi^{i_k}_{n_k}\mathbf{1}, z) a^{-L_V(0)} v\rangle\\
&\quad = \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{n_1} \cdots \xi_k^{n_k} R(\langle a^{L_V(0)'} v', \phi^{i_1}(\xi_1+z) \cdots \phi^{i_k}(\xi_k+z) a^{-L_V(0)} v\rangle)\\
&\quad = \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{n_1} \cdots \xi_k^{n_k} R(\langle v', a^{L_V(0)} \phi^{i_1}(\xi_1+z) \cdots \phi^{i_k}(\xi_k+z) a^{-L_V(0)} v\rangle)\\
&\quad = \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{n_1} \cdots \xi_k^{n_k} a^{\mathrm{wt}\, \phi^{i_1} + \cdots \mathrm{wt}\, \phi^{i_k}} R(\langle v', \phi^{i_1}(a\xi_1+az) \cdots \phi^{i_k}(a\xi_k+az) v\rangle)\\
&\quad = \mathrm{Res}_{\zeta_1=0} \cdots \mathrm{Res}_{\zeta_k=0} \zeta_1^{n_1} \cdots \zeta_k^{n_k} a^{\mathrm{wt}\, \phi^{i_1} + \cdots \mathrm{wt}\, \phi^{i_k} - k - n_1 - \cdots - n_k} \cdot\\
&\qquad\qquad \cdot R(\langle v', \phi^{i_1}(\zeta_1+az) \cdots \phi^{i_k}(\zeta_k+az) v\rangle)\\
&\quad = \langle v', Y_V(a^{L_V(0)} \phi^{i_1}_{n_1} \cdots \phi^{i_k}_{n_k}\mathbf{1}, az) v\rangle).
\end{aligned}$$

This formula implies the $L(0)$-bracket formula.

From Condition 2 and the definition of Y_V, we obtain

$$\frac{d}{dz} Y_V(\phi^{i_1}_{n_1} \cdots \phi^{i_k}_{n_k}\mathbf{1}, z) = [L_V(-1), Y_V(\phi^{i_1}_{n_1} \cdots \phi^{i_k}_{n_k}\mathbf{1}, z)].$$

From Property 9 and the definition of Y_V, we obtain

$$\frac{d}{dz} Y_V(\phi^{i_1}_{n_1} \cdots \phi^{i_k}_{n_k}\mathbf{1}, z) = Y_V(L_V(-1)\phi^{i_1}_{n_1} \cdots \phi^{i_k}_{n_k}\mathbf{1}, z).$$

Applying both sides of this formula to $\mathbf{1}$, taking the limit $z \to 0$ and then using the creation property, we obtain

$$L_V(-1)\phi^{i_1}_{n_1} \cdots \phi^{i_k}_{n_k}\mathbf{1} = \lim_{z\to 0} \frac{d}{dz} Y_V(\phi^{i_1}_{n_1} \cdots \phi^{i_k}_{n_k}\mathbf{1}, z)\mathbf{1}.$$

The $L(-1)$-derivative property is proved.

Let $\{e_n\}_{n\in\mathbb{Z}}$ be a homogeneous basis of V and $\{e'_n\}_{n\in\mathbb{Z}}$ its dual basis in V'. Then we have

$$\begin{aligned}
&\langle v', Y_V(\phi^{i_1}_{n_1} \cdots \phi^{i_k}_{n_k}\mathbf{1}, z_1) Y_V(\phi^{j_1}_{m_1} \cdots \phi^{j_l}_{m_l}\mathbf{1}, z_2) v\rangle\\
&\quad = \sum_{n\in\mathbb{Z}} \langle v', Y_V(\phi^{i_1}_{n_1} \cdots \phi^{i_k}_{n_k}\mathbf{1}, z_1) e_n\rangle \langle e'_n, Y_V(\phi^{j_1}_{m_1} \cdots \phi^{j_l}_{m_l}\mathbf{1}, z_2) v\rangle\\
&\quad = \sum_{n\in\mathbb{Z}} \mathrm{Res}_{\zeta_1=0} \cdots \mathrm{Res}_{\zeta_k=0} \zeta_1^{n_1} \cdots \zeta_k^{n_k} \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_l=0} \xi_1^{m_1} \cdots \xi_l^{m_l} \cdot\\
&\qquad\qquad \cdot R(\langle v', \phi^{i_1}(\zeta_1+z_1) \cdots \phi^{i_k}(\zeta_k+z_1) e_n\rangle) R(\langle e'_n, \phi^{j_1}(\xi_1+z_2) \cdots \phi^{j_l}(\xi_l+z_2) v\rangle)\\
&\quad = \mathrm{Res}_{\zeta_1=0} \cdots \mathrm{Res}_{\zeta_k=0} \zeta_1^{n_1} \cdots \zeta_k^{n_k} \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_l=0} \xi_1^{m_1} \cdots \xi_l^{m_l} \cdot\\
&\qquad\qquad \cdot \sum_{n\in\mathbb{Z}} R(\langle v', \phi^{i_1}(\zeta_1+z_1) \cdots \phi^{i_k}(\zeta_k+z_1) e_n\rangle) R(\langle e'_n, \phi^{j_1}(\xi_1+z_2) \cdots \phi^{j_l}(\xi_l+z_2) v\rangle).
\end{aligned} \tag{4.8}$$

By Condition 6, when $|z_1| > \cdots > |z_{k+l}| > 0$,

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} & R(\langle v', \phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)e_n\rangle)R(\langle e'_n, \phi^{j_1}(z_{k+1})\cdots\phi^{j_l}(z_{k+l})v\rangle)\\
&= \sum_{n\in\mathbb{Z}}\langle v', \phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)e_n\rangle\langle e'_n, \phi^{j_1}(z_{k+1})\cdots\phi^{j_l}(z_{k+l})v\rangle\\
&= \langle v', \phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)\phi^{j_1}(z_{k+1})\cdots\phi^{j_l}(z_{k+l})v\rangle \qquad (4.9)
\end{aligned}$$

is absolutely convergent to the rational function

$$R(\langle v', \phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)\phi^{j_1}(z_{k+1})\cdots\phi^{j_l}(z_{k+l})v\rangle) \qquad (4.10)$$

in $z_1, \dots, z_{k+l}$. On the other hand, since the only possible poles of (4.10) are $z_i - z_j = 0$ for $i \neq j$ and $z_i = 0$, there is a unique expansion of such a rational function in the region $|z_1|, \dots, |z_k| > |z_{k+1}|, \dots, |z_{k+l}| > 0$, $z_i \neq z_j$ for $i \neq j$, $i, j = 1, \dots, k$ and $i, j = k+1, \dots, k+l$ such that each term is a product of two rational functions, one in $z_1, \dots, z_k$ and the other in $z_{k+1}, \dots, z_{k+l}$. Since the left-hand side of (4.9) is a series of the same form and is absolutely convergent in the region $|z_1| > \cdots > |z_{k+l}| > 0$ to (4.10), it must be absolutely convergent in the larger region $|z_1|, \dots, |z_k| > |z_{k+1}|, \dots, |z_{k+l}| > 0$, $z_i \neq z_j$ for $i \neq j$, $i, j = 1, \dots, k$ and $i, j = k+1, \dots, k+l$ to (4.10).

Substituting $\zeta_i + z_1$ for z_i for $i = 1, \dots, k$ and $\xi_j + z_2$ for z_{k+j} for $j = 1, \dots, l$, we see that

$$\sum_{n\in\mathbb{Z}} R(\langle v', \phi^{i_1}(\zeta_1 + z_1)\cdots\phi^{i_k}(\zeta_k + z_1)e_n\rangle)R(\langle e'_n, \phi^{j_1}(\xi_1 + z_2)\cdots\phi^{j_l}(\xi_l + z_2)v\rangle)$$

is absolutely convergent to

$$R(\langle v', \phi^{i_1}(\zeta_1 + z_1)\cdots\phi^{i_k}(\zeta_k + z_1)\phi^{j_1}(\xi_1 + z_2)\cdots\phi^{j_l}(\xi_l + z_2)v\rangle)$$

when $|\zeta_1 + z_1|, \dots, |\zeta_k + z_1| > |\xi_1 + z_2|, \dots, |\xi_l + z_2| > 0$, $\zeta_i \neq \zeta_j$ for $i, j = 1, \dots, k$ and $\xi_i \neq \xi_j$ for $i, j = 1, \dots, l$. When $|z_1| > |z_2| > 0$, we can always find sufficiently small neighborhood of 0 such that when $\zeta_1, \dots, \zeta_k, \xi_1, \dots, \xi_l$ are in this neighborhood, $|\zeta_1 + z_1|, \dots, |\zeta_k + z_1| > |\xi_1 + z_2|, \dots, |\xi_l + z_2| > 0$ holds. Thus we see that when $|z_1| > |z_2| > 0$, the right-hand side of (4.8) is absolutely convergent to

$$\begin{aligned}
\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_k=0}\zeta_1^{n_1}\cdots\zeta_k^{n_k}\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\cdot&\\
\cdot R(\langle v', \phi^{i_1}(\zeta_1 + z_1)\cdots\phi^{i_k}(\zeta_k + z_1)\phi^{j_1}(\xi_1 + z_2)\cdots\phi^{j_l}(\xi_l + z_2)v\rangle).& \qquad (4.11)
\end{aligned}$$

This is a rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$. In particular, the left-hand side of (4.8), that is,

$$\langle v', Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z_1)Y_V(\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z_2)v\rangle, \qquad (4.12)$$

is absolutely convergent in the region $|z_1| > |z_2| > 0$ to this rational function.

We have proved the rationality of the product of two vertex operators. We are ready to prove the commutativity. The calculation above also shows that

$$\langle v', Y_V(\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z_2)Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z_1)v\rangle \tag{4.13}$$

is absolutely convergent to the rational function

$$\begin{aligned}&\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_k=0}\zeta_1^{n_1}\cdots\zeta_k^{n_k}\cdot\\ &\qquad\cdot R(\langle v', \phi^{j_1}(\xi_1+z_2)\cdots\phi^{j_l}(\xi_l+z_2)\phi^{i_1}(\zeta_1+z_1)\cdots\phi^{i_k}(\zeta_k+z_1)v\rangle),\end{aligned} \tag{4.14}$$

in the regions $|z_2| > |z_1| > 0$, respectively. By Property 12, the rational functions (4.11) and (4.14) are equal. Thus (4.12) and (4.13) are absolutely convergent in the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively, to a common rational function with the only possible poles at $z_1 = z_2$, $z_1 = 0$ and $z_2 = 0$.

We now prove the associativity. For $i_1, \dots, i_k, j_1, \dots, j_l \in I$, $m_1, \dots, m_l \in \mathbb{Z}$, $v \in V$ and $v' \in V'$, using the expansion of $\phi^{i_1}(\xi_1), \dots, \phi^{i_k}(\xi_k)$ and the definition of Y_V, we have

$$\begin{aligned}&\langle v', Y_V(\phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z)v\rangle\\ &\quad=\sum_{p_1,\dots,p_k\in\mathbb{Z}}\langle v', Y_V(\phi^{i_1}_{p_1}\cdots\phi^{i_k}_{p_k}\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z)v\rangle z_1^{-p_1-1}\cdots z_k^{-p_k-1}\\ &\quad=\sum_{p_1,\dots,p_k\in\mathbb{Z}}\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_k=0}\zeta_1^{p_1}\cdots\zeta_k^{p_k}\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\cdot\\ &\qquad\qquad\cdot R(\langle v', \phi^{i_1}(\zeta_1+z)\cdots\phi^{i_k}(\zeta_k+z)\phi^{j_1}(\xi_1+z)\cdots\phi^{j_l}(\xi_l+z)v\rangle)z_1^{-p_1-1}\cdots z_k^{-p_k-1}.\end{aligned} \tag{4.15}$$

We now expand

$$R(\langle v', \phi^{i_1}(\zeta_1+z)\cdots\phi^{i_k}(\zeta_k+z)\phi^{j_1}(\xi_1+z)\cdots\phi^{j_l}(\xi_l+z)v\rangle)$$

as a Laurent series $\sum_{l\in\mathbb{Z}} f_l(\zeta_1, \dots, \zeta_{k-1}, \xi_1, \dots, \xi_l, z)\zeta_k^{-l-1}$ in ζ_k in the region $|z|, |\zeta_1|, \dots, |\zeta_{k-1}| > |\zeta_k| > |\xi_1|, \dots, |\xi_l|$, where $f_l(\zeta_1, \dots, \zeta_{k-1}, \xi_1, \dots, \xi_l, z)$ are rational functions in $\zeta_1, \dots, \zeta_{k-1}, \xi_1, \dots, \xi_l$ and z. Then in the region that the Laurent series expansion holds, we have

$$\begin{aligned}&\sum_{p_k\in\mathbb{Z}}\mathrm{Res}_{\zeta_k=0}\zeta_k^{p_k}\left(\sum_{l\in\mathbb{Z}} f_l(\zeta_1, \dots, \zeta_{k-1}, \xi_1, \dots, \xi_l, z)\zeta_k^{-l-1}\right)z_k^{-p_k-1}\\ &\qquad=\sum_{p_k\in\mathbb{Z}} f_{p_k}(\zeta_1, \dots, \zeta_{k-1}, \xi_1, \dots, \xi_l, z)z_k^{-p_k-1}\\ &\qquad=R(\langle v', \phi^{i_1}(\zeta_1+z)\cdots\phi^{i_{k-1}}(\zeta_{k-1}+z)\phi^{i_k}(z_k+z)\phi^{j_1}(\xi_1+z)\cdots\phi^{j_l}(\xi_l+z)v\rangle).\end{aligned} \tag{4.16}$$

Repeating this step for the variables $\zeta_{k-1}, \dots, \zeta_1$, we see that the right-hand side of (4.15) is equal to the expansion of

$$\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}R(\langle v', \phi^{i_1}(z_1+z)\cdots\phi^{i_k}(z_k+z)\phi^{j_1}(\xi_1+z)\cdots\phi^{j_l}(\xi_l+z)v\rangle) \tag{4.17}$$

as a Laurent series in $z_1 \dots, z_k$ in the region $|z| > |z_1| > \cdots > |z_k| > 0$. Thus the left-hand side of (4.15) is absolutely convergent to (4.17) in the region for this Laurent series expansion. In particular, in the region $|z| > |z_1| > \cdots > |z_k| > 0$,

$$\begin{aligned}
&\langle v', Y_V(\phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z)v\rangle\\
&\quad = \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\cdot\\
&\qquad\qquad \cdot R(\langle v', \phi^{i_1}(z_1+z)\cdots\phi^{i_k}(z_k+z)\phi^{j_1}(\xi_1+z)\cdots\phi^{j_l}(\xi_l+z)v\rangle). \quad (4.18)
\end{aligned}$$

Now we have

$$\begin{aligned}
&\langle v', Y_V(Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z_1-z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z_2)v\rangle\\
&\quad = \sum_{n\in\mathbb{Z}}\langle v', Y_V(e_n, z_2)v\rangle\langle e'_n, Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z_1-z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}\rangle\\
&\quad = \sum_{n\in\mathbb{Z}}\langle v', Y_V(e_n, z_2)v\rangle\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_k=0}\zeta_1^{n_1}\cdots\zeta_k^{n_k}\cdot\\
&\qquad\qquad \cdot R(\langle e'_n, \phi^{i_1}(\zeta_1+z_1-z_2)\cdots\phi^{i_k}(\zeta_k+z_1-z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}\rangle). \quad (4.19)
\end{aligned}$$

But by (4.18), in the region $|z_2| > |\zeta_1+z_1-z_2| > \cdots > |\zeta_k+z_1-z_2| > 0$, we have

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}}\langle v', Y_V(e_n, z_2)v\rangle\langle e'_n, \phi^{i_1}(\zeta_1+z_1-z_2)\cdots\phi^{i_k}(\zeta_k+z_1-z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}\rangle\\
&\quad = \langle v', Y_V(\phi^{i_1}(\zeta_1+z_1-z_2)\cdots\phi^{i_k}(\zeta_k+z_1-z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z_2)v\rangle\\
&\quad = \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\cdot\\
&\qquad\qquad \cdot R(\langle v', \phi^{i_1}(\zeta_1+z_1)\cdots\phi^{i_k}(\zeta_k+z_1)\phi^{j_1}(\xi_1+z_2)\cdots\phi^{j_l}(\xi_l+z_2)v\rangle).
\end{aligned} \quad (4.20)$$

The right-hand side of (4.20) is a rational function in $\zeta_1, \dots, \zeta_k$, z_1 and z_2 with the only possible poles $\zeta_i - \zeta_j = 0$, for $i \neq j$, $\zeta_i + z_1 = 0$, $\zeta_i + z_1 - z_2 = 0$ and $z_2 = 0$. There is a unique expansion of such a rational function in the region $|z_2| > |\zeta_1 + z_1 - z_2|, \dots, |\zeta_k + z_1 - z_2| > 0$, $\zeta_i \neq \zeta_j$ for $i \neq j$, $i, j = 1, \dots, k$, such that each term is a product of two rational functions, one in z_2 and the other in $\zeta_1, \dots, \zeta_k$ and z_1. Since

$$\sum_{n\in\mathbb{Z}}\langle v', Y_V(e_n, z_2)v\rangle R(\langle e'_n, \phi^{i_1}(\zeta_1+z_1-z_2)\cdots\phi^{i_k}(\zeta_k+z_1-z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}\rangle)$$

is a series of the same form and is equal to the left-hand side of (4.20) in the region $|z_2| > |\zeta_1 + z_1 - z_2| > \cdots > |\zeta_k + z_1 - z_2| > 0$, it must be absolutely convergent to the right-hand side of (4.20) in the larger region $|z_2| > |\zeta_1 + z_1 - z_2|, \dots, |\zeta_k + z_1 - z_2| > 0$. Thus we obtain

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}}\langle v', Y_V(e_n, z_2)v\rangle R(\langle e'_n, \phi^{i_1}(\zeta_1+z_1-z_2)\cdots\phi^{i_k}(\zeta_k+z_1-z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}\rangle)\\
&\quad = \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\cdot\\
&\qquad\qquad \cdot R(\langle v', \phi^{i_1}(\zeta_1+z_1)\cdots\phi^{i_k}(\zeta_k+z_1)\phi^{j_1}(\xi_1+z_2)\cdots\phi^{j_l}(\xi_l+z_2)v\rangle)
\end{aligned} \quad (4.21)$$

in the region $|z_2| > |\zeta_1 + z_1 - z_2|, \ldots, |\zeta_k + z_1 - z_2| > 0$. Thus when $|z_2| > |z_1 - z_2| > 0$, the right-hand side of (4.19) is absolutely convergent to

$$\begin{aligned} &\mathrm{Res}_{\zeta_1=0} \cdots \mathrm{Res}_{\zeta_k=0} \zeta_1^{n_1} \cdots \zeta_k^{n_k} \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_l=0} \xi_1^{m_1} \cdots \xi_l^{m_l} \cdot \\ &\qquad \cdot R(\langle v', \phi^{i_1}(\zeta_1 + z_1) \cdots \phi^{i_k}(\zeta_k + z_1) \phi^{j_1}(\xi_1 + z_2) \cdots \phi^{j_l}(\xi_l + z_2) v \rangle), \end{aligned} \tag{4.22}$$

which is proved above to be equal to the left hand side of (4.8) in the region $|z_1| > |z_2| > 0$. The associativity is proved.

To prove the uniqueness, we need only show that any grading-restricted vertex superalgebra structure on V with the vacuum $\mathbf{1}$ must have the vertex operator map defined by (4.7). But this is clear from the motivation that we discussed before the definition (4.7) of the vertex operator map Y_V. ∎

We call the grading-restricted vertex algebra given in Theorem 4.5 the *grading-restricted vertex algebra generated by* ϕ^i, $i \in I$. The maps ϕ^i, $i \in I$, are called the *generating fields* of the grading-restricted vertex algebra V.

Remark 4.6. In the proof of Theorem 4.5, we gave a proof of the associativity using the the definition (4.7) of the vertex operators. But the associativity can also be obtained by quoting Proposition 3.6.1 in [FHL].

Proof of Theorem 1.3. By Proposition 1.2, Conditions 1–5 needed in Theorem 4.5 are satisfied by $S(\hat{\mathfrak{h}}_-)$, $a(x)$ for $a \in \mathfrak{h}$, $L_{S(\hat{\mathfrak{h}}_-)}(0)$ and $L_{S(\hat{\mathfrak{h}}_-)}(-1)$. By Theorem 4.5, Theorems 1.3 is proved. ∎

Proof of Theorem 2.3. By Proposition 2.2, Conditions 1–5 needed in Theorem 4.5 are satisfied by V_L, $a(x)$ for $a \in \mathfrak{h}$, $Y_{V_L}(e^\alpha, x)$ for $\alpha \in L$, $L_{V_L}(0)$ and $L_{V_L}(-1)$. By Theorem 4.5, Theorems 2.3 is proved. ∎

5 Some properties of grading-restricted vertex algebras

5.1 Operator product expansion

Let V be a grading-restricted vertex algebra. For $u_1, u_2, v \in V$ and $v' \in V'$, by definition,

$$\langle v', Y_V(u_1, z_1) Y_V(u_2, z_2) v \rangle$$

is absolutely convergent in the region $|z_1| > |z_2| > 0$ and

$$\langle v', Y_V(Y_V(u_1, z_1 - z_2) u_2, z_2) v \rangle$$

is absolutely convergent in the region $|z_2| > |z_1 - z_2| > 0$. Since $(V')^*$ is canonically isomorphic to $\overline{V} = \prod_{n\in\mathbb{Z}} V_{(n)}$, $Y_V(u_1, z_1)Y_V(u_2, z_2)v$ (when $|z_1| > |z_2| > 0$) and $Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v$ (when $|z_2| > |z_1 - z_2| > 0$) as elements of $(V')^*$ will be viewed as elements of $\overline{V}$. Since v is arbitrary, $Y_V(u_1, z_1)Y_V(u_2, z_2)$ (when $|z_1| > |z_2| > 0$) and $Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)$ (when $|z_2| > |z_1 - z_2| > 0$) are maps from V to $\overline{V}$. Then we obtain the *associativity*

$$Y_V(u_1, z_1)Y_V(u_2, z_2) = Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2) \tag{5.1}$$

in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$. Since $Y_V(u_1, x) \in V((x))$, we have $Y_V(u_1, x)u_2 = \sum_{n\in\mathbb{Z}}(Y_V)_n(u_1)u_2x^{-n-1}$ for $(Y_V)_n(u_1)u_2 \in V$ and there exists $N \in \mathbb{N}$ such that $(Y_V)_n(u_1)u_2 = 0$ for $n > N$. Then we have

$$Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2) = \sum_{n\leq N} Y_V((Y_V)_n(u_1)u_2, z_2)(z_1 - z_2)^{-n-1}.$$

in the region $|z_2| > |z_1 - z_2| > 0$. From this expansion and (5.1), we obtain

$$Y_V(u_1, z_1)Y_V(u_2, z_2) = \sum_{n\leq N} Y_V((Y_V)_n(u_1)u_2, z_2)(z_1 - z_2)^{-n-1} \tag{5.2}$$

in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$. The formula (5.2) is called the *operator product expansion* of the fields or vertex operators $Y_V(u_1, z_1)$ and $Y_V(u_2, z_2)$. The terms that are singular in the right-hand side of (5.2) are

$$\sum_{n=0}^{N} Y_V((Y_V)_n(u_1)u_2, z_2)(z_1 - z_2)^{-n-1}.$$

These singular terms are the only useful terms in the calculations of the commutators of the fields or vertex operators $Y_V(u_1, z_1)$ and $Y_V(u_2, z_2)$. So physicists usually write the operator product expansion with only these singular terms as

$$Y_V(u_1, z_1)Y_V(u_2, z_2) \sim \sum_{n=0}^{N} Y_V((Y_V)_n(u_1)u_2, z_2)(z_1 - z_2)^{-n-1}. \tag{5.3}$$

Example 5.1. By Theorem 1.3, $S(\hat{\mathfrak{h}}_-)$ has a structure of a grading-restricted vertex algebra. As usual, we use $Y_{S(\hat{\mathfrak{h}}_-)}$ to denote its vertex operator map. It is easy to see from the definition of the vertex operator map in the preceding section, we see that $Y_{S(\hat{\mathfrak{h}}_-)}(a(-1)\mathbf{1}, x) = a(x)$ for $a \in \mathfrak{h}$. Then the operator product expansion of $a(z_1)$ and $b(z_2)$ is

$$\begin{aligned} a(z_1)b(z_2) &= Y_{S(\hat{\mathfrak{h}}_-)}(a(-1)\mathbf{1}, z_1)Y_{S(\hat{\mathfrak{h}}_-)}(b(-1)\mathbf{1}, z_2) \\ &= Y_{S(\hat{\mathfrak{h}}_-)}(Y_{S(\hat{\mathfrak{h}}_-)}(a(-1)\mathbf{1}, z_1 - z_2)b(-1)\mathbf{1}, z_2) \end{aligned} \tag{5.4}$$

in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$. But

$$\begin{aligned} Y_{S(\hat{\mathfrak{h}}_-)}(a(-1)\mathbf{1}, z_1 - z_2)b(-1)\mathbf{1} &= a(z_1 - z_2)b(-1)\mathbf{1} \\ &= \sum_{n\in\mathbb{Z}} a(n)b(-1)\mathbf{1}(z_1 - z_2)^{-n-1} \\ &= (a, b)\mathbf{1}(z_1 - z_2)^{-2} + \sum_{n\in-\mathbb{Z}_+} a(n)b(-1)\mathbf{1}(z_1 - z_2)^{-n-1}. \end{aligned} \tag{5.5}$$

Substituting the right-hand side of (5.5) into the right-hand side of (5.9), we obtain the explicit form

$$a(z_1)b(z_2) = (a, b)(z_1 - z_2)^{-2} + \sum_{n\in-\mathbb{Z}_+} Y_{S(\hat{\mathfrak{h}}_-)}(a(n)b(-1)\mathbf{1}, z_2)(z_1 - z_2)^{-n-1} \tag{5.6}$$

of the operator product expansion of $a(z_1)$ and $b(z_2)$ Since the only singular term in $z_1 - z_2$ in the right-hand side of (5.6) is $(a, b)(z_1 - z_2)^{-2}$, we obtain

$$a(z_1)b(z_2) \sim (a, b)(z_1 - z_2)^{-2}. \tag{5.7}$$

The last formula can also be calculated using the commutator formula (1.1). Apply both sides of (1.1) to $v \in S(\hat{\mathfrak{h}}_-)$ and then rewrite the resulting formula as

$$a(x_1)b(x_2)v - (a, b)(x_1 - x_2)^{-2}v = b(x_2)a(x_1)v - (a, b)(-x_2 + x_1)^{-2}v. \tag{5.8}$$

Note that the left-hand side (5.8) has only finitely many negative powers of x_2 and the right-hand side of (5.8) has only finitely many negative powers of x_1. Thus both sides of (5.8) have finitely many negative powers of both x_1 and x_2. Let $f(x_1, x_2)$ be this Laurent series with finitely many negative powers of x_1 and x_2. We can write $f(x_1, x_2)$ as $f(x_2 + (x_1 - x_2), x_2)$ and expand it as a Laurent series in x_2 and $x_1 - x_2$ with only nonnegative powers of $x_1 - x_2$. We use $f(x_2 + (x_1 - x_2), x_2)$ to denote this expansion. So we obtain the operator product expansion

$$a(x_1)b(x_2)v = (a, b)(x_1 - x_2)^{-2}v + f(x_2 + (x_1 - x_2), x_2).$$

Since the expansion of $f(x_2 + (x_1 - x_2), x_2)$ contain only nonnegative powers of $x_1 - x_2$ and v is arbitrary, we obtain (5.7).

Example 5.2. By Theorem 2.3, V_L has a structure of a grading-restricted vertex algebra. We use Y_{V_L} to denote its vertex operator map. Then from the definition of the vertex operator map in the preceding section, we see that $Y_{V_L}(a(-1)\mathbf{1}, x) = a(x)$ for $a \in \mathfrak{h}$ and $Y_{V_L}(e^\alpha, x)$ is exactly the vertex operator associated with $e^\alpha = 1 \otimes e^\alpha \in V_L$. So the operator product expansion of $a(z_1)$ and $Y_{V_L}(e^\alpha, x)$ is

$$\begin{aligned} a(z_1)Y_{V_L}(e^\alpha, x) &= Y_{V_L}(a(-1)\mathbf{1}, z_1)Y_{V_L}(e^\alpha, z_2) \\ &= Y_{V_L}(Y_{V_L}(a(-1)\mathbf{1}, z_1 - z_2)e^\alpha, z_2) \end{aligned} \tag{5.9}$$

in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$. But

$$\begin{aligned} Y_{V_L}(a(-1)\mathbf{1}, z_1 - z_2)e^{\alpha} &= a(z_1 - z_2)e^{\alpha} \\ &= \sum_{n\in\mathbb{Z}} a(n)e^{\alpha}(z_1 - z_2)^{-n-1} \\ &= (a, \alpha)e^{\alpha}(z_1 - z_2)^{-1} + \sum_{n\in -\mathbb{Z}_+} a(n)e^{\alpha}(z_1 - z_2)^{-n-1}. \end{aligned} \tag{5.10}$$

Substituting the right-hand side of (5.10) into the right-hand side of (5.9), we obtain the explicit form

$$a(z_1)Y_{V_L}(e^{\alpha}, x) = (a, \alpha)Y_{V_L}(e^{\alpha}, z_2)(z_1 - z_2)^{-1} + \sum_{n\in -\mathbb{Z}_+} Y_{V_L}(a(n)e^{\alpha}, z_2)(z_1 - z_2)^{-n-1} \tag{5.11}$$

of the operator product expansion of $a(z_1)$ and $Y_{V_L}(e^{\alpha}, x)$ Since the only singular term in $z_1 - z_2$ in the right-hand side of (5.11) is $(a, \alpha)Y_{V_L}(e^{\alpha}, z_2)(z_1 - z_2)^{-1}$, we obtain

$$a(z_1)Y_{V_L}(e^{\alpha}, x) \sim (a, \alpha)Y_{V_L}(e^{\alpha}, z_2)(z_1 - z_2)^{-1}.$$

This last formula can also be obtained using the commutator formula (2.2) using the same method as in the preceding example.

5.2 The Jacobi identity

Let $\delta(x) = \sum_{n\in\mathbb{Z}} x^n$ be the formal delta function. Then we have the basic property of $\delta(x)$: For any formal Laurent series $f(x)$ with coefficients in a vector space such that $f(x)\delta(x)$ and $f(1)$ is well defined,

$$f(x)\delta(x) = f(1)\delta(x).$$

We need to consider the following three formal delta functions:

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right),\ x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right), x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right).$$

In these formal expressions, we always expand a binomial as a formal Laurent series in nonnegative powers in the second formal variable. It is easy to check directly that the following identity holds:

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) = x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right), \tag{5.12}$$

$$x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right). \tag{5.13}$$

Let V be a grading-restricted vertex algebra. For $u_1, u_2, v \in V$ and $v' \in V'$, the duality property says that (3.1), (3.2) and (3.3) are absolutely convergent in the regions $|z_1| > |z_2| >$

0, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$. This common rational function can be written explicitly as $\frac{f(z_1,z_2)}{z_1^r z_2^s (z_1-z_2)^t}$, where $f(z_1, z_2)$ is a polynomial in z_1 and z_2 and $r, s, t \in \mathbb{N}$. We multiply the Laurent polynomial $\frac{f(x_1,x_2)}{x_1^r x_2^s x_0^t}$ in the formal variable x_0, x_1 and x_2 to both sides of (5.12) to obtain the identity

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\frac{f(x_1,x_2)}{x_1^r x_2^s x_0^t} - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\frac{f(x_1,x_2)}{x_1^r x_2^s x_0^t} = x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\frac{f(x_1,x_2)}{x_1^r x_2^s x_0^t}. \tag{5.14}$$

Using the basic property of the formal delta function, we can rewrite (5.15) as

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)&\frac{f(x_1,x_2)}{x_1^r x_2^s (x_1-x_2)^t} - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\frac{f(x_1,x_2)}{x_1^r x_2^s (-x_2+x_1)^t} \\ &= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\frac{f(x_1,x_2)}{(x_2+x_0)^r x_2^s x_0^t}. \end{aligned} \tag{5.15}$$

Note that $\frac{1}{(x_1-x_2)^t}$ in $\frac{f(x_1,x_2)}{x_1^r x_2^s (x_1-x_2)^t}$ is expanded in nonnegative powers of x_2. We already know that (3.1) is absolutely convergent in the region $|z_1| > |z_2| > 0$ to $\frac{f(z_1,z_2)}{z_1^r z_2^s (z_1-z_2)^t}$. In other words, the expansion of $\frac{f(z_1,z_2)}{z_1^r z_2^s (z_1-z_2)^t}$ as a Laurent series in z_1 and z_2 in the region $|z_1| > |z_2| > 0$ is exactly (3.1). This is the same as saying that $\frac{f(x_1,x_2)}{x_1^r x_2^s (x_1-x_2)^t}$ as a formal Laurent series in x_1 and x_2 obtained by expanding $\frac{1}{(x_1-x_2)^t}$ in nonnegative powers of x_2 is exactly

$$\langle v', Y_V(u_1, x_1)Y_V(u_2, x_2)v\rangle. \tag{5.16}$$

So we can replace $\frac{f(z_1,z_2)}{z_1^r z_2^s (z_1-z_2)^t}$ in (5.15) by (5.16). Similarly, we can replace $\frac{f(x_1,x_2)}{x_1^r x_2^s (-x_2+x_1)^t}$ and $\frac{f(x_1,x_2)}{(x_2+x_0)^r x_2^s x_0^t}$ in (5.15) by

$$\langle v', Y_V(u_2, x_2)Y_V(u_1, x_1)v\rangle$$

and

$$\langle v', Y_V(Y_V(u_1, x_0)u_2, x_2)v\rangle,$$

respectively. Thus we obtain

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)&\langle v', Y_V(u_1, x_1)Y_V(u_2, x_2)v\rangle - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\langle v', Y_V(u_2, x_2)Y_V(u_1, x_1)v\rangle \\ &= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\langle v', Y_V(Y_V(u_1, x_0)u_2, x_2)v\rangle. \end{aligned}$$

Since v' and v are arbitrary, we obtain the following Jacobi identity:

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)&Y_V(u_1, x_1)Y_V(u_2, x_2) - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)Y_V(u_2, x_2)Y_V(u_1, x_1) \\ &= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_V(Y_V(u_1, x_0)u_2, x_2). \end{aligned} \tag{5.17}$$

5.3 Skew-symmetry

Replacing u_1, u_2, x_1, x_2 and x_0 in (5.17) by u_2, u_1, x_2, x_1 and $-x_0$, respectively, we obtain

$$
\begin{aligned}
-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_V(u_2,x_2)Y_V(u_1,x_1)+x_0^{-1}\delta\left(\frac{-x_1+x_2}{-x_0}\right)Y_V(u_2,x_1)Y_V(u_2,x_1)\\
=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_V(Y_V(u_2,-x_0)u_1,x_1). \qquad (5.18)
\end{aligned}
$$

Since the left-hand sides of (5.18) and (5.17) are equal, the right-hand sides are also equal. So we obtain

$$
x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_V(Y_V(u_1,x_0)u_2,x_2)=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_V(Y_V(u_2,-x_0)u_1,x_1). \qquad (5.19)
$$

Using (5.13) and (5.12) in the right-hand side of (5.19), we see that (5.19) becomes

$$
x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_V(Y_V(u_1,x_0)u_2,x_2)=x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_V(Y_V(u_2,-x_0)u_1,x_2+x_0). \qquad (5.20)
$$

From the $L(-1)$-derivative property

$$
\frac{d}{dx_2}Y_V(u_2,x_2)=Y_V(L_V(-1)u_2,x_2),
$$

we obtain

$$
\frac{d^n}{dx_2^n}Y_V(u_2,x_2)=Y_V(L_V(-1)^nu_2,x_2) \qquad (5.21)
$$

for $n\in\mathbb{N}$. For $f(x_2)\in V((x_2))$, we have the formal Taylor's theorem

$$
f(x_2+x_0)=\sum_{n\in\mathbb{N}}\frac{x_0^n}{n!}\frac{d^n}{dx_2^n}f(x_2). \qquad (5.22)
$$

Applying both sides of (5.20) to $\mathbf{1}$, using the formal Taylor's theorem (5.22) with $f(x_2)=Y_V(Y_V(u_2,-x_0)u_1,x_2+x_0)\mathbf{1}$, using (5.21), taking Res_{x_1}, letting $x_2=0$ and then replacing x_0 by x, we obtain the skew-symmetry

$$
Y_V(u_1,x)u_2=\sum_{n\in\mathbb{N}}\frac{x^n}{n!}Y_V(L_V(-1)^nu_2,-x)u_1=e^{xL_V(-1)}Y_V(u_2,-x)u_1. \qquad (5.23)
$$

5.4 Commutator and associator formula

For a formal Laurent series $f(x)$ in x, we use $\mathrm{Res}_xf(x)$ to denote the coefficient of x^{-1} term in $f(x)$. Now taking Res_{x_0} on both sides of the Jacobi identity (5.17), we obtain the commutator formula for vertex operators:

$$
\begin{aligned}
Y_V(u_1,x_1)Y_V(u_2,x_2)-Y_V(u_2,x_2)Y_V(u_1,x_1)\\
=\mathrm{Res}_{x_0}x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_V(Y_V(u_1,x_0)u_2,x_2). \qquad (5.24)
\end{aligned}
$$

Taking Res_{x_1} on both sides of the Jacobi identity (5.17) and using (5.13) and the basic property of the formal delta function, we have

$$\begin{aligned}
&Y_V(Y_V(u_1,x_0)u_2,x_2)\\
&\quad= \mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) Y_V(u_1,x_1)Y_V(u_2,x_2)\\
&\qquad - \mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right) Y_V(u_2,x_2)Y_V(u_1,x_1)\\
&\quad= \mathrm{Res}_{x_1} x_1^{-1}\delta\left(\frac{x_0+x_2}{x_1}\right) Y_V(u_1,x_1)Y_V(u_2,x_2)\\
&\qquad - \mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right) Y_V(u_2,x_2)Y_V(u_1,x_1)\\
&\quad= \mathrm{Res}_{x_1} x_1^{-1}\delta\left(\frac{x_0+x_2}{x_1}\right) Y_V(u_1,x_0+x_2)Y_V(u_2,x_2)\\
&\qquad - \mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right) Y_V(u_2,x_2)Y_V(u_1,x_1)\\
&\quad= Y_V(u_1,x_0+x_2)Y_V(u_2,x_2)\\
&\qquad - \mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right) Y_V(u_2,x_2)Y_V(u_1,x_1).
\end{aligned} \tag{5.25}$$

Moving the first term in the right-hand side of (5.25) to the left-hand side, we obtain the associator formula for vertex operators:

$$\begin{aligned}
&Y_V(Y_V(u_1,x_0)u_2,x_2) - Y_V(u_1,x_0+x_2)Y_V(u_2,x_2)\\
&\quad= -\mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right) Y_V(u_2,x_2)Y_V(u_1,x_1).
\end{aligned} \tag{5.26}$$

5.5 Weak commutativity and weak associativity

Since $Y_V(u_1,x_0)u_2$ is a formal Laurent series with only finitely many negative powers in x_0, there exists $N \in \mathbb{Z}_+$ such that $x_0^N Y_V(u_1,x_0)u_2 \in V[[x_0]]$. Multiplying $(x_1-x_2)^N$ to the right-hand side of the commutator formula (5.24), using the basic property of the formal delta function and using the fact that Res_{x_0} of a formal power series is 0, we obtain

$$\begin{aligned}
&\mathrm{Res}_{x_0}(x_1-x_2)^N x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) Y_V(Y_V(u_1,x_0)u_2,x_2)\\
&\quad= \mathrm{Res}_{x_0} x_0^N x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) Y_V(Y_V(u_1,x_0)u_2,x_2)\\
&\quad= 0.
\end{aligned} \tag{5.27}$$

Thus $(x_1-x_2)^N$ multiplied to the left-hand side of (5.24) is also 0. So we obtained the weak commutativity:

$$(x_1-x_2)^N Y_V(u_1,x_1)Y_V(u_2,x_2) = (x_1-x_2)^N Y_V(u_2,x_2)Y_V(u_1,x_1). \tag{5.28}$$

Similarly, since $Y_V(u_1, x_1)v$ is a formal Laurent series with only finitely many negative powers in x_1, there exists $N \in \mathbb{Z}_+$ such that $x_1^N Y_V(u_1, x_1)v \in V[[x_1]]$. Multiplying $(x_0+x_2)^N$ to the right-hand side of the associator formula (5.26), applying the result to v, using the basic property of the formal delta function and using the fact that Res_{x_1} of a formal power series is 0, we obtain

$$\begin{aligned}
-\mathrm{Res}_{x_1}&(x_0+x_2)^N x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right) Y_V(u_2, x_2)Y_V(u_1, x_1)v \\
&= -\mathrm{Res}_{x_1} x_1^N x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right) Y_V(u_2, x_2)Y_V(u_1, x_1)v \\
&= 0.
\end{aligned} \tag{5.29}$$

Thus $(x_1 - x_2)^N$ multiplied to the left-hand side of (5.24) and then applied to v is also 0. So we obtained the weak associativity:

$$(x_0+x_2)^N Y_V(Y_V(u_1, x_0)u_2, x_2)v = (x_0+x_2)^N Y_V(u_1, x_0+x_2)Y_V(u_2, x_2)v. \tag{5.30}$$

Weak commutativity and weak associativity can also be obtained directly from the duality property in the definition of grading-restricted vertex algebra.

5.6 Conformal element and Virasoro operators

Let ω be a conformal element of V (see Definition 3.5). Then

$$Y_V(\omega, x)\omega = L_V(-1)\omega x^{-1} + 2\omega x^{-2} + \frac{c}{2}\mathbf{1}x^{-4} + G(x), \tag{5.31}$$

where $G(x) \in V[[x]]$. Using the commutator formula (5.24) with $u_1 = u_2 = \omega$, we obtain

$$\begin{aligned}
Y_V&(\omega, x_1)Y_V(u_2, x_2) - Y_V(\omega, x_2)Y_V(u_1, x_1) \\
&= \mathrm{Res}_{x_0} x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) Y_V(Y_V(\omega, x_0)\omega, x_2) \\
&= \mathrm{Res}_{x_0} x_0^{-1} x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) Y_V(L_V(-1)\omega, x_2) + 2\mathrm{Res}_{x_0} x_0^{-2} x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) Y_V(\omega, x_2) \\
&\quad + \frac{c}{2}\mathrm{Res}_{x_0} x_0^{-4} x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) Y_V(\mathbf{1}, x_2) + \mathrm{Res}_{x_0} x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) Y_V(G(x_0), x_2) \\
&= x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) \frac{\partial}{\partial x_2} Y_V(\omega, x_2) + 2x_1^{-1}\frac{\partial}{\partial x_2}\delta\left(\frac{x_2}{x_1}\right) Y_V(\omega, x_2) + \frac{c}{12} x_1^{-1}\frac{\partial^3}{\partial x_2^3}\delta\left(\frac{x_2}{x_1}\right),
\end{aligned} \tag{5.32}$$

where in the last equality, we have used the $L(-1)$-derivative property, the formal Taylor's theorem (5.22) applied to $x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)$ and the fact $G(x_0) \in V[[x_0]]$.

Writing

$$Y(\omega, x) = \sum_{\in\in\mathbb{Z}} L_V(n)x^{-n-2}$$

and then taking the coefficients of $x_1^{-m-2}x_2^{-n-2}$ in (5.32), we obtain the Virasoro relations

$$\begin{aligned}
&L_V(m)L_V(n) - L_V(n)L_V(m)\\
&\quad = \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{m+1}x_2^{n+1}(Y_V(\omega, x_1)Y_V(u_2, x_2) - Y_V(\omega, x_2)Y_V(u_1, x_1))\\
&\quad = \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{m+1}x_2^{n+1}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)\frac{\partial}{\partial x_2}Y_V(\omega, x_2)\\
&\quad\quad + 2\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{m+1}x_2^{n+1}x_1^{-1}\frac{\partial}{\partial x_2}\delta\left(\frac{x_2}{x_1}\right)Y_V(\omega, x_2)\\
&\quad\quad + \frac{c}{12}\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{m+1}x_2^{n+1}x_1^{-1}\frac{\partial^3}{\partial x_2^3}\delta\left(\frac{x_2}{x_1}\right)\\
&\quad = \mathrm{Res}_{x_2}x_2^{m+n+2}\frac{\partial}{\partial x_2}Y_V(\omega, x_2) + 2(m+1)\mathrm{Res}_{x_2}x_2^{m+n+1}Y_V(\omega, x_2)\\
&\quad\quad + \frac{c}{12}(m+1)m(m-1)\mathrm{Res}_{x_2}x_2^{m+n-1}\\
&\quad = (-m-n-2)L_V(m+n) + 2(m+1)L_V(m+n) + \frac{c}{12}(m+1)m(m-1)\delta_{m+n,0}\\
&\quad = (m-n)L_V(m+n) + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \qquad (5.33)
\end{aligned}$$

It is also easy to see by reversing the proof above that if the Virasoro relations (5.33) holds, then (5.31) holds. Thus we can replace (5.31) in Definition 3.5 by (5.33).

Proof of Theorem 1.5. By the definition of $Y_{S(\hat{\mathfrak{h}}_-)}$, we have

$$u^i(-1)^2\mathbf{1} = \mathrm{Res}_{x_0}x_0^{-1}u^i(x_0)u^i(-1)\mathbf{1} = \mathrm{Res}_{x_0}x_0^{-1}Y_{S(\hat{\mathfrak{h}}_-)}(u^i(-1)\mathbf{1}, x_0)u^i(-1)\mathbf{1}.$$

Using this and the first equality in (5.25), we have

$$\begin{aligned}
&Y_{S(\hat{\mathfrak{h}}_-)}(u^i(-1)^2\mathbf{1}, x_2)\\
&\quad = \mathrm{Res}_{x_0}x_0^{-1}Y_{S(\hat{\mathfrak{h}}_-)}(Y_{S(\hat{\mathfrak{h}}_-)}(u^i(-1)\mathbf{1}, x_0)u^i(-1)\mathbf{1}, x_2)\\
&\quad = \mathrm{Res}_{x_0}x_0^{-1}\mathrm{Res}_{x_1}x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y_{S(\hat{\mathfrak{h}}_-)}(u^i(-1)\mathbf{1}, x_1)Y_{S(\hat{\mathfrak{h}}_-)}(u^i(-1)\mathbf{1}, x_2)\\
&\quad\quad - \mathrm{Res}_{x_0}x_0^{-1}\mathrm{Res}_{x_1}x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right)Y_{S(\hat{\mathfrak{h}}_-)}(u^i(-1)\mathbf{1}, x_2)Y_{S(\hat{\mathfrak{h}}_-)}(u^i(-1)\mathbf{1}, x_1)\\
&\quad = \mathrm{Res}_{x_1}(x_1 - x_2)^{-1}u^i(x_1)u^i(x_2) - \mathrm{Res}_{x_1}(-x_2 + x_1)^{-1}u^i(x_2)u^i(x_1)\\
&\quad = \left(\sum_{m\in\mathbb{N}}u^i(-m-1)x_2^m\right)u^i(x_2) + u^i(x_2)\left(\sum_{m\in-\mathbb{Z}_+}u^i(-m-1)x_2^m\right)\\
&\quad = {}^\circ_\circ u^i(x_2)u^i(x_2){}^\circ_\circ. \qquad (5.34)
\end{aligned}$$

Uisng (5.34) and the definition of $T(x)$, we obtain $Y_{S(\hat{\mathfrak{h}}_-)}(\omega, x) = T(x)$. By (1.4),

$$Y_{S(\hat{\mathfrak{h}}_-)}(\omega, x)\omega = L_{S(\hat{\mathfrak{h}}_-)}(-1)\omega x^{-1} + 2\omega x^{-2} + \frac{\dim\mathfrak{h}}{2}\mathbf{1}x^{-4} + G(x).$$

The other property for $L_{S(\hat{\mathfrak{h}}_-)}(-1)$ and $L_{S(\hat{\mathfrak{h}}_-)}(0)$ can be verified using the formula (1.3). We omit the proofs here. ∎

6 Meromorphic open-string vertex algebra

The following definition is from [Hua7]:

Definition 6.1. A *meromorphic open-string vertex algebra* is a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ (graded by weights) equipped with a linear map

$$\begin{aligned} Y_V : V &\to (\mathrm{End}\, V)[[x, x^{-1}]] \\ u &\mapsto Y_V(u, x), \end{aligned}$$

or equivalently, a linear map

$$\begin{aligned} Y_V : V \otimes V &\to V[[x, x^{-1}]] \\ u \otimes v &\mapsto Y_V(u, x)v \end{aligned}$$

called *vertex operator map* a *vacuum* $\mathbf{1} \in V$, satisfying the following conditions:

1. *Lower bound condition*: When n is sufficiently negative, $V_{(n)} = 0$.
2. Properties for the vacuum: $Y_V(\mathbf{1}, x) = 1_V$ (the *identity property*) and for $u \in V$, $Y_V(u, x)\mathbf{1} \in V[[x]]$ and $\lim_{x\to 0} Y_V(u, x)\mathbf{1} = u$ (the *creation property*).
3. *Rationality*: For $u_1, \dots, u_n, v \in V$ and $v' \in V'$, the series

$$\begin{aligned} &\langle v', Y_V(u_1, z_1) \cdots Y_V(u_n, z_n)v\rangle \\ &\quad = \langle v', Y_V(u_1, x_1) \cdots Y_V(u_n, x_n)v\rangle\Big|_{x_1=z_1,\dots,x_n=z_n} \end{aligned} \tag{6.1}$$

converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function in $z_1, \dots, z_n$ with the only possible poles at $z_i = 0$ for $i = 1, \dots, n$ and $z_i = z_j$ for $i \neq j$. For $u_1, u_2, v \in V$ and $v' \in V'$, the series

$$\begin{aligned} &\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle \\ &\quad = \langle v', Y_V(Y_V(u_1, x_0)u_2, x_2)v\rangle\Big|_{x_0=z_1-z_2,\; x_2=z_2} \end{aligned} \tag{6.2}$$

converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to a rational function with the only possible poles at $z_1 = 0$, $z_2 = 0$ and $z_1 = z_2$.
4. *Associativity*: For $u_1, u_2, v \in V$, $v' \in V'$, we have

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle = \langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle \tag{6.3}$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

5. **d**-*bracket property*: Let $\mathbf{d}_V$ be the grading operator on V, that is, $\mathbf{d}_V u = mu$ for $m \in \mathbb{Z}$ and $u \in V_{(m)}$. For $u \in V$,

$$[\mathbf{d}_V, Y_V(u, x)] = Y_V(\mathbf{d}_V u, x) + x\frac{d}{dx}Y_V(u, x). \tag{6.4}$$

6. The *D-derivative property* and the *D-commutator formula*: Let $D_V : V \to V$ be defined by

$$D_V(u) = \lim_{x\to 0} \frac{d}{dx}Y_V(u, x)\mathbf{1}$$

for $u \in V$. Then for $u \in V$,

$$\begin{aligned} \frac{d}{dx}Y_V(u, x) &= Y_V(D_V u, x) \\ &= [D_V, Y_V(u, x)]. \end{aligned} \tag{6.5}$$

A meromorphic open-string vertex algebra is said to be *grading restricted* if $\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$. *Homomorphisms*, *isomorphisms*, *subalgebras* of meromorphic open-string vertex algebras are defined in the obvious way.

We shall denote the meromorphic open-string vertex algebra defined above by $(V, Y_V, \mathbf{1})$ or simply by V. For $u \in V$, we call the map $Y_V(u, x) : V \to V[[x, x^{-1}]]$ the *vertex operator associated to u*.

A grading-restricted vertex algebra is a grading-restricted meromorphic open-string vertex algebra.

We now recall the notion of open-string vertex algebra from [HK]:

Definition 6.2. An *open-string vertex algebra* is an $\mathbb{R}$-graded vector space $V = \coprod_{n\in\mathbb{R}} V_{(n)}$ (graded by *weights*) equipped with a *vertex map*

$$\begin{aligned} Y^O : V \times \mathbb{R}_+ &\to \mathrm{Hom}(V, \overline{V}) \\ (u, r) &\mapsto Y^O(u, r) \end{aligned}$$

such that for $r \in \mathbb{R}_+$ the map given by $u \mapsto Y^O(u, r)$ is linear, or equivalently,

$$\begin{aligned} Y^O : (V \otimes V) \times \mathbb{R}_+ &\to \overline{V} \\ (u \otimes v, r) &\mapsto Y^O(u, r)v \end{aligned}$$

such that for $r \in \mathbb{R}_+$ the map given by $u\otimes \mapsto Y^O(u, r)v$ is linear, a *vacuum* $\mathbf{1} \in V$ and an operator $D \in \mathrm{End}\, V$ of weight 1, satisfying the following conditions:

1. *Vertex map weight property*: For $n_1, n_2 \in \mathbb{R}$, there exist a finite subset $N(n_1, n_2) \subset \mathbb{R}$ such that the image of $\left(\coprod_{n\in n_1+\mathbb{Z}} V_{(n)} \otimes \coprod_{n\in n_2+\mathbb{Z}} V_{(n)}\right) \times \mathbb{R}_+$ under Y^O is in $\prod_{n\in N(n_1,n_2)+\mathbb{Z}} V_{(n)}$.

2. Properties for the vacuum: For any $r \in \mathbb{R}_+$, $Y^O(\mathbf{1}, r) = 1_V$ (the *identity property*) and $\lim_{r\to 0} Y^O(u, r)\mathbf{1}$ exists and is equal to u (the *creation property*).

3. *Local-truncation property for D'*: Let $D' : V' \to V'$ be the adjoint of D. Then for any $v' \in V'$, there exists a positive integer k such that $(D')^k v' = 0$.

4. *Convergence properties*: For $v_1, \dots, v_n, v \in V$ and $v' \in V'$, the series
$$\langle v', Y^O(v_1, r_1) \cdots Y^O(v_n, r_n)v\rangle$$
converges absolutely when $r_1 > \cdots > r_n > 0$. For $v_1, v_2, v \in V$ and $v' \in V'$, the series
$$\langle v', Y^O(Y^O(v_1, r_0)v_2, r_2)v\rangle$$
converges absolutely when $r_2 > r_0 > 0$.

5. *Associativity*: For $v_1, v_2, v \in V$ and $v' \in V'$,
$$\langle v', Y^O(v_1, r_1)Y^O(v_2, r_2)v\rangle = \langle v', Y^O(Y^O(v_1, r_1 - r_2)v_2, r_2)v\rangle$$
for $r_1, r_2 \in \mathbb{R}$ satisfying $r_1 > r_2 > r_1 - r_2 > 0$.

6. $\mathbf{d}$-*bracket property*: Let $\mathbf{d}$ be the grading operator on V, that is, $\mathbf{d}u = mu$ for $m \in \mathbb{R}$ and $u \in V_{(m)}$. For $u \in V$, $Y^O(u, r)$ as a function of $r \in \mathbb{R}_+$ valued in $\mathrm{Hom}(V, \overline{V})$ is differentiable (that is, for $v \in V$, $v' \in V'$, $\langle v', Y^O(u, r)v\rangle$ as a function of r is differentiable) and
$$[\mathbf{d}, Y^O(u, r)] = Y^O(\mathbf{d}u, r) + r\frac{d}{dr}Y^O(u, r). \tag{6.6}$$

7. *D-derivative property*: We still use D to denote the natural extension of D to $\mathrm{Hom}(\overline{V}, \overline{V})$. For $u \in V$,
$$\frac{d}{dr}Y^O(u, r) = [D, Y^O(u, r)] = Y^O(Du, r). \tag{6.7}$$

The open-string vertex algebra defined above is denoted $(V, Y^O, \mathbf{1}, D)$ or simply V.

A meromorphic open-string vertex algebra is indeed an open-string vertex algebra.

Let $\mathfrak{h}$ be a vector space over $\mathbb{C}$ equipped with a nondegenerate bilinear form $(\cdot, \cdot)$. The Heisenberg algebra $\hat{\mathfrak{h}}$ associated with $\mathfrak{h}$ and $(\cdot, \cdot)$ is the vector space $\mathfrak{h} \otimes [t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$ equipped with the bracket operation defined by
$$[a \otimes t^m, b \otimes t^n] = m(a, b)\delta_{m+n,0}\mathbf{k},$$
$$[a \otimes t^m, \mathbf{k}] = 0,$$
for $a, b \in \mathfrak{h}$ and $m, n \in \mathbb{Z}$. It is a $\mathbb{Z}$-graded Lie algebra. In particular, we have the universal enveloping algebra $U(\mathfrak{h})$ of $\mathfrak{h}$. The universal enveloping algebra $U(\mathfrak{h})$ is constructed as a quotient of the tensor algebra $T(\mathfrak{h})$ of the vector space $\mathfrak{h}$. We have a triangle decomposition
$$\hat{\mathfrak{h}} = \hat{\mathfrak{h}}_- \oplus \hat{\mathfrak{h}}_0 \oplus \hat{\mathfrak{h}}_+,$$

where

$$\begin{aligned}\hat{\mathfrak{h}}_- &= \mathfrak{h}\otimes t^{-1}\mathbb{C}[t^{-1}],\\ \hat{\mathfrak{h}}_+ &= \mathfrak{h}\otimes t\mathbb{C}[t],\\ \hat{\mathfrak{h}}_0 &= \mathfrak{h}\otimes\mathbb{C}\oplus\mathbb{C}\mathbf{k}\\ &\simeq \mathfrak{h}\oplus\mathbb{C}\mathbf{k},\\ \mathfrak{h} &\simeq \mathfrak{h}\otimes\mathbb{C}\end{aligned}$$

are subalgebras of $\hat{\mathfrak{h}}$.

The meromorphic open-string vertex algebras and left modules in the present paper are constructed from left modules for a quotient algebra $N(\hat{\mathfrak{h}})$ of the tensor algebra $T(\hat{\mathfrak{h}})$ such that $U(\hat{\mathfrak{h}})$ is a quotient of $N(\hat{\mathfrak{h}})$. Let I be the two-sided ideal of $T(\hat{\mathfrak{h}})$ generated by elements of the form

$$\begin{gathered}(a\otimes t^m)\otimes(b\otimes t^n)-(b\otimes t^n)\otimes(a\otimes t^m)-m(a,b)\delta_{m+n,0}\mathbf{k},\\ (a\otimes t^k)\otimes(b\otimes t^0)-(b\otimes t^0)\otimes(a\otimes t^k),\\ (a\otimes t^l)\otimes\mathbf{k}-\mathbf{k}\otimes(a\otimes t^l)\end{gathered}$$

for $m\in\mathbb{Z}_+$, $n\in-\mathbb{Z}_+$, $k\in\mathbb{Z}\setminus\{0\}$ and $l\in\mathbb{Z}$. Let $N(\hat{\mathfrak{h}})=T(\hat{\mathfrak{h}})/I$. By definition, we see that $U(\hat{\mathfrak{h}})$ is a quotient algebra of $N(\hat{\mathfrak{h}})$.

We have the following Poincaré-Birkhof-Witt type result for $N(\hat{\mathfrak{h}})$;

Proposition 6.3. *As a vector space, $N(\hat{\mathfrak{h}})$ is linearly isomorphic to*

$$T(\hat{\mathfrak{h}}_-)\otimes T(\hat{\mathfrak{h}}_+)\otimes T(\mathfrak{h})\otimes T(\mathbb{C}\mathbf{k}) \tag{6.8}$$

where $T(\hat{\mathfrak{h}}_-)$, $T(\hat{\mathfrak{h}}_+)$, $T(\mathfrak{h})$ and $T(\mathbb{C}\mathbf{k})$ are the tensor algebras of the vector spaces $\hat{\mathfrak{h}}_-$, $\hat{\mathfrak{h}}_+$, $\mathfrak{h}$ and $\mathbb{C}\mathbf{k}$, respectively.

Now we construct left modules for $N(\hat{\mathfrak{h}})$. Let M be a left $T(\mathfrak{h})$-module. We define the action of $\mathbf{k}$ on M to be 1 and the actions of elements of $\hat{\mathfrak{h}}_+$ on M to be 0. Then M is also a left module for the subalgebra $N(\hat{\mathfrak{h}}_+\oplus\hat{\mathfrak{h}}_0)$ of $N(\hat{\mathfrak{h}})$ generated by elements of $\hat{\mathfrak{h}}_+$ and $\hat{\mathfrak{h}}_0$. We consider the induced left module $N(\hat{\mathfrak{h}})\otimes_{N(\hat{\mathfrak{h}}_+\oplus\hat{\mathfrak{h}}_0)}M$. By Proposition 6.3, we see that $N(\hat{\mathfrak{h}})\otimes_{N(\hat{\mathfrak{h}}_+\oplus\hat{\mathfrak{h}}_0)}M$ is linearly isomorphic to $T(\hat{\mathfrak{h}}_-)\otimes M$. We shall identify $N(\hat{\mathfrak{h}})\otimes_{N(\hat{\mathfrak{h}}_+\oplus\hat{\mathfrak{h}}_0)}M$ with $T(\hat{\mathfrak{h}}_-)\otimes M$. The left $N(\hat{\mathfrak{h}})$-module structure on $T(\hat{\mathfrak{h}}_-)\otimes M$ can be obtained explicitly by using the commutator relations defining the algebra $N(\hat{\mathfrak{h}})$ and the left $N(\hat{\mathfrak{h}}_+\oplus\hat{\mathfrak{h}}_0)$-module structure on M.

For a left $N(\hat{\mathfrak{h}})$-module, we denote the representation images of $a\otimes t^n\in\hat{\mathfrak{h}}$ for $a\in\mathfrak{h}$ and $n\in\mathbb{Z}$ acting on the left module by $a(n)$. Then a left $N(\hat{\mathfrak{h}})$-module $T(\hat{\mathfrak{h}}_-)\otimes M$ constructed from a left $T(\mathfrak{h})$-module M is spanned by elements of the form $a_1(-n_1)\cdots a_k(-n_k)w$, where $a_1,\dots,a_k\in\hat{\mathfrak{h}}$, $n_1,\dots,n_k\in\mathbb{Z}_+$ and $w\in M$.

Given a left $N(\hat{\mathfrak{h}})$-module, we define a *normal ordering* map $\mathop{\circ}\limits_{\circ}\cdot\mathop{\circ}\limits_{\circ}$ from the space of operators on the left module spanned by operators of the form $a_1(n_1)\cdots a_k(n_k)$ to itself by rearranging the order of $a_1(n_1), \dots, a_k(n_k)$ in $a_1(n_1)\cdots a_k(n_k)$ by moving $a_i(n_i)$ with $n_i < 0$ to the left, $a_i(n_i)$ with $n_i > 0$ to the middle and $a_i(n_i)$ with $n_i = 0$ to the right but keeping the orders of $a_i(n_i)$ with $n_i < 0$, with $n_i < 0$ or with $n_i = 0$. For example,

$$\begin{aligned}&{}^{\circ}_{\circ} a_1(-1)a_2(0)a_3(4)a_4(0)a_5(-3)a_6(-1)a_7(10) {}^{\circ}_{\circ}\\ &\quad = a_1(-1)a_5(-3)a_6(-1)a_3(4)a_7(10)a_2(0)a_4(0).\end{aligned}$$

More explicitly,

$${}^{\circ}_{\circ} a_1(n_1)\cdots a_k(n_k) {}^{\circ}_{\circ} = a_{\sigma(1)}(n_{\sigma(1)})\cdots a_{\sigma(k)}(n_{\sigma(k)}),$$

where $\sigma \in S_k$ is the unique permutation such that

$$\begin{gathered}\sigma(1) < \cdots < \sigma(\alpha),\\ \sigma(\alpha+1) < \cdots < \sigma(\beta),\\ \sigma(\beta+1) < \cdots < \sigma(k),\\ n_{\sigma(1)}, \dots, n_{\sigma(\alpha)} < 0,\\ n_{\sigma(\alpha+1)}, \dots, n_{\sigma(\beta)} > 0,\\ n_{\sigma(\beta+1)}, \dots, n_{\sigma(k)} = 0,\end{gathered}$$

for some integers α and β satisfying $0 \le \alpha \le \beta \le k$.

Given an induced left $N(\hat{\mathfrak{h}})$-module $W = T(\hat{\mathfrak{h}}_-)\otimes M$, $a_1, \dots, a_k \in \mathfrak{h}$ and $m_1, \dots, m_k \in \mathbb{Z}_+$, we define the vertex operator $Y_W(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}, x)$ associated to $a_1(-m_1)\cdots a_k(-m_k)\mathbf{1} \in T(\hat{\mathfrak{h}}_-)$ by

$$\begin{aligned}&Y_W(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}, x)\\ &\quad = {}^{\circ}_{\circ} \frac{1}{(n_1-1)!}\left(\frac{d^{m_1-1}}{dx^{m_1-1}}a_1(x)\right)\cdots\frac{1}{(m_k-1)!}\left(\frac{d^{m_k-1}}{dx^{m_k-1}}a_k(x)\right) {}^{\circ}_{\circ},\end{aligned} \tag{6.9}$$

where

$$a_i(x) = \sum_{n\in\mathbb{Z}} a_i(n)x^{-n-1}$$

for $i = 1, \dots, k$ and $a_i(n)$ for $i = 1, \dots, k$ and $n \in \mathbb{Z}$ are the representation images of $a_i \otimes t^n$ on W.

Theorem 6.4. *The triple $(T(\hat{\mathfrak{h}}_-), Y_{T(\hat{\mathfrak{h}}_-)}, \mathbf{1})$ defined above is a meromorphic open-string vertex algebra. In the case that $\mathfrak{h}$ is finite dimensional, $(T(\hat{\mathfrak{h}}_-), Y_{T(\hat{\mathfrak{h}}_-)}, \mathbf{1})$ is a grading-restricted meromorphic open-string vertex algebra.*

7 A quick guide to the representation theory of Lie algebras

The material in this section are notes I wrote before. They provide a guide to the theory of finite-dimensional Lie algebras. I will not lecture on this material systematically in the class. Instead I will briefly discuss some of them when I need

to quote some results on finite-dimensional Lie algebras. The main reference for this section is [Hum].

Definition 7.1. A *Lie algebra* is a vector space L over a field $\mathcal{F}$ equipped with a *bracket operation* $[\cdot,\cdot] : L \otimes L \to L$ satisfying the following conditions:

1. The *skew-symmetry*: For $x, y \in L$,

$$[x, y] = -y, x].$$

2. The *Jacobi identity*: For $x, y, z \in L$,

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

A *homomorphism* from a Lie algebra L_1 to another Lie algebra L_2 is a linear map f from L_1 to L_2 such that for $x, y \in L_1$, $f([x, y]_1) = [f(x), f(y)]_2$, where $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$ are the bracket operations for L_1 and L_2, respectively. An *isomorphism* from a Lie algebra to another Lie algebra is an invertible homomorphism of Lie algebras.

Example 7.2. Let A be an associative algebra. We define a bracket operation $[\cdot,\cdot]$ by $[a, b] = ab - ba]$ for $a, b \in A$. Then A equipped with this bracket operation is a Lie algebra. In particular, for a vector space M, the space End M of all linear operators on M is an associative algebra. Then we have a Lie algebra structure on End M. We shall denote this Lie algebra by $\mathfrak{gl}(M)$.

Definition 7.3. Let L be a Lie algebra. A *reppresentation* of L is a vector space M and a homomorphism ρ of Lie algebra from L to $\mathfrak{gl}(M)$. The vector space M equipped with the representation ρ is called a *module for* L or an *L-module*. For an L-module, we shall denote $\rho(x)y$ for $x \in L$ and $y \in M$ by xy. A *homomorphism* of L-modules from an L-module M_1 to another L-module M_2 is a linear map from M_1 to M_2 such that $f(xy) = xf(y)$ for $x \in L$ and $y \in M_1$. An *isomorphism* from an L-module to another L-module is an invertible homomorphism of L-modules.

Definition 7.4. Let L be a Lie algebra. A *subalgebra* of a Lie algebra L is a subspace N of L such that the bracket operation $[\cdot,\cdot]$ for L maps $N \times N$ to N. An ideal of L is a subalgebra I of L such that $[x, y] \in I$ for $x \in I$ and $y \in L$. Let I be an ideal of L. Then L/I has a natural structure of a Lie algebra and is called the *quotient of L by I*. L is said to be *simple* if the only ideals of L are 0 and L and in addition, $[L, L] \neq 0$.

Definition 7.5. Let L be a Lie algebra. Let $L^{(1)} = [L, L]$, $L^{(2)} = [L^{(1)}, L^{(1)}]$, $\ldots$, $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$, $\ldots$. The Lie algebra L is said to be *solvable* if $L^{(i)} = 0$ for some i.

Proposition 7.6. *Let L be a Lie algebra.*

1. *If L is solvable, then all subalgebras and homomorphism images of L are sovable.*

2. *If I is a solvable ideal of L such that L/I is also solvable, then L is solvable.*

3. *If I and J are solvable ideals of L, then so is $I+J$.*

4. *There is a unique maximal solvable ideal of L.*

Proof. Part 1 follows immediately from the definitions.

Let I be a solvable ideal of L such that L/I is also solvable. Then there exists $m \in \mathbb{Z}_+$ such that $(L/I)^{(m)} = 0$, or equivalently, $L^{(m)} \subset I$ and there exists $n \in \mathbb{Z}_+$ such that $I^{(n)} = 0$. Since $L^{(m)} \subset I$ and $I^{(n)} = 0$, we have $L^{(m+n)}) = (L^{(m)})^{(n)} \subset I^{(n)} = 0$, proving that L is solvable.

By the standard homomorphism theorem, we know that $(I+J)/J$ is isomorphic to $I/(I \cap J)$. Since $I/(I \cap J)$ is a homomorphism image of I, it is solvable. So $(I+J)/J$ is also solvable. But J is also aolvable. By Part 2, we see that $I+J$ is solvable.

Let S be a maximal solvable ideal (a solvable ideal such that any solvable ideal containing S must be equal to S). Let I be any solvable ideal. Then $S+I$ is also a solvable ideal. Since $S+I$ contains S, $S+I=S$ or equivalently, $I \subset S$. Thus such an S is unique. ∎

Definition 7.7. The unique maximal solvable ideal of L given in the proposition above is called the *radical* of L and is denoted Rad L. A Lie algebra is said to be *semisimple* if its radical is 0.

Definition 7.8. Let L be a vector space. The *tensor algebra generated by M* is the space

$$T(L) = \coprod_{n \in \mathbb{N}} L^{\otimes n},$$

where $\mathbb{N}$ is the set of nonnegative integers and $L^{\otimes n}$ is the tensor product of n copies of L (when $n = 0$, $L^{\otimes 0} = \mathcal{F}$), with the tensor product of elements as the multiplication. Let L be a Lie algebra. The the quotient of $T(L)$ by the two sided ideal I of $T(L)$ generated by elenets of the form $x \otimes y - y \otimes x - [x, y]$ for $x, y \in L$ is an associative algebra. This associative algebra is called *universal enveloping algebra of L* and is denoted by $U(L)$.

We shall use $x_1 \cdots x_n$ for $x_1, \dots, x_n \in L$ to denote the element $x_1 \otimes \cdots \otimes x_n + I$ of $U(L)$. Then we see that $U(L)$ is spanned by elements of this form. In particular, elements of the form x for $x \in L$ form a subspace of $U(L)$ linearly isomorphic to L.

Proposition 7.9. *A vector space M is an L-module if and only if it is a $U(L)$-module.*

Proof. Let M be an L-module. For $x_1 \cdots x_n \in U(L)$ and $y \in M$, we define

$$(x_1 \cdots x_n)y = x_1(\cdots(x_n y)\cdots).$$

Since M is an L-module, it is easy to see that this is well defined, that is, if $x_1 \cdots x_n$ is equal to a linear combination of elements of the same form, then the action of this element on y

defined above and by using the linear combination give the same result. It is also easy to see that this action gives a $U(L)$-module structure on M.

Conversely, given a $U(L)$-module M, since L can be viewed as a subspace of $U(L)$, we have an action of L on M. Using the definition of $U(L)$ and the meaning of $U(L)$-module, we see that M with this action of L is an L-module. ∎

Definition 7.10. Let L be a Lie algebra. Let $L^1 = [L, L]$, $L^2 = [L, L^1]$, $\ldots$, $L^i = [L, L^{i-1}]$, $\ldots$. A Lie algebra is said to be *nilpotent* if $L^i = 0$ for some $i \in \mathbb{Z}_+$.

It is clear that $L^1 = L^{(1)}$ and $L^{(i)} \subset L^i$ for $i \geq 1$. So we have:

Proposition 7.11. *Let L be a Lie algebra. Then L is solvable if L or $[L, L]$ is nilpotent.* ∎

Theorem 7.12 (Cartan's criterion)**.** *Let M be a finite-dimensional vector space and L be a subalgebra of the Lie algebra $\mathfrak{gl}(M)$. If $\mathrm{Tr}xy = 0$ for all $x \in [L, L]$ and $y \in L$, then L is solvable.*

The proof is omitted. See [Hum] for a proof.

Definition 7.13. Let L be a Lie algebra. A representation $\rho : L \to \mathfrak{gl}(M)$ is said to be *faithful* if $\ker \rho = 0$. In this case, the L-module M is also said to be *faithful*.

Definition 7.14. Let L be a finite-diemnsional Lie algebra and $\rho : L \to \mathfrak{gl}(M)$ a faithful representation of L. Define a bilinear form $\beta_\rho : L \otimes L \to \mathcal{F}$ by $\beta_\rho(x, y) = \mathrm{Tr}\rho()\rho(y)$ for $x, y \in L$. Let L be a finite-dimensional Lie algebra. The *Killing form* of L is the bilinear form $\kappa = \beta_{\mathrm{ad}}$ for the adjoint representation ad on L itself defined by $(\mathrm{ad}\ x)y = [x, y]$ for $x, y \in L$.

Exercise 7.15. Verify that the bilinear form β_ρ is associative, that is, $\beta_\rho([x, y], z) = \beta_\rho(x, [y, z])$ for $x, y, z \in L$.

Proposition 7.16. *Let L be a finite-diemnsional semisimple Lie algebra and $\rho : L \to \mathfrak{gl}(M)$ a finite-diemnsional faithful representation of L. Then β_ρ is nondegenerate, that is, $\beta_\rho(x, y) = 0$ for all $y \in L$ implies $x = 0$.*

Proof. Let $S = \{x \in L \mid \beta_\rho(x, y) = 0 \text{ forall } y \in L\}$ (S is called the *radical* of β_ρ). We need to show that $S = 0$.

Since $\rho(S)$ is a subalgebra of $\mathfrak{gl}(M)$, we can apply Cartan's criterion to $\rho(S)$. Since $\mathrm{Tr}xy = \beta_\rho(x, y) = 0$ for $x \in \rho(S)$ and $y \in \rho(L)$, we certainly have $\mathrm{Tr}xy = 0$ for $x \in [\rho(S), \rho(S)]$ and $y \in \rho(S)$. Thus $\rho(S)$ is solvable. Since ρ is faithful, S is isomorphic to $\rho(S)$ and is therefore also solvable. Since L is semisimple, $S = 0$. ∎

Since β_ρ is nondegenerate, it gives an isomorphism from L to the dual space L^* of L by $x \in L \mapsto \beta(x, \cdot)$. Let $\{x_1, \ldots, x_n\}$ be a basis of L and $\{x_1^*, \ldots, x_n^*$ the dual basis. By definition, we have

$$x_i^*(x_j) = \delta_{ij}$$

for $i, j = 1, \ldots, n$. Using the inverse of the isomorphism from L to L^*, The basis $\{x_1^*, \ldots, x_n^*$ corresponds to another basis $\{y_1, \ldots, y_n\}$ of L and satisfies

$$\beta_\rho(x_i, y_j) = \delta_{ij}$$

for $i, j = 1, \ldots, n$. We shall also call this basis the *dual basis of* $\{x_1, \ldots, x_n\}$ *with respect to the bilinear form* β_ρ or simply the dual basis of $\{x_1, \ldots, x_n\}$.

Definition 7.17. Let L be a finite-diemnsional semisimple Lie algebra and $\rho : L \to \mathfrak{gl}(M)$ a finite-diemnsional faithful representation of L, or equivalently, M is an L-module. The *Casimir element* of M is

$$\Omega_M = \sum_{i=1}^{n} \rho(x_i)\rho(y_i) \in \text{End } M.$$

Exercise 7.18. Verify that the definition of the Casimir element above is independent of the choice of the basis $\{x_1, \ldots, x_n\}$.

Proposition 7.19. *Suppose that ρ is a faithful representation of L. Then the Casimir element commutes with $\rho(x)$ for $x \in L$.*

Proof. Let

$$[x, x_i] = \sum_{j=1}^{n} a_{ij} x_j$$

and

$$[x, y_i] = \sum_{j=1}^{n} b_{ij} y_j$$

for $i = 1, \ldots, n$. Then we have

$$\begin{aligned}
a_{ik} &= \sum_{j=1}^{n} a_{ij}\delta_{jk} \\
&= \sum_{j=1}^{n} a_{ij}\beta_\rho(x_j, y_k) \\
&= \beta_\rho([x, x_i], y_k) \\
&= -\beta_\rho([x_i, x], y_k) \\
&= -\beta_\rho(x_i, [x, y_k]) \\
&= -\sum_{j=1}^{n} b_{kj}\beta_\rho(x_i, y_j) \\
&= -\sum_{j=1}^{n} b_{kj}\delta_{ij} \\
&= -b_{ki}
\end{aligned}$$

for $i, k = 1, \ldots, n$. Thus

$$\begin{aligned}
[\rho(x), \Omega_M] &= \sum_{i=1}^{n} [\rho(x), \rho(x_i)\rho(y_i)] \\
&= \sum_{i=1}^{n} \rho(x)\rho(x_i)\rho(y_i) - \sum_{i=1}^{n} \rho(x_i)\rho(y_i)\rho(x) \\
&= \sum_{i=1}^{n} (\rho(x)\rho(x_i)\rho(y_i) - \rho(x_i)\rho(x)\rho(y_i)) + \sum_{i=1}^{n} (\rho(x_i)\rho(x)\rho(y_i)) - \rho(x_i)\rho(y_i)\rho(x) \\
&= \sum_{i=1}^{n} [\rho(x), \rho(x_i)]\rho(y_i) + \sum_{i=1}^{n} \rho(x_i)[\rho(x), \rho(y_i)] \\
&= \sum_{i=1}^{n} \rho([x, x_i]\rho(y_i) + \sum_{i=1}^{n} \rho(x_i)\rho([x, y_i] \\
&= \sum_{i,j=1}^{n} a_{ij}\rho(x_j)\rho(y_i) + \sum_{i,j=1}^{n} b_{ij}\rho(x_i)\rho(y_j) \\
&= \sum_{i,j=1}^{n} (a_{ij} + b_{ji})\rho(x_j)\rho(y_i) \\
& \quad 0,
\end{aligned}$$

proving that Ω_M commutes with $\rho(x)$ for $x \in L$. ∎

Let M_1 and M_2 be modules for a Lie algebra L. We now give a *tensor product module* of M_1 and M_2: Consider the tensor product vector space $M_1 \otimes M_2$. Define an action of L on $M_1 \otimes M_2$ by

$$x(y_1 \otimes y_2) = xy_1 \otimes y_2 + x_1 \otimes xy_2$$

for $x \in L$, $x_1 \in M_1$ and $y_2 \in M_2$.

Exercise 7.20. Verify that $M_1 \otimes M_2$ with this action of L is indeed an L-module.

Let M_1 and M_2 be modules for a Lie algebra L. We next give an L-module structure on the vector space $\mathrm{Hom}(M_1, M_2)$ of all linear maps from M_1 to M_2: Define an action of L on $\mathrm{Hom}(M_1, M_2)$ by

$$(xf)(y_1) = xf(y_1) - f(xy_1)$$

for $x \in L$, $f \in \mathrm{Hom}(M_1, M_2)$ and $y_1 \in M_1$.

Exercise 7.21. Verify that $\mathrm{Hom}(M_1, M_2)$ with this action of L is indeed an L-module.

Let M be an L-module. We now consider then special case that $M_1 = M$ and $M_2 = \mathcal{F}$ with the trivial L-module structure (the action of elements of L on $\mathcal{F}$ is 0). Then $M^* =$

$\mathrm{Hom}(M, \mathcal{F})$ and we obtain an L-module structure on M^*. This is called the *contragredient module of M*. We can also give the action of the action of L on M^* directly by

$$(xf)(y) = -f(xy)$$

for $x \in L$, $f \in M^*$ and $y \in M$.

In the rest of this section, we assume that F is algebraic closed of characteristic 0.

Theorem 7.22 (Weyl)**.** *A finite-dimensional module for a finite-dimensional semisimple Lie algebra is completely reducible.*

Proof. We need only prove that an exact sequence

$$0 \to M_1 \to M \to M_2 \to 0$$

of finite-dimensional L-modules are completely reducible. Equivalently, we need only prove that for a finite-dimensional L-module M and a finite-dimensional L-submodule M_1, there exists a finite-dimensional L-module M_2 such that M is isomorphic to $M_1 \oplus M_2$.

If indeed we can find such M_2, then the projection p from M to M_1 is a homomorphism of L-modules. The projection p can be characterized as the linear map from M to M_1 such that $p|_{M_1} = I_{M_1}$ and $\ker p$ is isomorphic to M_2. So to prove the theorem, we need only to find a homomorphism of L-modules from M to M_1 such that its restriction to M_1 is the identity and its kernel is isomorphic to M_2.

To find such a homomorphism of L-modules from M to M_1, we consider $\mathrm{Hom}(M, M_1)$. We have given an L-module structure to this space. Such a homomorphism, if it exists, must belong to the subspace $\mathcal{M}$ of $\mathrm{Hom}(M, M_1)$ consisting of elements whose restriction to M_1 is proportional to the identity operator on M_1. On the other hand, we certainly do not want elements in this subspace whose restrictions to M_1 are 0. Let $\mathcal{M}_1$ be the space of all such elements. We claim that $\mathcal{M}$ is an L-submodule of $\mathrm{Hom}(M, M_1)$ and $\mathcal{M}_1$ is an L-submodule of $\mathcal{M}$. In fact, for $f \in \mathcal{M}$, there exists $\lambda \in \mathcal{F}$ such that $f|_{M_1} = \lambda I_{M_1}$. Then for $x \in L$ and $y \in M_1$, $(xf)(y) = xf(y) - f(xy) = \lambda xy - \lambda xy = 0$. Thus $(xf)|_{M_1} = 0$. The same proof also shows that $\mathcal{M}_1$ is an L-submodule of $\mathcal{M}$. Note that $\mathcal{M}/\mathcal{M}_1$ is one-dimensional because modulo elements of $\mathcal{M}_1$, elements of $\mathcal{M}$ are determined completely by its restrictions on M_1. If $\mathcal{M}$ can be decomposed as a direct sum of the L-submodule $\mathcal{M}_1$ and a one-dimensional L-submodule of $\mathcal{M}_1 \subset \mathrm{Hom}(M, M_1)$, then we can choose the homomorphism we are looking for to be a basis of this one-dimensional subspace of $\mathrm{Hom}(M, M_1)$.

We now prove that $\mathcal{M}$ can be decomposed as a direct sum of the L-submodule $\mathcal{M}_1$ and a one-dimensional L-submodule of $\mathcal{M}$. We have proved that $(xf)|_{M_1} = 0$ for $x \in L$. So $x\mathcal{M} \subset \mathcal{M}_1$ for $x \in L$. Thus L acts on the one-dimensional L-module $\mathcal{M}/\mathcal{M}_1$ trivially. In particular, the L-module $\mathcal{M}/\mathcal{M}_1$ is isomorphic to the trivial L-module $\mathcal{F}$.

We use induction on the dimension of $\mathcal{M}$. When the diemnsion of $\mathcal{M}$ is 1, $\mathcal{M}$ can certainly be decomposed as a direct sum of the L-submodule $\mathcal{M}_1 = 0$ and one-dimensional L-submodule $\mathcal{M}$ of $\mathcal{M}$. Now assume that when the dimension of $\mathcal{M}$ is less than k, the decompostion holds. We now consider the case that the dimension of $\mathcal{M}$ is k. If $\mathcal{M}_1$ is not irreducible, then there exists a nonzero proper L-submodule $\mathcal{M}_1'$ of $\mathcal{M}_1$. Then the dimension of $\mathcal{M}/\mathcal{M}_1'$ is less than k and $(\mathcal{M}/\mathcal{M}_1')/(\mathcal{M}_1/\mathcal{M}_1')$ is one-dimensional. By induction assumption, There is a one-dimensional L-submodule of $\mathcal{M}/\mathcal{M}_1'$ such that $\mathcal{M}/\mathcal{M}_1'$ is the direct sum of $\mathcal{M}_1/\mathcal{M}_1'$ and this one-dimensional L-submodule. But any L-submodule of $\mathcal{M}/\mathcal{M}_1'$ is of the form $\tilde{\mathcal{M}}/\mathcal{M}_1'$ where $\tilde{\mathcal{M}}$ is an L-submodule of $\mathcal{M}$. Now $\mathcal{M}_1'$ is an L-submodule of $\tilde{\mathcal{M}}$ such that $\tilde{\mathcal{M}}/\mathcal{M}_1'$ is one-dimensional. So we can use our induction assumption again to obtain a one-dimensional L-submodule X such that $\tilde{\mathcal{M}}$ is the direct sum of $\mathcal{M}_1'$ and X. We know that $\tilde{\mathcal{M}}/\mathcal{M}_1' \cap \mathcal{M}_1/\mathcal{M}_1' = 0$, $X \subset \tilde{\mathcal{M}}$ and $X \cap \mathcal{M}_1' = 0$. So $X \cap \mathcal{M}_1 = 0$. Thus

$$\dim \mathcal{M} = \dim \mathcal{M}_1 + 1 = \dim \mathcal{M}_1 + \dim X.$$

Since both $\mathcal{M}_1$ and X are L-submodules of $\mathcal{M}$ and their intersection is 0, their direct sum must be $\mathcal{M}$.

We still need prove the case that $\mathcal{M}_1$ is irreducible. If ρ is not faithful, then we consider the quotient $L/\ker\rho$. The representation ρ induces a faithful representation of $L/\ker\rho$. Since L is semisimple, Rad $L = 0$. The quotient as a homomorphism image of L is also semisimple. The complete reducibility of $\mathcal{M}$ as an L-module is equivalent to the complete reducibility of $L/\ker\rho$. Thus we can assume that ρ is faithful. Since the Casimir element Ω_M commutes with $\rho(x)$ for $x \in L$, Ω_M is in fact a homomorphism of L-modules from $\mathcal{M}$ to itself. In particualr, $\Omega_M(\mathcal{M}_1) \subset \mathcal{M}_1$ and $\ker\Omega_M$ is an L-submodule of $\mathcal{M}$. Since L acts on $\mathcal{M}/\mathcal{M}_1$ trivially, so does Ω_M. So $\mathrm{Tr}\Omega_M = 0$ on $\mathcal{M}/\mathcal{M}_1$. But since $\mathcal{M}_1$ is irreducible, Ω_M acts as a scalar on $\mathcal{M}_1$. This scalar cannot be 0 since $\mathrm{Tr}\Omega_M = \dim L$. Hence $\ker\Omega_M$ must be a one-dimensional L-submodule of $\mathcal{M}$ such that $\ker\Omega_M \cap \mathcal{M}_1 = 0$. Thus $\mathcal{M}$ is the direct sum of $\mathcal{M}_1$ and $\ker\Omega_M$. ∎

Let $Z(L)$ be the center of L, that is

$$Z(L) = \{x \in L \mid [x, y] = 0 \text{ for } y \in L\}.$$

Then by definition,

$$Z(L) = \ker \mathrm{ad}_L$$

and $Z(L)$ is a solvable ideal of L.

Lemma 7.23. *A Lie algebra L is semisimple if and only if all abelian ideals of L are 0.*

Proof. Any abelian ideal of L is a solvable ideal of L and hence is in Rad L. Thus Rad $L = 0$ implies that all abelian ideals of L are 0.

Conversely, assume that all abelian ideals of L are 0. Since Rad L is a solvable ideal of L, there exists $n \in \mathbb{N}$ such that $(\text{Rad } L)^{(n)} = 0$ and $(\text{Rad } L)^{(n-1)} \neq 0$ if $n \neq 0$. If $n \neq 0$, then Rad $L)^{(n-1)}$ is a nonzero abelian ideal of L. Contradiction. So $n = 0$, that is, Rad $L = 0$. ∎

Theorem 7.24. *A Lie algebra L is semisimple if and only if its Killing form is nondegenerate.*

Proof. Assume that L is semisimple. Then Rad $L = 0$. Let S be the radical of the Killing form κ, that is,

$$S = \{x \in L \mid \kappa(x, y) = 0 \text{ forall } y \in L\}.$$

Then for $x \in S$ and $y \in L$ (in particular for $y \in [S, S]$), $\kappa(x, y) = 0$. By Cartan's criterion, $\mathrm{ad}_L\, S$ is solvable. Since L is semisimple, $\ker \mathrm{ad}_L = Z(L) \subset \mathrm{Rad}\, L = 0$. So S is also solvable. Thus $S \subset \mathrm{Rad}\, L = 0$, proving that κ is nondegenrate.

Conversely, assuming that the radical S of the Killing form κ is 0, we want to prove that L is semisimple. We prove that all abelian ideals of L are 0. Let I be an abelian ideal of L. For $x \in I$ and $y \in L$, $((\mathrm{ad}_L\, x)(\mathrm{ad}_L\, y))^2$ maps L to $[I, I]$. Since I is abelain, $[I, I] = 0$. Thus $((\mathrm{ad}_L\, x)(\mathrm{ad}_L\, y))^2 = 0$. Since the eigenvalues of any nilpotent operator are 0, we have

$$\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}_L\, x)(\mathrm{ad}_L\, y) = 0.$$

So $x \in S = 0$ and thus $x = 0$, proving that $I = 0$. ∎

Before we discuss construct representations of semisimple Lie algebras, we need the following result from linear algebra:

Theorem 7.25. *Let T be a linear operator on a finite-dimensional vector space M. Then there exist a unique diagonalizable (or semisimple) operator T_s and a unique nilpotent operator T_n on M such that $T = T_s + T_n$.*

Proof. Choose an ordered basis $\mathcal{B} = \{u_1, \ldots, u_n\}$ such that under this basis, the matrix $[T]_{\mathcal{B}}$ of T is a Jordan canonical form. Then $[T]_{\mathcal{B}} = S + N$ where S is a diagonal matrix whose diagonal entris are eigenvalues of T and N is a nilpotent Jordan canonical form whose eigenvalues are 0. Let T_s and T_n be the linear operators whose matrices under the basis $\mathcal{B}$ are S and N, respectively. Then we have $T = T_s + T_n$. Clearly, T_s and T_n are unique.

We can also obtain T_s and T_n and the decomposition $T = T_s + T_n$ using generalized eigenspaces of T as follows: Let $a_1, \ldots, a_k$ be distinct eigenvalues of T and $M_{a_1}, \ldots, M_{a_k}$ the corrsponding eigenspaces. Then $M = \oplus_{i=1}^k M_{a_i}$. Define $T_s : M \to M$ by $T_s(u) = a_i u$ for $u \in M_{a_i}$. Then T_s is certainly diagonalizable or semisimple. Let $T_n = T - T_s$. It is easy to see that T_n is nilpotent. By definition, we have $T = T_s + T_n$. ∎

Now we discuss representations of

$$\mathfrak{sl}(2, \mathcal{F}) = \{A \in M_{2\times 2} \mid \mathrm{Tr} A = 0\}$$

with the bracket operation defined by

$$[A, B] = AB - BA$$

for $A, B \in \mathfrak{sl}(2, \mathcal{F})$. The Lie algebra $\mathfrak{sl}(2, \mathcal{F})$ has a basis consisting the elements

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Their brackets or commutators are given by

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h.$$

Exercise 7.26. Prove that $\mathfrak{sl}(2,\mathcal{F})$ is semisimple.

Since $\mathfrak{sl}(2,\mathcal{F})$ is semisimple, we need only discuss finite-dimensional irreducible $\mathfrak{sl}(2,\mathcal{F})$-modules and how arbitrary finite-dimensional $\mathfrak{sl}(2,\mathcal{F})$-modules decompose into these finite-dimensional irreducible $\mathfrak{sl}(2,\mathcal{F})$-modules. We shall discuss only finite-dimensional irreducible $\mathfrak{sl}(2,\mathcal{F})$-modules below.

First we need:

Lemma 7.27. *Let $\rho : \mathfrak{sl}(2,\mathcal{F}) \to \mathfrak{gl}(M)$ be a representation of $\mathfrak{sl}(2,\mathcal{F})$. Then $\rho(h)$ is semisimple.*

Proof. Since any $\mathfrak{sl}(2,\mathcal{F})$-module is completely reducible, M is a direct sum of irreducible $\mathfrak{sl}(2,\mathcal{F})$-submodules of M. To prove that $\rho(h)$ is semisimple, it is enough to prove that the restriction of $\rho(h)$ to each of these irreducible $\mathfrak{sl}(2,\mathcal{F})$-submodules is semisimple. So we can assume that M is irreducible.

From the formulas for $[h,x]$ and $[h,y]$, we see that $\mathrm{ad}_{\mathfrak{sl}(2,\mathcal{F})}\ h$ is semisimple. Then since ρ is a homomorphism of Lie algebras, $\mathrm{ad}_{\rho(\mathfrak{sl}(2,\mathcal{F}))}\ \rho(h)$ is semisimple.

Since $\mathfrak{sl}(2,\mathcal{F})$ is semisimple (actually it is simple), we have $[\mathfrak{sl}(2,\mathcal{F}),\mathfrak{sl}(2,\mathcal{F})] = \mathfrak{sl}(2,\mathcal{F})$. Then we have $[\rho(\mathfrak{sl}(2,\mathcal{F})),\rho(\mathfrak{sl}(2,\mathcal{F}))] = \rho(\mathfrak{sl}(2,\mathcal{F}))$. Thus we have

$$\rho(\mathfrak{sl}(2,\mathcal{F})) = [\rho(\mathfrak{sl}(2,\mathcal{F})),\rho(\mathfrak{sl}(2,\mathcal{F}))] \subset [\mathfrak{gl}(M),\mathfrak{gl}(M)] = \mathfrak{sl}(M).$$

In particular, $\rho(h) \in \mathfrak{sl}(M)$. If we let $\rho(h) = \rho(h)_s + \rho(h)_n$ be the Jordan decomposition of the linear operator $\rho(h)$ on M, by definition, $\mathrm{Tr}\rho(h)_n = 0$, that is, $\rho(h)_n \in \mathfrak{sl}(M)$. Thus we also have $\rho(h)_s \in \mathfrak{sl}(M)$.

Let $\mathcal{B} = \{u_1,\dots,u_n$ be a basis of M such that under this basis, the matrix $[\rho(h)]_{\mathcal{B}}$ of $\rho(h)$ is a Jordan canonical form. Then the matrix $[\rho(h)_s]_{\mathcal{B}}$ of $\rho(h)_s$ under $\mathcal{B}$ is a diagonal matrix diag $(a_1,\dots,a_n)$ where $a_1,\dots,a_n$ are eigenvalues of $\rho(h)$. Take a basis of $\mathfrak{gl}(M)$ to be the set of linear operators $T_{ij} \in \}\updownarrow(M)$ whose matrices under the basis $\mathcal{B}$ of M are E_{ij} for $i,j = 1,\dots,n$ where E_{ij} is the matrix whose only nonzero entry is 1 at the i-th row and the j-th column. Then it is easy to verify by direct calculations that E_{ij} are generalized eigenvectors of the action of $[\rho(h)]_{\mathcal{B}}$ on the space $M_{n\times n}$ of $n\times n$ matrices by the bracket operation with eigenvalues $a_i - a_j$, that is,

$$[\overbrace{(([\rho(h)]_{\mathcal{B}} - (a_i-a_j)I_n),\cdots,[(([\rho(h)]_{\mathcal{B}} - (a_i-a_j)I_n)}^{k},E_{ij}]\cdots] = 0$$

for sufficiently large k. Also, E_{ij} are eigenvectors of the action of $[\rho(h)_s]_{\mathcal{B}} = \mathrm{diag}\ (a_1,\dots,a_n)$ on the space $M_{n\times n}$ of $n\times n$ matrices by the bracket operation with eigenvalues $a_i - a_j$. Thus for the corresponding linear operators on M, we also have

$$(\mathrm{ad}_{\mathfrak{gl}(M)}\ \rho(h))^k T_{ij} = [\overbrace{(\rho(h) - (a_i-a_j)I_M),\cdots,[(\rho(h) - (a_i-a_j)I_M)}^{k},T_{ij}]\cdots] = 0$$

and T_{ij} are eigenvectors for $\mathrm{ad}_{\mathfrak{gl}(M)}\ \rho(h)_s$ with eigenvalues $a_i - a_j$. Since $\mathrm{ad}_{\mathfrak{gl}(M)}\ \rho(h)$ maps $\rho(\mathfrak{sl}(2,\mathcal{F}))$ to itself, $\rho(\mathfrak{sl}(2,\mathcal{F}))$ is also a direct sum of generalized eigenspace of the operator $\mathrm{ad}_{\mathfrak{gl}(M)}\ \rho(h)$. In particular, $\mathrm{ad}_{\mathfrak{gl}(M)}\ \rho(h)_s$ also maps $\rho(\mathfrak{sl}(2,\mathcal{F}))$ to itself. Moreover, the discussion above shows that $\mathrm{ad}_{\mathfrak{gl}(M)}\ \rho(h)_s$ restricted to $\rho(\mathfrak{sl}(2,\mathcal{F}))$ is semisimple, $\mathrm{ad}_{\mathfrak{gl}(M)}\ \rho(h) - \mathrm{ad}_{\mathfrak{gl}(M)}\ \rho(h)_s$ restricted to $\rho(\mathfrak{sl}(2,\mathcal{F}))$ is nilpotent and $\mathrm{ad}_{\mathfrak{gl}(M)}\ \rho(h)_s$ commutes with $\mathrm{ad}_{\mathfrak{gl}(M)}\ \rho(h) - \mathrm{ad}_{\mathfrak{gl}(M)}\ \rho(h)_s$. Thus $\mathrm{ad}_{\mathfrak{gl}(M)}\ \rho(h)_s|_{\rho(\mathfrak{sl}(2,\mathcal{F})))}$ and $(\mathrm{ad}_{\mathfrak{gl}(M)}\ \rho(h) - \mathrm{ad}_{\mathfrak{gl}(M)}\ \rho(h)_s)|_{\rho(\mathfrak{sl}(2,\mathcal{F})))}$ are the semisimple and nilpotent parts, respectively, of $\mathrm{ad}_{\rho(\mathfrak{sl}(2,\mathcal{F}))}\ \rho(h) = \mathrm{ad}_{\mathfrak{gl}(M)}\ \rho(h)|_{\rho(\mathfrak{sl}(2,\mathcal{F})))}$. But we already showed that $\mathrm{ad}_{\rho(\mathfrak{sl}(2,\mathcal{F}))}\ \rho(h)$ is semisimple. So

$$\mathrm{ad}_{\rho(\mathfrak{sl}(2,\mathcal{F}))}\ \rho(h) = \mathrm{ad}_{\mathfrak{gl}(M)}\ \rho(h)_s|_{\rho(\mathfrak{sl}(2,\mathcal{F})))}.$$

Since $\mathfrak{sl}(2,\mathcal{F})$ is semisimple, $\mathrm{ad}_{\rho(\mathfrak{sl}(2,\mathcal{F}))}$ is faithful. Hence we have $\rho(h) = \rho(h)_s$, proving that $\rho(h)$ is semisimple. ∎

Let M be a finite-dimensional $\mathfrak{sl}(2,\mathcal{F})$-module. Then M is the direct sum of the eigenspaces M_{λ_i} of h with eigenvalues λ_i of h, respectively, for $i = 1,\ldots,k$. For $\lambda \neq \lambda_i$, we let $M_\lambda = 0$. Then we have

$$M = \coprod_{\lambda\in\mathcal{F}} M_\lambda.$$

Definition 7.28. The eigenvalues λ_i for $i = 1,\ldots,k$ are called *weights* of h or *weights* of the corresponding eigenvectors and the eigenspaces M_{λ_i} for $i = 1,\ldots,k$ are called *weight spaces* of h. An nonzero element $v \in M$ is called a *maximal vector* if $xv = 0$.

Theorem 7.29. *Let M be a finite-dimensional irreducible $\mathfrak{sl}(2,\mathcal{F})$-module. Let $m = \dim M - 1$. Then we have:*

1. $M = \coprod_{i=0}^m M_{m-2i}$ *and* $\dim M_{m-2i} = 1$ *for* $i = 0,\ldots,m$.
2. *Up to nonzero scalar multiples, M has a unique maximal vector in M_m.*
3. *Let $v_0 \in M_m$ be a maximal vector of M, $v_{-1} = 0$ and $v_i = \frac{1}{i!}y^i v_0$ for $i \in \mathbb{N}$. Then $v_i \neq 0$ and the action of $\mathfrak{sl}(2,\mathcal{F})$ on M is given by $hv_i = (m-2i)v_i$, $yv_i = (i+1)v_i$ and $xv_i = (m-i+1)v_i$ for $i \in \mathbb{N}$. In particular, up to isomorphisms, there exists at most one irreducible $\mathfrak{sl}(2,\mathcal{F})$-module of dimension $m+1$ for $m \in \mathbb{N}$.*

Proof. Since the action of h on M is semisimple, we have $M = \coprod_{\lambda\in\mathcal{F}} M_\lambda$. Since M is finite dimensional, there must be $\lambda \in \mathcal{F}$ such that $M_\lambda \neq 0$ but $M_{\lambda+2} = 0$. Take any nonzero element $v_0 \in M_\lambda$. Then

$$hxv_0 = xhv_0 + 2xv_0 = \lambda xv_0 + 2xv_0 = (\lambda+2)xv_0$$

and thus $xv_0 \in M_{\lambda+2} = 0$. So v_0 is a maximal vector. Let $v_i = \frac{1}{i!}y^i v_0$ for $i \in \mathbb{N}$. Using the bracket formulas for x, y and h, we have $hv_i = (\lambda - 2i)v_i$, $yv_i = (i+1)v_{i+1}$ and $xv_i = (\lambda - i + 1)v_{i-1}$ for $i \in \mathbb{N}$.

Since M is finite dimensional, there must be $m \in \mathbb{N}$ such that $v_0,\ldots,v_m \neq 0$ but $v_{m+1} = 0$. Since $v_{m+1} = 0$, $v_i = 0$ for $i \geq m+1$. Since $v_0,\ldots,v_m$ are eigenvectors for h with distinct eigenvalues, they must be linearly independent. Also $v_0,\ldots,v_m$ span a vector space which is invariant under the action of x, y and h. So $v_0, \dots, v_m$ span a submodule of M. Since M is irreducible, M must be equal to this submodule. So we see that M has a basis $\{v_0, \dots, v_m\}$. Since $0 = xv_{m+1} = (\lambda - m - 1 + 1)v_m$ and $v_m \neq 0$, we obtain $\lambda = m$. Part 1 follows immediately.

Assume that there is another maximal vector u. Then $u = \alpha_0 v_0 + \cdots + \alpha_m v_m$ and

$$xu = \alpha_0 xv_0 + \cdots + \alpha_m xv_m = \alpha_1 mv_0 + \alpha_2(m-1)v_1 + \cdots + \alpha_m v_{m-1}.$$

Since u is a maximal vector,

$$\alpha_1 mv_0 + \alpha_2(m-1)v_1 + \cdots + \alpha_m v_{m-1} = xu = 0.$$

Thus we have $\alpha_1 = \cdots = \alpha_m$ and $u = \alpha_0 v_0$, proving Part 2.

Part 3 follows immediately. ∎

The theorem above gives the classification of irreducible $\mathfrak{sl}(2, \mathcal{F})$-modules. We still need to establish the existence. To establish the existence, we need the following Poincar'e-Birkhoff-Witt theorem in the case of finite-dimensional Lie algebras:

Theorem 7.30 (Poincaré-Birkhoff-Witt)**.** *Let L be a finite-dimensional Lie algebra and $\{u_1, \dots, u_n$ an ordered basis of L. Then elements of the form*

$$u_{i_1} \cdots u_{i_k}$$

for $k \in \mathbb{N}$ and $1 \leq i_1 \leq\leq i_k \leq n$ form a basis of $U(L)$ (when $k = 0$, the element is 1).

We omit the proof here. See [Hum].

We also need the following construction of "induced modules:"

Let L be a finite-dimensional Lie algebra and L_1 a subalgebra of L. Then $U(L_1)$ can be embedded into $U(L)$ as a subalgebra. Let M_1 be an L_1-module. Then $U(L) \otimes M_1$ is a $U(L)$-module. Let I be the $U(L)$-submodule of $U(L) \otimes M_1$ generated by elements of the form $ab \otimes c - a \otimes bc$ for $a \in U(L)$, $b \in U(L_1)$ and $c \in M_1$ where bc is the action of b on c. Then $(U(L) \otimes M_1)/I$ is also a $U(L)$-module and thus an L-module. This L-module is denoted by $\mathrm{Ind}_{U(L_1)}^{U(L)} M_1$ or $U(L) \otimes_{U(L_1)} M_1$ and is called an *induced module.*

Proposition 7.31. *Let L be a finite-dimensional Lie algebra and L_1 and L_2 are subalgebras o L such that $L = L_1 \oplus L_2$. Then the universal enveloping algebra $U(L)$ is linearly isomorphic to $U(L_1) \otimes U(L_2)$.*

Proof. We choose an ordered basis $\{u_1, \dots, u_k\}$ of L_1 and an ordered basis $\{v_1, \dots, v_l\}$ of L_2. Then $\{u_1, \dots, u_k, v_1, \dots, v_l\}$ is a basis of L. By the Poincar'e-Birkhoff-Witt theorem, elements of the form

$$u_{i_1} \cdots u_{i_p} v_{j_1} \cdots v_{j_q}$$

for $p, q \in \mathbb{N}$, $1 \leq i_1 \leq \cdots \leq i_p \leq k$ and $1 \leq j_1 \leq \cdots \leq j_q \leq l$ form a basis of $U(L)$. But also by the Poincar'e-Birkhoff-Witt theorem, the set of elements of the form

$$u_{i_1} \cdots u_{i_p} \otimes v_{j_1} \cdots v_{j_q}$$

for $p, q \in \mathbb{N}$, $1 \leq i_1 \leq \cdots \leq i_p \leq k$ and $1 \leq j_1 \leq \cdots \leq j_q \leq l$ form a basis of $U(L_1) \otimes U(L_2)$. It follows that $U(L)$ is linearly isomorphic to $U(L_1) \otimes U(L_2)$. ∎

Now come back to $\mathfrak{sl}(2, \mathcal{F})$-modules. Let $L_1 = \mathcal{F}x + \mathcal{F}h$ and $L_2 = \mathcal{F}y$. Then $\mathfrak{sl}(2, \mathcal{F}) = L_1 \oplus L_2$. Consider a one-dimensional vector space $\mathcal{F}v_0$ with a basis v_0. For $m \in \mathbb{N}$, we define an action of L_1 on on $\mathcal{F}v_0$ by $xv_0 = 0$ and $hv_0 = mv_0$. It is easy to see that this action gives an L_1-module structure to $\mathcal{F}v_0$. Now we have the induced module $U(\mathfrak{sl}(2, \mathcal{F})) \otimes_{U(L_1)} \mathcal{F}v_0$ for $\mathfrak{sl}(2, \mathcal{F})$. By the proposition above, this induced module is linearly isomorphic to the vector space $(U(L_2) \otimes U(L_1)) \otimes_{U(L_1)} \mathcal{F}v_0$ which in turn is linearly isomorphic to the vector space $U(L_2) \otimes \mathcal{F}v_0$. From the definition of L_2 we see that $U(L_2) \otimes \mathcal{F}v_0$ has a basis consiting of $\frac{1}{i!}y^i \otimes v_0$. From this basis, we see that the induced module $U(\mathfrak{sl}(2, \mathcal{F})) \otimes_{U(L_1)} \mathcal{F}v_0$ is infinite dimensional and is not what we are interested. What we are interested are finite dimensional and irreducible.

To obtain irreducible modules, we consider maximal submodules of $U(\mathfrak{sl}(2, \mathcal{F})) \otimes_{U(L_1)} \mathcal{F}v_0$. In fact, let J be the sum of all submodules of $U(\mathfrak{sl}(2, \mathcal{F})) \otimes_{U(L_1)} \mathcal{F}v_0$ which does not contain $1 \otimes v_0$. Then J is also a submodule. It is maximal because any submodule larger than this one must contain the element $1 \otimes v_0$ and thus is equal to $U(\mathfrak{sl}(2, \mathcal{F})) \otimes_{U(L_1)} \mathcal{F}v_0$. Thus we obtain an irreducible $\mathfrak{sl}(2, \mathcal{F})$-module $(U(\mathfrak{sl}(2, \mathcal{F})) \otimes_{U(L_1)} \mathcal{F}v_0)/J$.

Moreover, we have:

Theorem 7.32. *The dimension of the irreducible* $\mathfrak{sl}(2, \mathcal{F})$*-module* $(U(\mathfrak{sl}(2, \mathcal{F})) \otimes_{U(L_1)} \mathcal{F}v_0)/J$ *is* $m + 1$.

Proof. We first prove that in this irreducible $\mathfrak{sl}(2, \mathcal{F})$-module, $w = y^{m+1} \otimes v_0 = 0$. It is easy to see that

$$\begin{aligned} xw = xy^{m+1} \otimes v_0 &= y^{m+1}x \otimes v_0 + m(m+1)y^m \otimes v_0 - (m+1)y^m h \otimes v_0 \\ &= y^{m+1} \otimes xv_0 + m(m+1)y^m \otimes v_0 - (m+1)y^m \otimes hv_0 \\ &= (m+1)y^m \otimes v_0 - m(m+1)y^m \otimes v_0 \\ &= 0. \end{aligned}$$

Thus w is also a maximal vector. But an $\mathfrak{sl}(2, \mathcal{F})$-module cannot have more than one linearly independent maximal vector (see exercise below). Since the weight of w is not m, we must have $w = 0$, that is, $y^{m+1} \otimes v_0 = 0$.

We now have $y^i \otimes v_0 = 0$ for $i \geq m + 1$. Since $U(\mathfrak{sl}(2, \mathcal{F})) \otimes_{U(L_1)} \mathcal{F}v_0$ is linearly isomorphic to the space $U(L_2) \otimes \mathcal{F}v_0$ which has a basis consisting of $\frac{1}{i!}y^i \otimes v_0$, we see that $(U(\mathfrak{sl}(2, \mathcal{F})) \otimes_{U(L_1)} \mathcal{F}v_0)/J$ is linearly spanned by elements of the form $\frac{1}{i!}y^i \otimes v_0$ for $i \leq m$. Thus $(U(\mathfrak{sl}(2, \mathcal{F})) \otimes_{U(L_1)} \mathcal{F}v_0)/J$ is finite dimensional. Since the weight of v_0 is m, by the theorem we proved before, $\dim(U(\mathfrak{sl}(2, \mathcal{F})) \otimes_{U(L_1)} \mathcal{F}v_0)/J = m + 1$. ∎

Exercise 7.33. Prove that maximal vectors for the module $(U(\mathfrak{sl}(2, \mathcal{F})) \otimes_{U(L_1)} \mathcal{F}v_0)/J$ are unique up to a nonzero scalar.

Corollary 7.34. *There is a bijection between the set* $\mathbb{N}$ *of nonnegative integers and the set of equivalence classes of finite-dimensional irreducible* $\mathfrak{sl}(2, \mathcal{F})$*-modules.*

Now we quickly discuss the representation theory of general finite-dimensional semisimple Lie algebras. We shall describe only the main constructions and state the main results without giving any proofs.

Let L be a finite-dimensional semisimple Lie algebra. Then there must be a semisimple element of L. A *toral subalgebra* of L is a subalgebra of L consisting of semisimple elements.

Proposition 7.35. *A toral subalgebra of* L *is an abelian Lie algebra.*

Now we take a maximal toral subalgebra H of L, that is, a toral sublagebra of L such that any toral subalgebra containing H must be H. Since H is abelian and commuting operators have same eigenvectors, L is a direct sum of common eigenspaces of elements of H. For any eigenvector x of elements of H, there exists $\alpha \in H^*$ such that

$$[h, x] = \alpha(h)x.$$

Let Φ be the space of all nonzero such $\alpha \in H^*$ and let

$$L_\alpha = \{x \in L \mid [h, x] = \alpha(h)x \text{ for } h \in H\}.$$

Then we have

$$L = L_0 \oplus \coprod_{\alpha \in \Phi} L_\alpha.$$

$0 \in H^*$ is in Φ and $H \subset L_\alpha$. It can be proved that $L_0 = H$. Thus we have

$$L = H \oplus \coprod_{\alpha \in \Phi} L_\alpha.$$

It can be proved that one can find a basis Δ of the real vector space E spanned by elements of Φ such that $\Delta \subset \Phi$ and any element of Φ can be written as a linear combination of elements of Δ with either nonnegative coefficients or nonpositive coefficients. Elements of Φ are called *roots*. Elements of Δ are called *simple roots*. We fix a choice of $\Delta = \{\alpha_1, \ldots, \alpha_l\}$.

Let M be an L-module. As in the case for $\mathfrak{sl}(2, \mathcal{F})$, the actions of elements of H on M must be semisimple. Since H is abelian, the actions of elements of H on M commute with each other. Thus

$$M = \coprod_{\lambda \in H^*} M_\lambda$$

where

$$M_\lambda = \{x \in M \mid hx = \lambda(h)x, \text{ for } h \in H\}$$

for $\lambda \in H^*$. When $M_\lambda \neq 0$, we say that λ is a *weight* of M and M_λ the *weight space* of weight λ. Let $\{\lambda_1, \ldots, \lambda_l\}$ be a basis of E determined by

$$\frac{2(\lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}$$

for $i, j = 1, \ldots, l$, where $(\cdot, \cdot)$ is the bilinear form on E induced from the Killing form on H. A weight λ is said to be *dominant* if it is a linear combination of $\lambda_1, \ldots, \lambda_l$ with nonnegative coefficients and is said to be *integral* if it is a linear combination of $\lambda_1, \ldots, \lambda_l$ with integral coefficients. Let Λ^+ be the set of dominant A weight is *dominant integral* if it is dominant and integral. Then we have the following result:

Theorem 7.36. *Let L be a finite-dimensional semisimple Lie algebra. Then there is a bijection from the set Λ^+ of dominant integral weights to the set of equivalence classes of finite-dimensional irreducible L-modules.*

The proof of this theorem is in spirit the same as the corresponding theorem above when $L = \mathfrak{sl}(2, \mathcal{F})$.

8 Affine Lie algebras (Wess-Zumino-Novikov-Witten models)

In this section, we will provide constructions of the vertex algebras associated to affine Lie algebras and their modules. We begin with the vertex algebras.

8.1 Construction of the grading-restricted vertex algebra $V(\ell, 0)$

(This subsection was written by Jason Saied.)

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with symmetric invariant bilinear form $(\cdot|\cdot)$. We define the affine Lie algebra $\hat{\mathfrak{g}}$ by

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k},$$

where $\mathbf{k}$ is central and

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + \delta_{m+n,0} m(a|b)\mathbf{k}.$$

(This is a central extension of the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. See [K, Section 7].) In this note, we will write $a(n)$ instead of $a \otimes t^n$. We write $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_- \oplus \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+$, where $\hat{\mathfrak{g}}_+$ is the span of all $a(n)$ with $a \in \mathfrak{g}$ and $n > 0$, $\hat{\mathfrak{g}}_-$ is the span of all $a(n)$ with $a \in \mathfrak{g}$ and $n < 0$, and $\hat{\mathfrak{g}}_0$ is the span of $\mathbf{k}$ and all $a(0)$ with $a \in \mathfrak{g}$.

Fix $\ell \in \mathbb{C}$. Let $\mathbb{C}_\ell$ be a copy of $\mathbb{C}$, with the structure of a module for $\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+$ by defining $a(n)1 = 0$ for all $a \in \mathfrak{g}$ and $n \geq 0$, and $\mathbf{k}1 = \ell$. Now define

$$V(\ell, 0) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+)} \mathbb{C}_\ell,$$

where $U(\cdot)$ is the universal enveloping algebra. (See [Hum, Section 17].) When ℓ is understood, we will use the notation $V := V(\ell, 0)$. V is a $U(\hat{\mathfrak{g}})$-module under left multiplication. (Note that V is nothing but the induced $U(\hat{g})$-module constructed from the $U(\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+)$-module $\mathbb{C}_\ell$.)

Let $\mathbf{1} := 1 \otimes 1 \in V$. Recall that $\mathbf{k1} = \ell\mathbf{1}$ and if $n \geq 0$, $a(n)\mathbf{1} = 0$.

Proposition 8.1. *V is canonically linearly isomorphic to $U(\hat{g}_-)$ by the map determined by*

$$a_1(n_1)\cdots a_k(n_k)\mathbf{1} \mapsto a_1(n_1)\cdots a_k(n_k),$$

where all $n_i < 0$.

Proof. By the Poincare-Birkhoff-Witt (PBW) Theorem (see [Hum, Section 17], the multiplication map

$$\phi : U(\hat{\mathfrak{g}}_-) \otimes U(\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+) \to U(\hat{\mathfrak{g}})$$

is a linear isomorphism. Then

$$V = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+)} \mathbb{C}_\ell \cong (U(\hat{\mathfrak{g}}_-) \otimes U(\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+)) \otimes_{U(\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+)} \mathbb{C}_\ell \cong U(\hat{\mathfrak{g}}_-) \otimes \mathbb{C}_\ell \cong U(\hat{\mathfrak{g}}_-),$$

where $\cong$ denotes linear isomorphism and all the maps are canonical. □

We put a grading on V by defining $(V)_{(n)}$ to be the span of all $a_1(n_1)a_2(n_2)\cdots a_l(n_l) \otimes 1$ such that $-n = n_1 + \cdots + n_l$. It is easy to check that applying the affine Lie algebra relations to an element of $(V)_{(n)}$ gives a sum of terms in $(V)_{(n)}$, so this is well-defined. We have

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)}.$$

Define the spaces

$$V' = \coprod_{n \in \mathbb{Z}} V_{(n)}^* \text{ and } \overline{V} = \prod_{n \in \mathbb{Z}} V_{(n)},$$

where $V_{(n)}^*$ is the dual space of $V_{(n)}$.

Proposition 8.2. *$V_{(n)} = 0$ for $n < 0$. $V_{(0)} = \mathbb{C}\mathbf{1}$. $V_{(n)}$ is finite-dimensional for all $n \in \mathbb{Z}$.*

Proof. The first part is clear from Proposition 8.1. It is also clear that $V_{(0)}$ is one-dimensional.

Let $n > 0$, and let $\{a_1, \ldots, a_r\}$ be a basis for $\mathfrak{g}$. Then $\{a_i(-m) : i, m \in \mathbb{Z}, 1 \leq i \leq r, m > 0\}$ is a basis for $\hat{\mathfrak{g}}_-$. Order the basis lexicographically, in i then m. Taking a PBW basis for $U(\hat{\mathfrak{g}}_-)$, we see that $V_{(n)}$ is spanned by

$$\{a_{i_1}(-n_1)\cdots a_{i_l}(-n_l)\mathbf{1} : 1 \leq i_1 \leq i_2 \cdots \leq i_l \leq r, 0 < n_i, n_1 + \cdots + n_l = n\}.$$

Since the positive integer n has only finitely many partitions and there are finitely many a_i, this is a finite set. □

We will now define the maps needed for our construction of the vertex algebra structure on V. For $a \in \mathfrak{g}$ and $z \in \mathbb{C}^*$, define

$$a(z) : V \to \overline{V}$$

by

$$a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}.$$

When working with formal series rather than complex variables, we will use the notation

$$a(x) = \sum_{n\in\mathbb{Z}} a(n)x^{-n-1},$$

where x is a formal variable.

Proposition 8.3. *Given $v' \in V'$ and $v \in V$, $\langle v', a(z)v\rangle$ is a rational function with the only possible pole at $z = 0$.*

Proof. It suffices to consider $v = a_1(-n_1)\cdots a_l(-n_l)\mathbf{1} \in V$ and $v' \in V^*_{(h)}$. We have

$$\langle v', a(z)a_1(-n_1)\cdots a_l(-n_l)\mathbf{1}\rangle = \sum_{n\in\mathbb{Z}} \langle v', a(n)a_1(-n_1)\cdots a_l(-n_l)\mathbf{1}\rangle z^{-n-1}$$

Since $v' \in V^*_{(h)}$, by how we defined our grading the term $\langle v', a(n)a_1(-n_1)\cdots a_l(-n_l)\mathbf{1}\rangle$ is 0 unless $n - n_1 - \cdots - n_l = h$. Then

$$\langle v', a(z)a_1(-n_1)\cdots a_l(-n_l)\rangle = \langle v', a(h + n_1 + \cdots + n_l)a_1(-n_1)\cdots a_l(-n_l)\rangle z^{-(h+n_1+\cdots+n_l)-1}.$$

This is a monomial in z or z^{-1}, so it is a rational function with the only possible pole at $z = 0$. □

This proves that for all $a \in \mathfrak{g}$, $a(z)$ is an analytic map from $\mathbb{C}^\times$ to $\mathrm{Hom}(V, \overline{V})$, as defined in [Hua9]. We now show that the maps $a(x)$ satisfy the conditions given in Section 4. Let $L(0)$ be the grading operator on V.

Claim 8.4. *The maps $a(x)$ satisfy Condition 1:*

$$[L(0), a(x)] = x\frac{d}{dx}a(x) + a(x).$$

Proof. We compute that for $n \in \mathbb{Z}$ and $n_i > 0$,

$$\begin{aligned}
&[L(0), a(n)]a_1(-n_1)\cdots a_l(-n_l)\\
&= L(0)a(n)a_1(-n_1)\cdots a_l(-n_l) - a(n)L(0)a_1(-n_1)\cdots a_l(-n_l)\\
&= (n_1 + \cdots + n_l - n)a(n)a_1(-n_1)\cdots a_l(-n_l) - (n_1 + \cdots + n_l)a(n)a_1(-n_1)\cdots a_l(-n_l)\\
&= -na(n)a_1(-n_1)\cdots a_l(-n_l),
\end{aligned}$$

so

$$[L(0), a(n)] = -na(n).$$

Then

$$\begin{aligned}
[L(0), a(x)] &= [L(0), \sum_{n\in\mathbb{Z}} a(n)x^{-n-1}] \\
&= \sum_{n\in\mathbb{Z}} [L(0), a(n)]x^{-n-1} \\
&= \sum_{n\in\mathbb{Z}} (-n)a(n)x^{-n-1} \\
&= \sum_{n\in\mathbb{Z}} (-n-1)a(n)x^{-n-1} + \sum_{n\in\mathbb{Z}} a(n)x^{-n-1} \\
&= x\frac{d}{dx}a(x) + a(x).
\end{aligned}$$

□

Further, it is clear that $a(n)$ is a homogeneous linear map of weight $-n$ and that $a(x) = \sum_{n\in\mathbb{Z}} a(n)x^{-n-1}$ is precisely the decomposition into homogeneous components given by Lemma 4.1.

Define a linear map $L(-1) \in \mathrm{End}(\hat{\mathfrak{g}})$ by $L(-1)(\mathbf{k}) = 0$ and $L(-1)(b(n)) = -nb(n-1)$. It is easy to check that this defines a derivation on $\hat{\mathfrak{g}}$ and therefore $U(\hat{\mathfrak{g}})$. We then observe that $L(-1)|_{U(\hat{\mathfrak{g}}_-)}$ maps into $U(\hat{\mathfrak{g}}_-)$, so we may view $L(-1)$ as a derivation on $U(\hat{\mathfrak{g}}_-)$. Using the isomorphism $V \cong U(\hat{\mathfrak{g}}_-)$, this allows us to define $L(-1)$ as a linear map on V by

$$L(-1)a_1(n_1)\cdots a_k(n_k)\mathbf{1} := L(-1)(a_1(n_1)\cdots a_k(n_k))\mathbf{1} \text{ and } L(-1)\mathbf{1} = 0.$$

Claim 8.5. *$L(-1)$ and the maps $a(x)$ satisfy Condition 2: $L(-1)\mathbf{1} = 0$ and*

$$[L(-1), a(x)] = \frac{d}{dx}a(x).$$

Proof. We compute that for $n \in \mathbb{Z}$ and $n_i > 0$, using the fact that $L(-1)$ is a derivation on $U(\hat{\mathfrak{g}})$,

$$\begin{aligned}
&[L(-1), a(n)]a_1(-n_1)\cdots a_l(-n_l)\mathbf{1} \\
&= L(-1)(a(n)a_1(-n_1)\cdots a_l(-n_l))\mathbf{1} - a(n)L(-1)(a_1(-n_1)\cdots a_l(-n_l))\mathbf{1} \\
&= -na(n-1)a_1(-n_1)\cdots a_l(-n_l)\mathbf{1} \\
&+ a(n)L(-1)(a_1(-n_1)\cdots a_l(-n_l))\mathbf{1} - a(n)L(-1)(a_1(-n_1)\cdots a_l(-n_l))\mathbf{1} \\
&= -na(n-1)a_1(-n_1)\cdots a_l(-n_l)\mathbf{1}.
\end{aligned}$$

Then $[L(-1), a(n)] = -na(n-1)$ as operators on V, so

$$[L(-1), \sum_{n\in\mathbb{Z}} a(n)x^{-n-1}] = \sum_{n\in\mathbb{Z}} [L(-1), a(n)]x^{-n-1} = \sum_{n\in\mathbb{Z}} (-n)a(n-1)x^{-n-1} = \frac{d}{dx}a(x).$$

□

Claim 8.6. *Condition 3 is satisfied:* $\lim_{x\to 0} a(x)\mathbf{1} = a(-1)$.

Proof. We note that

$$a(x)\mathbf{1} = \sum_{n\in\mathbb{Z}} a(n)\mathbf{1}x^{-n-1} = \sum_{n<0} a(n)\mathbf{1}x^{-n-1}.$$

This has only nonnegative powers of x, and as $x \to 0$, we are left with only the constant term $a(-1)$. □

Remark 8.7. *Condition 4 follows from Proposition 8.1.*

We now introduce a lemma on formal series which will come in handy. Recall that $\delta(x) = \sum_{n\in\mathbb{Z}} x^n$.

Lemma 8.8 ([LL], Proposition 2.3.7)**.** *If* $m > n \geq 0$*, then*

$$(x_1 - x_2)^m \left(\frac{\partial}{\partial x_1}\right)^n x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = 0.$$

Exercise 8.9. Prove this lemma on your own.

Claim 8.10. *For* $a, b \in \mathfrak{g}$*, we have the following identity of formal series:*

$$(x_1 - x_2)^2 a(x_1)b(x_2) = (x_1 - x_2)^2 b(x_1)a(x_1).$$

Proof. We have

$$\begin{aligned}
&a(x_1)b(x_2)\\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} a(n)b(m)x_1^{-n-1}x_2^{-m-1}\\
&= b(x_2)a(x_1) + \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} ([a,b](m+n) + \delta_{m+n,0}n(a|b)\mathbf{k})x_1^{-n-1}x_2^{-m-1}\\
&= b(x_2)a(x_1) + \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} [a,b](m+n)x_1^{-n-1}x_2^{-m-1} - \sum_{m\in\mathbb{Z}} n(a|b)\mathbf{k}x_1^{m-1}x_2^{-m-1}.
\end{aligned}$$

Then it suffices to show that when multiplied by $(x_1 - x_2)^2$, the second and third terms are zero. We compute the following, applying Lemma 8.8 in the final line:

$$\begin{aligned}
(x_1 - x_2)^2 \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} [a,b](m+n)x_1^{-n-1}x_2^{-m-1} &= (x_1 - x_2)^2 \sum_{p\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} [a,b](p)x_1^{m-p-1}x_2^{-m-1}\\
&= (x_1 - x_2)^2 [a,b](x_1) \sum_{m\in\mathbb{Z}} x_1^m x_2^{-m-1}\\
&= [a,b](x_1)(x_1 - x_2)^2 \sum_{m\in\mathbb{Z}} x_2^{-1}\left(\frac{x_1}{x_2}\right)\\
&= [a,b](x_1)(x_1 - x_2)^2 x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\\
&= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}(x_1-x_2)^2\sum_{m\in\mathbb{Z}} m(a|b)\mathbf{k}x_1^{m-1}x_2^{-m-1} &= \mathbf{k}(a|b)(x_1-x_2)^2\sum_{m\in\mathbb{Z}} m x_1^{m-1}x_2^{-m-1}\\ &= \mathbf{k}(a|b)x_2^{-1}(x_1-x_2)^2\sum_{m\in\mathbb{Z}}\frac{m}{x_2}\left(\frac{x_1}{x_2}\right)^{m-1}\\ &= \mathbf{k}(a|b)(x_1-x_2)^2x_2^{-1}\frac{\partial}{\partial x_1}\delta\left(\frac{x_1}{x_2}\right)\\ &= 0.\end{aligned}$$

□

Then Theorem 4.5 gives a grading-restricted vertex algebra structure on $V(\ell,0)$.

8.2 $V(\ell,0)$ as a vertex operator algebra

(This subsection was written jointly with with Jason Saied.)

We show in this subsection that $V=V(\ell,0)$ has a conformal element and therefore is a vertex operator algebra. See Definition 3.5. In this section, we assume that the invariant bilinear form on $\hat{\mathfrak{g}}$ is positive definite (for example, in the case that $\hat{\mathfrak{g}}$ is semisimple and the form is obtained from the Killing form).

Define $\Omega\in U(\mathfrak{g})$ by

$$\Omega=\sum_{i=1}^{\dim\mathfrak{g}}u^iu^i,$$

where $\{u^i : 1\le i\le \dim\mathfrak{g}\}$ is an orthonormal basis for $\mathfrak{g}$ with respect to the form $(\cdot,\cdot)$. Ω is called the *Casimir element* of $\mathfrak{g}$. (In Definition 7.17, we have introduced a Casimir element associated to a representation of a finite-dimensional Lie algebra.)

We add the assumption that Ω acts on $\mathfrak{g}$ by a scalar $2h^\vee$, where $h^\vee\in\mathbb{C}$. In particular, this assumption is satisfied if $\mathfrak{g}$ is a simple Lie algebra. In fact, $\mathfrak{g}$ with the adjoint action is a faithful module of $\mathfrak{g}$. By Proposition 7.19, Ω acting on $\mathfrak{g}$ commutes with ad a for every $a\in\mathfrak{g}$. Since $\mathfrak{g}$ is simple, Ω must act as a scalar, which we denote by $2h^\vee$. (See [Hum], Section 6.)

We then define $\omega\in V_{(2)}$ by

$$\omega=\frac{1}{2(\ell+h^\vee)}\sum_{i=1}^{\dim\mathfrak{g}}u^i(-1)u^i(-1)\mathbf{1}=\frac{1}{2(\ell+h^\vee)}\sum_{i=1}^{\dim\mathfrak{g}}u^i(-1)^2\mathbf{1}.$$

Theorem 8.11. *$V(\ell,0)$ is a VOA with conformal element ω and central charge* $\dfrac{\ell\dim\mathfrak{g}}{\ell+h^\vee}$.

Proof. We need to calculate $Y_{V(\ell,0)}(\omega,x)\omega$. We first calculate

$$Y_{V(\ell,0)}(u^i(-1)^2\mathbf{1},x)\omega$$

for $i, j = 1, \dots, \dim \mathfrak{g}$. By the definition of $Y_{V(\ell,0)}$, we have

$$u^i(-1)^2\mathbf{1} = \mathrm{Res}_{x_0} x_0^{-1} u^i(x_0)u^i(-1)\mathbf{1} = \mathrm{Res}_{x_0} x_0^{-1} Y_{V(\ell,0)}(u^i(-1)\mathbf{1}, x_0)u^i(-1)\mathbf{1}.$$

Using this and the first equality in (5.25), we have

$$\begin{aligned}
&Y_{V(\ell,0)}(u^i(-1)^2\mathbf{1}, x_2)\\
&\quad = \mathrm{Res}_{x_0} x_0^{-1} Y_{V(\ell,0)}(Y_{V(\ell,0)}(u^i(-1)\mathbf{1}, x_0)u^i(-1)\mathbf{1}, x_2)\\
&\quad = \mathrm{Res}_{x_0} x_0^{-1} \mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_{V(\ell,0)}(u^i(-1)\mathbf{1}, x_1)Y_{V(\ell,0)}(u^i(-1)\mathbf{1}, x_2)\\
&\qquad - \mathrm{Res}_{x_0} x_0^{-1} \mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) Y_{V(\ell,0)}(u^i(-1)\mathbf{1}, x_2)Y_{V(\ell,0)}(u^i(-1)\mathbf{1}, x_1)\\
&\quad = \mathrm{Res}_{x_1}(x_1 - x_2)^{-1}u^i(x_1)u^i(x_2) - \mathrm{Res}_{x_1}(-x_2 + x_1)^{-1}u^i(x_2)u^i(x_1)\\
&\quad = \left(\sum_{m\in\mathbb{N}} u^i(-m-1)x_2^m\right) u^i(x_2) + u^i(x_2)\left(\sum_{m\in -\mathbb{Z}_+} u^i(-m-1)x_2^m\right). \qquad (8.10)
\end{aligned}$$

Using $\omega \in V_{(2)}$, $V_{(n)}(\ell, 0) = 0$ for $n < 0$ and $u^i(p)u^i(q) = u^i(q)u^i(p)$ for $p \neq -q$ and $i = 1, \dots, \dim \mathfrak{g}$, we obtain

$$\begin{aligned}
&Y_{V(\ell,0)}(u^i(-1)^2\mathbf{1}, x_2)\omega\\
&\quad = \left(\sum_{m\in\mathbb{N}} u^i(-m-1)x_2^m\right) u^i(x_2)\omega + u^i(x_2)\left(\sum_{m\in -\mathbb{Z}_+} u^i(-m-1)x_2^m\right)\omega\\
&\quad = \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}} u^i(-m-1)u^i(n)\omega x_2^{m-n-1} + \sum_{m\in -\mathbb{Z}_+}\sum_{n\in\mathbb{Z}} u^i(n)u^i(-m-1)\omega x_2^{m-n-1}\\
&\quad = \sum_{0\leq m\leq n\leq 2} u^i(-m-1)u^i(n)\omega x_2^{m-n-1} + \sum_{-3\leq m\leq n\leq m+3\leq 2} u^i(n)u^i(-m-1)\omega x_2^{m-n-1}\\
&\qquad + F_i(x_2)\\
&\quad = \sum_{m=0}^{2} u^i(-m-1)u^i(2)\omega x_2^{m-3} + \sum_{m=0}^{1} u^i(-m-1)u^i(1)\omega x_2^{m-2} + u^i(-1)u^i(0)\omega x_2^{-1}\\
&\qquad + \sum_{n=-3}^{0} u^i(n)u^i(2)\omega x_2^{-n-4} + \sum_{n=-2}^{1} u^i(n)u^i(1)\omega x_2^{-n-3} + \sum_{n=-1}^{2} u^i(n)u^i(0)\omega x_2^{-n-2} + F_i(x_2)\\
&\quad = 2u^i(-3)u^i(2)\omega x_2^{-1} + 2u^i(-2)u^i(1)\omega x_2^{-1} + 2u^i(-1)u^i(0)\omega x_2^{-1} + 2u^i(-2)u^i(2)\omega x_2^{-2}\\
&\qquad + 2u^i(-1)u^i(1)\omega x_2^{-2} + u^i(0)^2\omega x_2^{-2} + 2u^i(-1)u^i(2)\omega x_2^{-3} + 2u^i(0)u^i(1)\omega x_2^{-3}\\
&\qquad + 2u^i(0)u^i(2)\omega x_2^{-4} + u^i(1)u^i(1)\omega x_2^{-4} + F_i(x_2), \qquad (8.11)
\end{aligned}$$

where $F_i(x_2) \in V(\ell, 0)[[x_2]]$ for $i = 1, \dots, \dim \mathfrak{g}$.

Using the commutator formula for the affine Lie algebra operators, the formulas

$$\begin{aligned}
[u^i, u^i] &= [u^j, u^j] = [[u^i, u^j], [u^i, u^j]] = 0, \\
(u^i, u^j) &= \delta_{ij}. \\
(u^i, [u^i, u^j]) &= ([u^i, u^i].u^j) = 0, \\
([u^i, u^j], u^j) &= (u^i, [u^j, u^j]) = 0, \\
([u^i, [u^i, u^j]], u^j) &= -([u^i, u^j], [u^i, u^j]),
\end{aligned}$$

the fact that $\{u^i\}_{i=1}^{\dim \mathfrak{g}}$ is an orthonormal basis and the invariance of the bilinear form $(\cdot, \cdot)$, we have

$$u^i(2)u^j(-1)^2\mathbf{1} = 0, \tag{8.12}$$

$$u^i(1)u^j(-1)^2\mathbf{1} = [[u^i, u^j], u^j](-1)\mathbf{1} + 2\ell\delta_{ij}u^j(-1)\mathbf{1}, \tag{8.13}$$

$$\begin{aligned}
u^i(0)u^j(-1)^2\mathbf{1} &= [u^i, u^j](-1)u^j(-1)\mathbf{1} + u^j(-1)[u^i, u^j](-1)\mathbf{1} \\
&= \sum_{k=1}^{\dim \mathfrak{g}}([u^i, u^j], u^k)u^k(-1)u^j(-1)\mathbf{1} + \sum_{k=1}^{\dim \mathfrak{g}}([u^i, u^j], u^k)u^j(-1)u^k(-1)\mathbf{1} \\
&= \sum_{k=1}^{\dim \mathfrak{g}}(u^i, [u^j, u^k])u^k(-1)u^j(-1)\mathbf{1} + \sum_{k=1}^{\dim \mathfrak{g}}(u^i, [u^j, u^k])u^j(-1)u^k(-1)\mathbf{1}.
\end{aligned} \tag{8.14}$$

Using the defintion of ω

$$\begin{aligned}
\sum_{i=1}^{\dim \mathfrak{g}}[[u^j, u^i], u^i] &= \sum_{i=1}^{\dim \mathfrak{g}}[u^i, [u^i, u^j]] = \Omega u^j = 2h^\vee u^j, \\
\sum_{j=1}^{\dim \mathfrak{g}}[[u^i, u^j], u^j] &= \sum_{j=1}^{\dim \mathfrak{g}}[u^j, [u^j, u^i]] = \Omega u^i = 2h^\vee u^i
\end{aligned}$$

and (8.12)– (8.14), we obtain

$$u^i(2)\omega = 0, \tag{8.15}$$

$$u^i(1)\omega = \frac{1}{2(\ell + h^\vee)}\sum_{j=1}^{\dim \mathfrak{g}}([[u^i, u^j], u^j](-1)\mathbf{1} + 2\ell\delta_{ij}u^i(-1)\mathbf{1}) = u^i(-1)\mathbf{1}, \tag{8.16}$$

$$\begin{aligned}
u^i(0)\omega &= \frac{1}{2(\ell + h^\vee)}\sum_{j=1}^{\dim \mathfrak{g}}\sum_{k=1}^{\dim \mathfrak{g}}\big((u^i, [u^j, u^k])u^k(-1)u^j(-1)\mathbf{1} + (u^i, [u^j, u^k])u^j(-1)u^k(-1)\mathbf{1}\big) \\
&= \frac{1}{2(\ell + h^\vee)}\sum_{j=1}^{\dim \mathfrak{g}}\sum_{k=1}^{\dim \mathfrak{g}}\big((u^i, [u^k, u^j])u^j(-1)u^k(-1)\mathbf{1} + (u^i, [u^j, u^k])u^j(-1)u^k(-1)\mathbf{1}\big) \\
&= 0.
\end{aligned} \tag{8.17}$$

From (8.15)–(8.17), we obtain

$$u^i(0)^2\omega = 0, \tag{8.18}$$

$$u^i(0)u^i(1)\omega = 0, \tag{8.19}$$

$$u^i(1)u^i(1)\omega = \ell\mathbf{1}. \tag{8.20}$$

Substituting (8.12)–(8.20) into the right-hand side of (8.11), summing over i and using the formula

$$L_{V(\ell,0)}(-1)u^i(-1)^2\mathbf{1} = u^i(-2)u^i(-1) + u^i(-1)u^i(-2)\mathbf{1} = 2u^i(-2)u^i(-1)\mathbf{1},$$

we obtain

$$\begin{aligned}
&\sum_{i=1}^{\dim\mathfrak{g}} Y_{V(\ell,0)}(u^i(-1)^2\mathbf{1}, x_2)\omega \\
&\quad = \sum_{i=1}^{\dim\mathfrak{g}} 2u^i(-2)u^i(-1)x_2^{-1} + \sum_{i=1}^{\dim\mathfrak{g}} 2u^i(-1)u^i(-1)\mathbf{1}x_2^{-2} + \sum_{i=1}^{\dim\mathfrak{g}} \ell\mathbf{1}x_2^{-4} + \sum_{i=1}^{\dim\mathfrak{g}} F_i(x_2) \\
&\quad = 2(\ell+h^\vee)L_{V(\ell,0)}(-1)\omega x_2^{-1} + 4(\ell+h^\vee)\omega x_2^{-2} + \ell\dim\mathfrak{g}\mathbf{1}x_2^{-4} + \sum_{i=1}^{\dim\mathfrak{g}} F_i(x_2).
\end{aligned} \tag{8.21}$$

Note that the formula Dividing both sides of (8.21) by $2(\ell+h^\vee)$ and let $G(x_2) = \frac{1}{2(\ell+h^\vee)}\sum_{i=1}^{\dim\mathfrak{g}} F_i(x_2),$, we obtain

$$Y_{V(\ell,0)}(\omega, x_2)\omega = L_{V(\ell,0)}(-1)\omega x_2^{-1} + 2\omega x_2^{-2} + \frac{c}{2}\mathbf{1}x_2^{-4} + G(x_2),$$

where

$$c = \frac{\ell\dim\mathfrak{g}}{\ell+h^\vee}.$$

Using the commutator formula for $Y_{V(\ell,0)}$, we have

$$\begin{aligned}
&[a(x_1), Y_{V(\ell,0)}(\omega, x_2)] \\
&\quad = [Y_{V(\ell,0)}(a(-1)\mathbf{1}, x_1), Y_{V(\ell,0)}(\omega, x_2)] \\
&\quad = \mathrm{Res}_{x_0} x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) Y_{V(\ell,0)}(Y_{V(\ell,0)}(a(-1)\mathbf{1}, x_0)\omega, x_2) \\
&\quad = \sum_{n\in\mathbb{Z}} \mathrm{Res}_{x_0} x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) Y_{V(\ell,0)}(a(n)\omega, x_2)x_0^{-n-1}
\end{aligned} \tag{8.22}$$

for $a \in \mathfrak{g}$. From (8.12)–(8.14) and the fact that $\{u^i\}_{i\in I}$ is a basis of $\mathfrak{g}$, by writing $a \in \mathfrak{g}$ as a linear combination of u^i for $i \in I$, we obtain

$$a(2)u^j(-1)^2\mathbf{1} = 0, \tag{8.23}$$

$$a(1)u^j(-1)^2\mathbf{1} = [[a, u^j], u^j](-1)\mathbf{1} + 2\ell(a, u^j)u^i(-1)\mathbf{1}, \tag{8.24}$$

$$a(0)u^j(-1)^2\mathbf{1} = \sum_{k=1}^{\dim\mathfrak{g}} (a, [u^j, u^k])u^k(-1)u^j(-1)\mathbf{1} + \sum_{k=1}^{\dim\mathfrak{g}} (a, [u^j, u^k])u^j(-1)u^k(-1)\mathbf{1}. \tag{8.25}$$

From (8.23)–(8.25), we obtain

$$a(2)\omega = 0, \tag{8.26}$$
$$a(1)\omega = a(-1)\mathbf{1}, \tag{8.27}$$
$$a(0)\omega = 0. \tag{8.28}$$

Using (8.26)–(8.28) and the formal Taylor's theorem, we see that the right-hand side of (8.22) becomes

$$\begin{aligned}
&\mathrm{Res}_{x_0} x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) Y_{V(\ell,0)}(a(-1)\mathbf{1}, x_2)x_0^{-2} \\
&= \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{N}} \mathrm{Res}_{x_0} \frac{x_0^k}{k!}\frac{d^k}{dx_2^k} x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) a(n)x_0^{-2}x_2^{-n-1} \\
&= \sum_{n\in\mathbb{Z}} \frac{d}{dx_2} x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) a(n)x_2^{-n-1}
\end{aligned} \tag{8.29}$$

Taking the coefficient of $x_1^{-m-1}x_2^{-2}$ for $m \in \mathbb{Z}$ on the left-hand sides of (8.22) and the right-hand side of (8.22), we obtain

$$[a(m), \mathrm{Res}_{x_2} x_2 Y_{V(\ell,0)}(\omega, x_2)] = ma(m). \tag{8.30}$$

Taking the coefficient of $x_1^{-m-1}x_2^{-1}$ for $m \in \mathbb{Z}$ on the left-hand sides of (8.22) and the right-hand side of (8.22), we obtain

$$[a(m), \mathrm{Res}_{x_2} Y_{V(\ell,0)}(\omega, x_2)] = ma(m-1). \tag{8.31}$$

Since $Y_{V(\ell,0)}(\omega, x_2)\mathbf{1} \in V(\ell,0)[[x_2]]$,

$$\mathrm{Res}_{x_2} x_2 Y_{V(\ell,0)}(\omega, x_2)\mathbf{1} = \mathrm{Res}_{x_2} Y_{V(\ell,0)}(\omega, x_2)\mathbf{1} = 0. \tag{8.32}$$

From (8.30)–(8.32) and the definitions of $L_{V(\ell,0)}(0)$ and $L_{V(\ell,0)}(-1)$, we obtain

$$\begin{aligned}
[a(m), \mathrm{Res}_{x_2} x_2 Y_{V(\ell,0)}(\omega, x_2)] &= [a(m), L_{V(\ell,0)}(0)], \\
[a(m), \mathrm{Res}_{x_2} Y_{V(\ell,0)}(\omega, x_2)] &= [a(m), L_{V(\ell,0)}(-1)], \\
\mathrm{Res}_{x_2} x_2 Y_{V(\ell,0)}(\omega, x_2)\mathbf{1} &= L_{V(\ell,0)}(0)\mathbf{1}, \\
\mathrm{Res}_{x_2} Y_{V(\ell,0)}(\omega, x_2)\mathbf{1} &= L_{V(\ell,0)}(-1)\mathbf{1}.
\end{aligned}$$

From these formulas, we obtain

$$\begin{aligned}
L_{V(\ell,0)}(0) &= \mathrm{Res}_{x_2} x_2 Y_{V(\ell,0)}(\omega, x_2), \\
L_{V(\ell,0)}(-1) &= \mathrm{Res}_{x_2} Y_{V(\ell,0)}(\omega, x_2).
\end{aligned}$$

Thus ω is a conformal element and $V(\ell, 0)$ is a vertex operator algebra. ∎

The proof of this theorem can be found in [LL], pages 210-213. Although they constructed the map $Y(v, x)$ in a different style, this proof does not refer to the particular construction of $Y(v, x)$.

8.3 Construction of the vertex operator algebra $L(\ell, 0)$

Even for simple $\mathfrak{g}$ and $\ell \in \mathbb{Z}_+$, the vertex operator algebra $V(\ell, 0)$ is in fact not the vertex algebra for the Wess-Zumino-Witten model associated. We need to take an irreducible quotient of $V(\ell, 0)$.

Let $I(\ell, 0)$ be the maximal proper submodule of the $\hat{\mathfrak{g}}$-module $V(\ell, 0)$. In fact, it is easy to see that $I(\ell, 0)$ exists: Consider all $\hat{\hat{\mathfrak{g}}}$-submodules of $V(\ell, 0)$ that do not contain $\mathbf{1}$. In particular, homogeneous elements of these $\hat{\mathfrak{g}}$-submodules have weights greater than 0. Take $I(\ell, 0)$ to be the sum of all such $\hat{\mathfrak{g}}$-submodules. Then $\mathbf{1} \notin I(\ell, 0)$ since $\mathbf{1}$ has weight 0. $I(\ell, 0)$ is maximal. Let $L(\ell, 0) = V(\ell, 0)/I(\ell, 0)$. Then as a $\hat{\hat{\mathfrak{g}}}$-module, $L(\ell, 0)$ is irreducible, that is, there is no $\hat{\hat{\mathfrak{g}}}$-submodule of $L(\ell, 0)$ that is not 0 or $L(\ell, 0)$ itself.

We take the vacuum of $L(\ell, 0)$ to be the equivalent class of the vacuum of $V(\ell, 0)$. We define the vertex operator map $Y_{L(\ell,0)} : L(\ell, 0) \otimes L(\ell, 0) \to L(\ell, 0)((x))$ by

$$Y_{L(\ell,0)}(u + I(\ell, 0), x)(v + I(\ell, 0)) = Y_{V(\ell,0)}(u, x)v + I(\ell, 0).$$

The vacuum $\mathbf{1}_{L(\ell,0)}$ is defined to $\mathbf{1}_{V(\ell,0)} + I(\ell, 0)$ and in the case $\ell + h^\vee \neq 0$, the conformal element $\omega_{L(\ell,0)}$ is defined to be $\omega_{V(\ell,0)} + I(\ell, 0)$.

Theorem 8.12. *The graded vector space $L(\ell, 0)$ equipped with $Y_{L(\ell,0)}$ and $\mathbf{1}$ is a grading-restricted vertex algebra. When $\ell + h^\vee \neq 0$, $L(\ell, 0)$ equipped with $Y_{L(\ell,0)}$, $\mathbf{1}_{L(\ell,0)}$ and $\omega_{L(\ell,0)}$ is a vertex operator algebra.*

Proof. We need only verify that $Y_{L(\ell,0)}$ is well defined; all the axioms can be verified using the properties of $V(\ell, 0)$. To prove that $Y_{L(\ell,0)}$ is well defined, we need only show that $Y_{V(\ell,0)}(u, x)v \in I(\ell, 0)((x))$ when one of u and v is in $I(\ell, 0)$. Since $I(\ell, 0)$ is a $\hat{\mathfrak{g}}$-submodule of $V(\ell, 0)$, we have $a(x)v \in I(\ell, 0)((x))$ for $a \in \mathfrak{g}$ and $v \in I(\ell, 0)$. Then by the definition of the vertex operator map $Y_{V(\ell,0)}$ (see (4.7), we have

$$\begin{aligned}&\langle v', Y_{V(\ell,0)}(a_1(m_1) \cdots a_k(m_k)\mathbf{1}, z)v\rangle \\ &\quad = \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0}\xi_1^{m_1} \cdots \xi_k^{m_k} R(\langle v', a_1(\xi_1 + z) \cdots a_k(\xi_k + z)v\rangle)\end{aligned} \tag{8.33}$$

for $a_1, \ldots, a_k \in \mathfrak{g}$, $m_1, \ldots, m_k \in \mathbb{Z}$, $v \in V(\ell, 0)$ and $v' \in V(\ell, 0)'$. To prove that $Y_{V(\ell,0)}(u, x)v \in I(\ell, 0)((x))$ for $u \in V(\ell, 0)$ and $v \in I(\ell, 0)$, we need only prove $Y_{V(\ell,0)}(a_1(m_1) \cdots a_k(m_k)\mathbf{1}, z)v \in I(\ell, 0)((x))$ for $a_1, \ldots, a_k \in \mathfrak{g}$, $m_1, \ldots, m_k \in \mathbb{Z}$ and $v \in I(\ell, 0)$. Let $I(\ell, 0)^0$ be the annihilator of $I(\ell, 0)$, that is, the subspace of $V(\ell, 0)^0$ containing all linear functionals v' on $V(\ell, 0)$ such that $\langle v', v\rangle = 0$ for all $v \in I(\ell, 0)$. Then $Y_{V(\ell,0)}(a_1(m_1) \cdots a_k(m_k)\mathbf{1}, z)v \in I(\ell, 0)((x))$ if and only if $\langle v', Y_{V(\ell,0)}(a_1(m_1) \cdots a_k(m_k)\mathbf{1}, z)v\rangle = 0$ for all $v' \in I(\ell, 0)^0$. From (8.33) and $a(x)v \in I(\ell, 0)((x))$ for $a \in \mathfrak{g}$ and $v \in I(\ell, 0)$, we see that the right-hand side of (8.33) is 0 for all $v' \in I(\ell, 0)^0$. Thus the left-hand side of (8.33) is also 0 for all $v' \in I(\ell, 0)^0$.

We also need to prove $Y_{V(\ell,0)}(u, x)v \in I(\ell, 0)((x))$ for $u \in I(\ell, 0)$ and $v \in V(\ell, 0)$. Since $L(-1)$ can be expresed as a linear combination of products of $a(n)$ for $a \in \mathfrak{g}$ and $n \in \mathbb{Z}$ and $I(\ell, 0)$ is a $\mathfrak{g}$-submodule of $V(\ell, 0)$, we have $L(-1)I(\ell, 0)v \in I(\ell, 0)$ for $v \in I(\ell, 0)$. Then by the skew-symmetry (5.23),

$$Y_{V(\ell,0)}(u, x)v = e^{xL(-1)}Y_{V(\ell,0)}(v, -x)u \in I(\ell, 0)((x)).$$

■

The vertex operator algebra underlying the Wess-Zumino-Witten model associated to a finite-dimensional simple Lie algebra $\mathfrak{g}$ and a level $\ell \in \mathbb{Z}_+$ is exactly $L(\ell, 0)$. In this case, there is an explicit formula $I(\ell, 0) = U(\hat{\mathfrak{g}})e_\theta(-1)^{\ell+1}\mathbf{1}$, where θ is the highest root of $\mathfrak{g}$ and e_θ is a root vector in $\mathfrak{g}_\theta$ (see [K] and [LL]).

9 Virasoro vertex operator algebras (minimal mdels)

In this section, we will introduce the basic ingredients of the Virasoro vertex operator algebras. But all the proofs will be left as exercises.

Let $\mathcal{L} = \coprod_{n\in\mathbb{Z}} \mathbb{C}L_n + \mathbb{C}\rfloor$ be a vector space with with a basis $\{L_n, \mathbf{c}\}$. We define a bracket operation for $\mathcal{L}$ by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}\mathbf{c},$$
$$[\mathbf{c}, L_n] = 0,$$

for $m, n \in \mathbb{Z}$.

Exercise 9.1. Prove that $\mathcal{L}$ equipped with the bracket operation is a Lie algebra.

The Lie algebra $\mathcal{L}$ is called the Virasoro algebra.The center element $\mathbf{c}$ usually acts as a fixed number c on an $\mathcal{L}$-module. THis number is called the central charge of the module. In Subsection 8.2, we have shown that $V(\ell, 0)$ is a $\mathcal{L}$-module for the Virasoro algebra $\mathcal{L}$ with the central charge $c = \frac{\ell \dim \mathfrak{g}}{\ell + h^\vee}$.

To construct a vertex operator algebra, we first need to construct an $\mathcal{L}$-module. We use the same induced module construction as we have done for the Heisenberg and affine Lie algebras. The Virasoro algebra has a triangle decomposition $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_0 \oplus \mathcal{L}_-$, where $\mathcal{L}_\pm = \coprod_{n\in\pm\mathbb{Z}_+} \mathbb{C}L_n$ and $\mathcal{L}_0 = \mathbb{C}l_0 \oplus \mathbb{C}\mathbf{c}$. Let $c \in \mathbb{C}$ and $\mathbb{C}_c$ be the one-dimensional vector space $\mathbb{C}$ with $\mathbf{c}$ acts on $\mathbb{C}_c$ as the number c. Let $\mathcal{L}_+$ and L_0 acts on $\mathbb{C}_c$ as 0. Then $\mathbb{C}_c$ becomes a module for the subalgebra $\mathcal{L}_+ \oplus \mathcal{L}_0$. Let $M(c, 0) = U(\mathcal{L}) \otimes_{\mathcal{L}_+\oplus\mathcal{L}_0} \mathbb{C}_c$ (where 0 denotes that L_0 acts on $\mathbb{C}_c$ as 0). Then $M(c, 0)$ is an $\mathcal{L}$-module. By the Poincaré-Birkhoff-Witt theorem, as a vector space $M(c, 0)$ is isomorphic to $U(\mathcal{L}_-) \otimes \mathbb{C}_c$ which is in turn linearly isomorphic to $U(\mathcal{L}_-)$. We shall denote the action of L_n on $M(c, 0)$ by $L(n)$ and the element $1 \otimes 1 \in U(\mathcal{L}) \otimes_{\mathcal{L}_+\oplus\mathcal{L}_0} \mathbb{C}_c$ by $\mathbf{1}$. Then $M(c, 0)$ is spanned by elements of the form $L(-n_1)\cdots L(-n_k)\mathbf{1}$ for $n_1, \dots, n_k \in \mathbb{Z}_+$.

For the vertex operator algebra, unlike in the affine Lie algebra case, we need to take a further quotient. Let $\langle L(-1)\mathbf{1}\rangle = U(\mathcal{L})L(-1)\mathbf{1}$ be the submodule of $M(c, 0)$ generated by $L(-1)\mathbf{1}$. Let $V(c, 0) = M(c, 0)/\langle L(-1)\mathbf{1}\rangle$. The action of L_n on $V(c, 0)$ is still denoted by $L(n)$. Then using the Virasoro braket relations, we see that $V(c, 0)$ is spanned by elements of the form $L(-n_1)\cdots L(-n_k)\mathbf{1}$ for $n_1, \dots, n_k \in \mathbb{Z}_+ + 1$.

Define the weight of $L(-n_1)\cdots L(-n_k)\mathbf{1}$ to be $n_1 + \cdots + n_k$. Then $V(c, 0) = \coprod_{n\in\mathbb{N}} V_{(n)}(c, 0)$, where $V_{(n)}(c, 0)$ is the subspace of $V(c, 0)$ consisting of elements of weight n. Let $T(x) =$

$\sum_{n\in\mathbb{Z}} L(n)x^{-n-2}$. This is the generating field for $V(c,0)$. We already have an operator $L(-1)$.

Exercise 9.2. Using the construction theorem in Section 4 to prove that $V(c,0)$ has a structure of grading-restricted vertex algebra. By Theorem 4.5, you need only prove that $V(c,0)$ together with the generating field $T(x)$, $L(-1)$ and $\mathbf{1}$ satisfies the five conditions in Section 4.

Exercise 9.3. Prove that $\omega = L(-2)\mathbf{1} \in V(c,0)$ is a conformal element of $V(c,0)$. In particular, $V(c,0)$ is a vertex operator algebra.

The vertex operator algebra for the minimal model is not $V(c,0)$. As in the case of Wess-Zumino-Novikov-Witten modules, we have to take a further irreducible quotient.

Let $I(c,0)$ be the maximal proper submodule of the $\mathcal{L}$-module $V(c,0)$.

Exercise 9.4. Prove that $I(c,0)$ exists.

Let $L(c,0) = V(c,0)/I(c,0)$. Then as a $\mathcal{L}$-module, $L(c,0)$ is irreducible.

Exercise 9.5. Define the vertex operator map $Y_{L(c,0)}$ for $L(c,0)$. Then prove that $L(c,0)$ equipped with $Y_{L(c,0)}$ and $\mathbf{1}$ is a vertex operator algebra.

10 Quantum vertex algebras

We have introduced meromorphic open-string vertex algebras with one example in Section 6. But in general, without the commutativity or the commutator formula, it is not easy to obtain substantial results. So from this point of view, it is natural to study meromorphic open-string vertex algebras satisfying a weak version of the commutativity. This is what I call quasi-commutative meromorphic open-string vertex algebras. This is in the spirit of quantum vertex algebras but the examples of quantum vertex algebras are much more complicated. Here we first introduce an analytic notion of quasi-commutative vertex algebra which is much stonger than the one introduced by Etingof and Kazhdan in [EK].

Definition 10.1. A *quasi-commutative meromorphic open-string vertex algebra* is a meromorphic open-string vertex algebra satisfying following *quasi-commutativity*: For $u_1, u_2 \in V$, there exist $c_i(x) \in \mathbb{C}[x, x^{-1}]$ and $u_1^i, u_2^i \in V$ for $i = 1, \dots, k$ such that for $v \in V$ and $v' \in V'$,

$$R(\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle = \sum_{i=1}^{k} c_i(z_2 - z_1)R(\langle v', Y_V(u_2^i, z_2)Y_V(u_1^i, z_1)v\rangle).$$

The same proof of the Jacobi identity in Subsection 5.2 gives us the *quasi-Jacobi identity*

$$\begin{aligned}
&{x_0}^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_V(u_1,x_1)Y_V(u_2,x_2)\\
&\quad -\sum_{i=1}^{k} x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)c_i(-x_0)Y_V(u_2^i,x_2)Y_V(u_1^i,x_1)\\
&= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_V(Y_V(u_1,x_0)u_2,x_2)
\end{aligned}\tag{10.34}$$

and the *quasi-commutator formula*

$$\begin{aligned}
&Y_V(u_1,x_1)Y_V(u_2,x_2)-\sum_{i=1}^{k} c_i(-x_0)Y_V(u_2^i,x_2)Y_V(u_1^i,x_1)\\
&= \mathrm{Res}_{x_0}x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_V(Y_V(u_1,x_0)u_2,x_2)
\end{aligned}\tag{10.35}$$

All the properties we proved in Section 5 can be generalize to such a quasi-commutative meromorphic open-string vertex algebra.

Problem 10.2. Construct an example of quasi-commutative meromorphic open-string vertex algebra.

We now introduce quantum vertex algebra in the sense of Etingof and Kazhdan. The idea is to view quantum vertex algebras as formal deformations of vertex algebras. We briefly explain what is a formal noncommutative deformation of a commutative associative algebra. Let A be a commutative associative algebra with the multiplication $\cdot$. Let h be a formal variable. We conside the $\mathbb{C}[[h]]$-module $A[[h]]$. A formal noncommutative deformation of A is $A[[h]]$ together with an associative multiplication $\cdot_h$ on $A[[h]]$ such that $a\cdot_h b = a\cdot b + hB$ for $a, b \in A$ and $B \in A[[h]]$. Note that we do not require $a \cdot_h b$ be commutative. We also note that A is in fact isomorphic to $A[[h]]/hA[[h]]$ as a commutative associative algebra. In general, we can consider a $\mathbb{C}[[h]]$-module A together with an associative multiplication such that the multiplication induced on A/hA is a commutative associative algebra. Then A is a noncommutative deformation of A/hA. Such formal deformations are also called deformation quantization.

We now want to apply the same idea to vertex algebras. Let V be a grading-restricted vertex algebra. We consider $V[[h]]$. Then a formal noncommutative deformation of V is roughly speaking $V[[h]]$ together with a vertex operator map $Y_V^h : V[[h]] \otimes V[[h]] \to V[[h]]((x))$ satisfying the associativity and all the other properties for a meromorphic open-string vertex algebra. But to get nice properties, we still need to give a quasi-commutativity type property.

For a vector space V_0, let $V = V_0[[h]]$. Let

$$V_h((x)) = \left\{\sum_{n\in\mathbb{Z}} a_n x^{-n-1} \mid \text{for every } M\in\mathbb{N},\ a_n \in h^M V \text{ for } n >> 0\right\}.$$

Given a $\mathbb{C}[[h]]$-linear map

$$\begin{aligned} \mathcal{S} : V \otimes V &\to V \otimes V \otimes \mathbb{C}((x))[[h]] \\ u_2 \otimes u_1 &\mapsto \mathcal{S}(x)(u_2 \otimes u_1), \end{aligned}$$

for $u_1, u_2 \in V$, there exists u_1^i, u_2^i and $c_i(x) \in \mathbb{C}((x))[[h]]$ for $i = 1, \ldots, k$ such that $\mathcal{S}(x)(u_2 \otimes u_1) = \sum_{i=1}^k u_2^i \otimes u_1^i \otimes c_i(x)$.

We define $\mathcal{S}_{21}(x) : V \otimes V \to V \otimes V \otimes \mathbb{C}((x))[[h]]$ by

$$\mathcal{S}_{21}(x)(u \otimes v) = \sigma_{12}(\mathcal{S}(x)(v \otimes u))$$

where σ_{12} is the permutation switch 1 and 2. We also define $\mathcal{S}_{12}(x) : V \otimes V \otimes V \to V \otimes V \otimes V \otimes \mathbb{C}((x))$ by

$$\mathcal{S}_{12}(x)(v_1 \otimes v_2 \otimes v_3) = \mathcal{S}(x)(v_1 \otimes v_2) \otimes v_3.$$

Similarly we also have $\mathcal{S}_{13}(x)$ and $\mathcal{S}_{23}(x)$. We also have $\mathcal{S}_{31}(x)$, $\mathcal{S}_{32}(x)$ and more generally $S_{ij}(x)$ on $V^{\otimes n}$ for $i, j = 1, \ldots n$.

Definition 10.3. A quantum vertex algebra is a $\mathbb{C}[[h]]$-module of the form $V = V_0[[h]]$, equipped with a vertex operator map

$$\begin{aligned} Y_V : (V_0 \otimes V_0)[[h]] &\to V_h((x)) \\ u \otimes v &\mapsto Y_V(u, x)v, \end{aligned}$$

a vacuum $\mathbf{1} \in V$ and a $\mathbb{C}[[h]]$-linear map

$$\begin{aligned} \mathcal{S} : V \otimes V &\to V \otimes V \otimes \mathbb{C}((x))[[h]] \\ u_2 \otimes u_1 &\mapsto \mathcal{S}(x)(u_2 \otimes u_1) = \sum_{i=1}^{k} u_1^i \otimes u_2^i \otimes c_i(x) \end{aligned}$$

such that $\mathcal{S}(x)(u_2 \otimes u_1) = u_2 \otimes u_1 \mod h$ satisfying the following axioms:

1. Axioms for the vacuum: (a) *Identity proerty*: $Y_V(\mathbf{1}, x) = 1_V$. (b) *Creation property*: For $v \in V$, $Y_V(v, x)\mathbf{1} \in V[[x]]$ and $\lim_{x \to 0} Y_V(v, x)\mathbf{1} = v$.

2. $L(-1)$-*derivative property* and $L(-1)$-*commutator formula*: Let $L_V(-1)$ be the $\mathbb{C}[[h]]$-linear map $L(-1) : V \to V$ defined by

$$L_V(-1)v = \lim_{x \to 0} \frac{d}{dx} Y_V(v, x)\mathbf{1}$$

for $v \in V$. Then

$$\frac{d}{dx} Y_V(v, x) = Y_V(L_V(-1)v, x) = [L_V(-1), Y_V(v, x)].$$

3. *Weak associativity*: For $u_1, u_2, v \in V$ and $M \in \mathbb{N}$, there exists $N \in \mathbb{Z}_+$ such that

$$(x_0+x_2)^N Y_V(Y_V(u,x_0)v,x_2) = (x_0+x_2)^N Y_V(u,x_0+x_2)Y_V(v,x_2) \mod h^M V.$$

4. *Weak $\mathcal{S}$-commutativity*: For $u_1, u_2 \in V$ and $M \in \mathbb{N}$, there exists $N \in \mathbb{Z}_+$ such that

$$\begin{aligned}(x_1-x_2)^N & Y_V(u_1,x_1)Y_V(u_2,x_2)\\ &= (x_1-x_2)^N \sum_{i=1}^{k} c_i(x_2-x_1)Y_V(u_2^i,x_1)Y_V(u_2^i,x_2) \mod h^M.\end{aligned}$$

5. *Unitarity* and *Yang-Baxter relation* for $\mathcal{S}$:

$$\mathcal{S}_{21}(x_2-x_1)\mathcal{S}(x_1-x_2) = 1_{V\otimes V}, \tag{10.36}$$

$$\mathcal{S}_{12}(x_1-x_2)\mathcal{S}_{13}(x_1-x_3)\mathcal{S}_{23}(x_2-x_3) = \mathcal{S}_{23}(x_2-x_3)\mathcal{S}_{13}(x_1-x_3)\mathcal{S}_{13}(x_1-x_2). \tag{10.37}$$

Remark 10.4. The heuristic meaning of the unitarity for $\mathcal{S}$: If $\mathcal{S}(x_1-x_2)$ can be written as $e^{i(x_1-x_2)K}$, where K is an operator satisfying $\sigma_{12}\overline{K}\sigma_{12} = K$, then

$$\mathcal{S}_{21}(x_2-x_1) = e^{i(x_2-x_1)\sigma_{12}K\sigma_{12}} = \overline{e^{i(x_1-x_2)\sigma_{12}\overline{K}\sigma_{12}}} = \overline{e^{i(x_1-x_2)K}} = \overline{\mathcal{S}(x_1-x_2)}.$$

So (10.36) becomes

$$\overline{\mathcal{S}(x_1-x_2)}\mathcal{S}(x_1-x_2) = 1_{V\otimes V}.$$

This says exactly that $\mathcal{S}(x_1-x_2)$ is unitary.

Remark 10.5. The heuristic meaning of the Yang-Baxter relation for $\mathcal{S}$: See Fig. (will give a picture for this).

Examples of quantum vertex algebras are constructed using modules for quantum affine Lie algebras such that when we set $h = 0$, we obtain the vertex algebra associated to affine Lie algebras constructed in Subsection 8.1. See [EK].

11 Modules

11.1 Definition and properties of modules

Roughly speaking, a module for a grading-restricted vertex algebra V is $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ eqipped with a vertex operator map $Y_W : V \otimes W \to W((x))$ and an operator $L_W(-1)$ satisfying all the axioms for a grading-restricted vertex algebra that still make sense. But we have to consider more general types of modules.

Definition 11.1. Let V be a grading-restricted vertex superalgebra. A *generalized V-module* is a $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ equipped with a *vertex operator map*

$$\begin{aligned} Y_W : V \otimes W &\to W((x)), \\ u \otimes w &\mapsto Y_W(u,x)w \end{aligned}$$

satisfying the following axioms:

1. Axioms for the gradings: There are operators $L_W(0)$, $L_W(0)_S$ and $L_W(0)_N$ on W such that $L_W(0) = L_W(0)_S + L_W(0)_N$, $L_W(0)_S v = nv$ for $v \in W_{[n]}$, $L_W(0)_N$ is nilpotent (for $w \in W$, there exists $K \in \mathbb{N}$ such that $(L_W(0)_N)^K w = 0$), and

$$[L_W(0), Y_W(v,x)] = x\frac{d}{dx}Y_W(v,x) + Y_W(L_V(0)v,x)$$

   for $v \in V$.

2. *Identity property*: Let 1_W be the identity operator on W. Then $Y_W(\mathbf{1}, z) = 1_W$.

3. $L(-1)$*-derivative property*: There exists $L_W(-1) : W \to W$ such that for $u \in V$,

$$\frac{d}{dz}Y_W(u,z) = Y_W(L_V(-1)u,z) = [L_W(-1), Y_W(u,z)].$$

4. *Duality*: For $u_1, u_2 \in V$, $w \in W$ and $w' \in W'$, the series

$$\begin{gathered} \langle w', Y_W(u_1,z_1)Y_W(u_2,z_2)w\rangle, \\ \langle w', Y_W(u_2,z_2)Y_W(u_1,z_1)w\rangle, \\ \langle w', Y_W(Y_V(u_1,z_1-z_2)u_2,z_2)w\rangle \end{gathered}$$

   are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.

A *lower-bounded generalized V-module* is a generalized V-module $(W, Y_W, L_W(0), L_W(-1))$ such that $W_{[n]} = 0$ when $\Re n$ is sufficiently negative. A *grading-restricted generalized V-module* is a lower-bounded generalized V-module $(W, Y_W, L_W(0), L_W(-1))$ such that $\dim W_{[n]} < \infty$. An *ordinary V-module* or simply a *V-module* is a grading-restricted generalized V-module $(W, Y_W, L_W(0), L_W(-1))$ such that $L_W(0)_N = 0$. When V is a vertex operator algebra, a *lower-bounded generalized V-module* or *grading-restricted generalized V-module* or an *ordinary V-module* is such a V-module when V is viewed as a grading-restricted vertex algebra such that $L(0) = \mathrm{Res}_x xY_W(\omega,x)$ and $L_W(-1) = \mathrm{Res}_x Y_W(\omega,x)$.

All the properties for grading-restricted vertex algebras and vertex operator algebras that still make sense also hold for modules. Here we state these properties without proofs. The proofs are the same as those for grading-restricted vertex algebras and vertex operator algebras.

Operator product expansion For $u_1, u_2 \in V$, there exists $N \in \mathbb{N}$ such that $Y_V(u_1, x)u_2 = \sum_{n \leq N} (Y_V)_n(u_1)u_2 x^{-n-1}$ (see Subsection 5.1). Then

$$\begin{aligned} Y_W(u_1, z_1)Y_W(u_2, z_2) &= \sum_{n \leq N} Y_W((Y_V)_n(u_1)u_2, z_2)(z_1 - z_2)^{-n-1} \\ &\sim \sum_{n=0}^{N} Y_W((Y_V)_n(u_1)u_2, z_2)(z_1 - z_2)^{-n-1}. \end{aligned}$$

The Jacobi identity For $u_1, u_2 \in V$,

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y_W(u_1, x_1)Y_W(u_2, x_2) - x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right)Y_W(u_2, x_2)Y_W(u_1, x_1) \\ = x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right)Y_W(Y_V(u_1, x_0)u_2, x_2). \end{aligned} \tag{11.38}$$

Commutator formula For $u_1, u_2 \in V$,

$$\begin{aligned} &Y_W(u_1, x_1)Y_W(u_2, x_2) - Y_W(u_2, x_2)Y_W(u_1, x_1) \\ &\quad = \mathrm{Res}_{x_0} x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right)Y_W(Y_V(u_1, x_0)u_2, x_2). \end{aligned} \tag{11.39}$$

Associator formula For $u_1, u_2 \in V$,

$$\begin{aligned} &Y_W(Y_V(u_1, x_0)u_2, x_2) - Y_W(u_1, x_0 + x_2)Y_W(u_2, x_2) \\ &\quad = -\mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right)Y_W(u_2, x_2)Y_W(u_1, x_1). \end{aligned} \tag{11.40}$$

Weak commutativity For $u_1, u_2 \in V$, there exists $N \in \mathbb{N}$ such that

$$(x_1 - x_2)^N Y_W(u_1, x_1)Y_W(u_2, x_2) = (x_1 - x_2)^N Y_W(u_2, x_1)Y_W(u_1, x_1). \tag{11.41}$$

Weak associativity For $u_1 \in V$ and $w \in W$, there exists $N \in \mathbb{N}$ such that

$$(x_0 + x_2)^N Y_W(Y_V(u_1, x_0)u_2, x_2)w = (x_0 + x_2)^N Y_W(u_1, x_0 + x_2)Y_W(u_2, x_2)w \tag{11.42}$$

for $u_2 \in V$.

Virasoro operators Let V be a vertex operator algebra with the conformal element ω. Write $Y_W(\omega, x) = \sum_{n \in \mathbb{Z}} L_W(n)x^{-n-2}$. Then

$$[L_W(m), L_W(n)] = (m - n)L_W(m + n) + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

for $m, n \in \mathbb{Z}$.

Definition 11.2. A *generalized V-submodule* of a generalized V-module W is a generalized V-module (W_0, Y_{W_0}) such that W_0 is a graded subspace of W and $Y_{W_0} = Y_W|_{V\otimes W_0}$. A generalized V-module W is said to be *irreducible* if there is no nonzero proper V-submodule of W. *Lower-bounded generalized V-submodules*, *grading-restricted generalized V-submodules*, *(ordinary) V-submodules* and the corresponding *irreducbible* ones are defined in the obvious way.

11.2 Modules for Heisenberg vertex operator algebras

Recall in Section 2, we have the group algebra $\mathbb{C}[L]$ for a lattice. We now consider the group algebra $\mathbb{C}[\mathfrak{h}]$. We use the same notation to denote e^α to denote the the basis element $\alpha \in \mathfrak{h}$ in $\mathbb{C}[\mathfrak{h}]$ but note that now α does not have to be in a lattice L. Note that for each $\alpha \in \mathfrak{h}$, $\mathbb{C}e^\alpha$ is a subspace of $\mathbb{C}[\mathfrak{h}]$.

Let $M(1, \alpha) = S(\hat{\mathfrak{h}}_-) \otimes \mathbb{C}e^\alpha$. We define the action of the Heisenberg algebra $\hat{\mathfrak{h}}$ on $M(1, \alpha)$ in the same way as in Section 2; $a \otimes t^n$ for $n \neq 0$ acts only on $S(\hat{\mathfrak{h}}_-)$ and $a \otimes t^0$ acts only on e^α by $a(0)e^\alpha = (a, \alpha)e^\alpha$, and $\mathbf{k}$ acts on $M(1, \alpha)$ as 1. THis explains our notation: 1 in $M(1, \alpha)$ is used to denote that the center $\mathbf{k}$ acts as 1. Then as in Section 2, it is easy to verify that $M(1, \alpha)$ becomes an $\hat{\mathfrak{h}}$-module. As in Section 2, we use $a(n)$ to denote the action of $a \otimes t^n$ on $M(1, \alpha)$. Then $M(1, \alpha)$ is spanned by elements of the form $a_1(-n_1) \cdots a_k(-n_k)e^\alpha$ for $k \in \mathbb{N}$, $a_1, \dots, a_k \in \mathfrak{h}$ and $n_1, \dots, n_k \in \mathbb{Z}_+$.

We define the weight of $a_1(-n_1) \cdots a_k(-n_k)e^\alpha$ to be $n_1 + \cdots + n_k + \frac{1}{2}(\alpha, \alpha)$. In particular, the weight of e^α is $\frac{1}{2}(\alpha, \alpha)$. Then $M(1, \alpha) = \coprod_{n \in \frac{1}{2}(\alpha,\alpha)+\mathbb{N}} M_{[n]}(\alpha)$ is a space graded by $\frac{1}{2}(\alpha, \alpha) + \mathbb{N}$. The same argument as for $S(\hat{\mathfrak{h}}_-)$ shows that $M(1, \alpha)$ is grading restricted, that is, $M_{[n]}(\alpha) = 0$ when $n - \frac{1}{2}(\alpha, \alpha) < 0$ and $\dim M_{[n]}(\alpha) < \infty$.

Let L be an even positive definite lattice. For $\alpha \in L$, $M(1, \alpha)$ is in fact a $\hat{\mathfrak{h}}$-submodule of the vertex operator algebra $V_L = S(\hat{\mathfrak{h}}_-) \otimes \mathbb{C}[L]$. Then we have a vertex operator map Y_{V_L} for V_L. By definition, $Y_{V_L}(u, x)v \in M(1, \alpha)$ for $u \in S(\hat{\mathfrak{h}}_-) \otimes \mathbb{C} \simeq S(\hat{\mathfrak{h}}_-)$ and $v \in M(1, \alpha)$. We define $Y_{M(1,\alpha)} = Y_{V_L}|_{S(\hat{\mathfrak{h}}_-)\otimes M(1,\alpha)}$. Then since V_L is a vertex operator algebra, all the axioms for $(M(1, \alpha), Y_{M(1,\alpha)}$ being a $S(\hat{\mathfrak{h}}_-)$-module are satisfied.

For $\alpha \in \mathfrak{h}$ but not in any even positive lattice, we cannot use grading-restricted vertex algebras associated to a lattice. This construction of $S(\hat{\mathfrak{h}}_-)$-module does not work . For such α, we will give the construction of an $S(\hat{\mathfrak{h}}_-)$-module structure on $M(1, \alpha)$ after we prove a construction theorem for modules. In this construction theorem, we shall need some formal series of operators called generator fields. These can be obtained from generalizations of the vertex operators $Y_{V_L}(e^\alpha, x)$ in Section 2. Here we define these generator fields and give their properties.

For $\alpha \in \mathfrak{h}$, we define $\psi^\alpha(x) : S(\hat{\mathfrak{h}}_-) \to M(1, \alpha)$ by

$$\begin{aligned}
&\psi^\alpha(x)a_1(-n_1) \cdots a_k(-n_k)\mathbf{1} \\
&\quad = \exp\left(-\sum_{n \in -\mathbb{Z}_+} \frac{\alpha(n)}{n} x^{-n}\right) \exp\left(-\sum_{n \in \mathbb{Z}_+} \frac{\alpha(n)}{n} x^{-n}\right) a_1(-n_1) \cdots a_k(-n_k)e^\alpha
\end{aligned}$$

for $a_1, \dots, a_k \in \mathfrak{h}$ and $n_1, \dots, n_k \in \mathbb{Z}_+$. In the case that $\alpha \in L$ for a even positive definite lattice,

$$\psi^\alpha(x) a_1(-n_1) \cdots a_k(-n_k)\mathbf{1} = Y_{V_L}(e^\alpha, x) a_1(-n_1) \cdots a_k(-n_k)\mathbf{1}.$$

Note that (2.12)–(2.15) still hold for $\alpha \in \mathfrak{h}$. Also by definition,

$$\begin{aligned}
&a(0)\psi^\alpha(x) a_1(-n_1) \cdots a_k(-n_k)\mathbf{1} \\
&\quad = (a, \alpha) \exp\left(-\sum_{n \in -\mathbb{Z}_+} \frac{\alpha(n)}{n} x^{-n}\right) \exp\left(-\sum_{n \in \mathbb{Z}_+} \frac{\alpha(n)}{n} x^{-n}\right) a_1(-n_1) \cdots a_k(-n_k) e^\alpha \\
&\quad = (a, \alpha)\psi^\alpha(x) a_1(-n_1) \cdots a_k(-n_k)\mathbf{1} + \psi^\alpha(x) a(0) a_1(-n_1) \cdots a_k(-n_k)\mathbf{1}.
\end{aligned}$$

From these formulas, by the same calculations as in Section 2, we obtain the following weak commutativity between $a(x_1)$ and $\psi^\alpha(x_2)$ for $a, \alpha \in \mathfrak{h}$:

$$(x_1 - x_2) a(x_1) \psi^\alpha(x_2) = (x_1 - x_2) \psi^\alpha(x_2) a(x_1).$$

We now give the theorem for $M(1, \alpha)$ but we prove only the irreducibility here; the proof of the (ordinary) module structure will be given in the next section.

Theorem 11.3. *For $\alpha \in \mathfrak{h}$, $M(1, \alpha)$ has a structure of irreducible (ordinary) $S(\hat{\mathfrak{h}}_-)$-module.*

Proof. As we mentioned, the proof that $M(1, \alpha)$ is a grading-restricted (ordinary) $S(\hat{\mathfrak{h}}_-)$-module will be given later. Here we show that $M(1, \alpha)$ is in fact irreducible.

Assume that W_0 is a nonzero $S(\hat{\mathfrak{h}}_-)$-submodule of $M(1, \alpha)$. From the commutator formula (12.55 with $u_1 = a(-1)\mathbf{1}$ and $u_2 = b(-1)\mathbf{1}$, we obtain

$$[a_W(m), b_W(n)] = m(a, b)\delta_{m+n,0}$$

for $m, n \in \mathbb{Z}$, where $a_W(m)$ and $b_W(n)$ are given by $Y_W(a(-1)\mathbf{1}, x) = \sum_{m \in \mathbb{Z}} a(m) x^{-m-1}$ and $Y_W(b(-1)\mathbf{1}, x) = \sum_{n \in \mathbb{Z}} b(n) x^{-n-1}$. In particular, we see that W is an $\hat{\mathfrak{h}}$-module. Since W is a $S(\hat{\mathfrak{h}}_-)$-submodule of $M(1, \alpha)$, this $\hat{\mathfrak{h}}$-module structure must be induced from the $\hat{\mathfrak{h}}$-module structure on $M(1, \alpha)$. So W is a $\hat{\mathfrak{h}}$-submodule of $M(1, \alpha)$.

Since $M(1, \alpha)$ is grading restricted, W as a submodle must also be grading-restricted. Since W is nonzero, there exists homogeneous $w \in W$ such that any element of W of weight less than $\mathrm{wt}\, w$ is 0. In particular, $\hat{\mathfrak{h}}_+$ annihilates w since elements of $\hat{\mathfrak{h}}_+$ lowers the weights. Since $M(1, \alpha) = S(\hat{\mathfrak{h}}_-) \otimes \mathbb{C}e^\alpha$, we see that the only elements annihilated by $\hat{\mathfrak{h}}_+$ are those in $\mathbb{C}e^\alpha$. Thus $w = \lambda e^\alpha$ for some $\lambda \in \mathbb{C}$. Then $W = M(1, \alpha)$. So W cannot be proper. ∎

11.3 A construction theorem for modules

(The material in subsection incorporated some of the writings by Jason Saied.)

We now give a construction theorem for modules. The theorem that we present below is a special case of Theorem 4.3 in [Hua10].

In the construction of grading-restricted vertex algebras in Section 4, we start with a grading-restricted vector space V, a set of generating fields $\phi^i(x)$ for $i \in I$, a vacuum $\mathbf{1}$ and an operator $L_V(-1)$ on V satisfying the five conditions in the section. For modules, we will start with a $\mathbb{C}$-graded vector space W, a set of generating fields $\phi^i_W(x)$ on W for $i \in I$ (here I is the same index set as the one for the algebra), an operator $L_W(-1)$ on W satisfying some conditions. But for modules, these are not enough. We have to introduce a set of what we will call "generator fields" $\psi^a_W(x)$ for $a \in A$.

Here we explain why we need these generator fields. In the proof of Lemma 4.4 in Section 4, we have to express an element v on which the generating fields act using these generating fields acting on the vacuum. But for a module, the generating fields $\phi^i_W(x)$ act on W and it is impossible to express $w \in$ using these generating fields acting on the vacuum. To give a construction theorem for modules, we have to introduce some fields which give elements in W when applied to the vacuum in the algebra V.

Note that in the properties of modules in the preceding subsection, there is no skew symmetry since it does not make sense. In fact skew-symmetry is what motivates the definition of what we call generator fields. We first give this motivation.

Let W be a generalized V-module. In [FHL], a linear map

$$\begin{aligned} Y_{WV}^W : W \times V &\to W((x)), \\ w \otimes v &\mapsto Y_{WV}^W(w, x)v \end{aligned}$$

is defined by $Y_{WV}^W(w,x)v = e^{xL_W(-1)}Y_W(v,-x)w$ for $v \in V$ and $w \in W$. Replacing x_0 and x_2 in the weak associativity (12.58) by x_1 and $-x_2$, we obtain

$$(x_1 - x_2)^N Y_W(Y_V(u_1, x_1)u_2, -x_2)w = (x_1 - x_2)^N Y_W(u_1, x_1 - x_2)Y_W(u_2, -x_2)w \quad (11.43)$$

for $u_1, u_2 \in V$ and $w \in W$. Applying $e^{x_2 L_W(-1)}$ to both sides of (11.43) from the left, using the definition of Y_{WV}^W and the $L(-1)$-conjugation property $e^{x_2 L_W(-1)} Y_W(u_1, x_1 - x_2) = Y_W(u_1, x_1)e^{x_2 L_W(-1)}$ (obtained by exponentiating the $L(-1)$-derivative property $[L_W(-1), Y_W(u_1, x_1)] = \frac{d}{dx} Y_W(u_1, x_1)$ and the formal Taylor's theorem) and then replacing u_1 by v and removing u_2, we see that (11.43) becomes

$$(x_1 - x_2)^N Y_{WV}^W(w, x_2)Y_V(v, x_1) = (x_1 - x_2)^N Y_W(v, x_1)Y_{WV}^W(w, x_2). \quad (11.44)$$

Note that (11.44) is of the same form as the weak commutativity (12.57) but with $Y_W(u_2, x_2)$ replaced by $Y_{WV}^W(w, x_2)$.

Let M be a subspace of W such that W is spanned by coefficients of formal series of the form $Y(v,x)w$ for $v \in V$ and $w \in W$. Then we say that W *is generated by* M and M is a set of *generators of* W. We also call $Y_{WV}^W(w,x)$ for $w \in M$ a *generator fields* of W. Let $\phi(x) = Y_V(v,x)$ be a generating field of V. We use $\psi(x)$ to denote the generator field $Y_{WV}^W(w,x)$. Then the weak commutativity (11.44) becomes

$$(x_1 - x_2)^N \phi(x_1)\psi(x_2) = (x_1 - x_2)^N \psi(x_2)\phi(x_1).$$

We shall use generator fields and this weak commutativity as additional data and properties in the construction theorem for modules.

Let V be a grading-restricted vertex algebra. We assume that V is a grading-restricted vertex algebra generated by $\phi^i(x) = Y_V(\phi^i_{-1}\mathbf{1}, x)$ for $i \in I$, where $\phi^i(x)$ is homogeneous with respect to weights, ϕ^i_{-1} is the constant term of $\phi^i(x)$, and $\phi^i_{-1}\mathbf{1} = \lim_{x\to 0} \phi^i(x)\mathbf{1}$ (see Section 4 and [Hua9]). Let $\mathrm{wt}\, \phi^i$ be the weight of $\phi^i_{-1}\mathbf{1}$.

We shall give a construction of lower-bounded generalized V-modules. The construction is based on the following data and assumptions:

Data 11.4. (a) Let $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ be a $\mathbb{C}$-graded vector space such that $W_{[n]} = 0$ if $\Re(n)$ is sufficiently negative.

(b) Let

$$\begin{aligned} \phi^i_W : W &\to W((x)) \\ w \mapsto \phi^i_W(x)w &= \sum_{n\in\mathbb{Z}} (\phi^i_W)_n w x^{-n-1} \end{aligned}$$

for $i \in I$ (the same index set I for the generating fields for V) be linear maps called *generating field maps*.

(c) Let

$$\begin{aligned} \psi^a_W : V &\to W((x)) \\ v \mapsto \psi^a_W(x)v &= \sum_{n\in\mathbb{Z}} (\psi^a_W)_n v x^{-n-1} \end{aligned}$$

for $a \in A$ be linear maps called *generator field maps*.

(d) Let $L_W(0), L_W(-1)$ be linear operators on W.

Assumption 11.5. The data given in Data 11.4 satisfy the following properties:

1. There exist semisimple and nilpotent operators $L_W(0)_S$ and $L_W(0)_N$ on W such that $L_W(0) = L_W(0)_S + L_W(0)_N$. For $w \in W_{[n]}$, $L(0)w = nw$. For $i \in I$,

$$[L_W(0), \phi^i_W(x)] = x\frac{d}{dx}\phi^i_W(x) + (\mathrm{wt}\phi^i)\phi^i_W(x).$$

For $a \in A$, there exists $\mathrm{wt}\, \psi^a_W \in \mathbb{C}$ and, when $L_W(0)_N\psi^a_W(x) \neq 0$, there exists $L_W(0)_N(a) \in A$ such that

$$L_W(0)\psi^a_W(x) - \psi^a_W(x)L_V(0) = x\frac{d}{dx}\psi^a_W(x) + (\mathrm{wt}\, \psi^a_W)\psi^a_W(x) + \psi^{L_W(0)_N(a)}_W(x),$$

where $\psi^{L_W(0)_N(a)}_W(x) = 0$ when $L_W(0)_N\psi^a_W(x) = 0$.

2. For $i \in I$ and $a \in A$,
$$[L_W(-1), \phi^i_W(x)] = \frac{d}{dx}\phi^i_W(x)$$
and
$$L_W(-1)\psi^a_W(x) - \psi^a_W(x)L_V(-1) = \frac{d}{dx}\psi^a_W(x).$$

3. For $a \in A$, $\psi^a_W(x)\mathbf{1} \in W[[x]]$ and the constant term $(\psi^a_W)_{-1}\mathbf{1} = \lim_{x\to 0}\psi^a_W(x)\mathbf{1}$ is homogeneous.

4. The vector space W is spanned by elements of the form
$$(\phi^{i_1}_W)_{n_1}\cdots(\phi^{i_k}_W)_{n_k}(\psi^a_W)_n v$$
for $i_1, \dots, i_k \in I$, $a \in A$, $n, n_1, \dots, n_k \in \mathbb{Z}$, and $v \in V$.

5. For $i, j \in I$, there exists $M_{ij} \in \mathbb{Z}_+$ such that
$$(x_1 - x_2)^{M_{ij}}[\phi^i_W(x_1), \phi^j_W(x_2)] = 0.$$

6. For $i \in I$ and $a \in A$, there exists $M_{ia} \in \mathbb{Z}_+$ such that
$$(x_1 - x_2)^{M_{ia}}\phi^i_W(x_1)\psi^a_W(x_2) = (x_1 - x_2)^{M_{ia}}\psi^a_W(x_2)\phi^i(x_1).$$

We have the following results analogous to Proposition 4.2:

Proposition 11.6. *Assume that the data given in Data 11.4 satisfy only the parts on ϕ^i_W for $i \in I$ of Conditions 1–4 in Assumption 11.5. Then Conditions 5 and 6 in Assumption 11.5 are equivalent to the following three conditions:*

7. *For $w' \in W'$, $w \in W$, $i_1, \dots, i_{k+l} \in I$ and $a \in A$, the series*
$$\langle w', \phi^{i_1}_W(z_1)\cdots\phi^{i_k}_W(z_k)\psi^a_W(z)\phi^{i_1}(z_{k+1})\cdots\phi^{i_k}(z_{k+l})v\rangle$$
is absolutely convergent in the region $|z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l}| > 0 > 0$ to a rational function
$$R(\langle w', \phi^{i_1}_W(z_1)\cdots\phi^{i_k}_W(z_k)\psi^a_W(z)\phi^{i_1}(z_{k+1})\cdots\phi^{i_k}(z_{k+l})v\rangle)$$
in $z_1, \dots, z_k, z, z_{k+1}, \dots, z_{k+l}$ with the only possible poles at $z_i = 0$ for $i = 1, \dots, k+l$, $z = 0$, $z_j = z_m$ for $j \neq m$ and $z_j = z$ for $j = 1, \dots, k+l$. In addition, the order of the pole $z_j = z_m$ depends only on $\phi^{i_j}_W$ and $\phi^{i_m}_W$, the order of the pole $z_j = z$ depends only on $\phi^{i_j}_W$ and ψ^a_W, the order of the pole $z_j = 0$ depends only $\phi^{i_j}_W$ and v and the order of $z = 0$ depends only on ψ^a_W and v.

8. *For $w \in V$, $w' \in V'$, $i_1, i_2 \in I$,*
$$R(\langle w', \phi^{i_1}_W(z_1)\phi^{i_2}_W(z_2)w\rangle) = R(\langle w', \phi^{i_2}_W(z_2)\phi^{i_1}_W(z_1)w\rangle).$$

9. *For $v \in V$, $w' \in W'$, $i \in I$ and $a \in A$,*

$$R(\langle w', \phi^i_W(z_1)\psi^a_W(z_2)v\rangle) = R(\langle w', \psi^a_W(z_2)\phi^i(z_1)w\rangle).$$

The proof of these two results is essentially the same as the proof of Proposition 4.2. So we omit it here.

Proposition 11.7. *The space V, the fields ϕ^i for $i \in I$, $L_V(-1)$ and $\mathbf{1}$ have the following properties:*

10. *For $a \in \mathbb{C}$, $i \in I$ and $a \in A$,*

$$\begin{aligned} e^{cL_W(0)}\phi^i_W(x)e^{-cL_W(0)} &= e^{c(\operatorname{wt}\phi^i)}\phi^i_W(e^c x),\\ e^{cL_W(0)}\psi^a_W(x)e^{-cL_V(0)} &= e^{c(\operatorname{wt}\phi^i)}\psi^i_W(e^c x).\end{aligned}$$

11. *For $i_1, \dots, i_k \in I$, $n_1, \dots, n_k \in \mathbb{Z}$, $a \in A$, $n \in \mathbb{Z}$ and $v \in V$,*

$$\begin{aligned} &L_W(0)(\phi^{i_1}_W)_{n_1}\cdots(\phi^{i_k}_W)_{n_k}(\psi^a_W)_n v\\ &\quad= \sum_{j=1}^k (\phi^{i_1}_W)_{n_1}\cdots(\phi^{i_{j-1}}_W)_{n_{j-1}}\cdot\\ &\qquad\qquad \cdot\left((-n_j-1)(\phi^{i_j}_W)_{n_j} + (\operatorname{wt}\phi^{i_j})(\phi^{i_j}_W)_{n_j}\right)(\phi^{i_{j+1}}_W)_{n_{j+1}}\cdots(\phi^{i_k}_W)_{n_k}(\psi^a_W)_n v\\ &\qquad + (\phi^{i_1}_W)_{n_1}\cdots(\phi^{i_k}_W)_{n_k}\left((-n-1)(\psi^a_W)_n + (\operatorname{wt}\psi^a_W)(\psi^a_W)_n + (\psi_W^{L_W(0)_N(a)})_n\right)v\\ &\qquad + (\phi^{i_1}_W)_{n_1}\cdots(\phi^{i_k}_W)_{n_k}(\psi^a_W)_n L_V(0)v\end{aligned}$$

and

$$\begin{aligned} &L_W(-1)(\phi^{i_1}_W)_{n_1}\cdots(\phi^{i_k}_W)_{n_k}(\psi^a_W)_n v\\ &\quad= \sum_{j=1}^k (\phi^{i_1}_W)_{n_1}\cdots(\phi^{i_{j-1}}_W)_{n_{j-1}}(-n_j(\phi^{i_j}_W)_{n_j-1})(\phi^{i_{j+1}}_W)_{n_{j+1}}\cdots(\phi^{i_k}_W)_{n_k}(\psi^a_W)_n v\\ &\qquad + (\phi^{i_1}_W)_{n_1}\cdots(\phi^{i_k}_W)_{n_k}(-n(\psi^a_W)_{n-1})v\\ &\qquad + (\phi^{i_1}_W)_{n_1}\cdots(\phi^{i_k}_W)_{n_k}(\psi^a_W)_n L_V(-1)v\end{aligned}$$

12. *For $c \in \mathbb{C}$, $z \in \mathbb{C}^\times$ satisfying $|z| > |a|$, $i \in I$ and $a \in A$, $e^{cL_W(-1)}\phi^i_W(z)e^{-cL_W(-1)} = \phi^i_W(z+c)$ and $e^{cL_W(-1)}\psi^i_W(z)e^{-cL_V(-1)} = \psi^i_W(z+c)$.*

13. *The operator $L_W(-1)$ has weight 1 and its adjoint $L_W(-1)'$ as an operator on W' has weight -1. In particular, $e^{zL_W(-1)'}w' \in W'$ for $z \in \mathbb{C}$ and $w' \in W'$.*

14. *For $v \in V$, $w' \in W'$ and $\sigma \in S_k$,*

$$R(\langle w', \varphi_1(z_1)\cdots\varphi_k(z_k)v\rangle) = R(\langle w', \varphi_{\sigma(1)}(z_{\sigma(1)})\cdots\varphi_{\sigma(k)}(z_{\sigma(k)})v\rangle),$$

where one of φ_j is ψ^a_W for some $a \in A$ and the others are either in $\{\phi^i\}_{i\in I}$ when they are to the right of ψ^a_W or in $\{\phi^i_W\}_{i\in I}$ when they are to the left of ψ^a_W.

These properties follows easily from Conditions 1–9 in Assumption 11.5 and Proposition 11.6. We also omit the proof of Proposition 11.7.

We define a vertex operator map

$$\begin{aligned} Y_W : V \otimes W &\to W((x)) \\ v \otimes w &\mapsto Y_W(v, x)w \end{aligned}$$

by

$$\begin{aligned} \langle w', &Y_W(\phi^{i_1}_{m_1} \cdots \phi^{i_k}_{m_k} \mathbf{1}, z)w\rangle \\ &= \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{m_1} \cdots \xi_k^{m_k} R\left(\langle w', \phi^{i_1}_W(\xi_1 + z) \cdots \phi^{i_k}_W(\xi_k + z)w\rangle\right) \end{aligned} \tag{11.45}$$

for $i_1, \dots, i_k \in I$, $m_1, \dots, m_k \in \mathbb{Z}$, $w \in W$ and $w' \in W'$. But as in the definition of the vertex operator map for the algebra V given by (4.7), we first have to prove that the vertex operator map Y_W is well defined. In fact, only in the proof of this well-definedness we need the generator fields ψ^a_W for $a \in A$.

Lemma 11.8. *If*

$$\sum_{p=1}^{q} \lambda_p \phi^{i^p_1}_{m^p_1} \cdots \phi^{i^p_k}_{m^p_k} \mathbf{1} = 0,$$

then

$$\sum_{p=1}^{q} \lambda_p \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{m^p_1} \cdots \xi_k^{m^p_k} R\left(\langle w', \phi^{i^p_1}_W(\xi_1 + z) \cdots \phi^{i^p_k}_W(\xi_k + z)w\rangle\right) = 0$$

for $w \in W$ and $w' \in W'$.

Proof. By Condition 4 in Assumption 11.5, we can take w to be of the form

$$(\phi^{j_1}_W)_{n_1} \cdots (\phi^{j_l}_W)_{n_l} (\psi^a_W)_n v$$

for $v \in V$. But by Consdition 4 in Section 4, we can take v to be of the form $\phi^{q_1}_{r_1} \cdots \phi^{q_s}_{r_s}\mathbf{1}$. Moreover, in this case,

$$\begin{aligned} R(\langle v', &\phi^{i^p_1}(z_1) \cdots \phi^{i^p_k}(z_k)(\phi^{j_1}_W)_{n_1} \cdots (\phi^{j_l}_W)_{n_l} (\psi^a_W)_n \phi^{q_1}_{r_1} \cdots \phi^{q_s}_{r_s}\mathbf{1}\rangle) \\ &= \mathrm{Res}_{\zeta_1=0} \cdots \mathrm{Res}_{\zeta_l=0} \mathrm{Res}_{\zeta} \mathrm{Res}_{\eta_1=0} \cdots \mathrm{Res}_{\eta_s=0} \zeta_1^{n_1} \cdots \zeta_l^{n_l} \zeta^n \eta_1^{r_1} \cdots \eta_s^{r_s} \cdot \\ &\qquad \cdot R(\langle v', \phi^{i^p_1}(z_1) \cdots \phi^{i^p_k}(z_k) \psi^a_W(\zeta) \phi^{j_1}(\zeta_1) \cdots \phi^{j_l}(\zeta_l) \phi^{q_1}(\zeta_1) \cdots \phi^{q_s}(\zeta_l)\mathbf{1}\rangle). \end{aligned}$$

Then the other steps of the proof is the same as that of (4.4), except that here we use pProperty 14 in Proposition 11.7. We omit these steps. ∎

Theorem 11.9. *The graded vector space W together with $Y_W : V \otimes W \to W((x))$ defined by (11.45) is a lower-bounded generalized V-module. Moreover, (W, Y_W) is the unique structure of a lower-bounded generalized V-module on W such that $Y_W(\phi^i_{-1}\mathbf{1}, x) = \phi^i(x)$ for $i \in I$.*

Note that since the definition of Y_W does not use ψ^a_W, the proof of Theorem 11.9 also does not need ψ^a_W. Thus the proof of Theorem 11.9 is the same as that of Theorem 4.5, except that ϕ^i are replaced by ϕ^i_W. We omit the proof.

Proof of the module structure in Theorem 11.3. We take $I = \mathfrak{h}$ and $\phi^a_{M(1,\alpha)}(x) = a(x)$ for $a \in \mathfrak{h}$. We also $A = \{\alpha\}$ and $\psi^a_{M(1,\alpha)}(x) = \psi^\alpha(x)$. See the preceding subsection. Then $M(1, \alpha)$, $a(x)$ for $a \in \mathfrak{h}$ and ψ^α satisfy Assumption 11.5. By Theorem 11.9, $M(1, \alpha)$ has a structure of (ordinary) $S(\hat{\mathfrak{h}}_-)$-module. ∎

11.4 Modules for $V(\ell, 0)$

(This subsection was written by Jason Saied, with some minor additions.)

We now wish to construct modules for $V = V(\ell, 0)$. To begin, let M be a finite-dimensional module for $\mathfrak{g}$. Similarly to above, make M into a module for $\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+$ by defining, for $a \in \mathfrak{g}$, $m \in M$, and $n > 0$, $a(0)m := am$ (the action of $a \in \mathfrak{g}$ on $m \in M$), $a(n)m := 0$, and $\mathbf{k}m := \ell m$. We then let $\widetilde{W}$ be the induced $\hat{\mathfrak{g}}$-module

$$\widetilde{W} := U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+)} M.$$

We will often omit the tensor product symbol when writing elements of $\widetilde{W}$: for $m \in M$, $1 \otimes m$ will be written as m.

Recall the Casimir element

$$\Omega = \sum_{i=1}^{\dim \mathfrak{g}} u^i u^i,$$

where $\{u^i : 1 \le i \le \dim \mathfrak{g}\}$ is an orthonormal basis for $\mathfrak{g}$ with respect to the form $(\cdot, \cdot)$. Since M is a $\mathfrak{g}$-module, Ω acts on M. We denote the action of ω on M by Ω_M and this is the Casimir operator on M introduced in Section 7. Since M is finite dimensional, it is a finite direct sum of irreducible $\mathfrak{g}$-modules by Theorem 7.22. On each irreducible $\mathfrak{g}$-submodule, by Proposition 7.19, Ω_M commutes with the action of elements of $\mathfrak{g}$ and thus must be proportional to the identity operator on this submodule. So M is a direct sum of eigenspaces of Ω_M. For an element m in an eigenspace of Ω_M, we define its weight, denoted by $\mathrm{wt}\, m$, to be the eigenvalue of Ω_M associated to the eigenvector m divided by $2(\ell + h^\vee)$. We then define $L(0)$ on $\widetilde{W}$ by defining

$$L_{\widetilde{W}}(0) a_1(n_1) \cdots a_k(n_k) m = -n_1 - \cdots - n_k + \mathrm{wt}\, m.$$

Our generating fields are the maps $a_W(x) : \widetilde{W} \to \widetilde{W}((x))$ for $a \in \mathfrak{g}$, where for $w \in \widetilde{W}$,

$$a_W(x) w := \sum_{n \in \mathbb{Z}} a(n) w x^{-n-1},$$

and $a(n)$ is the multiplication operator.

With the exception of those involving the as-yet-undefined generator fields, the conditions needed to prove that $\widetilde{W}$ is a module can be proven in the same way as they were proven for V. Rather than constructing the generator fields directly, we will construct W in a different (more abstract) way involving several quotients, then prove the two constructions are isomorphic.

Define

$$\widetilde{M} := U(\hat{\mathfrak{g}}) \otimes (M \otimes \mathbb{C}[t, t^{-1}]) \otimes V.$$

This has the structure of a $U(\hat{\mathfrak{g}})$-module by left multiplication. We will often omit the tensor products when the context is clear.

Note that by the PBW Theorem, $\widetilde{M}$ is spanned by elements of the form

$$\mathbf{k}^r a_1(n_1) \cdots a_k(n_k) \otimes (m \otimes t^n) \otimes v$$

with $a_i \in \mathfrak{g}$, $n_i, n \in \mathbb{Z}$, $r \in \mathbb{Z}_+$, $m \in M$, and $v \in V$.

Put a grading on $\widetilde{M}$ by defining, for homogeneous $v \in V$,

$$\begin{aligned} &L_W(0)\mathbf{k}^r a_1(n_1) \cdots a_k(n_k) \otimes (m \otimes t^n) \otimes v \\ &\quad = (-n_1 - \cdots - n_k + \operatorname{wt} m - n - 1 + \operatorname{wt} v)\mathbf{k}^r a_1(n_1) \cdots a_k(n_k) \otimes (m \otimes t^n) \otimes v \end{aligned}$$

and extending linearly. As above, one must check that this grading is well-defined, in that any other expression for an element of $\widetilde{M}$ will be given the same conformal weight. This follows because the only relations are the affine Lie algebra relations in $U(\hat{\mathfrak{g}})$, and it is easy to check that they preserve conformal weight.

Remark 11.10. *Note that we are abusing notation by calling this map $L_W(0)$ when we are not acting on the module W yet. We choose to do so, rather than changing the name of the map every time we take a quotient.*

We also define $L_W(-1)$, motivated by condition 2 of Assumption 11.5, by extending

$$[L_W(-1), a_W(n)] = -n a_W(n-1),$$

and

$$L_W(-1)(m \otimes t^p) - (m \otimes t^p)L_V(-1) = -p(m \otimes t^{p-1}).$$

We will construct a module for V by imposing the following relations on $\widetilde{M}$.

1. For $v \in V, a \in \mathfrak{g}, n \in \mathbb{Z}, m \in M$, and $p \in \mathbb{Z}$,

   $$(a(n)(m \otimes t^p) - (m \otimes t^p)a(n))v = (a(n-1)(m \otimes t^{p+1}) - (m \otimes t^{p+1})a(n-1))v.$$

   Let J_1 be the $U(\hat{\mathfrak{g}})$-submodule of $\widetilde{M}$ generated by these relations.

   We will abbreviate these relations as

   $$[a(n), m \otimes t^p] = [a(n-1), m \otimes t^{p+1}].$$

They are sufficient for Condition 6 of Assumption 11.5. In fact, the relations equivalent to Condition 6 are

$$2(a(n)(m \otimes t^p) - (m \otimes t^p)a(n))v = (a(n-1)(m \otimes t^{p+1}) - (m \otimes t^{p+1})a(n-1))v + (a(n+1)(m \otimes t^{p-1}) - (m \otimes t^{p-1})a(n+1))v$$

or

$$[a(n), m \otimes t^p] = [a(n-1), m \otimes t^{p+1}] + [a(n+1), m \otimes t^{p-1}].$$

2. For $m \in M$ and $p \geq 0$,

$$(m \otimes t^p)\mathbf{1} = 0.$$

Let J_2 be the $U(\hat{\mathfrak{g}})$-submodule of $\widetilde{M}$ generated by these relations. They are necessary for condition 3 of Assumption 11.5.

3. For $a \in \mathfrak{g}$ and $m \in M$,

$$a(0)(m \otimes t^{-1})\mathbf{1} = ((am) \otimes t^{-1})\mathbf{1}.$$

We also impose the relation that for $m \in M$ and $v \in V$,

$$\mathbf{k}(m \otimes t^{-1})v = \ell(m \otimes t^{-1})v.$$

Let J_3 be the $U(\hat{\mathfrak{g}})$-submodule generated by these relations. They are needed to take the module relations of M into account. No relation for $a(n)$, $n > 0$, is needed, because it will be implied by the above.

Define $\widetilde{M_2} = \widetilde{M}/(J_1 + J_2 + J_3)$.

Notice that J_1, J_2, and J_3 are generated by $L_W(0)$-homogeneous elements, so the submodule generated by all of them is $L_W(0)$-graded, implying that in the quotient, $L_W(0)$ is still well-defined. We also have the following result for $L_W(-1)$.

Proposition 11.11. *For $i = 1, 2, 3$, $L_W(-1)J_i \subseteq J_i$. $L_W(-1)$ descends to an operator on $\widetilde{M_2}$.*

Proof. For J_1, this is a tedious but routine calculation.

For J_2, we simply compute that

$$L_W(-1)(m \otimes t^p)\mathbf{1} = -p(m \otimes t^{p-1})\mathbf{1}.$$

The right-hand side is visibly a multiple of a generator of J_2, except for the $p = 0$ case where we simply get 0.

For J_3, we have

$$L_W(-1)a(0)(m \otimes t^p)\mathbf{1} = 0 - pa(0)(m \otimes t^{p-1})\mathbf{1} \in J_3.$$

Then $L_W(-1)$ preserves $J_1 + J_2 + J_3$, so it is a well-defined operator on the quotient $W = \widetilde{M}/(J_1 + J_2 + J_3)$. □

We have the following corollary of our relations.

Corollary 11.12. *In* $\widetilde{M}_2$*,*

$$a(n)(m \otimes t^p)\mathbf{1} = a(0)(m \otimes t^{p+n})\mathbf{1},$$

for $n \geq 0, m \in M, a \in \mathfrak{g}$*, and* $p \in \mathbb{Z}$*.*

Proof. Using the relations from J_1, we have

$$a(n)(m \otimes t^p)\mathbf{1} = a(n-1)(m \otimes t^{p+1})\mathbf{1} + (m \otimes t^p)a(n)\mathbf{1} - (m \otimes t^{p+1})a(n-1)\mathbf{1}.$$

If $n \geq 1$, the latter two terms are 0 due to the relations in V, giving

$$a(n)(m \otimes t^p)\mathbf{1} = a(n-1)(m \otimes t^{p+1})\mathbf{1}.$$

The claim follows by induction. □

Proposition 11.13. $\widetilde{M}_2$ *is spanned by elements of the form*

$$a_1(n_1) \cdots a_k(n_k)(m \otimes t^p)\mathbf{1}$$

for $a_i \in \mathfrak{g}$*,* $n_i \leq 0$*,* $p \leq -1$*,* $m \in M$*. With respect to the* $L_W(0)$*-grading,* $\widetilde{M}_2$ *is lower-bounded and has finite-dimensional graded components.*

Proof. The second claim easily follows from the first.

Recall that $\widetilde{M}$ and therefore $\widetilde{M}_2$ is spanned by elements of the form

$$w = \mathbf{k}^r a_1(n_1) \cdots a_k(n_k)(m \otimes t^n)b_1(s_1) \cdots b_l(s_l)\mathbf{1}$$

with $a_i, b_i \in \mathfrak{g}$, $n_i, s_i, n \in \mathbb{Z}$, $r \in \mathbb{Z}_+$, $m \in M$, and $v \in V$. By the relations of J_3, we may assume $r = 0$.

Rewrite the relations of J_1 as

$$(m \otimes t^p)a(n) = a(n)(m \otimes t^p) + (m \otimes t^{p-1})a(n+1) - a(n+1)(m \otimes t^{p-1}).$$

This allows us to write w as a sum of terms with either a shorter V component or a V component with lower conformal weight. By induction, since an element of V with a sufficiently low conformal weight is equal to zero, w is a sum of terms of the form

$$a_1(n_1) \cdots a_k(n_k)(m \otimes t^p)\mathbf{1}$$

where $a_i \in \mathfrak{g}$, $n_i, p \in \mathbb{Z}$, $r \in \mathbb{Z}_+$, and $m \in M$.

By the PBW Theorem, we need only consider elements of the form

$$a_1(n_1) \cdots a_k(n_k)b_1(0) \cdots b_l(0)c_1(r_1) \cdots c_s(r_s)(m \otimes t^p)\mathbf{1},$$

where $n_1, \cdots, n_k < 0$ and $r_1, \cdots, r_s \geq 0$. Using Corollary 11.12, this is equal to

$$a_1(n_1) \cdots a_k(n_k)b_1(0) \cdots b_l(0)c_1(0) \cdots c_s(0)(m \otimes t^{p+r_1+\cdots+r_s})\mathbf{1}.$$

By the relations of J_2, this is 0 unless

$$p + r_1 + \cdots + r_s \leq -1.$$

□

Theorem 11.14. *$\widetilde{M_2}$ is a module for $V(\ell,0)$ as a grading-restricted vertex algebra, with generating fields*

$$a_W(x) = \sum_{n\in\mathbb{Z}} a(n)x^{-n-1}$$

for $a \in \mathfrak{g}$ and generator fields

$$\psi_W^m(x) = \sum_{n\in\mathbb{Z}} (m\otimes t^n)x^{-n-1}$$

for $m \in M$.

Proof. We must only check the conditions in Assumption 11.5. Conditions 1,2, and 5 follow as in the proof that $V(\ell,0)$ is a grading-restricted vertex algebra. Condition 3 follows from the relations of J_2. Condition 4 follows from the previous proposition.

Finally, recall the relations of J_1: for $a \in \mathfrak{g}, m \in M$, and $p, n \in \mathbb{Z}$,

$$[a(n), m\otimes t^p] = [a(n-1), m\otimes t^{p+1}].$$

It is an easy exercise to see that this is simply the component form of

$$(x_1-x_2)^2 a_W(x_1)\psi_W^m(x_2) = (x_1-x_2)^2\psi_W^m(x_2)a_V(x_1),$$

giving Condition 6. □

There is one problem with the module $\widetilde{M_2}$: it is not compatible with the vertex operator algebra structure on $V(\ell,0)$. We will remedy this issue with another quotient. Let $Y_{\widetilde{M_2}}$ be the vertex operator of $V(\ell,0)$ acting on its module $\widetilde{M_2}$, and write

$$Y_{\widetilde{M_2}}(\omega,x) = \sum_{n\in\mathbb{Z}} L_n x^{-n-1}.$$

Now let J_4 be the $U(\hat{\mathfrak{g}})$-submodule of $\widetilde{M_2}$ generated by the relation

$$L_{-1}(m\otimes t^p)\mathbf{1} = L_W(-1)(m\otimes t^p)\mathbf{1},$$

where $m \in M$ and $p \in \mathbb{Z}$. We can then define

$$W = \widetilde{M_2}/J_4.$$

Proposition 11.15. *$L_W(0)$ and $L_W(-1)$ preserve J_4. Both maps descend to operators on W.*

Proof. First we show that $L_W(0)$ preserves J_4 by showing that the generators of J_4 are graded with respect to $L_W(0)$. We calculate

$$\begin{aligned} L_W(0)L_W(-1)(m\otimes t^p)\mathbf{1} &= -pL_W(0)(m\otimes t^{p-1})\mathbf{1} \\ &= (\mathrm{wt}\, m-(p-1)-1)(-p(m\otimes t^{p-1})\mathbf{1}) \\ &= (\mathrm{wt}\, m-p)L_W(-1)(m\otimes t^p)\mathbf{1}. \end{aligned}$$

Since $\omega \in V_{(2)}$, by the $L_W(0)$ property of $Y_{\widetilde{M_2}}$ we have

$$\begin{aligned}[L_W(0), L_{-1}] &= \mathrm{Res}_x[L_W(0), Y_{\widetilde{M_2}}(\omega, x)] \\ &= \mathrm{Res}_x x\frac{d}{dx}Y_{\widetilde{M_2}}(\omega, x) + 2Y_{\widetilde{M_2}}(\omega, x) \\ &= L_{-1},\end{aligned}$$

so

$$\begin{aligned}L_W(0)L_{-1}(m\otimes t^p)\mathbf{1} &= L_{-1}L_W(0)(m\otimes t^p)\mathbf{1} + L_{-1}(m\otimes t^p)\mathbf{1} \\ &= (\mathrm{wt}\, m - p)L_{-1}(m\otimes t^p)\mathbf{1}.\end{aligned}$$

So

$$L_W(0)(L_{-1} - L_W(-1))(m\otimes t^p)\mathbf{1} = (\mathrm{wt}\, m - p)(L_{-1} - L_W(-1))(m\otimes t^p)\mathbf{1}.$$

The proof that $L_W(-1)$ preserves J_4 is similar, using the $L_W(-1)$ property of $Y_{\widetilde{M_2}}$ instead. □

Theorem 11.16. *W is a module for $V = V(\ell, 0)$ as a vertex operator algebra.*

Proof. The Conditions 1-6 in Assumption 11.5 all follow from the relevant versions for $\widetilde{M_2}$, giving us the structure of a module for $V = V(\ell, 0)$ as a grading-restricted vertex algebra. To claim that W is a module for V as a vertex operator algebra, we must only verify that $L_W(0) = L_0$ and $L_W(-1) = L_{-1}$ as operators on W. Proposition 11.15 gives $L_W(-1) = L_{-1}$. We still need to prove $L_W(0) = L_0$

The same calculations as those from (8.22)–(8.30) gives

$$[a_W(m), \mathrm{Res}_{x_2} x_2 Y_W(\omega, x_2)] = m a_W(m). \tag{11.46}$$

The same calculation as given by (8.10) gives

$$\begin{aligned}&Y_V(u^i(-1)^2\mathbf{1}, x_2) \\ &\quad= \left(\sum_{m\in\mathbb{N}} u^i_W(-m-1)x_2^m\right) u^i_W(x_2) + u^i_W(x_2)\left(\sum_{m\in-\mathbb{Z}_+} u^i_W(-m-1)x_2^m\right).\end{aligned} \tag{11.47}$$

Applying both sides of (11.47) to an element $w \in M$ and using $u^i_W(n)w = 0$ for $n > 0$, we obtain

$$\begin{aligned}&Y_V(u^i(-1)^2\mathbf{1}, x_2)w \\ &\quad= \left(\sum_{m\in\mathbb{N}} u^i_W(-m-1)x_2^m\right) u^i_W(x_2)w + u^i_W(x_2)\left(\sum_{m\in-\mathbb{Z}_+} u^i_W(-m-1)x_2^m\right) w \\ &\quad= \left(\sum_{m\in\mathbb{N}} u^i_W(-m-1)x_2^m\right) \sum_{n\in-\mathbb{N}} u^i_W(n)x_2^{-n-1}w + u^i_W(x_2)u^i_W(0)x_2^{-1}w.\end{aligned} \tag{11.48}$$

Taking the coefficients of X_2^{-2} on both sides of (11.48), we obtain

$$\operatorname{Res}_{x_2} x_2 Y_V(u^i(-1)^2\mathbf{1}, x_2)w = u^i_W(0)u^i_W(0)w = u^i u^i w. \tag{11.49}$$

Summing over $i = 1, \ldots, \dim \mathfrak{g}$ on both sides of (11.49) and then dividing the results by $2(\ell + h^\vee)$, we obtain

$$\operatorname{Res}_{x_2} x_2 Y_W(\omega, x_2)\mathbf{1} = (\operatorname{wt} w)w = L_W(0)w.$$

Thus we proved that

$$[a(m), \operatorname{Res}_{x_2} x_2 Y_W(\omega, x_2)] = [a(m), L_W(0)],$$
$$\operatorname{Res}_{x_2} x_2 Y_W(\omega, x_2)\mathbf{1} = L_W(0)\mathbf{1}.$$

From these formulas, we obtain

$$L_W(0) = \operatorname{Res}_{x_2} x_2 Y_W(\omega, x_2).$$

□

Proposition 11.17. *W is spanned by elements of the form*

$$a_1(n_1) \cdots a_k(n_k)(m \otimes t^{-1})\mathbf{1}$$

for $a_i \in \mathfrak{g}$, $n_i < 0$, and $m \in M$. With respect to the $L_W(0)$-grading, W is lower-bounded and has finite-dimensional graded components.

Proof. The second claim follows immediately from Proposition 11.13 and the definition of W. For the first claim, recall that by Proposition 11.13, W is spanned by elements of the form

$$w = a_1(n_1) \cdots a_k(n_k)(m \otimes t^p)\mathbf{1}$$

for $a_i \in \mathfrak{g}$, $n_i \le 0$, $p \le -1$, and $m \in M$. Recall that

$$L_W(-1)(m \otimes t^p)\mathbf{1} = -p(m \otimes t^{p-1})\mathbf{1}.$$

Then for $p < -1$, $(m \otimes t^p)\mathbf{1}$ is proportional to

$$L_W(-1)^{-p-1}(m \otimes t^{-1})\mathbf{1},$$

so up to a scalar, w has the form

$$a_1(n_1) \cdots a_k(n_k)(L_W(-1))^{-p-1}(m \otimes t^{-1})\mathbf{1}.$$

By the relations of J_4, we may rewrite this as

$$a_1(n_1) \cdots a_k(n_k)(L_{-1})^{-p-1}(m \otimes t^{-1})\mathbf{1}.$$

Recall the construction of the vertex operator Y_W in Theorem 11.9. Since ω is expressed using only operators of the form $a(n)$ ($a \in \mathfrak{g}, n \in \mathbb{Z}$) acting on $\mathbf{1}$, the formula for Y_W indicates that the components of $Y_W(\omega, x)$ should be expressible only in terms of linear combinations of products of operators of the form $a(n)$ ($a \in \mathfrak{g}, n \in \mathbb{Z}$). (Explicitly, the formula for the components of $Y_W(\omega, x)$ in terms of the operators $a(n)$ is given in formula (6.2.44) of [LL].) This will allow us to rewrite w as a linear combination of terms in the desired form. We will sketch this procedure now.

First, use the affine Lie algebra relations to move all operators of the form $a(n)$, $n > 0$, to the right of other operators of the form $a(s)$. Then as in Proposition 11.13, use Corollary 11.12 to get rid of the operators $a(n)$ with $n > 0$, possibly increasing the power of t in the terms. Since p started as -1, the powers of t arising here will either be nonnegative, making the whole term 0 due to the relations of J_2, or -1. Then the nonzero part of the expression is a linear combination of terms of the form

$$a_1(n_1) \cdots a_k(n_k)(m \otimes t^{-1})\mathbf{1}$$

for $a_i \in \mathfrak{g}$, $n_i \leq 0$, and $m \in M$. Again using the affine Lie algebra relations, move the operators of the form $a(0)$ to the right of the other operators of the form $a(n)$, and apply the relations of J_3 to get rid of the operators $a(0)$ (possibly changing m to another element of M in the process). This leaves us only with terms of the desired form. □

We typically use the notation

$$a_1(n_1) \cdots a_k(n_k)m$$

to represent the element

$$a_1(n_1) \cdots a_k(n_k)(m \otimes t^{-1})\mathbf{1} \in W.$$

This is how we reconcile the given construction of the module W with the object $\widetilde{W}$ defined earlier.

12 Intertwining operators

Intertwining operators among modules are the main objects to study in conformal field theories. As for any quantum field theory, a chiral conformal field theories are determined by its chiral correlation functions. Intertwining operators correspond to three-point correlation functions on genus-zero Riemann surfaces (conformally equivalent to the Riemann sphere $\mathbb{C} \cup \{0\}$ by the uniformization theorem). Axioms for conformal field theories require that n-point correlation functions on Riemann surfaces of arbitrary genera be obtained from three point correlation functions together with the vacuum and the Virasoro operators. Thus the construction and study of chiral conformal field theories are the same as the study of intertwining operators together with the Virasoro operators and the vacuum.

The vacuum is in a vertex operator algebra. The Virasoro operators act on modules for the vertex operator algebra. But intertwining operators involve three modules, corresponding to three points on the Riemann sphere $\mathbb{C} \cup \{0\}$. In this section we introduce intertwining operators and give their basic properties. The deep results are the convergence, associativity, modular invariance of intertwining operators and their consequences including the Verlinde formula and modular tensor category structures. We will not be able to prove them this semester. But we shall discuss some of them briefly below. The precise formulations and proofs will be given in next semester's continuation of this course. I will also continue to write these lecture notes.

12.1 Definition

Intertwining operators were introduced in mathematics in [FHL]. Here we use a modified version of the definition given in [Hua9] by adding logarithmic terms since our modules are in general generalized modules. We define the notion of intertwining operators for grading-restricted vertex algebras. In the case that V is a vertex operator algebra, the definition is completely the same.

Definition 12.1. Let V be a grading-restricted vertex algebra and W_1, W_2, W_3 lower-bounded generalized V-modules (grading-restricted generalized V-modules and ordinary V-modules are special cases). An *intertwining operator of type* $\binom{W_3}{W_1 W_2}$ is a linear map

$$\begin{aligned} \mathcal{Y} : W_1 \otimes W_2 &\to W_3\{x\}[\log x] \\ w_1 \otimes w_2 &\mapsto \mathcal{Y}(w_1, x)w_2 \end{aligned}$$

(where $W_3\{x\}[\log x]$ is the space of formal series of the form $\sum_{k=0}^{K} \sum_{n \in \mathbb{C}} a_{n,k} x^n (\log x)^k$ for $a_{n,k} \in W_3$ and x and $\log x$ is formal variables such that $\frac{d}{dx} \log x = x^{-1}$) satisfying the following axioms:

1. *$L(0)$-bracket formula*: For $w_1 \in W_1$,

$$L_{W_3}(0)\mathcal{Y}(w_1, x) - \mathcal{Y}(w_1, x)L_{W_2}(0) = \frac{d}{dx}\mathcal{Y}(w_1, x) + Y_W(L_{W_1}(0)w_1, x).$$

2. *$L(-1)$-derivative property*: For $w_1 \in W_1$,

$$\frac{d}{dx}\mathcal{Y}(w_1, x) = Y_W(L_{W_1}(-1)w_1, x) = L_{W_3}(-1)\mathcal{Y}(w_1, x) - \mathcal{Y}(w_1, x)L_{W_2}(-1).$$

3. *Duality with vertex operators*: For $u \in V$, $w_1 \in W_1$, and $w_2 \in W_2$, $w_3' \in W_3'$, for any single-valued branch $l(z_2)$ of the logarithm of z_2 in the region $z_2 \neq 0$, $0 \leq \arg z_2 \leq 2\pi$, the series

$$\langle w_3', Y_{W_3}(u, z_1)\mathcal{Y}(w_1, x_2)w_2\rangle \Big|_{x_2^n = e^{nl(z_2)}, n \in \mathbb{C}}, \tag{12.50}$$

$$\langle w_3', \mathcal{Y}(w_1, x_2)Y_{W_2}(u, z_1)w_2\rangle \Big|_{x_2^n = e^{nl(z_2)}, n \in \mathbb{C}}, \tag{12.51}$$

$$\langle w_3', \mathcal{Y}(Y_{W_1}(u, z_1 - z_2)w_1, x_2)w_2\rangle \Big|_{x_2^n = e^{nl(z_2)}, n \in \mathbb{C}} \tag{12.52}$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common analytic function in z_1 and z_2 and can be analytically extended to a multivalued analytic functions with the only possible poles $z_1 = 0$ and $z_1 = z_2$ and the only possible branch point $z_2 = 0$.

The set of all intertwining operators of type $\binom{W_3}{W_1 W_2}$ clearly form a vector space. The dimension of this space is called the *fusion rule of type* $\binom{W_3}{W_1 W_2}$ and is denoted by $N_{W_1 W_2}^{W_3}$.

12.2 Examples for Heisenberg vertex operator algebra

We have constructed $S(\hat{\mathfrak{h}}_-)$-modules $M(1, \alpha)$ for $\alpha \in \mathfrak{h}$ in Subsection 11.2 and 11.3. Let L be a positive definite even lattice in $\mathfrak{h}$ and $\alpha_1, \alpha_2 \in L$. Then we have three $S(\hat{\mathfrak{h}}_-)$-modules $M(1, \alpha_1)$, $M(1, \alpha_2)$ and $M(1, \alpha_1 + \alpha_2)$. Recall from Sections 2 and Section 4 that we have a lattice vertex operator algebra V_L with the vertex operator Y_{V_L}. Note that $M(1, \alpha_1) \subset V_L$ and $M(1, \alpha_2) \subset V_L$. We define

$$\begin{aligned} \mathcal{Y} : M(1, \alpha_1) \otimes M(1, \alpha_2) &\to M(1, \alpha_1 + \alpha_2)\{x\}, \\ w_1 \otimes w_2 &\mapsto \mathcal{Y}(w_1, x)w_2 \end{aligned}$$

by $\mathcal{Y}(w_1, x)w_2 = Y_{V_L}(w_1, x)w_2$. Then the map $\mathcal{Y}$ is an intertwining operator of type $\binom{M(1, \alpha_1+\alpha_2)}{M(1,\alpha_1) M(1,\alpha_2)}$. In fact, from the definition of the vertex operator Y_{V_L}, we know that $\mathcal{Y}(w_1, x)w_2$ is in $M(1, \alpha_1 + \alpha_2)((x))$. The axioms of intertwining operators are satisfied because Y_{V_L} satisfies these axioms.

These intertwining operators are not typical enough because $\mathcal{Y}(w_1, x)w_2$ is in fact a Laurent series, not a series with nonintegral powers of the variable. To give examples with nonintegral powers, we have to consider $S(\hat{\mathfrak{h}}_-)$-modules which are graded by nonintegers. We can choose $\alpha_1, \alpha_2 \in \mathfrak{h}$ such that $\frac{1}{2}(\alpha_1, \alpha_1)$ and $\frac{1}{2}(\alpha_2, \alpha_2)$ are not integers. In this case, the intertwining operators can be constructed in the same way. But we omit the construction and discussion here.

12.3 Basic properites

From the definition of intertwining operator, we can derive some of their basic properties in the same way as those of vertex operator lagebras and modules. But we should note that there is one crucial difference between intertwining operators and vertex operators for an algebra or a module: Intertwining operators are in general not Laurent series and thus give multivalued analytic functions, not rational functions. This makes the study of intertwining operators much more difficult than vertex operators for an algebra or a module. On the other hand, when we discuss only one intertwining operators, as in the definition of intertwining operator above, the multivalued analytic functions we have to work with is not too far away from rational functions. So in this subsection, we discuss only those properties involving one intertwining operator. The major results in conformal field theory and in the representation theory of vertex operator algebras are about properties of two or more intertwining operators.

In the next two subsections, we will discuss the statements of two major properties. The proofs of these major properties will be discussed in next semester's course, which will be a continuation of this course.

Let $\mathcal{Y}$ be an intertwining operator of type $\binom{W_3}{W_1W_2}$. Then we have the following properties:

Correlation function of one intertwining operator and one vertex operator for a module Let

$$F(\langle w_3', Y_{W_3}(u, z_1)\mathcal{Y}(w_1, x_2)w_2\rangle),$$
$$F(\langle w_3', \mathcal{Y}(w_1, x_2)Y_{W_2}(u, z_1)w_2\rangle),$$
$$F(\langle w_3', \mathcal{Y}(Y_{W_1}(u, z_1 - z_2)w_1, x_2)w_2\rangle)$$

be the multivalued function obtained by analytically extending the sums of the series (12.50), (12.51) and (12.52). Then they are of the form

$$\sum_{k=0}^{K}\sum_{i=1}^{N}\frac{g_{i,k}(z_1, z_2)}{z_1^{m_{i,k}}(z_1 - z_2)^{n_{i,k}}z_2^{p_{i,k}}}z_2^{r_{i,k}}(\log z_2)^k \tag{12.53}$$

for polynomials $g_{i,k}(z_1, z_2)$ of z_1 and z_2, $m_{i,k}, n_{i,k}, p_{i,k} \in \mathbb{N}$, $r_i \in \mathbb{C}$ satisfying $0 \leq \Re(r_i) < 1$ for $i = 1, \dots, N$. In the case that w_1, w_2 and w_3' are homogeneous, N can be taken to be 1 and $r_{1,k}$ can be taken to be $-\mathrm{wt}\, w_3' + \mathrm{wt}\, w_1 + \mathrm{wt}\, w_2$.

Operator product expeansion For $u \in V$ and $w_1 \in W_1$, there exists $N \in \mathbb{N}$ such that $Y_{W_1}(u, x)w = \sum_{n\leq N}(Y_W)_n(u) = wx^{-n-1}$. Then

$$\begin{aligned} Y_{W_3}(u, z_1)\mathcal{Y}(w_1, z_2) &= \sum_{n\leq N}\mathcal{Y}((Y_{W_1})_n(u)w_1, z_2)(z_1 - z_2)^{-n-1} \\ &\sim \sum_{n=0}^{N}\mathcal{Y}((Y_{W_1})_n(u)w_1, z_2)(z_1 - z_2)^{-n-1}. \end{aligned}$$

The Jacobi identity For $u \in V$ and $w_1 \in W_1$,

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y_{W_3}(u, x_1)\mathcal{Y}(w_1, x_2) - x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right)\mathcal{Y}(w_1, x_2)Y_{W_2}(u, x_1) \\ = x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right)\mathcal{Y}(Y_{W_1}(u, x_0)w_1, x_2). \end{aligned} \tag{12.54}$$

Commutator formula For $u_1, u_2 \in V$,

$$\begin{aligned} Y_{W_3}(u, x_1)\mathcal{Y}(w_1, x_2) - \mathcal{Y}(w_1, x_2)Y_{W_2}(u, x_1) \\ = \mathrm{Res}_{x_0}x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right)\mathcal{Y}(Y_{W_1}(u, x_0)w_1, x_2). \end{aligned} \tag{12.55}$$

Associator formula For $u_1, u_2 \in V$,

$$\begin{aligned}\mathcal{Y}(Y_{W_1}(u, x_0)w_1, x_2) - Y_{W_3}(u, x_0 + x_2)\mathcal{Y}(w_1, x_2)\\ = -\mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right)\mathcal{Y}(w_1, x_2)Y_{W_2}(u, x_1).\end{aligned} \tag{12.56}$$

Weak commutativity For $u \in V$ and $w_1 \in W_1$, there exists $N \in \mathbb{N}$ such that

$$(x_1 - x_2)^N Y_{W_1}(u, x_1)\mathcal{Y}(w_1, x_2) = (x_1 - x_2)^N \mathcal{Y}(w_1, x_1)Y_{W_2}(u, x_1). \tag{12.57}$$

Weak associativity For $u \in V$ and $w_2 \in W_2$, there exists $N \in \mathbb{N}$ such that

$$(x_0 + x_2)^N \mathcal{Y}(Y_{W_1}(u, x_0)w_1, x_2)w_2 = (x_0 + x_2)^N Y_{W_3}(u, x_0 + x_2)\mathcal{Y}(w_1, x_2)w_2 \tag{12.58}$$

for $w_1 \in W_1$.

The skew-symmetry isomorphism Given an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$ and $p \in \mathbb{Z}$, define

$$\begin{aligned}\Omega_p(\mathcal{Y}) : W_2 \otimes W_1 &\to W_3\{x\}[\log x]\\ w_2 \otimes w_1 &\mapsto \Omega_n(\mathcal{Y})(w_2, x)w_1\end{aligned}$$

by

$$\Omega_n(\mathcal{Y})(w_2, x)w_1 = e^{xL_{W_3}(-1)}\mathcal{Y}(w_1, y)w_2|_{y^n = e^{n(2p+1)\pi 1}x^n, \log y = \log x + (2p+1)\pi i}$$

for $w_1 \in W_1$ and $w_2 \in W_2$. Then $\Omega_p(\mathcal{Y})$ is an intertwining operator of type $\binom{W_3}{W_2 W_1}$. Moreover, for $p \in \mathbb{Z}$, Ω_p is a linear isomorphism from the space of intertwining operators of type $\binom{W_3}{W_1 W_2}$ to the space of intertwining operators of type $\binom{W_3}{W_2 W_1}$.

The proofs of these properties are analogous to the proofs of the corresponding properties for grading-restricted vertex algebras and are omitted here.

12.4 Tensor products of modules

We have mentioned above that for two V-modules W_1 and W_2, $W_1 \otimes W_2$ is not a V-module. But tensor products for V-modules are important. They describe interations of the quantum objects whose state spaces are W_1 and W_2. Mathematically tensor products also give us new V-modules. Using intertwining operators, we can introduce a notion of tensor product of two V-modules. Such a tensor product does not always exist. In order to prove the existence, V must satisfies certain conditions. In this subsection we give the definition of tensor product V-module of two V-modules. But we will not discuss the existence of the tensor product V-modules.

Our definition of tensor product V-module is given in terms of intertwining operators. To motivate our definition of tensor product V-module, we first give a definition of tensor product of two vector spaces using analogues of intertwining operators. Let W_1, W_2 and W_3 be vector spaces. A bilinear map $I : W_1 \times W_2 \to W_3$ is called an intertwining operator of type $\binom{W_3}{W_1 W_2}$. We call a pair $W_3, I)$ consisting of a vector space W_3 and an intertwining operator I of type $\binom{W_3}{W_1 W_2}$ a product of W_1 and W_2. We define a tensor product vector space of W_1 and W_2 to be a product $(W_1 \otimes W_2, \otimes)$ such that the following universal property holds: Given any product (W_3, I) of W_1 and W_2, there exists a unique linear map $f : W_1 \otimes W_2 \to W_3$ such that $I = f \circ \otimes$.

Exercise 12.2. Let $\mathbb{C}(W_1 \times W_2)$ be the free vector space generated by the direct product $W_1 \times W_2$. Let $W_1 \otimes W_2$ be the quotient vector space $\mathbb{C}(W_1 \times W_2)/J$, where J is the subspace of $W_1 \times W_2$ spanned by elemenets of the form $(\lambda w_1, w_2) - (w_1, \lambda w_2)$, $\lambda(w_1, w_2) - (\lambda w_1, w_2)$, $(w_1 + \tilde{w}_1, w_2) - (w_1, w_2) - (\tilde{w}_1, w_2)$ and $(w_1, w_2 + \tilde{w}_2) - (w_1, w_2) - (w_1, \tilde{w}_2)$ for $w_1, \tilde{w}_1 \in W_1$, $w_2, \tilde{w} \in W_2$ and $\lambda \in \mathbb{C}$. We use $w_1 \otimes w_2$ to denote the coset $(w_1, w_2) + J$. Let $\otimes : W_1 \times W_2 \to W_1 \otimes W_2$ be the projection map. Prove that $\otimes$ is an intertwining operator of type $\binom{W_1 \otimes W_2}{W_1 W_2}$ and $(W_1 \otimes W_2, \otimes)$ is a tensor product vector space of W_1 and W_2.

We now give the definition of tensor product V-module of two V-modules. For simplicity, we work with the category of lower bounded generalized V-modules. For other categories of V-modules, the definition is the same except that we replace the words "lower bounded generalized V-module" by the names for the types of V-modules in the other categories.

One crucial new feature for the tensor product V-module is that it involves $z \in \mathbb{C}^\times$.

Definition 12.3. Let $z \in \mathbb{C}^\times$ and W_1 and W_2 lower-bounded generalized V-modules. A *$P(z)$-product of W_1 and W_2* is a pair (W_3, I) consisiting of a lower-bounded generalized V-module W_3 and the value $I = \mathcal{Y}(\cdot, z)\cdot : W_1 \otimes W_2 \to \overline{W}_3$ of an intertwining operator $\mathcal{Y}(\cdot, x)\cdot : W_1 \otimes W_2 \to W_3\{x\}[\log x]$ at z (with the choice of the value $\log z = \log |z| + i \arg z$ where $0 \leq \arg z < 2\pi$). A *$P(z)$-tensor product of W_1 and W_2* is a $P(z)$-product $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$ such that the following universal property holds: Given any $P(z)$-product (W_3, I) of W_1 and W_2, there exists a unique module map $f : W_1 \boxtimes_{P(z)} W_2 \to W_3$ such that $I = \bar{f} \circ \otimes$, where $\bar{f} : \overline{W_1 \boxtimes_{P(z)} W_2} \to \overline{W}_3$ is the unique extenstion of f to $\overline{W_1 \boxtimes_{P(z)} W_2}$ (note that f as a module map must preserve wegihts).

The first question about the $P(z)$-tensor product is its existence. For vector spaces, the existence is easy (see 12.2). But for V-modules, it is not trivial in general. As we mentioned above, in general the $P(z)$-tensor product might not exist. Under certain conditions, the existence of $P(z)$-tensor product was proved in [HL] and [Hua6].

The category of V-mdules form a braided tensor category under certain conditions on V or a modular tensor category under stronger conditions. The two difficult part of the construction is the construction of the associativity isomorphism and the proof of the rigidity. These two difficult parts corresponding to the associativity of intertwining operators and the modular invariance of intertwining operators, respectively. See [Hua1], [Hua3], [Hua4], [Hua5], [Hua6] and [HLZ3].

12.5 The first major property: Associativity

We shall give the statement of, but not prove, the associativity of intertwining operators. When some other conditions are satisfied, this associativity is equivalent to the operator product expansion of intertwining operators. In a work [MS] of Moore and Seiberg, the operator product expansion (called chiral vertex operators by Moore and Seiberg) is one of the two major assumptions used to derive Verlinde formula which was conjectured by E. Verlinde and a set of polynomial equations which led to the notion of modular tensor category.

Before we formulate the associativity, we first have to formulate the convergence of products and iterates of intertwining operators.

Convergence and extension property of products of n intertwining operators Let $W_0, W_1, \dots W_{n+1}$, $\widetilde{W}_1, \dots, \widetilde{W}_{n-1}$ be lower-boundedd generalized V-modules and $\mathcal{Y}_1, \dots, \mathcal{Y}_i$, $\dots, \mathcal{Y}_n$ intertwining operators of types $\binom{W_0}{W_1\, \widetilde{W}_1}, \dots, \binom{\widetilde{W}_{i-1}}{W_i\, \widetilde{W}_i}, \dots, \binom{\widetilde{W}_{n-1}}{W_n\, W_{n+1}}$, respectively. For $w_1 \in W_1, \dots, w_{n+1} \in W_{n+1}$ and $w_0' \in W_0'$, the series

$$\langle w_0', \mathcal{Y}_1(w_1, z_1) \cdots \mathcal{Y}_n(w_n, z_n) w_{n+1}\rangle$$

in complex variables $z_1, \dots z_n$ is absolutely convergent in the region $|z_1| > \cdots > |z_n| > 0$ and its sum can be analytically continued to a multivalued analytic function

$$F(\langle u_1', \mathcal{Y}_1(w_1, z_1) \cdots \mathcal{Y}_n(w_n, z_n) u_{n+1}\rangle)$$

on the region

$$\{(z_1, \dots, z_n) \mid z_i \neq 0, z_i - z_j \neq 0 \text{ for } i \neq j\} \subset \mathbb{C}^n$$

and the only possible singular points $z_i = 0, \infty$ and $z_i = z_j$ are regular singular points. (A regular singular point of a multivalued analytic function is a point on which the function is not defined but in the neighborhood of the point, the function can be expanded as a series in powers of the variables and a polynomial in the logarithms of the variables.)

Using the skew-symmetry isomorphism above, we see that the iterate of two intertwining operators can be written as the product of two intertwining operators. So we have the following result:

Proposition 12.4. *Let W_1, W_2, W_3, W_4, W be lower-bounded generalized V-modules. Assume that the convergence and extension property of products of 2 intertwining operators holds. Then for lower-bounded generalized V-module W and intertwining operators $\mathcal{Y}_3$ and $\mathcal{Y}_4$ of types $\binom{W_4}{W_6\, W_3}$ and $\binom{W_6}{W_1\, W_2}$, respectively, the series*

$$\langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2) w_2, z_2) w_3\rangle$$

is absolutely convergent in the region $|z_2| > |z_1 - z_2| > 0$ for $w_1 \in W_1, w_2 \in W_2$, $w_3 \in W_3$ and $w_4' \in W_4'$ and can be analytically extended to a multivalued analytic function

$$F(\langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2) w_2, z_2) w_3\rangle)$$

on the region

$$\{(z_1, z_2) \mid z_1, z_2 \neq 0, z_1 - z_2 \neq 0\} \subset \mathbb{C}^2$$

with the only possible singular points $z_1 = 0, \infty, z_2 = 0, \infty$ *and* $z_i = z_j$ *are regular singular points.*

The proof of this proposition is easy but we also omit it here.

We are ready to state precisely the associativity or the operator product expansion of intertwining operators.

Associativity of intertwining operators Let W_1, W_2, W_3, W_4, W_5 be lower-bounded generalized V-modules and $\mathcal{Y}_1$ and $\mathcal{Y}_2$ intertwining operators of types $\binom{W_4}{W_1\, W_5}$ and $\binom{W_5}{W_2\, W_3}$, respectively. There exist a lower-bounded generalized V-module W_6 and intertwining operators $\mathcal{Y}_3$ and $\mathcal{Y}_4$ of the types $\binom{W_4}{W_6\, W_3}$ and $\binom{W_6}{W_1\, W_2}$, respectively, such that for $w_1 \in W_1, w_2 \in W_2$, $w_3 \in W_3$ and $w_4' \in W_4'$,

$$F(\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle) = F(\langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3\rangle).$$

Another important property following immediately from the associativity of twisted intertwining operators and the skew-symmetry isomorphism is the commutativity of twisted intertwining operators:

Commutativity of twisted intertwining operators Let W_1, W_2, W_3, W_4, W_5 be lower-bounded generalized V-modules and $\mathcal{Y}_1$ and $\mathcal{Y}_2$ intertwining operators of types $\binom{W_4}{W_1\, W_5}$ and $\binom{W_5}{W_2\, W_3}$, respectively. There exist a lower-bounded generalized V-module W_6 and intertwining operators $\mathcal{Y}_3$ and $\mathcal{Y}_4$ of the types $\binom{W_4}{W_2\, W_6}$ and $\binom{W_6}{W_1\, W_3}$, respectively, such that for $w_1 \in W_1, w_2 \in W_2$, $w_3 \in W_3$ and $w_4' \in W_4'$,

$$F(\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle) = F(\langle w_4', \mathcal{Y}_3(w_2, z_2)\mathcal{Y}_4(w_1, z_1)w_3\rangle).$$

The associativity of intertwining operators was proved when V satisfies certain conditions. See [Hua1], [Hua3], [HLZ1], [HLZ2], [Hua6] and [Hua11].

12.6 The second major property: The modular invariance

To formulate the modular invariace, we first have to introduce geometrically-modified intertwining operators and the convergence of q-traces of products of such operators.

Given an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1\, W_2}$ and $w_1 \in W_1$, we have an operator (actually a series with linear maps from W_2 to W_3 as coefficients) $\mathcal{Y}_1(w_1, z)$. The corresponding geometrically-modified operator is

$$\mathcal{Y}_1(\mathcal{U}(q_z)w_1, q_z),$$

where $q^z = e^{2\pi i z}$, $\mathcal{U}(q_z) = (2\pi i q_z)^{L(0)} e^{-L^+(A)}$ and $A_j \in \mathbb{C}$ for $j \in \mathbb{Z}_+$ are defined by

$$\frac{1}{2\pi i} \log(1 + 2\pi i y) = \left(\exp \left(\sum_{j \in \mathbb{Z}_+} A_j y^{j+1} \frac{\partial}{\partial y} \right) \right) y.$$

See [Hua4] for details.

In general we have to take pseudo-traces instead of just q-traces of products of intertwining operators. For simplicity, Here we discuss only q-traces. In this case, we consider only (ordinary) V-mdoules.

Convergence and extension property of q-traces of products of n geometrically-modified intertwining operators Let W_i and $\tilde{W}_i$ for $i = 1, \dots, n$ be (ordinary) V-modules, and $\mathcal{Y}_i$ for $i = 1, \dots, n$ intertwining operators of types $\binom{\tilde{W}_{i-1}}{W_i \tilde{W}_i}$, respectively, where we use the convention $\tilde{W}_0 = \tilde{W}_n$. For $w_i \in W_i$, $i = 1, \dots, n$,

$$\mathrm{Tr}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(q_{z_1}) w_1, q_{z_1}) \cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n}) w_n, q_{z_n}) q_\tau^{L(0) - \frac{c}{24}}$$

is absolutely convergent in the region $1 > |q_{z_1}| > \ldots > |q_{z_n}| > |q_\tau| > 0$ and can be extended to a multivalued analytic function

$$\overline{F}_{\mathcal{Y}_1, \dots, \mathcal{Y}_n}(w_1, \dots, w_n; z_1, \dots, z_n; \tau).$$

in the region $\Im(\tau) > 0$, $z_i \neq z_j + l + m\tau$ for $i \neq j$, $l, m \in \mathbb{Z}$.

Modular invariance of intertwining operators For (ordinary) V-modules W_i and $w_i \in W_i$ for $i = 1, \dots, n$, let $\mathcal{F}_{w_1, \dots, w_n}$ be the vector space spanned by functions of the form

$$\overline{F}^{\phi}_{\mathcal{Y}_1, \dots, \mathcal{Y}_n}(w_1, \dots, w_n; z_1, \dots, z_n; \tau)$$

for all (ordinary) V-modules $\tilde{W}_i$ for $i = 1, \dots, n+1$, all intertwining operators $\mathcal{Y}_i$ of types $\binom{\tilde{W}_{i-1}}{W_i \tilde{W}_i}$ for $i = 1, \dots, n$, respectively. Then for

$$\left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \in SL(2, \mathbb{Z}),$$

$$\overline{F}_{\mathcal{Y}_1, \dots, \mathcal{Y}_n} \left(\left(\frac{1}{\gamma\tau + \delta} \right)^{L(0)} w_1, \dots, \left(\frac{1}{\gamma\tau + \delta} \right)^{L(0)} w_n; \frac{z_1}{\gamma\tau + \delta}, \dots, \frac{z_n}{\gamma\tau + \delta}; \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right)$$

is in $\mathcal{F}_{w_1, \dots, w_n}$.

The modular invariance of intertwining operators was proved when V satisfies certain conditions. See [Hua4].

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