Associative algebras for (logarithmic) twisted modules for a vertex operator algebra

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Outline

1. Background
2. Twisted modules
3. Associative algebras for g-twisted modules
4. Functors between module categories

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Orbifold conformal field theories

- Orbifold conformal field theories play an important role in mathematics. The moonshine module constructed by Frenkel, Lepowsky and Meurman is the first example of orbifold conformal field theories. The mirror symmetry for Calabi-Yau manifolds was studied first in physics using the conjectures on the existence of the Calabi-Yau superconformal field theories and on the construction of orbifold conformal field theories.
- **Conjecture [H.]**: Let V be a vertex operator algebra satisfying suitable conditions and G a group of automorphisms of G. Then twisted intertwining operators among irreducible g-twisted V-modules for $g \in G$ form a twisted intertwining operator algebra satisfying the modular invariance property.

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- **Conjecture [Kirillov, H.]**: Let V and G be as above. Then the category of g-twisted V-modules for $g \in G$ is a G-crossed category in the sense of Turaev.
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Vertex operator algebras

- Data: $V=\coprod_{n\in\mathbb{Z}}V_{(n)}$, $Y_V: V\otimes V\to V((x))$, $\mathbf{1}\in V_{(0)}$ (the vacuum), $\omega\in V_{(2)}$ (the conformal element).
- The main axioms: The Jacobi identity: For $u, v\in V$,

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_V(u,x_1)Y_V(v,x_2)$$
$$-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_V(v,x_2)Y_V(u,x_1)$$
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Vertex operator algebras

- Or, equivalently, the duality property: For $u_1, u_2, v \in V$ and $v' \in V' \in \coprod_{n\in\mathbb{Z}} V^*_{(n)}$,

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are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function of the form

$$\frac{g(z_1, z_2)}{z_1^m z_2^n (z_1 - z_2)^t}$$

for a polynomial $g(z_1, z_2)$, $m, n, t \in \mathbb{N}$.

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Automorphisms of a vertex operator algebra

- Let V be a vertex operator algebra. An automorphism of V is an invertible linear map $g : V \to V$ preserving the grading, the vacuum and the conformal element, and satisfying the condition $gY_V(u, x)v = Y_V(gu, x)gv$.
- Examples: An element of the Monster group gives an automorphism of the moonshine module vertex operator algebra $V^\natural$. Such an automorphism is of finite order.
- An element of a simply connected finite-dimensional Lie group gives an automorphism of the vertex operator algebra associated to the affine Lie algebra of the Lie algebra of the Lie group. Such an automorphism is in general of infinite order. Moreover, in general it might not act on the vertex operator algebra semisimply.

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Definition of twisted module

- Let V be a vertex operator algebra and g an automorphism of V.
- Frenkel-Lepowsky-Meurman and Lepowsky, mid 1980's : g-twisted modules when g is of finite order.
- H., 2009: g-twisted modules in the general case. Important new feature: The twisted vertex operators might involve the logarithm of the variable (logarithmic g-twisted module).
- Data for a (grading-restricted generalized) g-twisted V-module: $W = \coprod_{n\in\mathbb{C},\alpha\in\mathbb{C}/\mathbb{Z}} W_{[n]}^{[\alpha]}$, $Y_W : V \otimes W \mapsto W\{x\}[\log x]$.

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- Main axioms:
  - The equivariance property: For $p \in \mathbb{Z}$, $z \in \mathbb{C}^\times$, $v \in V$ and $w \in W$, $Y^{g;p+1}(gv, z)w = Y^{g;p}(v, z)w$, where for $p \in \mathbb{Z}$, $Y^{g;p}(v, z)w = Y^g(v, x)w|_{x^n = e^{nl_p(z)},\ \log x = l_{p(z)}}$, $l_p(z) = \log z + 2\pi p\sqrt{-1}$.
  - The duality property: For $u, v \in V$, $w \in W$ and $w' \in W'$, there exist $a_{ijkl} \in \mathbb{C}$, $m_i, n_j \in \mathbb{R}$, $t \in \mathbb{N}$ such that the series

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Jacobi identity for a component of twisted vertex operator map

- The definition of g-twisted V-modules uses the duality property for the twisted vertex operator map Y^g, not a Jacobi identity.
- But as in the theory of intertwining operator algebras, there can be a Jacobi identity for suitable coefficients of twisted vertex operators.
- Bakalov, 2015: Gave an associator formula for Y_0^g and derived from this formula a Jacobi identity for Y_0^g where Y_0^g is given by $Y^g(u, x)v = \sum_{i=0}^{k} Y_i^g(u, x)v(\log x)^i$.
- Multiplicative Jordan-Chevalley decomposition: $g = \sigma e^{2\pi i \mathcal{N}}$ where σ is semisimple and $\mathcal{N}$ is nilpotent.

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- $V = \coprod_{\alpha\in\mathbb{C}/\mathbb{Z}} V^{[\alpha]}$ where $V^{[\alpha]}$ is the eigenspace of σ with the eigenvalue $e^{2\pi\alpha\sqrt{-1}}$.
- For $u \in V^{[\alpha]}$, $v \in V$,

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\begin{aligned}
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&=x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\left(\frac{x_2+x_0}{x_1}\right)^a\\
&\cdot Y_0^g\left(Y_V\left(\left(1+\frac{x_0}{x_2}\right)^{\mathcal{N}}u,x_0\right)v,x_2\right), \qquad (1)
\end{aligned}
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where $a \in \mathbb{C}$ such that $\Re\{a\} \in [0,1)$ and $a+\mathbb{Z}=\alpha$.

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The equivalence between the duality property and the Jacobi identity

- Bakalov stated that one can derive the Jacobi identity from the definition of twisted module using the duality as the main axiom. But no proof was given.
- H.-Yang: The duality property for Y^g and the Jacobi identity for Y_0^g are equivalent.
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- In the theory of vertex operator algebras, we are interested only in (g-twisted) modules with gradings and are lower bounded with respect to the gradings. For these (g-twisted) modules, there are lowest weight spaces. In the case that a (g-twisted) module is irreducible, the lowest weight space determines the module completely.
- We want to construct an associative algebra from the vertex operator algebra V and an automorphism g of V such that the lowest weight space of a g-twisted module is a module for this algebra. Since the action of such an algebra on the lowest weight space must preserve weights, such an algebra is called g-twisted zero-mode algebra.

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- H.-Yang: Constructed $A_g(V)$ in the case that the order of g is finite or infinite.
- Here we give the definition of an isomorphic algebra $\tilde{A}_g V)$ that generalizes the definition of Zhu's algebra given by H: For $u \in V^{[0]}$ and $v \in V$,

$$u \bullet_g v = \mathrm{Res}_y y^{-1} Y\left((1+y)^{\mathcal{N}} u, \frac{1}{2\pi i}\log(1+y)\right) v$$

and for $u \in V^{[\alpha]}$ $(\alpha \neq 0)$ and $v \in V$,

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- Let $\tilde{O}_g(V)$ be the subspace of V spanned by elements of the form

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- Let $\tilde{A}_g(V) = V/\tilde{O}_g(V)$.
- **Conjecture**: $Z_g(V)$ and $\tilde{A}_g(V)$ are isomorphic. Yang almost has a proof of this conjecture.

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Functors between module categories

- **Theorem**: Let W be a lower bounded generalized g-twisted V-module. Let $\Omega_g(W)$ be the subspace of W on which the action of the components of the vertex operators of negative weights are 0. Then $\Omega_g(V)$ is a $Z_g(V)$-module and also an $\tilde{A}_g(V)$-module. Moreover, the $Z_g(V)$-module structure and the $\tilde{A}_g(V)$-module structure on $\Omega^g(W)$ are compatible with the action of g.

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- **Theorem**: The functor Ω_g from the category of lower bounded g-twisted generalized V-modules to the category of graded $Z_g(V)$-modules given by $W \mapsto \Omega_g(W)$ has a right inverse, that is, there exists functors H_g from the categories of graded $Z_g(V)$-modules to the category of lower bounded g-twisted generalized V-modules such that $\Omega_g \circ H_g = 1$, where 1 is the identity functors on the categories of $Z_g(V)$-modules.
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