Vertex operator algebras, the Verlinde conjecture and modular tensor categories

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Abstract

Let V be a simple vertex operator algebra satisfying the following conditions: (i) $V_{(n)} = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ and the contragredient module V' is isomorphic to V as a V-module. (ii) Every $\mathbb{N}$-gradable weak V-module is completely reducible. (iii) V is C_2-cofinite. We announce a proof of the Verlinde conjecture for V, that is, of the statement that the matrices formed by the fusion rules among irreducible V-modules are diagonalized by the matrix given by the action of the modular transformation $\tau \mapsto -1/\tau$ on the space of characters of irreducible V-modules. We discuss some consequences of the Verlinde conjecture, including the Verlinde formula for the fusion rules, a formula for the matrix given by the action of $\tau \mapsto -1/\tau$ and the symmetry of this matrix. We also announce a proof of the rigidity and nondegeneracy property of the braided tensor category structure on the category of V-modules when V satisfies in addition the condition that irreducible V-modules not equivalent to V has no nonzero elements of weight 0. In particular, the category of V-modules has a natural structure of modular tensor category.

0 Introduction

In 1987, by comparing fusion algebras with certain algebras obtained in the study of conformal field theories on genus-one Riemann surfaces, Verlinde [V] conjectured that the matrices formed by the fusion rules are diagonalized by the matrix given by the action of the modular transformation $\tau \mapsto -1/\tau$ on the space of characters of a rational conformal field theory. From this conjecture, Verlinde obtained the famous Verlinde formulas for the fusion rules and, more generally, for the dimensions of conformal blocks on Riemann surfaces of arbitrary genera. In the particular case of the conformal field theories associated to affine Lie algebras (the Wess-Zumino-Novikov-Witten models), the Verlinde formulas give a surprising formula for the dimensions of the spaces of sections of the "generalized theta divisors"; this has given rise to a great deal of excitement and new mathematics. See the works [TUY] by Tsuchiya-Ueno-Yamada, [BL] by Beauville-Laszlo, [F] by Faltings and [KNR] by Kumar-Narasimhan-Ramanathan for details and proofs of this particular case of the Verlinde formulas.

In 1988, Moore and Seiberg [MS1] showed on a physical level of rigor that the Verlinde conjecture is a consequence of the axioms for rational conformal field theories. This result of Moore and Seiberg is based on certain polynomial equations which they derived from the axioms for rational conformal field theories [MS1] [MS2]. Moore and Seiberg further demonstrated that these polynomial equations are actually conformal-field-theoretic analogues of the tensor category theory for group representations. This work of Moore and Seiberg greatly advanced our understanding of the structure of conformal field theories. In particular, the notion of modular tensor category was later introduced to summarize the properties of the Moore-Seiberg polynomial equations and has played a central role in the developments of conformal field theories and three-dimensional topological field theories. See for example [T] and [BK] for the theory of modular tensor categories, their applications and references to many important works done by mathematicians and physicists.

The work of Moore and Seiberg gave a conceptual understanding of the Verlinde conjecture and the modular tensor categories arising in conformal field theories. However, it is a very hard problem to mathematically construct theories satisfying the axioms for rational conformal field theories. In fact, these axioms for rational conformal field theories are much stronger than the Verlinde conjecture and the modular tensor category structures. In the general theory of vertex operator algebras, introduced and studied first by Borcherds [B] and Frenkel-Lepowsky-Meurman [FLM], a mathematical version of the notion of fusion rule was introduced and studied by Frenkel, Lepowsky and the author in [FHL] using intertwining operators, and the modular transformations were given by Zhu's modular invariance theorem [Z]. Using these notions and some natural conditions, including in particular Zhu's C_2-cofiniteness condition, one can formulate a general version of the Verlinde conjecture in the framework of the theory of vertex operator algebras. Further results on intertwining operators and modular invariance were obtained in [HL1]–[HL4] by Huang-Lepowsky, in [H1], [H2] and [H3] by the author, in [DLM] by Dong-Li-Mason and in [M] by Miyamoto. But these results were still not enough for the proof of this general version of the Verlinde conjecture. The main obstructions were the duality and modular invariance properties for genus-zero and genus-one *multi-point* correlation functions constructed from intertwining operators for a vertex operator algebra satisfying the conditions mentioned above. These properties have recently been proved in [H4] and [H5].

In this paper, we announce a proof of the general version of the Verlinde conjecture above. Our theorem assumes only that the vertex operator algebra that we consider satisfies certain natural grading, finiteness and reductivity properties (see Section 2). We also discuss some consequences of our theorem, including the Verlinde formula for the fusion rules, a formula for the matrix given by the action of $\tau \mapsto -1/\tau$ and the symmetry of this matrix. For the details, see [H6]. We also announce a proof of the rigidity and nondegenracy condition of the braided tensor category structure on the category of modules for such a vertex operator algebra constructed by Lepowsky and the author [HL1]–[HL4] [H1] [H4], when V satisfies in addition the condition that irreducible modules not equivalent to the algebra (as a module) has no nonzero elements of weight 0. In particular, the category of modules for such a vertex operator algebra has a natural structure of modular tensor category.

This paper is organized as follows: In Section 1, we give the definitions of fusion rule, of the fusing and of the braiding isomorphisms in terms of matrix elements, and of the corresponding action of the modular transformation. These are the basic ingredients needed in the formulations of the main results given in Sections 2 and 3 and they are in fact based on substantial mathematical results in [H1], [H2], [H3], and in [Z] and [DLM], respectively. Our main theorems on the Verlinde conjecture, on the Verlinde formula for the fusion rules, on the formula for the matrix given by the action of $\tau \mapsto -1/\tau$, and on the symmetry of this matrix, are stated in Section 2. A very brief sketch of the proof of the Verlinde conjecture is given in this section. In Section 3, our main theorem on the modular tensor category structure is stated and a sketch of the proof is given.

Acknowledgment I am grateful to Jim Lepowsky and Robert Wilson for comments. The author is partially supported by NSF grant DMS-0401302.

1 Fusion rules, fusing and braiding isomorphisms and modular transformations

We assume that the reader is familiar with the basic definitions and results in the theory of vertex operator algebras as introduced and presented in [B] and [FLM]. We shall use the notations, terminology and formulations in [FLM], [FHL] and [LL].

Let V be a simple vertex operator algebra, V' the contragredient module of V, and $C_2(V)$ the subspace of V spanned by $u_{-2}v$ for $u, v \in V$. In the present paper, we shall always assume that V satisfies the following conditions:

1. $V_{(n)} = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ and V' is isomorphic to V as a V-module.
2. Every $\mathbb{N}$-gradable weak V-module is completely reducible.
3. V is C_2-cofinite, that is, $\dim V/C_2(V) < \infty$.

We recall that an $\mathbb{N}$-gradable weak V-module is a vector space that admits an $\mathbb{N}$-grading $W = \coprod_{n\in\mathbb{N}} W_{[n]}$, equipped with a vertex operator map

$$\begin{array}{rcl} Y : V \otimes W & \to & W[[z, z^{-1}]] \\ u \otimes w & \mapsto & Y(u, z)w = \displaystyle\sum_{n\in\mathbb{Z}} u_n z^{-n-1} \end{array}$$

satisfying all axioms for V-modules except that the condition $L(0)w = nw$ for $w \in W_{(n)}$ is replaced by $u_k w \in W_{[m-k-1+n]}$ for $u \in V_{(m)}$ and $w \in W_{[n]}$. Condition 2 is equivalent to the statement that every finitely-generated $\mathbb{N}$-gradable weak V-module is a V-module and every V-module is completely reducible.

From [DLM], we know that there are only finitely many inequivalent irreducible V-modules. Let $\mathcal{A}$ be the set of equivalence classes of irreducible V-modules. We denote the equivalence class containing V by e. For each $a \in \mathcal{A}$, we choose a representative W^a of a. Note that the contragredient module of an irreducible module is also irreducible (see [FHL]). So we have a map

$$\begin{array}{rcl} {}' : \mathcal{A} & \to & \mathcal{A} \\ a & \mapsto & a'. \end{array}$$

From [AM] and [DLM], we know that irreducible V-modules are in fact graded by rational numbers. Thus for $a \in \mathcal{A}$, there exist $h_a \in \mathbb{Q}$ such that $W^a = \coprod_{n \in h_a + \mathbb{N}} W^a_{(n)}$.

Let $\mathcal{V}_{a_1 a_2}^{a_3}$ for $a_1, a_2, a_3 \in \mathcal{A}$ be the space of intertwining operators of type $\binom{W^{a_3}}{W^{a_1} W^{a_2}}$ and $N_{a_1 a_2}^{a_3}$ for $a_1, a_2, a_3 \in \mathcal{A}$ the fusion rule, that is, the dimension of the space of intertwining operators of type $\binom{W^{a_3}}{W^{a_1} W^{a_2}}$. For any $\mathcal{Y} \in \mathcal{V}_{a_1 a_2}^{a_3}$, we know from [FHL] that for $w_{a_1} \in W^{a_1}$ and $w_{a_2} \in W^{a_2}$

$$\mathcal{Y}(w_{a_1}, x) w_{a_2} \in x^{\Delta(\mathcal{Y})} W^{a_3}[[x, x^{-1}]], \tag{1.1}$$

where

$$\Delta(\mathcal{Y}) = h_{a_3} - h_{a_1} - h_{a_2}.$$

From [GN], [L], [AN], [H5], we also know that the fusion rules $N_{a_1 a_2}^{a_3}$ for $a_1, a_2, a_3 \in \mathcal{A}$ are all finite. For $a \in \mathcal{A}$, let $\mathcal{N}(a)$ be the matrix whose entries are $N_{a a_1}^{a_2}$ for $a_1, a_2 \in \mathcal{A}$, that is,

$$\mathcal{N}(a) = (N_{a a_1}^{a_2}).$$

We also need matrix elements of fusing and braiding isomorphisms. In the proof of the Verlinde conjecture, we need to use several bases of one space of intertwining operators. We shall use $p = 1, 2, 3, 4, 5, 6, \dots$ to label different bases. For $p = 1, 2, 3, 4, 5, 6, \dots$ and $a_1, a_2, a_3 \in \mathcal{A}$, let $\{\mathcal{Y}_{a_1 a_2; i}^{a_3; (p)} \mid i = 1, \dots, N_{a_1 a_2}^{a_3}\}$, be a basis of $\mathcal{V}_{a_1 a_2}^{a_3}$. For $a_1, \dots, a_6 \in \mathcal{A}$, $w_{a_1} \in W^{a_1}$, $w_{a_2} \in W^{a_2}$, $w_{a_3} \in W^{a_3}$, and $w'_{a_4} \in (W^{a_4})'$, using the differential equations satisfied by the series

$$\langle w'_{a_4}, \mathcal{Y}_{a_1 a_5; i}^{a_4; (1)}(w_{a_1}, x_1) \mathcal{Y}_{a_2 a_3; j}^{a_5; (2)}(w_{a_2}, x_2) w_{a_3} \rangle \big|_{x_1^n = e^{n \log z_1},\ x_2^n = e^{n \log z_2},\ n \in \mathbb{Q}}$$

and

$$\langle w'_{a_4}, \mathcal{Y}_{a_6 a_3; k}^{a_4; (3)}(\mathcal{Y}_{a_1 a_2; l}^{a_6; (4)}(w_{a_1}, x_0) w_{a_2}, x_2) w_{a_3} \rangle \big|_{x_0^n = e^{n \log(z_1 - z_2)},\ x_2^n = e^{n \log z_2},\ n \in \mathbb{Q}},$$

it was proved in [H4] that these series are convergent in the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively. Note that for any a_1, a_2, a_3, a_4, a_5, $a_6 \in \mathcal{A}$, $\{\mathcal{Y}_{a_1 a_5; i}^{a_4; (1)} \otimes \mathcal{Y}_{a_2 a_3; j}^{a_5; (2)} \mid i = 1, \dots, N_{a_1 a_5}^{a_4}, j = 1, \dots, N_{a_2 a_3}^{a_5}\}$ and $\{\mathcal{Y}_{a_6 a_3; l}^{a_4; (3)} \otimes \mathcal{Y}_{a_1 a_2; k}^{a_6; (4)} \mid l = 1, \dots, N_{a_6 a_3}^{a_4}, k = 1, \dots, N_{a_1 a_2}^{a_6}\}$ are bases of $\mathcal{V}_{a_1 a_5}^{a_4} \otimes \mathcal{V}_{a_2 a_3}^{a_5}$ and $\mathcal{V}_{a_6 a_3}^{a_4} \otimes \mathcal{V}_{a_1 a_2}^{a_6}$, respectively. The associativity of intertwining operators proved and studied in [H1], [H3] and [H4] says that there exist

$$F(\mathcal{Y}_{a_1 a_5; i}^{a_4; (1)} \otimes \mathcal{Y}_{a_2 a_3; j}^{a_5; (2)}; \mathcal{Y}_{a_6 a_3; l}^{a_4; (3)} \otimes \mathcal{Y}_{a_1 a_2; k}^{a_6; (4)}) \in \mathbb{C}$$

for $a_1, \dots, a_6 \in \mathcal{A}$, $i = 1, \dots, N_{a_1a_5}^{a_4}$, $j = 1, \dots, N_{a_2a_3}^{a_5}$, $k = 1, \dots, N_{a_6a_3}^{a_4}$, $l = 1, \dots, N_{a_1a_2}^{a_6}$ such that

$$\begin{aligned}
&\langle w'_{a_4}, \mathcal{Y}_{a_1a_5;i}^{a_4;(1)}(w_{a_1}, x_1)\mathcal{Y}_{a_2a_3;j}^{a_5;(2)}(w_{a_2}, z_2)w_{a_3}\rangle\big|_{x_1^n=e^{n\log z_1},\ x_2^n=e^{n\log z_2},\ n\in\mathbb{Q}}\\
&\quad = \sum_{a_6\in\mathcal{A}}\sum_{k=1}^{N_{a_6a_3}^{a_4}}\sum_{l=1}^{N_{a_1a_2}^{a_6}} F(\mathcal{Y}_{a_1a_5;i}^{a_4;(1)}\otimes\mathcal{Y}_{a_2a_3;j}^{a_5;(2)};\mathcal{Y}_{a_6a_3;l}^{a_4;(3)}\otimes\mathcal{Y}_{a_1a_2;k}^{a_6;(4)})\cdot\\
&\quad\quad \cdot\langle w'_{a_4}, \mathcal{Y}_{a_6a_3;k}^{a_4;(3)}(\mathcal{Y}_{a_1a_2;l}^{a_6;(4)}(w_{a_1}, z_1-z_2)w_{a_2}, z_2)w_{a_3}\rangle\big|_{x_0^n=e^{n\log(z_1-z_2)},\ x_2^n=e^{n\log z_2},\ n\in\mathbb{Q}}
\end{aligned}\tag{1.2}$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$, for $a_1, \dots, a_5 \in \mathcal{A}$, $w_{a_1} \in W^{a_1}$, $w_{a_2} \in W^{a_2}$, $w_{a_3} \in W^{a_3}$, $w'_{a_4} \in (W^{a_4})'$, $i = 1, \dots, N_{a_1a_5}^{a_4}$ and $j = 1, \dots, N_{a_2a_3}^{a_5}$. The numbers

$$F(\mathcal{Y}_{a_1a_5;i}^{a_4;(1)}\otimes\mathcal{Y}_{a_2a_3;j}^{a_5;(2)};\mathcal{Y}_{a_6a_3;k}^{a_4;(3)}\otimes\mathcal{Y}_{a_1a_2;l}^{a_6;(4)})$$

together give a matrix which represents a linear isomorphism

$$\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}}\mathcal{V}_{a_1a_5}^{a_4}\otimes\mathcal{V}_{a_2a_3}^{a_5}\to\coprod_{a_1,a_2,a_3,a_4,a_6\in\mathcal{A}}\mathcal{V}_{a_6a_3}^{a_4}\otimes\mathcal{V}_{a_1a_2}^{a_6},$$

called the *fusing isomorphism*, such that these numbers are the matrix elements.

By the commutativity of intertwining operators proved and studied in [H2], [H3] and [H4], for any fixed $r \in \mathbb{Z}$, there exist

$$B^{(r)}(\mathcal{Y}_{a_1a_5;i}^{a_4;(1)}\otimes\mathcal{Y}_{a_2a_3;j}^{a_5;(2)};\mathcal{Y}_{a_2a_6;l}^{a_4;(3)}\otimes\mathcal{Y}_{a_1a_3;k}^{a_6;(4)})\in\mathbb{C}$$

for $a_1, \dots, a_6 \in \mathcal{A}$, $i = 1, \dots, N_{a_1a_5}^{a_4}$, $j = 1, \dots, N_{a_2a_3}^{a_5}$, $k = 1, \dots, N_{a_2a_6}^{a_4}$, $l = 1, \dots, N_{a_1a_3}^{a_6}$, such that the analytic extension of the single-valued analytic function

$$\langle w'_{a_4}, \mathcal{Y}_{a_1a_5;i}^{a_4;(1)}(w_{a_1}, x_1)\mathcal{Y}_{a_2a_3;j}^{a_5;(2)}(w_{a_2}, x_2)w_{a_3}\rangle\big|_{x_1^n=e^{n\log z_1},\ x_2^n=e^{n\log z_2},\ n\in\mathbb{Q}}$$

on the region $|z_1| > |z_2| > 0$, $0 \le \arg z_1, \arg z_2 < 2\pi$ along the path

$$t \mapsto \left(\frac{3}{2} - \frac{e^{(2r+1)\pi it}}{2}, \frac{3}{2} + \frac{e^{(2r+1)\pi it}}{2}\right)$$

to the region $|z_2| > |z_1| > 0$, $0 \leq \arg z_1, \arg z_2 < 2\pi$ is

$$\sum_{a_6\in\mathcal{A}} \sum_{k=1}^{N_{a_2a_6}^{a_4}} \sum_{l=1}^{N_{a_1a_3}^{a_6}} B^{(r)}(\mathcal{Y}_{a_1a_5;i}^{a_4;(1)} \otimes \mathcal{Y}_{a_2a_3;j}^{a_5;(2)}; \mathcal{Y}_{a_2a_6;k}^{a_4;(3)} \otimes \mathcal{Y}_{a_1a_3;l}^{a_6;(4)}) \cdot$$
$$\cdot \langle w'_{a_4}, \mathcal{Y}_{a_2a_6;k}^{a_4;(3)}(w_{a_2}, z_1)\mathcal{Y}_{a_1a_3;l}^{a_6;(4)}(w_{a_1}, z_2)w_{a_3}\rangle|_{x_1^n=e^{n\log z_1},\, x_2^n=e^{n\log z_2},\, n\in\mathbb{Q}}.$$

The numbers

$$B^{(r)}(\mathcal{Y}_{a_1a_5;i}^{a_4;(1)} \otimes \mathcal{Y}_{a_2a_3;j}^{a_5;(2)}; \mathcal{Y}_{a_2a_6;k}^{a_4;(3)} \otimes \mathcal{Y}_{a_1a_3;l}^{a_6;(4)})$$

together give a linear isomorphism

$$\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_5}^{a_4} \otimes \mathcal{V}_{a_2a_3}^{a_5} \to \coprod_{a_1,a_2,a_3,a_4,a_6\in\mathcal{A}} \mathcal{V}_{a_2a_6}^{a_4} \otimes \mathcal{V}_{a_1a_3}^{a_6},$$

called the *braiding isomorphism*, such that these numbers are the matrix elements.

We need an action of S_3 on the space

$$\mathcal{V} = \coprod_{a_1,a_2,a_3\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_3}.$$

For $r \in \mathbb{Z}$, $a_1, a_2, a_3 \in \mathcal{A}$, consider the isomorphisms $\Omega_r : \mathcal{V}_{a_1a_2}^{a_3} \to \mathcal{V}_{a_2a_1}^{a_3}$ and $A_r : \mathcal{V}_{a_1a_2}^{a_3} \to \mathcal{V}_{a_1a'_3}^{a'_2}$ given in (7.1) and (7.13) in [HL2]. For $a_1, a_2, a_3 \in \mathcal{A}$, $\mathcal{Y} \in \mathcal{V}_{a_1a_2}^{a_3}$, we define

$$\begin{aligned}
\sigma_{12}(\mathcal{Y}) &= e^{\pi i\Delta(\mathcal{Y})}\Omega_{-1}(\mathcal{Y})\\
&= e^{-\pi i\Delta(\mathcal{Y})}\Omega_0(\mathcal{Y}),\\
\sigma_{23}(\mathcal{Y}) &= e^{\pi i h_{a_1}}A_{-1}(\mathcal{Y})\\
&= e^{-\pi i h_{a_1}}A_0(\mathcal{Y}).
\end{aligned}$$

We have the following:

Proposition 1.1 *The actions σ_{12} and σ_{23} of* (12) *and* (23) *on $\mathcal{V}$ generate a left action of S_3 on $\mathcal{V}$.*

We now choose a basis $\mathcal{Y}_{a_1a_2;i}^{a_3}$, $i = 1, \ldots, N_{a_1a_2}^{a_3}$, of $\mathcal{V}_{a_1a_2}^{a_3}$ for each triple $a_1, a_2, a_3 \in \mathcal{A}$. For $a \in \mathcal{A}$, we choose $\mathcal{Y}_{ea;1}^{a}$ to be the vertex operator Y_{W^a}

defining the module structure on W^a and we choose $\mathcal{Y}^a_{ae;1}$ to be the intertwining operator defined using the action of σ_{12},

$$\begin{aligned}\mathcal{Y}^a_{ae;1}(w_a, x)u &= \sigma_{12}(\mathcal{Y}^a_{ea;1})(w_a, x)u\\ &= e^{xL(-1)}\mathcal{Y}^a_{ea;1}(u, -x)w_a\\ &= e^{xL(-1)}Y_{W^a}(u, -x)w_a\end{aligned}$$

for $u \in V$ and $w_a \in W^a$. Since V' as a V-module is isomorphic to V, we have $e' = e$. From [FHL], we know that there is a nondegenerate invariant bilinear form $(\cdot, \cdot)$ on V such that $(\mathbf{1}, \mathbf{1}) = 1$. We choose $\mathcal{Y}^e_{aa';1} = \mathcal{Y}^{e'}_{aa';1}$ to be the intertwining operator defined using the action of σ_{23} by

$$\mathcal{Y}^{e'}_{aa';1} = \sigma_{23}(\mathcal{Y}^a_{ae;1}),$$

that is,

$$(u, \mathcal{Y}^{e'}_{aa';1}(w_a, x)w'_a) = e^{\pi i h_a}\langle \mathcal{Y}^a_{ae;1}(e^{xL(1)}(e^{-\pi i}x^{-2})^{L(0)}w_a, x^{-1})u, w'_a\rangle$$

for $u \in V$, $w_a \in W^a$ and $w'_{a'} \in (W^a)'$. Since the actions of σ_{12} and σ_{23} generate the action of S_3 on $\mathcal{V}$, we have

$$\mathcal{Y}^e_{a'a;1} = \sigma_{12}(\mathcal{Y}^e_{aa';1})$$

for any $a \in \mathcal{A}$. When $a_1, a_2, a_3 \neq e$, we choose $\mathcal{Y}^{a_3}_{a_1a_2;i}$, $i = 1, \ldots, N^{a_3}_{a_1a_2}$, to be an arbitrary basis of $\mathcal{V}^{a_3}_{a_1a_2}$. Note that for each element $\sigma \in S_3$, $\{\sigma(\mathcal{Y})^{a_3}_{a_1a_2;i} \mid i = 1, \ldots, N^{a_3}_{a_1a_2}\}$ is also a basis of $\mathcal{V}^{a_3}_{a_1a_2}$.

We now discuss modular transformations. Let $q_\tau = e^{2\pi i\tau}$ for $\tau \in \mathbb{H}$ ($\mathbb{H}$ is the upper-half plane). We consider the q_τ-traces of the vertex operators Y_{W^a} for $a \in \mathcal{A}$ on the irreducible V-modules W^a of the following form:

$$\mathrm{Tr}_{W^a} Y_{W^a}(e^{2\pi izL(0)}u, e^{2\pi iz})q_\tau^{L(0)-\frac{c}{24}} \tag{1.3}$$

for $u \in V$. In [Z], under some conditions slightly different from (mostly stronger than) those we assume in this paper, Zhu proved that these q-traces are independent of z, are absolutely convergent when $0 < |q_\tau| < 1$ and can be analytically extended to analytic functions of τ in the upper-half plane. We shall denote the analytic extension of (1.3) by

$$E(\mathrm{Tr}_{W^a} Y_{W^a}(e^{2\pi izL(0)}u, e^{2\pi iz})q_\tau^{L(0)-\frac{c}{24}}).$$

In [Z], under his conditions alluded to above, Zhu also proved the following modular invariance property: For

$$\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL(2, \mathbb{Z}),$$

let $\tau' = \frac{a\tau+b}{c\tau+d}$. Then there exist unique $A_{a_1}^{a_2} \in \mathbb{C}$ for $a_1, a_2 \in \mathcal{A}$ such that

$$\begin{aligned} & E\left(\mathrm{Tr}_{W^{a_1}} Y_{W^{a_1}}\left(e^{\frac{2\pi i z}{c\tau+d}L(0)}\left(\frac{1}{c\tau+d}\right)^{L(0)} u, e^{\frac{2\pi i z}{c\tau+d}}\right) q_{\tau'}^{L(0)-\frac{c}{24}}\right) \\ & \quad = \sum_{a_2 \in \mathcal{A}} A_{a_1}^{a_2} E(\mathrm{Tr}_{W^{a_2}} Y_{W^{a_2}}(e^{2\pi i z L(0)} u, e^{2\pi i z}) q_\tau^{L(0)-\frac{c}{24}}) \end{aligned}$$

for $u \in V$. In [DLM], Dong, Li and Mason, among many other things, improved Zhu's results above by showing that the results of Zhu above also hold for vertex operator algebras satisfying the conditions (slightly weaker than what) we assume in this paper. In particular, for

$$\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right) \in SL(2, \mathbb{Z}),$$

there exist unique $S_{a_1}^{a_2} \in \mathbb{C}$ for $a_1 \in \mathcal{A}$ such that

$$\begin{aligned} & E\left(\mathrm{Tr}_{W^{a_1}} Y_{W^{a_1}}\left(e^{-\frac{2\pi i z}{\tau}L(0)}\left(-\frac{1}{\tau}\right)^{L(0)} u, e^{-\frac{2\pi i z}{\tau}}\right) q_{-\frac{1}{\tau}}^{L(0)-\frac{c}{24}}\right) \\ & \quad = \sum_{a_2 \in \mathcal{A}} S_{a_1}^{a_2} E(\mathrm{Tr}_{W^{a_2}} Y_{W^{a_2}}(e^{2\pi i z L(0)} u, e^{2\pi i z}) q_\tau^{L(0)-\frac{c}{24}}) \end{aligned}$$

for $u \in V$. When $u = \mathbf{1}$, we see that the matrix $S = (S_{a_1}^{a_2})$ actually acts on the space of spanned by the vacuum characters $\mathrm{Tr}_{W^a} q_\tau^{L(0)-\frac{c}{24}}$ for $a \in \mathcal{A}$.

2 The Verlinde conjecture and consequences

In [H6], we proved the following general version of the Verlinde conjecture in the framework of vertex operator algebras (cf. Section 3 in [V] and Section 4 in [MS1]):

Theorem 2.1 *Let V be a vertex operator algebra satisfying the following conditions:*

1. $V_{(n)} = 0$ *for* $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ *and* V' *is isomorphic to* V *as a* V*-module.*
2. *Every* $\mathbb{N}$*-gradable weak* V*-module is completely reducible.*
3. V *is* C_2*-cofinite, that is,* $\dim V/C_2(V) < \infty$.

Then for $a \in \mathcal{A}$,

$$F(\mathcal{Y}^{a}_{ae;1} \otimes \mathcal{Y}^{e}_{a'a;1}; \mathcal{Y}^{a}_{ea;1} \otimes \mathcal{Y}^{e}_{aa';1}) \neq 0$$

and

$$\sum_{a_1, a_3 \in \mathcal{A}} (S^{-1})^{a_1}_{a_4} N^{a_3}_{a_1 a_2} S^{a_5}_{a_3} = \delta^{a_5}_{a_4} \frac{(B^{(-1)})^2(\mathcal{Y}^{a_4}_{a_4 e;1} \otimes \mathcal{Y}^{e}_{a'_2 a_2;1}; \mathcal{Y}^{a_4}_{a_4 e;1} \otimes \mathcal{Y}^{e}_{a'_2 a_2;1})}{F(\mathcal{Y}^{a_2}_{a_2 e;1} \otimes \mathcal{Y}^{e}_{a'_2 a_2;1}; \mathcal{Y}^{a_2}_{e a_2;1} \otimes \mathcal{Y}^{e}_{a_2 a'_2;1})}, \tag{2.1}$$

where $(B^{(-1)})^2(\mathcal{Y}^{a_4}_{a_4 e;1} \otimes \mathcal{Y}^{e}_{a'_2 a_2;1}; \mathcal{Y}^{a_4}_{a_4 e;1} \otimes \mathcal{Y}^{e}_{a'_2 a_2;1})$ *is the corresponding matrix elements of the square of the braiding isomorphism. In particular, the matrix* S *diagonalizes the matrices* $\mathcal{N}(a_2)$ *for all* $a_2 \in \mathcal{A}$.

Sketch of the proof. Moore and Seiberg showed in [MS1] that the conclusions of the theorem follow from the following formulas (which they derived by assuming the axioms of rational conformal field theories): For $a_1, a_2, a_3 \in \mathcal{A}$,

$$\begin{aligned}\sum_{i=1}^{N^{a_3}_{a_1 a_2}} \sum_{k=1}^{N^{a_2}_{a'_1 a_3}} & F(\mathcal{Y}^{a_2}_{a_2 e;1} \otimes \mathcal{Y}^{e}_{a'_3 a_3;1}; \mathcal{Y}^{a_2}_{a'_1 a_3;k} \otimes \mathcal{Y}^{a'_1}_{a_2 a'_3;i}) \cdot \\ & \cdot F(\mathcal{Y}^{a_2}_{a'_1 a_3;k} \otimes \sigma_{123}(\mathcal{Y}^{a'_1}_{a_2 a'_3;i}); \mathcal{Y}^{a_2}_{e a_2;1} \otimes \mathcal{Y}^{e}_{a'_1 a_1;1}) \\ = {} & N^{a_3}_{a_1 a_2} F(\mathcal{Y}^{a_2}_{a_2 e;1} \otimes \mathcal{Y}^{e}_{a'_2 a_2;1}; \mathcal{Y}^{a_2}_{e a_2;1} \otimes \mathcal{Y}^{e}_{a_2 a'_2;1})\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\sum_{a_4 \in \mathcal{A}} & S^{a_4}_{a_1} (B^{(-1)})^2(\mathcal{Y}^{a_4}_{a_4 e;1} \otimes \mathcal{Y}^{e}_{a'_2 a_2;1}; \mathcal{Y}^{a_4}_{a_4 e;1} \otimes \mathcal{Y}^{e}_{a'_2 a_2;1})(S^{-1})^{a_3}_{a_4} \\ & = \sum_{i=1}^{N^{a_3}_{a_1 a_2}} \sum_{k=1}^{N^{a_2}_{a'_1 a_3}} F(\mathcal{Y}^{a_2}_{a_2 e;1} \otimes \mathcal{Y}^{e}_{a'_3 a_3;1}; \mathcal{Y}^{a_2}_{a'_1 a_3;k} \otimes \mathcal{Y}^{a'_1}_{a_2 a'_3;i}) \cdot \\ & \qquad \cdot F(\mathcal{Y}^{a_2}_{a'_1 a_3;k} \otimes \sigma_{123}(\mathcal{Y}^{a'_1}_{a_2 a'_3;i}); \mathcal{Y}^{a_2}_{e a_2;1} \otimes \mathcal{Y}^{e}_{a'_1 a_1;1}).\end{aligned} \tag{2.3}$$

So the main work is to prove these two formulas. The proofs of these formulas in [H6] are based in turn on the proofs of a number of other formulas and on nontrivial applications of a number of results in the theory of vertex operator algebras, so here we can only outline what is used in the proofs.

The proof of the first formula (2.2) uses mainly the works of Lepowsky and the author [HL1]–[HL4] and of the author [H1] [H2] [H3] and [H4] on the tensor product theory, intertwining operator algebras and the construction of genus-zero chiral conformal field theories. The main technical result used is the associativity for intertwining operators proved in [H1] and [H4] for vertex operator algebras satisfying the three conditions stated in the theorem. Using the associativity for intertwining operators repeatedly to express the correlation functions obtained from products of three suitable intertwining operators as linear combinations of the correlation functions obtained from iterates of three intertwining operators in two ways, we obtain a formula for the matrix elements of the fusing isomorphisms. Then using certain properties of the matrix elements of the fusing isomorphisms and their inverses proved in [H6], we obtain the first formula (2.2).

The proof of the second formula (2.3) heavily uses the results obtained in [H5] on the convergence and analytic extensions of the q_τ-traces of products of what we call "geometrically-modified intertwining operators", the genus-one associativity, and the modular invariance of these analytic extensions of the q_τ-traces, where $q_\tau = e^{2\pi i\tau}$. These results allows us to (rigorously) establish a formula which corresponds to the fact that the modular transformation $\tau \mapsto -1/\tau$ changes one basic Dehn twist on the Teichmüller space of genus-one Riemann surfaces to the other. Calculating the matrices corresponding to the Dehn twists and substituting the results into this formula, we obtain (2.3).

As in [MS1], the conclusions of the theorem follow immediately from (2.2) and (2.3). ∎

Remark 2.2 Note that finitely generated $\mathbb{N}$-gradable weak V-modules are what naturally appear in the proofs of the theorems on genus-zero and genus-one correlation functions. Thus Condition 2 is natural and necessary because the Verlinde conjecture concerns V-modules, not finitely generated $\mathbb{N}$-gradable weak V-modules. Condition 3 would be a consequence of the finiteness of the dimensions of genus-one conformal blocks, if the conformal field theory had been constructed, and is thus natural and necessary. For vertex operator algebras associated to affine Lie algebras (Wess-Zumino-Novikov-Witten models) and vertex operator algebras associated to the Virasoro algebra (minimal models), Condition 2 can be verified easily by reformulating the corresponding complete reducibility results in terms of the representation theory of affine Lie algebras and the Virasoro algebra. For these vertex operator algebras, Condition 3 can also be easily verified by using results in the representation theory of affine Lie algebras and the Virasoro algebra. In fact, Condition 3 was stated to hold for these algebras in Zhu's paper [Z] and was verified by Dong-Li-Mason [DLM] (see also [AN] for the case of minimal models).

Using the fact that $N_{ea_1}^{a_2} = \delta_{a_1}^{a_2}$ for $a_1, a_2 \in \mathcal{A}$, we can easily derive the following formulas from Theorem 2.1 (cf. Section 3 in [V]):

Theorem 2.3 *Let V be a vertex operator algebra satisfying the conditions in Section 1. Then we have $S_e^a \neq 0$ for $a \in \mathcal{A}$ and*

$$N_{a_1a_2}^{a_3} = \sum_{a_4\in\mathcal{A}} \frac{S_{a_1}^{a_4} S_{a_2}^{a_4} S_{a_4}^{a_3'}}{S_0^{a_4}}. \tag{2.4}$$

Theorem 2.4 *For $a_1, a_2 \in \mathcal{A}$,*

$$S_{a_1}^{a_2} = \frac{S_e^e((B^{(-1)})^2(\mathcal{Y}_{a_2e;1}^{a_2} \otimes \mathcal{Y}_{a_1'a_1;1}^{e}; \mathcal{Y}_{a_2e;1}^{a_2} \otimes \mathcal{Y}_{a_1'a_1;1}^{e}))}{F(\mathcal{Y}_{a_1e;1}^{a_1} \otimes \mathcal{Y}_{a_1'a_1;1}^{e}; \mathcal{Y}_{ea_1;1}^{a_1} \otimes \mathcal{Y}_{a_1a_1';1}^{e}) F(\mathcal{Y}_{a_2e;1}^{a_2} \otimes \mathcal{Y}_{a_2'a_2;1}^{e}; \mathcal{Y}_{ea_2;1}^{a_2} \otimes \mathcal{Y}_{a_2a_2';1}^{e})}. \tag{2.5}$$

Using (2.5) and certain properties of the matrix elements of the fusing and braiding isomorphisms proved in [H6], we can prove the following:

Theorem 2.5 *The matrix $(S_{a_1}^{a_2})$ is symmetric.*

3 Rigidity, nondegeneracy property and modular tensor categories

A tensor category with tensor product bifunctor $\boxtimes$ and unit object V is rigid if for every object W in the category, there are right and left dual objects W^* and ${}^*W$ together with morphisms $e_W : W^* \boxtimes W \to V$, $i_W : V \to W \boxtimes W^*$,

$e'_W : W \boxtimes {}^*W \to V$ and $i'_W : V \to {}^*W \boxtimes W$ such that the compositions of the morphisms in the sequence

$$\begin{array}{rcl} W \longrightarrow & V \boxtimes W & \xrightarrow{i_W \boxtimes I_W} (W \boxtimes W^*) \boxtimes W \longrightarrow \\ \longrightarrow W \boxtimes (W^* \boxtimes W) & \xrightarrow{I_W \boxtimes e_W} & W \boxtimes V \longrightarrow W \end{array}$$

and three similar sequences are equal to the identity I_W on W. Rigidity is a standard notion in the theory of tensor categories. A rigid braided tensor category together with a twist (a natural isomorphism from the category to itself) satisfying natural conditions (see [T] and [BK] for the precise conditions) is called a ribbon category. A semisimple ribbon category with finitely many inequivalent irreducible objects is a modular tensor category if it has the following nondegeneracy property: The $m \times m$ matrix formed by the traces of the morphism $c_{W_iW_j} \circ c_{W_jW_i}$ in the ribbon category for irreducible modules $W_1, \dots, W_m$ is invertible. The term "modular tensor category" was first suggested by I. Frenkel to summarize Moore-Seiberg's theory of polynomial equations. See [T] and [BK] for details of the theory of modular tensor categories.

The results in the proceeding section give the following:

Theorem 3.1 *Let V be a simple vertex operator algebra satisfying the following conditions:*

1. *$V_{(n)} = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$, $W_{(0)} = 0$ for any irreducible V-module which is not equivalent to V.*

2. *Every $\mathbb{N}$-gradable weak V-module is completely reducible.*

3. *V is C_2-cofinite, that is, $\dim V/C_2(V) < \infty$.*

Then the braided tensor category structure on the category of V-modules constructed in [HL1]–[HL4], [H1] and [H4] is rigid, has a natural structure of ribbon category and has the nondegeneracy property. In particular, the category of V-modules has a natural structure of modular tensor category.

Sketch of the proof. Note that Condition 1 implies that V' is equivalent to V as a V-module. Thus Condition 1 is stronger than Condition 1 in the preceding section. In particular, we can use all the results in the proceeding section. This slightly stronger Condition 1 is needed in the proof of the rigidity and nondegeneracy property.

We take both the left and right dual of a V-module W to be the contragredient module W' of W. Since our tensor category is semisimple, to prove the rigidity, we need only discuss irreducible modules. For any V-module $W = \coprod_{n\in\mathbb{Q}} W_{(n)}$, we use $\overline{W}$ to denote its algebraic completion $\prod_{n\in\mathbb{Q}} W_{(n)}$. For $a \in \mathcal{A}$, using the universal property (see Definition 3.1 in [HL3] and Definition 12.1 in [HL4]) for the tensor product module $(W^a)' \boxtimes W^a$, we know that there exists a unique module map $\hat{e}_a : (W^a)' \boxtimes W^a \to V$ such that

$$\overline{\hat{e}}_a(w'_a \boxtimes w_a) = \mathcal{Y}^e_{a'a;1}(w'_a, 1)w_a$$

for $w_a \in W^a$ and $w'_a \in (W^a)'$, where $w'_a \boxtimes w_a \in \overline{(W^a)' \boxtimes W^a}$ is the tensor product of w_1 and w_2, $\overline{\hat{e}}_a : \overline{(W^a)' \boxtimes W^a} \to \overline{V}$ is the natural extension of $\hat{e}_a$ to $\overline{(W^a)' \boxtimes W^a}$. Similarly, we have a module map from $W^a \boxtimes (W^a)'$ to V. Since $W^a \boxtimes (W^a)'$ is completely reducible and the fusion rule $N^V_{W^a (W^a)'}$ is 1, there is a V-submodule of $W^a \boxtimes (W^a)'$ which is isomorphic to V under the module map from $W^a \boxtimes (W^a)'$ to V. Thus we obtain a module map $i_a : V \to W^a \boxtimes (W^a)'$ which maps V isomorphically to this submodule of $W^a \boxtimes (W^a)'$. Now

$$\begin{array}{lllll} W^a \longrightarrow & V \boxtimes W^a & \xrightarrow{i_a \boxtimes I_{W^a}} & (W^a \boxtimes (W^a)') \boxtimes W^a & \longrightarrow \\ \longrightarrow & W^a \boxtimes ((W^a)' \boxtimes W^a) & \xrightarrow{I_{W^a} \boxtimes \hat{e}_a} & W^a \boxtimes V & \longrightarrow W_a \end{array} \tag{3.1}$$

is a module map from an irreducible module to itself. So it must be the identity map multiplied by a number. One can calculate this number explicitly and it is equal to

$$F(\mathcal{Y}^a_{ae;1} \otimes \mathcal{Y}^e_{a'a;1}; \mathcal{Y}^a_{ea;1} \otimes \mathcal{Y}^e_{aa';1}).$$

From Theorem 2.1, this number is not 0. Let

$$e_a = \frac{1}{F(\mathcal{Y}^a_{ae;1} \otimes \mathcal{Y}^e_{a'a;1}; \mathcal{Y}^a_{ea;1} \otimes \mathcal{Y}^e_{aa';1})}\hat{e}_a$$

Then the map obtained from (3.1) by replacing $\hat{e}_a$ by e_a is the identity. Similarly, we can prove that all the other maps in the definition of rigidity are also equal to the identity. Thus the tensor category is rigid.

For any $a \in \mathcal{A}$, we define the twist on W^a to $e^{2\pi i h_a}$. Then it is easy to verify that the rigid braided tensor category with this twist is a ribbon category.

To prove the nondegeneracy property, we use the formula (2.5). Now it is easy to calculate in the tensor category the trace of $c_{W^{a_2},W^{a_1}} \circ c_{W^{a_1},W^{a_2}}$ for $a_1, a_2 \in \mathcal{A}$, where $c_{W^{a_1},W^{a_2}} : W^{a_1} \boxtimes W^{a_2} \to W^{a_2} \boxtimes W^{a_1}$ is the braiding isomorphism. The result is

$$\frac{((B^{(-1)})^2(\mathcal{Y}^{a_2}_{a_2e;1} \otimes \mathcal{Y}^{e}_{a_1'a_1;1}; \mathcal{Y}^{a_2}_{a_2e;1} \otimes \mathcal{Y}^{e}_{a_1'a_1;1}))}{F(\mathcal{Y}^{a_1}_{a_1e;1} \otimes \mathcal{Y}^{e}_{a_1'a_1;1}; \mathcal{Y}^{a_1}_{ea_1;1} \otimes \mathcal{Y}^{e}_{a_1a_1';1})F(\mathcal{Y}^{a_2}_{a_2e;1} \otimes \mathcal{Y}^{e}_{a_2'a_2;1}; \mathcal{Y}^{a_2}_{ea_2;1} \otimes \mathcal{Y}^{e}_{a_2a_2';1})}.$$

By (2.5), this is equal to

$$\frac{S^{a_2}_{a_1}}{S^{e}_{e}},$$

and these numbers form an invertible matrix. The other data and axioms for modular tensor categories can be given or proved trivially. Thus the tensor category is modular. The details will be given in [H7]. ∎

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