Vertex operator algebras, fusion rules and modular transformations

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Abstract

We discuss a recent proof by the author of a general version of the Verlinde conjecture in the framework of vertex operator algebras and the application of this result to the construction of modular tensor tensor category structure on the category of modules for a vertex operator algebra.

0 Introduction

One of the most important discoveries by physicists in two-dimensional conformal field theory is the famous relation between the fusion rules and the action of the modular transformation $\tau \mapsto -1/\tau$ on the space of vacuum characters. It states that this action of the modular transformation diagonalizes the matrices formed by the fusion rules. This relation was first conjectured by Verlinde [V] based on a comparison between the fusion algebra of a rational conformal field theory and an algebra arising from the study of the genus-one part of the theory. Assuming the axioms for rational conformal field theories, Moore and Seiberg [MS1] proved this Verlinde conjecture by deriving a fundamental set of polynomial equations. Moore and Seiberg [MS2] also observed that the genus-zero part of these polynomial equations is analogous to braided tensor categories. The theory of complete sets of such polynomial equations was then called "modular tensor category" which was first suggested by I. Frenkel. Later, this notion of modular tensor category was reformulated precisely using the language of tensor categories (see, for example, [T] and [BK] for expositions and references on modular tensor categories).

Given a modular tensor category, we have fusion rules and an action of modular transformations. It is not difficult to show (see, for example, [BK]) that for the fusion rules and the action of the modular transformation $\tau \mapsto -1/\tau$ given in this way, the Verlinde conjecture holds. But this version of the Verlinde conjecture is not the original one because the action of the modular transformation is the one constructed from the modular tensor category, not the one on the space of the vacuum characters of the corresponding conformal field theory. The missing piece is an identification of the two actions of the same modular transformation $\tau \mapsto -1/\tau$, or equivalently, is a mathematical construction of the modular tensor category associated to a rational conformal field theory. Moreover, the starting point of the original work [MS1] and [MS2] of Moore and Seiberg is the axioms for rational conformal field theories, which are actually even more difficult to prove than the Verlinde conjecture. It is therefore desirable to formulate and prove a general and mathematical version of the Verlinde conjecture which should give precise and natural conditions on the vertex operator algebras (chiral algebras) such that the Verlinde conjecture holds for these algebras.

Recently, the author was able to formulate and prove such a general and mathematical version of the Verlinde conjecture. Using this result, the author has also proved the rigidity and nondegeneracy of the semisimple braided tensor category structure constructed by Lepowsky and the author on the category of modules for the vertex operator algebra. In particular, the modular tensor category structure on the category of modules for the vertex operator algebra is mathematically constructed. See [H7] and [H9] for details and see also [H8] for an announcement of the results. In the present paper, we shall discuss this general and mathematical version of the Verlinde conjecture, its proof and its application to the proofs of the rigidity and nondegerneracy of the braided tensor category structure mentioned above.

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1 Vertex operator algebras and fusion rules

Vertex (operator) algebras were introduced in 1986 by Borcherds [B] in connection with representations of affine Lie algebras and the moonshine module for the Monster finite simple group, which was conjectured to exist by Conway and Norton [CN] and constructed using vertex operators by Frenkel, Lepowsky and Meurman [FLM1] [FLM2]. These algebras are essentially equivalent to chiral algebras (see, for example, [MS2]) in physics, which were first studied systematically by Belavin, Polyakov and Zamolodchikov [BPZ] in 1984, though without the name chiral algebra. Here we explain briefly the basic concepts in the theory of vertex operator algebras.

A vertex operator algebra is a graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ equipped with a vertex operator map $Y : V \otimes V \to V((z))$ (the space of formal Laurent series in z with finitely many negative power terms), a vacuum $\mathbf{1} \in V$ and a conformal element $\omega \in V$. These data must satisfy the following axioms which are very natural from the point of view of conformal field theory in physics: One formulation of the main axioms is: For $u_1, u_2, v \in V$, $v' \in V' = \coprod_{n\in\mathbb{Z}} V_{(n)}^*$, the series

$$\langle v', Y(u_1, z_1)Y(u_2, z_2)v\rangle$$
$$\langle v', Y(u_2, z_2)Y(u_1, z_1)v\rangle$$
$$\langle v', Y(Y(u_1, z_1 - z_2)u_2, z_2)v\rangle$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$. Other axioms include:

$$\dim V_{(n)} < \infty$$

for $n \in \mathbb{Z}$,

$$V_{(n)} = 0$$

when n is sufficiently negative (these are called grading-restriction conditions);

$$Y(\mathbf{1}, z) = 1, \quad \lim_{z\to 0} Y(u, z)\mathbf{1} = u;$$

let $L(n) : V \to V$ be defined by $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$, then

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

(c is called the central charge of V),

$$\frac{d}{dz}Y(u, z) = Y(L(-1)u, z)$$

and

$$L(0)u = nu \qquad \text{for } u \in V_{(n)}$$

(n is called the weight of u and is denoted wt u). For $u \in V$, we write $Y(u, z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1}$. Then for homogeneous $u \in V$, the maps $u_n : V \to V$ have weights wt $u - n - 1$, that is, they map $V_{(m)}$ to $V_{(\text{wt } u-n-1+m)}$.

For vertex operator algebras, we have the following important condition which was first introduced by Zhu [Z]:

C_2-cofiniteness condition: Let V be a vertex operator algebra and $C_2(V)$ be the subspace of V spanned by elements of the form $u_{-2}v$ for $u, v \in V$. Then we say that V is *C_2-cofinite* or satisfies the *C_2-cofiniteness condition* if $\dim V/C_2(V) < \infty$.

In other words, the C_2-cofiniteness condition says that the coefficients of the terms in the first power of z in $Y(u, z)v$ for all $u, v \in V$ span the whole space V except for a finite-dimensional subspace. This condition is an easy consequence of the condition that the spaces of genus-one conformal blocks are finite-dimensional. The author showed in [H6] that together with some other conditions, the C_2-cofiniteness condition also implies the finiteness of the dimensions of the spaces of genus-one conformal blocks. In practice, it is much easier to verify this condition than to prove the finiteness of the dimensions of the spaces of genus-one conformal blocks. For vertex operator algebras associated to the Wess-Zumino-Novikov-Witten models, vertex operator algebras associated to minimal models, lattice vertex operator algebras and the moonshine module vertex operator algebra, the C_2-cofiniteness condition was stated to hold in Zhu's paper [Z] and was verified by Dong-Li-Mason [DLM] (see also [AN] for the case of minimal models).

A V-module can be defined simply as a $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{(n)}$ equipped with a vertex operator map $Y_W : V \otimes W \to W[[z, z^{-1}]]$ satisfying all those axioms for V which still make sense. We also need the notion of $\mathbb{N}$-gradable weak V-module. An $\mathbb{N}$-gradable weak V-module is an $\mathbb{N}$-graded vector space $W = \coprod_{n\in\mathbb{N}} W_{[n]}$ and a vertex operator map

$$\begin{aligned} Y : V \otimes W &\to W[[z, z^{-1}]] \\ u \otimes w &\mapsto Y(u, z)w = \sum_{n\in\mathbb{Z}} u_n z^{-n-1} \end{aligned}$$

satisfying all axioms for V-modules except that the condition $L(0)w = nw$ for $w \in W_{(n)}$ is replaced by $u_k w \in W_{[m-k-1+n]}$ for $u \in V_{(m)}$ and $w \in W_{[n]}$.

For V-modules W_1, W_2 and W_3, an intertwining operator of type $\binom{W_3}{W_1 W_2}$ is a linear map $\mathcal{Y} : W_1 \otimes W_2 \to W_3\{z\}$, where $W_3\{z\}$ is the space of all series in complex powers of z with coefficients in W_3, satisfying all those axioms for V which still make sense. For example, for $u \in V$, $w_1 \in W_1$, $w_2 \in W_2$ and $w_3' \in W_3' = \coprod_{n\in\mathbb{C}} (W_3)^*_{(n)}$,

$$\langle w_3', Y_{W_3}(u, z_1)\mathcal{Y}(w_1, z_2)w_2\rangle$$
$$\langle w_3', \mathcal{Y}(w_1, z_2)Y_{W_2}(u, z_1)w_2\rangle$$
$$\langle w_3', \mathcal{Y}(Y_{W_1}(u, z_1 - z_2)w_1, z_2)w_2\rangle$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, to a common (multivalued) analytic function in z_1 and z_2 with the only possible singularities (branch points) at $z_1, z_2 = 0$ and $z_1 = z_2$. Also

$$\frac{d}{dz}\mathcal{Y}(w_1, z) = \mathcal{Y}(L(-1)w_1, z).$$

We denote the space of intertwining operator of type $\binom{W_3}{W_1 W_2}$ by $\mathcal{V}_{W_1 W_2}^{W_3}$. The dimension of $\mathcal{V}_{W_1 W_2}^{W_3}$ is the fusion rule $N_{W_1 W_2}^{W_3}$.

2 Fusing and braiding isomorphisms

In the definitions of vertex operator algebra, module and intertwining operator, we see that the main axioms are all about products and iterates of vertex operators or products and iterates of one vertex operator and one intertwining operator. It is natural to expect that products and iterates of intertwining operators should have similar properties. Indeed, it was proved by the author in [H1] and [H5] that for vertex operator algebras satisfying suitable finiteness and reductivity conditions, intertwining operators satisfy associativity and commutativity properties. These properties give fusing and braiding isomorphisms.

Note that the associativity for intertwining operators is a strong version of the operator product expansion of “chiral vertex operators” (which is equivalent to intertwining operators for vertex operator algebras). In fact, in the important work [MS1] and [MS2] of Moore and Seiberg, in addition to other axioms for rational conformal field theories, the operator product expansion of chiral vertex operators was a basic assumption. In physics, though there were calculations for particular operators in particular examples, operator product expansion was not a mathematical theorem and calculations based on it are by no means simple. The associativity for intertwining operators proved in [H1] and [H5] under suitable conditions is in fact a strong version of operator product expansion for intertwining operators in the following sense: The associativity states that the product of two intertwining operators is equal to the iterate of two other intertwining operators in a suitable region. If we expand the intertwining operator inside the other intertwining operator in the iterate, we immediately obtain the operator product expansion of intertwining operators. On the other hand, in [H1], the author proved associativity for intertwining operators under the assumption that the convergence and extension property and some other conditions hold. Operator product expansion for chiral vertex operators (intertwining operators) assumed in [MS1] and [MS2] actually implies the convergence and extension property. So the operator product expansion together with some other conditions implies associativity for intertwining operators. Since the associativity immediately gives the operator product expansion, but on the other hand, only together with some other conditions the operator product expansion gives the associativity, we see that the associativity is indeed stronger than the operator product expansion.

Let V be a simple vertex operator algebra of central charge c and V' the contragredient module of V as a V-module. In the remaining part of the present paper, we shall always assume that V satisfies the following conditions:

1. $V_{(n)} = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ and V' is isomorphic to V.

2. Every $\mathbb{N}$-gradable weak V-module is completely reducible.

3. V is C_2-cofinite, that is, $\dim V/C_2(V) < \infty$.

Note that finitely generated $\mathbb{N}$-gradable weak V-modules are what naturally appear in the proofs of the theorems on genus-zero and genus-one correlation functions. Thus Condition 2 is natural and necessary because the Verlinde conjecture concerns V-modules, not finitely generated $\mathbb{N}$-gradable weak V-modules. For vertex operator algebras associated to affine Lie algebras (Wess-Zumino-Novikov-Witten models) and vertex operator algebras associated to the Virasoro algebra (minimal models), Condition 2 can be verified easily by reformulating the corresponding complete reducibility results in terms of the representation theory of affine Lie algebras and the Virasoro algebra.

From [DLM], we know that there are only finitely many inequivalent irreducible V-modules. Let $\mathcal{A}$ be the set of equivalence classes of irreducible V-modules. We denote the equivalence class containing V by e. For each $a \in \mathcal{A}$, we choose a representative W^a of a. Note that the contragredient module of an irreducible module is also irreducible (see [FHL]). So we have a map

$$\begin{array}{rcl} {}' : \mathcal{A} & \to & \mathcal{A} \\ a & \mapsto & a'. \end{array}$$

From [AM] and [DLM], we know that irreducible V-modules are in fact graded by rational numbers. Thus for $a \in \mathcal{A}$, there exist $h_a \in \mathbb{Q}$ such that $W^a = \coprod_{n \in h_a + \mathbb{N}} W^a_{(n)}$.

Let $\mathcal{V}_{a_1a_2}^{a_3}$ for $a_1, a_2, a_3 \in \mathcal{A}$ be the space of intertwining operators of type $\binom{W^{a_3}}{W^{a_1}W^{a_2}}$ and $N_{a_1a_2}^{a_3}$ for $a_1, a_2, a_3 \in \mathcal{A}$ the fusion rule, that is, the dimension of the space of intertwining operators of type $\binom{W^{a_3}}{W^{a_1}W^{a_2}}$. For any $\mathcal{Y} \in \mathcal{V}_{a_1a_2}^{a_3}$, we know from [FHL] that for $w_{a_1} \in W^{a_1}$ and $w_{a_2} \in W^{a_2}$

$$\mathcal{Y}(w_{a_1}, x)w_{a_2} \in x^{\Delta(\mathcal{Y})} W^{a_3}[[x, x^{-1}]], \tag{2.1}$$

where

$$\Delta(\mathcal{Y}) = h_{a_3} - h_{a_1} - h_{a_2}.$$

From [GN], [L], [AN], [H6], we also know that the fusion rules $N_{a_1a_2}^{a_3}$ for $a_1, a_2, a_3 \in \mathcal{A}$ are all finite. For $a \in \mathcal{A}$, let $\mathcal{N}(a)$ be the matrix whose entries are $N_{aa_1}^{a_2}$ for $a_1, a_2 \in \mathcal{A}$, that is,

$$\mathcal{N}(a) = (N_{aa_1}^{a_2}).$$

We also need matrix elements of fusing and braiding isomorphisms. In the proof of the Verlinde conjecture, we need to use several bases of one space of intertwining operators. We shall use $p = 1, 2, 3, 4, 5, 6, \dots$ to label different bases. For $p = 1, 2, 3, 4, 5, 6, \dots$ and $a_1, a_2, a_3 \in \mathcal{A}$, let $\{\mathcal{Y}_{a_1a_2;i}^{a_3;(p)} \mid i = 1, \dots, N_{a_1a_2}^{a_3}\}$, be a basis of $\mathcal{V}_{a_1a_2}^{a_3}$. For $a_1, \dots, a_6 \in \mathcal{A}$, $w_{a_1} \in W^{a_1}$, $w_{a_2} \in W^{a_2}$, $w_{a_3} \in W^{a_3}$, and $w'_{a_4} \in (W^{a_4})'$, using the differential equations satisfied by the series

$$\langle w'_{a_4}, \mathcal{Y}_{a_1a_5;i}^{a_4;(1)}(w_{a_1}, x_1)\mathcal{Y}_{a_2a_3;j}^{a_5;(2)}(w_{a_2}, x_2)w_{a_3}\rangle|_{x_1^n = e^{n\log z_1},\, x_2^n = e^{n\log z_2},\, n\in\mathbb{Q}}$$

and

$$\langle w'_{a_4}, \mathcal{Y}^{a_4;(3)}_{a_6a_3;k}(\mathcal{Y}^{a_6;(4)}_{a_1a_2;l}(w_{a_1}, x_0)w_{a_2}, x_2)w_{a_3}\rangle|_{x_0^n=e^{n\log(z_1-z_2)},\ x_2^n=e^{n\log z_2},\ n\in\mathbb{Q}},$$

it was proved in [H5] that these series are convergent in the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively. Note that for any a_1, a_2, a_3, a_4, a_5, $a_6 \in \mathcal{A}$, $\mathcal{Y}^{a_4;(1)}_{a_1a_5;i} \otimes \mathcal{Y}^{a_5;(2)}_{a_2a_3;j}$ and $\mathcal{Y}^{a_4;(3)}_{a_6a_3;l} \otimes \mathcal{Y}^{a_6;(4)}_{a_1a_2;k}$ are bases of $\mathcal{V}^{a_4}_{a_1a_5} \otimes \mathcal{V}^{a_5}_{a_2a_3}$ and $\mathcal{V}^{a_4}_{a_6a_3} \otimes \mathcal{V}^{a_6}_{a_1a_2}$, respectively. The associativity of intertwining operators proved and studied in [H1], [H4] and [H5] says that there exist

$$F(\mathcal{Y}^{a_4;(1)}_{a_1a_5;i} \otimes \mathcal{Y}^{a_5;(2)}_{a_2a_3;j}; \mathcal{Y}^{a_4;(3)}_{a_6a_3;l} \otimes \mathcal{Y}^{a_6;(4)}_{a_1a_2;k}) \in \mathbb{C}$$

for $a_1, \dots, a_6 \in \mathcal{A}$, $i = 1, \dots, N^{a_4}_{a_1a_5}$, $j = 1, \dots, N^{a_5}_{a_2a_3}$, $k = 1, \dots, N^{a_4}_{a_6a_3}$, $l = 1, \dots, N^{a_6}_{a_1a_2}$ such that

$$\begin{aligned}
&\langle w'_{a_4}, \mathcal{Y}^{a_4;(1)}_{a_1a_5;i}(w_{a_1}, x_1)\mathcal{Y}^{a_5;(2)}_{a_2a_3;j}(w_{a_2}, z_2)w_{a_3}\rangle|_{x_1^n=e^{n\log z_1},\ x_2^n=e^{n\log z_2},\ n\in\mathbb{Q}}\\
&\quad= \sum_{a_6\in\mathcal{A}} \sum_{k=1}^{N^{a_4}_{a_6a_3}} \sum_{l=1}^{N^{a_6}_{a_1a_2}} F(\mathcal{Y}^{a_4;(1)}_{a_1a_5;i} \otimes \mathcal{Y}^{a_5;(2)}_{a_2a_3;j}; \mathcal{Y}^{a_4;(3)}_{a_6a_3;l} \otimes \mathcal{Y}^{a_6;(4)}_{a_1a_2;k}) \cdot\\
&\qquad \cdot\langle w'_{a_4}, \mathcal{Y}^{a_4;(3)}_{a_6a_3;k}(\mathcal{Y}^{a_6;(4)}_{a_1a_2;l}(w_{a_1}, z_1 - z_2)w_{a_2}, z_2)w_{a_3}\rangle|_{x_0^n=e^{n\log(z_1-z_2)},\ x_2^n=e^{n\log z_2},\ n\in\mathbb{Q}}
\end{aligned} \tag{2.2}$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$, for $a_1, \dots, a_5 \in \mathcal{A}$, $w_{a_1} \in W^{a_1}$, $w_{a_2} \in W^{a_2}$, $w_{a_3} \in W^{a_3}$, $w'_{a_4} \in (W^{a_4})'$, $i = 1, \dots, N^{a_4}_{a_1a_5}$ and $j = 1, \dots, N^{a_5}_{a_2a_3}$. The numbers

$$F(\mathcal{Y}^{a_4;(1)}_{a_1a_5;i} \otimes \mathcal{Y}^{a_5;(2)}_{a_2a_3;j}; \mathcal{Y}^{a_4;(3)}_{a_6a_3;k} \otimes \mathcal{Y}^{a_6;(4)}_{a_1a_2;l})$$

are the matrix elements of the fusing isomorphism, that is, these numbers together give a matrix which represents a linear isomorphism, called fusing isomorphism, from

$$\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}^{a_4}_{a_1a_5} \otimes \mathcal{V}^{a_5}_{a_2a_3}$$

to

$$\coprod_{a_1,a_2,a_3,a_4,a_6\in\mathcal{A}} \mathcal{V}^{a_4}_{a_6a_3} \otimes \mathcal{V}^{a_6}_{a_1a_2}.$$

By the commutativity of intertwining operators proved and studied in [H2], [H4] and [H5], there exist

$$B^{(r)}(\mathcal{Y}^{a_4;(1)}_{a_1a_5;i} \otimes \mathcal{Y}^{a_5;(2)}_{a_2a_3;j}; \mathcal{Y}^{a_4;(3)}_{a_2a_6;l} \otimes \mathcal{Y}^{a_6;(4)}_{a_1a_3;k}) \in \mathbb{C}$$

for $r\in \mathbb{Z}$, $a_1, \dots, a_6\in \mathcal{A}$, $i=1, \dots, N_{a_1a_5}^{a_4}$, $j=1, \dots, N_{a_2a_3}^{a_5}$, $k=1, \dots, N_{a_2a_6}^{a_4}$, $l=1, \dots, N_{a_1a_3}^{a_6}$, such that the analytic extension of the single-valued analytic function

$$\langle w'_{a_4}, \mathcal{Y}_{a_1a_5;i}^{a_4;(1)}(w_{a_1}, x_1)\mathcal{Y}_{a_2a_3;j}^{a_5;(2)}(w_{a_2}, x_2)w_{a_3}\rangle|_{x_1^n=e^{n\log z_1},\ x_2^n=e^{n\log z_2},\ n\in \mathbb{Q}}$$

on the region $|z_1|>|z_2|>0$, $0\le \arg z_1, \arg z_2<2\pi$ along the path

$$t\mapsto \left(\frac{3}{2}-\frac{e^{(2r+1)\pi it}}{2}, \frac{3}{2}+\frac{e^{(2r+1)\pi it}}{2}\right)$$

to the region $|z_2|>|z_1|>0$, $0\le \arg z_1, \arg z_2<2\pi$ is

$$\sum_{a_6\in \mathcal{A}}\sum_{k=1}^{N_{a_2a_6}^{a_4}}\sum_{l=1}^{N_{a_1a_3}^{a_6}} B^{(r)}(\mathcal{Y}_{a_1a_5;i}^{a_4;(1)}\otimes \mathcal{Y}_{a_2a_3;j}^{a_5;(2)}; \mathcal{Y}_{a_2a_6;k}^{a_4;(3)}\otimes \mathcal{Y}_{a_1a_3;l}^{a_6;(4)})\cdot$$
$$\cdot\langle w'_{a_4}, \mathcal{Y}_{a_2a_6;k}^{a_4;(3)}(w_{a_2}, z_1)\mathcal{Y}_{a_1a_3;l}^{a_6;(4)}(w_{a_1}, z_2)w_{a_3}\rangle|_{x_1^n=e^{n\log z_1},\ x_2^n=e^{n\log z_2},\ n\in \mathbb{Q}}.$$

The numbers

$$B^{(r)}(\mathcal{Y}_{a_1a_5;i}^{a_4;(1)}\otimes \mathcal{Y}_{a_2a_3;j}^{a_5;(2)}; \mathcal{Y}_{a_2a_6;k}^{a_4;(3)}\otimes \mathcal{Y}_{a_1a_3;l}^{a_6;(4)})$$

are the matrix elements of the braiding isomorphism.

We need an action of S_3 on the space

$$\mathcal{V}=\coprod_{a_1,a_2,a_3\in \mathcal{A}}\mathcal{V}_{a_1a_2}^{a_3}.$$

For $a_1, a_2, a_3\in \mathcal{A}$, we have isomorphisms $\Omega_{-r}: \mathcal{V}_{a_1a_2}^{a_3}\to \mathcal{V}_{a_2a_1}^{a_3}$ and $A_{-r}: \mathcal{V}_{a_1a_2}^{a_3}\to \mathcal{V}_{a_1a_3'}^{a_2'}$ for $r\in \mathbb{Z}$ (see [HL2]). For $a_1, a_2, a_3\in \mathcal{A}$, $\mathcal{Y}\in \mathcal{V}_{a_1a_2}^{a_3}$, we define

$$\begin{aligned}\sigma_{12}(\mathcal{Y}) &= e^{\pi i\Delta(\mathcal{Y})}\Omega_{-1}(\mathcal{Y})\\ &= e^{-\pi i\Delta(\mathcal{Y})}\Omega_0(\mathcal{Y}),\\ \sigma_{23}(\mathcal{Y}) &= e^{\pi ih_{a_1}}A_{-1}(\mathcal{Y})\\ &= e^{-\pi ih_{a_1}}A_0(\mathcal{Y}).\end{aligned}$$

Then the actions σ_{12} and σ_{23} of (12) and (23) on $\mathcal{V}$ defined above generate a left action of S_3 on $\mathcal{V}$.

We now choose a basis $\mathcal{Y}_{a_1a_2;i}^{a_3}$, $i=1, \dots, N_{a_1a_2}^{a_3}$, of $\mathcal{V}_{a_1a_2}^{a_3}$ for each triple $a_1, a_2, a_3\in \mathcal{A}$. For $a\in \mathcal{A}$, we choose $\mathcal{Y}_{ea;1}^{a}$ to be the vertex operator Y_{W^a}

defining the module structure on W^a and we choose $\mathcal{Y}_{ae;1}^a$ to be the intertwining operator defined using the action of σ_{12},

$$\begin{aligned}
\mathcal{Y}_{ae;1}^a(w_a, x)u &= \sigma_{12}(\mathcal{Y}_{ea;1}^a)(w_a, x)u \\
&= e^{xL(-1)}\mathcal{Y}_{ea;1}^a(u, -x)w_a \\
&= e^{xL(-1)}Y_{W^a}(u, -x)w_a
\end{aligned}$$

for $u \in V$ and $w_a \in W^a$. Since V' as a V-module is isomorphic to V, we have $e' = e$. From [FHL], we know that there is a nondegenerate invariant bilinear form $(\cdot, \cdot)$ on V such that $(\mathbf{1}, \mathbf{1}) = 1$. We choose $\mathcal{Y}_{aa';1}^e = \mathcal{Y}_{aa';1}^{e'}$ to be the intertwining operator defined using the action of σ_{23} by

$$\mathcal{Y}_{aa';1}^{e'} = \sigma_{23}(\mathcal{Y}_{ae;1}^a),$$

that is,

$$(u, \mathcal{Y}_{aa';1}^{e'}(w_a, x)w_a') = e^{\pi i h_a}\langle \mathcal{Y}_{ae;1}^a(e^{xL(1)}(e^{-\pi i}x^{-2})^{L(0)}w_a, x^{-1})u, w_a'\rangle$$

for $u \in V$, $w_a \in W^a$ and $w_{a'}' \in (W^a)'$. Since the actions of σ_{12} and σ_{23} generate the action of S_3 on $\mathcal{V}$, we have

$$\mathcal{Y}_{a'a;1}^e = \sigma_{12}(\mathcal{Y}_{aa';1}^e)$$

for any $a \in \mathcal{A}$. When $a_1, a_2, a_3 \neq e$, we choose $\mathcal{Y}_{a_1a_2;i}^{a_3}$, $i = 1, \dots, N_{a_1a_2}^{a_3}$, to be an arbitrary basis of $\mathcal{V}_{a_1a_2}^{a_3}$. Note that for each element $\sigma \in S_3$, $\{\sigma(\mathcal{Y})_{a_1a_2;i}^{a_3} \mid i = 1, \dots, N_{a_1a_2}^{a_3}\}$, is also a basis of $\mathcal{V}_{a_1a_2}^{a_3}$.

3 Modular Invariance

We discuss modular invariance briefly in this section. Let $q_\tau = e^{2\pi i\tau}$ for $\tau \in \mathbb{H}$ ($\mathbb{H}$ is the upper-half plane). We consider the q_τ-traces of the vertex operators Y_{W^a} for $a \in \mathcal{A}$ on the irreducible V-modules W^a of the following form:

$$\mathrm{Tr}_{W^a} Y_{W^a}(e^{2\pi izL(0)}u, e^{2\pi iz})q_\tau^{L(0)-\frac{c}{24}} \tag{3.1}$$

for $u \in V$ (recall that c is the central charge of V). In [Z], under some conditions slightly different from (mostly stronger than) those we assume in this paper, Zhu proved that these q-traces are independent of z, are absolutely convergent when $0 < |q_\tau| < 1$ and can be analytically extended to analytic functions of τ in the upper-half plane. We shall denote the analytic extension of (3.1) by

$$E(\mathrm{Tr}_{W^a} Y_{W^a}(e^{2\pi i z L(0)}u, e^{2\pi i z}) q_\tau^{L(0)-\frac{c}{24}}).$$

In [Z], under his conditions mentioned above, Zhu also proved the following modular invariance property: For

$$\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL(2, \mathbb{Z}),$$

let $\tau' = \frac{a\tau+b}{c\tau+d}$. Then there exist unique $A_{a_1}^{a_2} \in \mathbb{C}$ for $a_1, a_2 \in \mathcal{A}$ such that

$$\begin{aligned} & E\left(\mathrm{Tr}_{W^{a_1}} Y_{W^{a_1}}\left(e^{\frac{2\pi i z}{c\tau+d}L(0)}\left(\frac{1}{c\tau+d}\right)^{L(0)} u, e^{\frac{2\pi i z}{c\tau+d}}\right) q_{\tau'}^{L(0)-\frac{c}{24}}\right) \\ & \quad = \sum_{a_2\in\mathcal{A}} A_{a_1}^{a_2} E(\mathrm{Tr}_{W^{a_2}} Y_{W^{a_2}}(e^{2\pi i z L(0)}u, e^{2\pi i z}) q_\tau^{L(0)-\frac{c}{24}}) \end{aligned}$$

for $u \in V$. In [DLM], Dong, Li and Mason, among many other things, improved Zhu's results above by showing that the results of Zhu above also hold for vertex operator algebras satisfying the conditions (slightly weaker than what) we assume in this paper. In particular, for

$$\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right) \in SL(2, \mathbb{Z}),$$

there exist unique $S_{a_1}^{a_2} \in \mathbb{C}$ for $a_1 \in \mathcal{A}$ such that

$$\begin{aligned} & E\left(\mathrm{Tr}_{W^{a_1}} Y_{W^{a_1}}\left(e^{-\frac{2\pi i z}{\tau}L(0)}\left(-\frac{1}{\tau}\right)^{L(0)} u, e^{-\frac{2\pi i z}{\tau}}\right) q_{-\frac{1}{\tau}}^{L(0)-\frac{c}{24}}\right) \\ & \quad = \sum_{a_2\in\mathcal{A}} S_{a_1}^{a_2} E(\mathrm{Tr}_{W^{a_2}} Y_{W^{a_2}}(e^{2\pi i z L(0)}u, e^{2\pi i z}) q_\tau^{L(0)-\frac{c}{24}}) \end{aligned}$$

for $u \in V$. When $u = \mathbf{1}$, we see that the matrix $S = (S_{a_1}^{a_2})$ actually acts on the space of spanned by the vacuum characters $\mathrm{Tr}_{W^a} q_\tau^{L(0)-\frac{c}{24}}$ for $a \in \mathcal{A}$.

4 The Verlinde conjecture and consequences

In [H7], the author proved the following general version of the Verlinde conjecture in the framework of vertex operator algebras (cf. Section 3 in [V] and Section 4 in [MS1]):

Theorem 4.1 *Let V be a vertex operator algebra satisfying the following conditions:*

1. *$V_{(n)} = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ and V' is isomorphic to V as a V-module.*
2. *Every $\mathbb{N}$-gradable weak V-module is completely reducible.*
3. *V is C_2-cofinite, that is, $\dim V/C_2(V) < \infty$.*

Then for $a \in \mathcal{A}$,

$$F(\mathcal{Y}^{a}_{ae;1} \otimes \mathcal{Y}^{e}_{a'a;1}; \mathcal{Y}^{a}_{ea;1} \otimes \mathcal{Y}^{e}_{aa';1}) \neq 0$$

and

$$\sum_{a_1, a_3 \in \mathcal{A}} (S^{-1})^{a_1}_{a_4} N^{a_3}_{a_1 a_2} S^{a_5}_{a_3} = \delta^{a_5}_{a_4} \frac{(B^{(-1)})^2(\mathcal{Y}^{a_4}_{a_4 e;1} \otimes \mathcal{Y}^{e}_{a'_2 a_2;1}; \mathcal{Y}^{a_4}_{a_4 e;1} \otimes \mathcal{Y}^{e}_{a'_2 a_2;1})}{F(\mathcal{Y}^{a_2}_{a_2 e;1} \otimes \mathcal{Y}^{e}_{a'_2 a_2;1}; \mathcal{Y}^{a_2}_{e a_2;1} \otimes \mathcal{Y}^{e}_{a_2 a'_2;1})}. \tag{4.1}$$

In particular, the matrix S diagonalizes the matrices $\mathcal{N}(a_2)$ for all $a_2 \in \mathcal{A}$.

The main work in the proof of this theorem is to prove the following formulas derived by Moore and Seiberg [MS1] from the axioms of rational conformal field theories: For $a_1, a_2, a_3 \in \mathcal{A}$,

$$\begin{aligned}\sum_{i=1}^{N^{a_3}_{a_1 a_2}} \sum_{k=1}^{N^{a_2}_{a'_1 a_3}} & F(\mathcal{Y}^{a_2}_{a_2 e;1} \otimes \mathcal{Y}^{e}_{a'_3 a_3;1}; \mathcal{Y}^{a_2}_{a'_1 a_3;k} \otimes \mathcal{Y}^{a'_1}_{a_2 a'_3;i}) \cdot \\ & \cdot F(\mathcal{Y}^{a_2}_{a'_1 a_3;k} \otimes \sigma_{123}(\mathcal{Y}^{a'_1}_{a_2 a'_3;i}); \mathcal{Y}^{a_2}_{e a_2;1} \otimes \mathcal{Y}^{e}_{a'_1 a_1;1}) \\ = {} & N^{a_3}_{a_1 a_2} F(\mathcal{Y}^{a_2}_{a_2 e;1} \otimes \mathcal{Y}^{e}_{a'_2 a_2;1}; \mathcal{Y}^{a_2}_{e a_2;1} \otimes \mathcal{Y}^{e}_{a_2 a'_2;1})\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\sum_{a_4 \in \mathcal{A}} & S^{a_4}_{a_1} (B^{(-1)})^2(\mathcal{Y}^{a_4}_{a_4 e;1} \otimes \mathcal{Y}^{e}_{a'_2 a_2;1}; \mathcal{Y}^{a_4}_{a_4 e;1} \otimes \mathcal{Y}^{e}_{a'_2 a_2;1})(S^{-1})^{a_3}_{a_4} \\ = {} & \sum_{i=1}^{N^{a_3}_{a_1 a_2}} \sum_{k=1}^{N^{a_2}_{a'_1 a_3}} F(\mathcal{Y}^{a_2}_{a_2 e;1} \otimes \mathcal{Y}^{e}_{a'_3 a_3;1}; \mathcal{Y}^{a_2}_{a'_1 a_3;k} \otimes \mathcal{Y}^{a'_1}_{a_2 a'_3;i}) \cdot \\ & \cdot F(\mathcal{Y}^{a_2}_{a'_1 a_3;k} \otimes \sigma_{123}(\mathcal{Y}^{a'_1}_{a_2 a'_3;i}); \mathcal{Y}^{a_2}_{e a_2;1} \otimes \mathcal{Y}^{e}_{a'_1 a_1;1}).\end{aligned} \tag{4.3}$$

The proof of the first formula (4.2) uses mainly the works of Lepowsky and the author [HL1]–[HL4] and of the author [H1] [H2] [H4] and [H5] on the tensor product theory, intertwining operator algebras and the construction of genus-zero chiral conformal field theories. The main technical result used is the associativity for intertwining operators proved in [H1] and [H5] for vertex operator algebras satisfying the three conditions stated in the theorem.

As is discussed in Section 2, assuming the convergence and extension property and some other conditions, the associativity for intertwining operators was proved in [H1]. In [H3], the commutativity for intertwining operators was proved using the associativity. In Lemma 4.1 in [H6], using the associativity and commutativity proved in [H1] and [H2], respectively, it was shown that one can analytically extend products or iterates of two intertwining operators to

$$M^2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1, z_2 \neq 0, z_1 \neq z_2\}.$$

Moreover, the construction of the map $(\Psi^{(1)}_{P(z_1,z_2)})^{-1}$ in (14.65) in [H1] can be reinterpreted as a proof of the fact that any four point correlation function can be obtained in this way. For multipoint correlation functions, the corresponding results can be proved using the same method or using the results for four point correlation functions above. This result shows that assuming the convergence and extension property and some other conditions, products or iterates of intertwining operators can be analytically extended to multivalued analytic functions defined on the moduli spaces of pointed genus-zero Riemann surfaces.

The convergence and extension property was proved in [H1] under conditions weaker than the conditions we assume above. The proof is based on the existence of systems of differential equations and the regularity of the singular points of these systems. Since we know only the existence, not the explicit forms, of these systems of differential equations, many powerful tools available for Knizhnik-Zamolodchikov equations in the Wess-Zumino-Novikov-Witten models are not available here anymore. However, the theory of intertwining operator algebras developed in [H3], [H4] and [H5] allows us to reduce the use of these differential equations to a minimum. For example, we do need the regularity of the singular points of these systems. But only the regularity of some ordinary differential equations induced from these systems are needed and such regularity can be proved easily.

Now using the associativity for intertwining operators repeatedly to express the correlation functions obtained from products of three suitable intertwining operators as linear combinations of the correlation functions obtained from iterates of three intertwining operators in two ways, we obtain a formula for the matrix elements of the fusing isomorphisms. Then using some properties of the matrix elements of the fusing isomorphisms and their inverses proved in [H7], we obtain the first formula (4.2).

The proof of the second formula (4.3) mainly uses the results obtained in [H6] on the convergence and analytic extensions of the q_τ-traces of products of what we call "geometrically-modified intertwining operators", the genus-one associativity, and the modular invariance of the space of these analytic extensions of the q_τ-traces, where, as in Section 3, $q_\tau = e^{2\pi\tau}$. In [Z], in addition to the convergence and modular invariance of q_τ-traces of vertex operators, Zhu also proved the convergence and modular invariance of the space of the q_τ-traces of products of more than one vertex operators acting on modules. In [DLM], Dong, Li and Mason generalized Zhu's result to the case of twisted modules. In [M], Miyamoto generalized Zhu's result to the case of products of one intertwining operator and arbitrarily many vertex operators. However, these results of Zhu, Dong-Li-Mason and Miyamoto do not give all genus-one correlation functions and thus are not enough for the construction conformal field theories. More specifically, the problem of proving the convergence, the modular invariance and duality properties for q_τ-traces of products of *more than one* intertwining operator was still open at that time. This problem is equivalent to the problem of constructing all the chiral genus-zero correlation functions and establishing all the desired properties.

The difficulty in the case of more than one intertwining operator is that the method of Zhu, further developed by Dong-Li-Mason and Miyamoto, cannot be adapted or generalized to this case, because there is no commutator formula for general intertwining operators. Here by commutator formulas for general intertwining operators, we mean a formula for

$$\mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(w_2, x_2) - \mathcal{Y}_3(w_2, x_2)\mathcal{Y}_4(w_1, x_1)$$

where $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Y}_3$ and $\mathcal{Y}_4$ are suitable intertwining operators. There is no such formula, even in the case of abelian intertwining operator algebras in the sense of Dong and Lepowsky [DL]. Without such commutator formula, one cannot prove a recurrence formula needed in the method of Zhu, Dong-Li-Mason and Miyamoto. Even the generalized commutator formula of Dong and Lepowsky for abelian intertwining operator algebras cannot be used to obtained such recurrence relations.

In [H6], the author solved these previously open problems. These problems actually form one of the main obstructions to a new range of applications of the theory of vertex operator algebras. It is the solution to these problems together with the results on the duality properties of genus-zero correlation functions discussed in the preceding section that allowed us to prove (4.3). The method in [H6] is completely different from the one in [Z], [DLM] and [M] in the case of products of more than one operators: Instead of the nonexistent commutator formula, the author used the associativity and commutativity for intertwining operators and the method of analytic extensions. The lack of a commutator formula was one of the subtle difficulties which was overcome in [H6].

To prove the second formula (4.3), using the results proved in [H6], we introduce two operators α and β on the space of linear maps from $\coprod_{a\in\mathcal{A}} W^a\otimes (W^a)'$ to the space of genus-one two-point correlation functions obtained from the analytic extensions of the q_τ-traces of iterates of suitable "geometrically-modified intertwining operators" introduced in [H6]: α is induced from the translation of one of the points by -1 and β is induced from the translation of one of the points by τ. The work in [H6] is crucial in proving that these operators are well-defined and have the desired properties. The matrix S also acts on the same space of linear maps and it is easy to prove the relation

$$S\alpha S^{-1}=\beta.$$

We then use the genus-one associativity and other properties of the genus-one correlation functions in [H6] to calculate α and β explicitly in terms of the matrix elements of the fusing and braiding isomorphisms. Substituting the explicit expressions of α and β into this relation, we obtain (4.3).

As in [MS1], the conclusions of the theorem follow immediately from (4.2) and (4.3).

All the consequences of the Verlinde conjecture derived by physicists now hold for vertex operator algebras satisfying the conditions in the theorem. In particular, we have the following Verlinde formula for fusion rules: For $a\in\mathcal{A}$, $S_e^a\neq 0$ and

$$N_{a_1a_2}^{a_3}=\sum_{a_4\in\mathcal{A}}\frac{S_{a_1}^{a_4}S_{a_2}^{a_4}S_{a_4}^{a_3'}}{S_e^{a_4}} \tag{4.4}$$

(cf. Section 3 in [V]). We also have the following well-known formula: For $a_1, a_2 \in \mathcal{A}$,

$$S_{a_1}^{a_2} = \frac{S_e^e((B^{(-1)})^2(\mathcal{Y}_{a_2e;1}^{a_2} \otimes \mathcal{Y}_{a_1'a_1;1}^{e}; \mathcal{Y}_{a_2e;1}^{a_2} \otimes \mathcal{Y}_{a_1'a_1;1}^{e}))}{F(\mathcal{Y}_{a_1e;1}^{a_1} \otimes \mathcal{Y}_{a_1'a_1;1}^{e}; \mathcal{Y}_{ea_1;1}^{a_1} \otimes \mathcal{Y}_{a_1a_1';1}^{e})F(\mathcal{Y}_{a_2e;1}^{a_2} \otimes \mathcal{Y}_{a_2'a_2;1}^{e}; \mathcal{Y}_{ea_2;1}^{a_2} \otimes \mathcal{Y}_{a_2a_2';1}^{e})}. \tag{4.5}$$

The formula (4.5) and certain properties of the matrix elements of the fusing and braiding isomorphisms proved in [H7] imply that the matrix $(S_{a_1}^{a_2})$ is symmetric.

5 Rigidity, nondegeneracy and modular tensor categories

A tensor category with tensor product bifunctor $\boxtimes$ and unit object V is rigid if for every object W in the category, there are right and left dual objects W^* and $^*W$ together with morphisms $e_W : W^* \boxtimes W \to V$, $i_W : V \to W \boxtimes W^*$, $e'_W : W \boxtimes^* W \to V$ and $i'_W : V \to^* W \boxtimes W$ such that the compositions of the morphisms in the sequence

$$\begin{array}{ccccc} W \longrightarrow & V \boxtimes W & \xrightarrow{i_W \boxtimes I_W} & (W \boxtimes W^*) \boxtimes W & \longrightarrow \\ \longrightarrow & W \boxtimes (W^* \boxtimes W) & \xrightarrow{I_W \boxtimes e_W} & W \boxtimes V & \longrightarrow W \end{array}$$

and three similar sequences are equal to the identity I_W on W. The rigidity is a standard notion in the theory of tensor categories. A rigid braided tensor category together with a twist (a natural isomorphism from the category to itself satisfying natural conditions) is called a ribbon category. A semisimple ribbon category with finitely many inequivalent irreducible objects is a modular tensor category if the following nondegeneracy condition or modularity is satisfied: The $m \times m$ matrix formed by the traces of the morphism $c_{W_iW_j} \circ c_{W_jW_i}$ in the ribbon category for irreducible modules $W_1, \dots, W_m$ is invertible. See [T] and [BK] for details of the notions in the theory of modular tensor categories.

Using the results discussed in the proceeding section, we obtain the following result:

Theorem 5.1 *Let V be a simple vertex operator algebra satisfying the following conditions:*

1. *$V_{(n)} = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$, $W_{(0)} = 0$ for any irreducible V-module which is not isomorphic to V.*

2. *Every $\mathbb{N}$-gradable weak V-module is completely reducible.*

3. *V is C_2-cofinite, that is, $\dim V/C_2(V) < \infty$.*

Then the braided tensor category structure on the category of V-modules constructed in [HL1]–[HL4], [H1] and [H5] is rigid, has a natural structure of ribbon category and satisfies the nondegeneracy condition. In particular, the category of V-modules has a natural structure of modular tensor category.

Note that Condition 1 implies that V' is isomorphic to V as a V-module. Thus Condition 1 in the theorem is slightly stronger than Condition 1 in Theorem 4.1.

We now discuss the proof of this theorem. We take both the left and right dual of a V-module W to be the contragredient module W' of W. Since our tensor category is semisimple, to prove the rigidity, we need only discuss irreducible modules. For $a \in \mathcal{A}$, using the universal property for the tensor product module $(W^a)' \boxtimes W^a$, we know that there exists a unique module map $\hat{e}_a : (W^a)' \boxtimes W^a \to V$ such that

$$\overline{\hat{e}}_a(w'_a \boxtimes w_a) = \mathcal{Y}^e_{a'a;1}(w'_a, 1)w_a$$

for $w_a \in W^a$ and $w'_a \in (W^a)'$, where $\overline{\hat{e}}_a : \overline{(W^a)' \boxtimes W^a} \to \overline{V}$ is the natural extension of $\hat{e}_a$. Similarly, we have a module map from $W^a \boxtimes (W^a)'$ to V. Since $W^a \boxtimes (W^a)'$ is completely reducible and the fusion rule $N^V_{W^a (W^a)'}$ is 1, there is a V-submodule of $W^a \boxtimes (W^a)'$ which is isomorphic to V under the module map from $W^a \boxtimes (W^a)'$ to V. Thus we obtain a module map $i_a : V \to W^a \boxtimes (W^a)'$ which maps V bijectively to this submodule of $W^a \boxtimes (W^a)'$. Now

$$\begin{aligned} W^a \longrightarrow V \boxtimes W^a \xrightarrow{i_a \boxtimes I_{W^a}} (W^a \boxtimes (W^a)') \boxtimes W^a \longrightarrow \\ \longrightarrow W^a \boxtimes ((W^a)' \boxtimes W^a) \xrightarrow{I_{W^a} \boxtimes \hat{e}_a} W^a \boxtimes V \longrightarrow W_a \end{aligned} \tag{5.1}$$

is a module map from an irreducible module to itself. So it must be the identity map multiplied by a number. One can calculate this number explicitly and it is equal to

$$F(\mathcal{Y}^a_{ae;1} \otimes \mathcal{Y}^e_{a'a;1}; \mathcal{Y}^a_{ea;1} \otimes \mathcal{Y}^e_{aa';1}).$$

From Theorem 4.1, this number is not 0. Let

$$e_a = \frac{1}{F(\mathcal{Y}^a_{ae;1} \otimes \mathcal{Y}^e_{a'a;1}; \mathcal{Y}^a_{ea;1} \otimes \mathcal{Y}^e_{aa';1})} \hat{e}_a.$$

Then the map obtained from (5.1) by replacing $\hat{e}_a$ by e_a is the identity. Similarly, we can prove that all the other maps in the definition of rigidity are also equal to the identity. So the tensor category is rigid.

For any $a \in \mathcal{A}$, we define the twist on W^a to $e^{2\pi i h_a}$. Then it is easy to verify that the rigid braided tensor category with this twist is a ribbon category.

To prove the nondegeneracy, we use the formula (4.5). Now it is easy to calculate in the tensor category the trace of $c_{W^{a_2}, W^{a_1}} \circ c_{W^{a_1}, W^{a_2}}$ for $a_1, a_2 \in \mathcal{A}$, where $c_{W^{a_1}, W^{a_2}} : W^{a_1} \boxtimes W^{a_2} \to W^{a_2} \boxtimes W^{a_1}$ is the braiding isomorphism. The result is

$$\frac{((B^{(-1)})^2(\mathcal{Y}^{a_2}_{a_2e;1} \otimes \mathcal{Y}^e_{a_1'a_1;1}; \mathcal{Y}^{a_2}_{a_2e;1} \otimes \mathcal{Y}^e_{a_1'a_1;1}))}{F(\mathcal{Y}^{a_1}_{a_1e;1} \otimes \mathcal{Y}^e_{a_1'a_1;1}; \mathcal{Y}^{a_1}_{ea_1;1} \otimes \mathcal{Y}^e_{a_1a_1';1}) F(\mathcal{Y}^{a_2}_{a_2e;1} \otimes \mathcal{Y}^e_{a_2'a_2;1}; \mathcal{Y}^{a_2}_{ea_2;1} \otimes \mathcal{Y}^e_{a_2a_2';1})}.$$

By (4.5), this is equal to

$$\frac{S^{a_2}_{a_1}}{S^e_e}$$

which form an invertible matrix. So the semisimple balanced rigid tensor category is nondegenerate. Thus the tensor category is modular. Details will be given in [H9].

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