A construction of lower-bounded generalized twisted modules for a grading-restricted vertex (super)algebra

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Vertex algebras in mathematics and physics

University at Albany, SUNY

Outline

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Definition

- Frenkel, Lepowsky and Meurman (1988): Twisted modules associated to automorphisms of finite order of a vertex operator algebra.
- H. (2009): Twisted module associated a general automorphism of a vertex operator algebra.
- Twisted vertex operators contain logarithm of the variable when the automorphism does not act semisimply.
- V: A grading-restricted vertex superalgebra.
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Definition

- Data of a g-twisted V-module:
  - A $\mathbb{C} \times \mathbb{Z}_2 \times \mathbb{C}/\mathbb{Z}$-graded vector space

$$W = \coprod_{n\in\mathbb{C}, s\in\mathbb{Z}_2, [\alpha]\in\mathbb{C}/\mathbb{Z}} W_{[n]}^{s;[\alpha]}.$$

  - A twisted vertex operator map:

$$Y_W^g : V \otimes W \to W\{x\}[\log x],$$
$$v \otimes w \mapsto Y_W^g(v, x)w = \sum_{n\in\mathbb{C}} \sum_{k\in\mathbb{N}} (Y_W^g)_{n,k} x^n (\log x)^k$$

  - Operators $L_W(0)$ and $L_W(-1)$ on W.
  - An action of g on W.

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- Axioms:
  - The equivariance property:
  $$(Y_W^g)^{p+1}(gv, z)w = (Y_W^g)^p(v, z)w.$$
  - The identity property: $Y_W^g(\mathbf{1}, x)w = w$.
  - The duality property: The series
  $$\langle w', (Y_W^g)^p(u, z_1)(Y_W^g)^p(v, z_2)w\rangle,$$
  $$(-1)^{|u||v|}\langle w', (Y_W^g)^p(v, z_2)(Y_W^g)^p(u, z_1)w\rangle,$$
  $$\langle w', (Y_W^g)^p(Y_V(u, z_1 - z_2)v, z_2)w\rangle$$
  are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$ (also $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$), respectively, to
  $$\sum_{i,j,k,l=0}^{N} a_{ijkl} e^{m_i l_p(z_1)} e^{n_j l_p(z_2)} l_p(z_1)^k l_p(z_2)^l (z_1 - z_2)^{-t}.$$

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- More axioms:
  - The $L(0)$-grading condition: For $w \in W_{[n]}$, $(L_W(0) - n)^K w = 0$.
  - The $L(0)$-commutator formula:

$$[L_W(0), Y_W^g(v,z)] = z\frac{d}{dz}Y_W^g(v,z) + Y_W^g(L_V(0)v,z).$$

  - The g-grading condition: $(g - e^{2\pi\alpha i})^\Lambda w = 0$.
  - The g-compatibility condition and $\mathbb{Z}_2$-fermion number compatibility condition: $gY_W^g(u,x)w = Y_W^g(gu,x)gw$ and $|Y_W^g(u,x)w| = |u| + |w|$.
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A Jacobi identity

- Jacobi identity for twisted vertex operators (a reformulation of the Jacobi identity obtained by Frenkel-Lepowsky-Meurman, Bakalov, H.-Yang): Write the automorphism g as $g = e^{2\pi i \mathcal{L}_g}$. Then

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W^g(u, x_1) Y_W^g(v, x_2)$$

$$- (-1)^{|u||v|} x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) Y_W^g(v, x_2) Y_W^g(u, x_1)$$

$$= x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right) Y_W^g\left(Y_V\left(\left(\frac{x_2 + x_0}{x_1}\right)^{\mathcal{L}_g} u, x_0\right) v, x_2\right).$$

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The problem

- The problem: Give a **general**, **direct** and **explicit** construction of (universal lower-bounded generalized) g-twisted V-modules based on algebraic assumptions on a graded space and some operators on the space.
- **General**: Not just for special classes of vertex algebras (lattices, tensor powers of a vertex algebra, etc.) or special classes of automorphisms (permutation automorphisms or automorphisms of finite order, etc.).
- **Direct**: Not from modules for the twisted Zhu's algebras given by Dong-Li-Mason in the finite order case, the twisted zero-mode algebras given by H.-Yang or the twisted Zhu's algebra given by H.-Yang in the general case.
- **Explicit**: Not just a proof of the existence.

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Why is it important to have such a construction?

- The universal g-twisted V-modules are analogues of the Verma modules for finite-dimensional Lie algebras.
- This construction should be useful even for the construction and study of (untwisted) modules for those $\mathcal{W}$ algebras discussed in this workshop when the $\mathcal{W}$ algebras are obtained as kernels of the screening operators.
- In fact, the exponentials of the screening operators are automorphisms of the vertex algebras one starts with. The kernels of the screening operators are the same as the fixed point subalgebras of these automorphisms. One way to construct modules for the fixed point subalgebras is to construct twisted modules for the larger vertex algebras and then take the generalized eigenspaces for the automorphisms.

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Orbifold theory conjectures

- **Conjecture**: For a suitable vertex operator algebra V and a finite group G of automorphisms of V, the category of g-twisted V-modules for all $g \in G$ has a natural structure of G-crossed modular tensor category satisfying additional properties.
- A stronger **Conjecture**: Twisted intertwining operators among g-twisted V-modules for $g \in G$ satisfy associativity, commutativity and modular invariance property.
- In the case of $g = 1$, this stronger conjecture is a theorem (H., 2002, 2003).
- This theorem in the case of $g = 1$ indeed gives a modular tensor category (H., 2005).

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Outline

Definition

- For $v \in V$ and $w \in W$,

$$(Y^g)_{WV}^{W}(w, x)v = (-1)^{|v||w|} e^{xL_W(-1)} Y_W^g(v, y)w \Big|_{y^n = e^{\pi n i} x^n,\ \log y = \log x + \pi i}$$

- The twist vertex operator map:

$$(Y^g)_{WV}^{W} : W \otimes V \to W\{x\}[\log x], \quad w \otimes v \mapsto (Y^g)_{WV}^{W}(w, x)v$$

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Main properties

- The duality property: The series

$$\langle w', (Y_W^g)^{p_1}(u, z_1)((Y^g)_{WV}^W)^{p_2}(w, z_2)v\rangle,$$
$$(-1)^{|u||w|}\langle w', ((Y^g)_{WV}^W)^{p_2}(w, z_2)Y_V(u, z_1)v\rangle,$$
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are absolutely convergent in the regions given by $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to suitable branches of the multivalued analytic function

$$\sum_{j,k,m,n=0}^{N} a_{jkmn} z_1^{r} z_2^{s_j} (z_1 - z_2)^{t_k} (\log z_2)^m (\log(z_1 - z_2))^n.$$

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Outline

The algebra and the space

- Assume that $V = \coprod_{\alpha \in P_V} V^{[\alpha]}$ where $V^{[\alpha]}$ is the generalized eigenspace for g with eigenvalue $e^{2\pi i\alpha}$ and is generated by $\phi^i(x)$ for $i \in I$ of weights $\mathrm{wt}\,\phi^i$ and $\mathbb{Z}_2$-fermion numbers $|\phi^i|$. Also assume that for $i \in \mathcal{I}$, there exist $\alpha^i \in P_V$ and $\mathcal{N}_g(i) \in I$ such that $e^{2\pi i\mathcal{S}_g}\phi^i(z)e^{-2\pi i\mathcal{S}_g} = e^{2\pi\alpha^i}\phi^i(z)$ and $[\mathcal{N}_g, \phi^i(x)] = \phi^{\mathcal{N}_g(i)}(x)$, where $\mathcal{S}_g$ and $\mathcal{N}_g$ are the semisimple and nilpotent, respectively, parts of $\mathcal{L}_g$.
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$$W = \coprod_{n\in\mathbb{C}, s\in\mathbb{Z}_2, [\alpha]\in\mathbb{C}/\mathbb{Z}} W_{[n]}^{s;[\alpha]}$$
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The fields and operators

- Generating twisted fields: For $i \in I$, $\phi^i_W(x) \in x^{\alpha^i}(\mathrm{End}\ W)[[x, x^{-1}]][\log x]$ such that for $w \in W$,
$$\phi^i_W(x)w = \sum_{k=0}^{K^i} \sum_{n \in \alpha^i + N - \mathbb{N}} (\phi^i_W)_{n,k} w x^{-n-1}(\log x)^k.$$
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where $\psi^{L_W(0)_N(a)}_W(x) = 0$ when $L_W(0)_N\psi^a_W(x) = 0$.
- For $i \in I$, $[L^g_W(-1), \phi^i_W(x)] = \frac{d}{dx}\phi^i_W(x)$ and for $a \in A$, $L^g_W(-1)\psi^a_W(x) - \psi^a(x)L_V(-1) = \frac{d}{dz}\psi^a_W(x)$.
- For $a \in A$, $\psi^a_W(x)\mathbf{1} \in W[[x]]$ and its constant terms $\lim_{x\to 0}\psi^a_W(x)\mathbf{1}$ is homogeneous with respect to weights, $\mathbb{Z}_2$-fermion number and g-weights.

The assumptions

- The vector space W is spanned by elements of the form $(\phi^{i_1}_W)_{n_1,l_1}\cdots(\phi^{i_k}_W)_{n_k,l_k}(\psi^a_W)_{n,l}v$ for $i_1,\dots,i_k\in I$, $a\in\mathcal{A}$ and $n_1\in\alpha^{i_1}+\mathbb{Z},\dots,n_k\in\alpha^{i_k}+\mathbb{Z}$, $n\in\mathbb{C}$, $l_1,\dots,l_k,l\in\mathbb{N}$, $v\in V$.
- Let $\mathcal{S}_g$ and $\mathcal{N}_g$ be the semisimple and nilpotent parts, respectively, of $\mathcal{L}_g$. (i) For $i\in I$, $g\phi^{i;p+1}_W(z)g^{-1}=\phi^{i;p}_W(z)$. (ii) For $i\in I$, $\phi^i_W(x)=x^{-\mathcal{N}_g}(\phi^i_W)_0(x)x^{\mathcal{N}_g}$ and for $a\in A$, $\psi^a_W(x)=(\psi^a_W)_0(x)e^{-\pi i\mathcal{N}_g}x^{-\mathcal{N}_g}$ where $(\phi^i_W)_0(x)$ and $(\psi^a_W)_0(x)$ are the constant terms of $\phi^i_W(x)$ and $\psi^a_W(x)$, respectively, viewed as a power series of $\log x$. (iii) For $i\in I$, $e^{2\pi i\mathcal{S}_g}\phi^i_W(z)e^{-2\pi i\mathcal{S}_g}=e^{2\pi\alpha^i}\phi^i_W(z)$ and $[\mathcal{N}_g,\phi^i_W(z)]=\phi^{\mathcal{N}_g(i)}_W(z)$. (iv) For $a\in A$, there exists $\alpha^a\in[0,1)$ such that $(\psi^a_W)_{n,0}\mathbf{1}$ for $n\in-\mathbb{N}-1$ are generalized eigenvectors for g with the eigenvalue $e^{2\pi i\alpha^a}$.

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The main assumptions

- For $i, j \in I$, there exists $M_{ij} \in \mathbb{Z}_+$ such that

$$(x_1 - x_2)^{M_{ij}} \phi^i_W(x_1) \phi^j_W(x_2) = (x_1 - x_2)^{M_{ij}} (-1)^{|\phi^i||\phi^j|} \phi^j_W(x_2) \phi^i_W(x_1).$$

- For $i \in I$ and $a \in A$, there exists $M_{ia} \in \mathbb{Z}_+$ such that

$$(x_1 - x_2)^{\alpha_i + M_{ia}} (x_1 - x_2)^{\mathcal{N}_g} \phi^i_W(x_1) (x_1 - x_2)^{-\mathcal{N}_g} \psi^a_W(x_2) = (-x_2 + x_1)^{\alpha_i + M_{ia}} (-1)^{|\phi^i||\psi^a|} \cdot \psi^a_W(x_2) (-x_2 + x_1)^{\mathcal{N}_g} \phi^i(x_1) (-x_2 + x_1)^{-\mathcal{N}_g}.$$

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The twisted vertex operator map

Theorem

For $i \in I$ and $p \in \mathbb{Z}$, let $\phi_W^{i;p}(z)$ be the p-th branch of $\phi_W^i(z)$. The series

$$\langle w', \phi_W^{i_1,p}(z_1) \cdots \phi_W^{i_k,p}(z_k) w \rangle$$

is absolutely convergent in the region $|z_1| > \cdots > |z_k| > 0$ to an analytic function of the form

$$\sum_{n_1,\dots,n_k=0}^{N} f_{n_1\cdots n_k}(z_1,\dots,z_k) e^{-\alpha_{i_1} l_p(z_1)} \cdots e^{-\alpha_{i_k} l_p(z_k)} \cdot (l_p(z_1))^{n_1} \cdots (l_p(z_k))^{n_k},$$

where $f_{n_1\cdots n_k}(z_1,\dots,z_k)$ are rational functions with the usual poles, and denoted by $F^p(\langle w', \phi_W^{i_1}(z_1) \cdots \phi_W^{i_k}(z_k) w \rangle)$.

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- For $w' \in W'$, $w \in W$, $i_1, \dots, i_k \in I$, $m_1, \dots, m_k \in \mathbb{Z}$,

$$\begin{aligned}\langle w', (Y_W^g)^p(\phi^{i_1}_{m_1} \cdots \phi^{i_k}_{m_k}\mathbf{1}, z)w\rangle \\ = \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0}\xi_1^{m_1} \cdots \xi_k^{m_k} \cdot \\ \cdot F^p(\langle w', \phi^{i_1}_W(\xi_1+z) \cdots \phi^{i_k}_W(\xi_k+z)w\rangle).\end{aligned}$$

- If $\sum_{\mu=1}^{M} \lambda_p \phi^{i_1^\mu}_{m_1^\mu} \cdots \phi^{i_k^\mu}_{m_k^\mu}\mathbf{1} = 0$, then for $w' \in W'$, $w \in W$

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In particular, Y_W^g is well defined.

The main theorem

Theorem

The pair (W, Y_W^g) is a grading-restricted generalized g-twisted V-module generated by $(\psi_W^a)_{n,k}v$ for $a \in A$, $n \in \alpha + \mathbb{Z}$, $v \in V^{[\alpha]}$ and $\alpha \in P_V$. Moreover, this is the unique generalized g-twisted V-module structure on W generated by $(\psi_W^a)_{n,k}v$ for $a \in A$, $n \in \alpha + \mathbb{Z}$, $v \in V^{[\alpha]}$ and $\alpha \in P_V$ such that $Y_W(\phi_{-1}^i \mathbf{1}, z) = \phi_W^i(z)$ for $i \in I$.

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Outline

Twisted affinization

- Let $\hat{V}^{[g]}_{\phi} = \coprod_{i\in I, k\in\mathbb{N}}(\mathbb{C}\mathcal{N}_g^k\phi^i_{-1}\mathbf{1}\otimes t^{\alpha^i}\mathbb{C}[t, t^{-1}])\oplus\mathbb{C}L_0\oplus\mathbb{C}L_{-1}$, where L_0 and L_{-1} are two abstract elements.
- Let $T(\hat{V}^{[g]}_{\phi})$ be the tensor algebra of $\hat{V}^{[g]}_{\phi}$.
- Let $\phi^i_{\hat{V}^{[g]}_{\phi}}(x) = \sum_{n\in\alpha^i+\mathbb{Z}}((x^{-\mathcal{N}_g}\phi^i_{-1}\mathbf{1})\otimes t^n)x^{-n-1}$ for $i\in I$.
- $\phi^i_{\hat{V}^{[g]}_{\phi}}(x)$ for $i\in I$ can be viewed as formal series of operators on $T(\hat{V}^{[g]}_{\phi})$. Also L_0 and L_{-1} can be viewed as operators, on $T(\hat{V}^{[g]}_{\phi})$.
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Weak commutativity for $\phi^{i}_{\hat{V}^{[g]}_{\phi}}(x)$

- For $i, j \in I$, let $M_{i,j}$ be the smallest positive integer such that $x_0^{M_{i,j}} Y_V(\phi^i_{-1}\mathbf{1}, x)\phi^j_{-1}\mathbf{1}$ is a power series in x.
- Let $J(\hat{V}^{[g]}_{\phi})$ be the ideal of $T(\hat{V}^{[g]}_{\phi})$ generated by the coefficients of the formal series

$$(x_1 - x_2)^{M_{ij}} \phi^{i}_{\hat{V}^{[g]}_{\phi}}(x_1)\phi^{j}_{\hat{V}^{[g]}_{\phi}}(x_2) - (-1)^{|\phi^i||\phi^j|}(x_1 - x_2)^{M_{ij}} \phi^{j}_{\hat{V}^{[g]}_{\phi}}(x_2)\phi^{i}_{\hat{V}^{[g]}_{\phi}}(x_1),$$

$$L_0\phi^{i}_{\hat{V}^{[g]}_{\phi}}(x) - \phi^{i}_{\hat{V}^{[g]}_{\phi}}(x)L_0 - x\frac{d}{dx}\phi^{i}_{\hat{V}^{[g]}_{\phi}}(x) - (\mathrm{wt}\ \phi^i)\phi^{i}_{\hat{V}^{[g]}_{\phi}}(x),$$

$$L_{-1}\phi^{i}_{\hat{V}^{[g]}_{\phi}}(x) - \phi^{i}_{\hat{V}^{[g]}_{\phi}}(x)L_{-1} - \frac{d}{dx}\phi^{i}_{\hat{V}^{[g]}_{\phi}}(x)$$

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$$(x_1 - x_2)^{M_{ij}} \phi^i_{\hat{V}^{[g]}_\phi}(x_1)\phi^j_{\hat{V}^{[g]}_\phi}(x_2) - (-1)^{|\phi^i||\phi^j|}(x_1 - x_2)^{M_{ij}} \phi^j_{\hat{V}^{[g]}_\phi}(x_2)\phi^i_{\hat{V}^{[g]}_\phi}(x_1),$$

$$L_0\phi^i_{\hat{V}^{[g]}_\phi}(x) - \phi^i_{\hat{V}^{[g]}_\phi}(x)L_0 - x\frac{d}{dx}\phi^i_{\hat{V}^{[g]}_\phi}(x) - (\mathrm{wt}\ \phi^i)\phi^i_{\hat{V}^{[g]}_\phi}(x),$$

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A space of generators

- Let M be a $\mathbb{Z}_2$-graded vector space (graded by $\mathbb{Z}_2$-fermion numbers).
- Assume that g acts on M and there is an operator $L_M(0)$ on M.
- Assume that there exist operators $\mathcal{L}_g$, $\mathcal{S}_g$, $\mathcal{N}_g$ such that on M, $g = e^{2\pi i \mathcal{L}_g}$ and $\mathcal{S}_g$ and $\mathcal{N}_g$ are the semisimple and nilpotent, respectively, parts of $\mathcal{L}_g$.
- Also assume that $L_M(0)$ can be decomposed as the sum of its semisimple part $L_M(0)_S$ and nilpotent part $L_M(0)_N$ and that the real parts of the eigenvalues of $L_M(0)$ has a lower bound. In particular, M is a direct sum of generalized eigenspaces for the operator $L_M(0)$.

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- We call the eigenvalue of a generalized eigenvector $w \in M$ for $L_M(0)$ the *weight* of w and denote it by $\mathrm{wt}\ w$.
- Let $\{w^a\}_{a\in A}$ be a basis of M consisting of vectors homogeneous in weights, $\mathbb{Z}_2$-fermion numbers and g-weights (eigenvalues of $\mathcal{L}_g$) such that for $a \in A$, either $L_M(0)_N w^a = 0$ or there exists $L_M(0)_N(a) \in A$ such that $L_M(0)_N w^a = w^{L_M(0)_N(a)}$.
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The graded space

- Let $\widetilde{M}^{[g]} = \coprod_{\alpha \in P_V} U(\hat{V}_\phi^{[g]}) \otimes (M \otimes t^\alpha \mathbb{C}[t, t^{-1}]) \otimes V^{[\alpha]}$. Then $\widetilde{M}^{[g]}$ is a left $U(\hat{V}_\phi^{[g]})$-module.
- for $i \in I$, we denote the action of $\phi^i_{\hat{V}_\phi^{[g]}}(x)$ on $\widetilde{M}^{[g]}$ by $\phi^i_{\widetilde{M}^{[g]}}(x)$. Then $\phi^i_{\widetilde{M}^{[g]}}(x)$ for $i \in I$ satisfy the same weak commutativity, $L(0)$-commutator formula and $L(-1)$-commutator formula as those for $\phi^i_{\hat{V}_\phi^{[g]}}(x)$.
- For $i \in I$, let $K^i \in \mathbb{N}$ such that $\mathcal{N}_g^{K^i+1} \phi^i_{-1} \mathbf{1} = 0$ and we denote the actions of the elements $\frac{(-1)^k}{k!}(\mathcal{N}_g^k \phi^i_{-1} \mathbf{1}) \otimes t^n$ for $n \in \alpha + \mathbb{Z}$ and $k = 0, \ldots, K^i$ of $U(\hat{V}_\phi^{[g]})$ on $\widetilde{M}_\ell^{[g]}$ by $(\phi^i_{\widetilde{M}^{[g]}})_{n,k}$.

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The graded space

- For $a \in A$, $\alpha \in P_V$, $n \in \alpha + \mathbb{Z}$ and $k \in \mathbb{N}$, we denote the linear map $v \mapsto \frac{(-1)^k}{k!}(w^a \otimes t^n) \otimes \mathcal{N}_g^k v$ from $V^{[\alpha]}$ to $\widetilde{M}^{[g]}$ by $(\psi^a_{\widetilde{M}^{[g]}})_{n,k}$ and extend it to a linear map from V to $\widetilde{M}^{[g]}$ by mapping $V^{[\alpha']}$ to 0 for $\alpha' \neq \alpha$.
- Then $\widetilde{M}^{[g]}$ is spanned by elements of the form

$$(\phi^{i_1}_{\widetilde{M}^{[g]}})_{n_1,k_1} \cdots (\phi^{i_l}_{\widetilde{M}^{[g]}})_{n_l,k_l} (L_{\widetilde{M}^{[g]}}(m))^q)(\psi^a_{\widetilde{M}^{[g]}})_{n,k} v,$$

for $i_1, \dots, i_l \in I$, $n_1 \in \alpha^{i_1} + \mathbb{Z}, \dots, n_l \in \alpha^{i_l} + \mathbb{Z}$, $0 \leq k_1 \leq K^{i_1}, \dots, 0 \leq k_l \leq K^{i_l}$, $m = 0, -1$, $q \in \mathbb{N}$, $a \in A$, $n \in \alpha + \mathbb{Z}$, $0 \leq k \leq K$, $v \in V^{[\alpha]}$, $\alpha \in P_V$, where $K \in \mathbb{N}$ satisfying $\mathcal{N}_g^{K+1} v = 0$.

The graded space

- For $a \in A$, $\alpha \in P_V$, $n \in \alpha + \mathbb{Z}$ and $k \in \mathbb{N}$, we denote the linear map $v \mapsto \frac{(-1)^k}{k!}(w^a \otimes t^n) \otimes \mathcal{N}_g^k v$ from $V^{[\alpha]}$ to $\widetilde{M}^{[g]}$ by $(\psi^a_{\widetilde{M}^{[g]}})_{n,k}$ and extend it to a linear map from V to $\widetilde{M}^{[g]}$ by mapping $V^{[\alpha']}$ to 0 for $\alpha' \neq \alpha$.
- Then $\widetilde{M}^{[g]}$ is spanned by elements of the form

$$(\phi^{i_1}_{\widetilde{M}^{[g]}})_{n_1,k_1} \cdots (\phi^{i_l}_{\widetilde{M}^{[g]}})_{n_l,k_l}(L_{\widetilde{M}^{[g]}}(m))^q)(\psi^a_{\widetilde{M}^{[g]}})_{n,k} v,$$

for $i_1, \ldots, i_l \in I$, $n_1 \in \alpha^{i_1} + \mathbb{Z}, \ldots, n_l \in \alpha^{i_l} + \mathbb{Z}$, $0 \leq k_1 \leq K^{i_1}, \ldots, 0 \leq k_l \leq K^{i_l}$, $m = 0, -1$, $q \in \mathbb{N}$, $a \in A$, $n \in \alpha + \mathbb{Z}$, $0 \leq k \leq K$, $v \in V^{[\alpha]}$, $\alpha \in P_V$, where $K \in \mathbb{N}$ satisfying $\mathcal{N}_g^{K+1} v = 0$.

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for $i_1, \dots, i_l \in I$, $n_1 \in \alpha^{i_1} + \mathbb{Z}, \dots, n_l \in \alpha^{i_l} + \mathbb{Z}$, $0 \leq k_1 \leq K^{i_1}, \dots, 0 \leq k_l \leq K^{i_l}$, $m = 0, -1$, $q \in \mathbb{N}$, $a \in A$, $n \in \alpha + \mathbb{Z}$, $0 \leq k \leq K$, $v \in V^{[\alpha]}$, $\alpha \in P_V$, where $K \in \mathbb{N}$ satisfying $\mathcal{N}_g^{K+1} v = 0$.

The graded space

- Let $\psi^{a}_{\widetilde{M}^{[g]}}(x)v = \sum_{k=0}^{K^v} \sum_{n\in\alpha+\mathbb{Z}} (\psi^{a}_{\widetilde{M}^{[g]}})_{n,k} v x^{-n-1} (\log x)^k$.
- Let $B \in \mathbb{R}$ such that $B \leq \Re(\mathrm{wt}\; w^a)$ for $a \in A$. Let $J_B(\widetilde{M}^{[g]})$ be the $U(\hat{V}^{[g]}_{\phi})$-submodule of $\widetilde{M}^{[g]}$ generated by elements of the following forms: (i) $(\psi^{a}_{\widetilde{M}^{[g]}})_{n,0}\mathbf{1}$ for $a \in A$, and $n \notin -\mathbb{N} - 1$; (ii) homogeneous elements such that the real parts of their weights are less than B.
- Since $J_B(\widetilde{M}^{[g]})$ is spanned by homogeneous elements, $\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]})$ is also graded. In addition, $\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]})$ is lower bounded with respect to the weight grading with a lower bound B.

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The graded space

- For $i \in I$ and $a \in A$, let $M_{i,a} \in \mathbb{Z}_+$ be the smallest of $m \in \mathbb{Z}$ such that $m > \mathrm{wt}\ \phi^i - 1 + \Re(\mathrm{wt}\ w^a) - B - \Re(\alpha^i)$.
- Let $J(\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]}))$ be the $U(\hat{V}_{\phi}^{[g]})$-submodule of $\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]})$ generated by the coefficients of the series

$$(x_1 - x_2)^{\alpha^i + M_{i,a}}(x_1 - x_2)^{\mathcal{N}_g}\phi^i_{\widetilde{M}^{[g]}}(x_1)(x_1 - x_2)^{-\mathcal{N}_g}\psi^a_{\widetilde{M}^{[g]}}(x_2)v$$
$$-(-1)^{|u||w|}(-x_2 + x_1)^{\alpha^i + M_{i,a}}\psi^a_{\widetilde{M}^{[g]}}(x_2)\cdot$$
$$\cdot(-x_2 + x_1)^{\mathcal{N}_g}\phi^i(x_1)(-x_2 + x_1)^{-\mathcal{N}_g}v,$$

$$L_{\widetilde{M}^{[g]}}(0)\psi^a_{\widetilde{M}^{[g]}}(x)v - \psi^a_{\widetilde{M}^{[g]}}(x)L_V(0)v$$
$$-x\frac{d}{dx}\psi^a_{\widetilde{M}^{[g]}}(x)v - (\mathrm{wt}\ w^a)\psi^a_{\widetilde{M}^{[g]}}(x)v - \psi^{L_M(0)_N a}_{\widetilde{M}^{[g]}}(x)v,$$

$$L_{\widetilde{M}^{[g]}}(-1)\psi^a_{\widetilde{M}^{[g]}}(x)v - \psi^a_{\widetilde{M}^{[g]}}(x)L_V(-1)v - \frac{d}{dx}\psi^a_{\widetilde{M}^{[g]}}(x)v$$

The graded space

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$$-(-1)^{|u||w|}(-x_2+x_1)^{\alpha^i+M_{i,a}}\psi^a_{\widetilde{M}^{[g]}}(x_2)\cdot$$
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$$-x\frac{d}{dx}\psi^a_{\widetilde{M}^{[g]}}(x)v-(\mathrm{wt}\ w^a)\psi^a_{\widetilde{M}^{[g]}}(x)v-\psi^{L_M(0)_N a}_{\widetilde{M}^{[g]}}(x)v,$$

$$L_{\widetilde{M}^{[g]}}(-1)\psi^a_{\widetilde{M}^{[g]}}(x)v-\psi^a_{\widetilde{M}^{[g]}}(x)L_V(-1)v-\frac{d}{dx}\psi^a_{\widetilde{M}^{[g]}}(x)v$$

The graded space

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$$-(-1)^{|u||w|}(-x_2 + x_1)^{\alpha^i + M_{i,a}}\psi^a_{\widetilde{M}^{[g]}}(x_2)\cdot$$
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$$-(-1)^{|u||w|}(-x_2 + x_1)^{\alpha^i + M_{i,a}}\psi^a_{\widetilde{M}^{[g]}}(x_2)\cdot$$
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$$\cdot (-x_2 + x_1)^{\mathcal{N}_g}\phi^i(x_1)(-x_2 + x_1)^{-\mathcal{N}_g}v,$$
$$L_{\widetilde{M}^{[g]}}(0)\psi^a_{\widetilde{M}^{[g]}}(x)v - \psi^a_{\widetilde{M}^{[g]}}(x)L_V(0)v$$
$$-x\frac{d}{dx}\psi^a_{\widetilde{M}^{[g]}}(x)v - (\mathrm{wt}\ w^a)\psi^a_{\widetilde{M}^{[g]}}(x)v - \psi^{L_M(0)_N a}_{\widetilde{M}^{[g]}}(x)v,$$
$$L_{\widetilde{M}^{[g]}}(-1)\psi^a_{\widetilde{M}^{[g]}}(x)v - \psi^a_{\widetilde{M}^{[g]}}(x)L_V(-1)v - \frac{d}{dx}\psi^a_{\widetilde{M}^{[g]}}(x)v$$

The graded space

- Let $\widehat{M}_B^{[g]} = (\widetilde{M}_\ell^{[g]}/J_B(\widetilde{M}^{[g]}))/J(\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]}))$.
- In $\widehat{M}_B^{[g]}$, we have

$$\begin{aligned}
&(x_1 - x_2)^{\alpha^i + M_{i,a}}(x_1 - x_2)^{\mathcal{N}_g}\phi^i_{\widehat{M}_B^{[g]}}(x_1)(x_1 - x_2)^{-\mathcal{N}_g}\psi^a_{\widehat{M}_B^{[g]}}(x_2)v \\
&\quad = (-1)^{|u||w|}(-x_2 + x_1)^{\alpha^i + M_{i,a}}\psi^a_{\widehat{M}_B^{[g]}}(x_2)\cdot \\
&\qquad \cdot(-x_2 + x_1)^{\mathcal{N}_g}\phi^i(x_1)(-x_2 + x_1)^{-\mathcal{N}_g}v, \\
&L_{\widehat{M}_B^{[g]}}(0)\psi^a_{\widehat{M}_B^{[g]}}(x)v - \psi^a_{\widehat{M}_B^{[g]}}(x)L_V(0)v \\
&\quad = x\frac{d}{dx}\psi^a_{\widehat{M}_B^{[g]}}(x) + (\mathrm{wt}\ w^a)\psi^a_{\widehat{M}_B^{[g]}}(x) + \psi^{L_M(0)_N a}_{\widehat{M}_B^{[g]}}(x), \\
&L_{\widehat{M}_B^{[g]}}(-1)\psi^a_{\widehat{M}_B^{[g]}}(x)v - \psi^a_{\widehat{M}_B^{[g]}}(x)L_V(-1)v = \frac{d}{dx}\psi^a_{\widehat{M}_B^{[g]}}(x)v.
\end{aligned}$$

The graded space

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- In $\widehat{M}_B^{[g]}$, we have

$$
\begin{aligned}
(x_1-x_2)^{\alpha^i+M_{i,a}}&(x_1-x_2)^{\mathcal{N}_g}\phi^i_{\widehat{M}_B^{[g]}}(x_1)(x_1-x_2)^{-\mathcal{N}_g}\psi^a_{\widehat{M}_B^{[g]}}(x_2)v\\
&= (-1)^{|u||w|}(-x_2+x_1)^{\alpha^i+M_{i,a}}\psi^a_{\widehat{M}_B^{[g]}}(x_2)\cdot\\
&\qquad\cdot(-x_2+x_1)^{\mathcal{N}_g}\phi^i(x_1)(-x_2+x_1)^{-\mathcal{N}_g}v,
\end{aligned}
$$

$$
\begin{aligned}
L_{\widehat{M}_B^{[g]}}(0)&\psi^a_{\widehat{M}_B^{[g]}}(x)v - \psi^a_{\widehat{M}_B^{[g]}}(x)L_V(0)v\\
&= x\frac{d}{dx}\psi^a_{\widehat{M}_B^{[g]}}(x) + (\mathrm{wt}\, w^a)\psi^a_{\widehat{M}_B^{[g]}}(x) + \psi^{L_M(0)_N a}_{\widehat{M}_B^{[g]}}(x),
\end{aligned}
$$

$$
L_{\widehat{M}_B^{[g]}}(-1)\psi^a_{\widehat{M}_B^{[g]}}(x)v - \psi^a_{\widehat{M}_B^{[g]}}(x)L_V(-1)v = \frac{d}{dx}\psi^a_{\widehat{M}_B^{[g]}}(x)v.
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The graded space

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\begin{aligned}
&(x_1 - x_2)^{\alpha^i + M_{i,a}}(x_1 - x_2)^{\mathcal{N}_g}\phi^i_{\widehat{M}_B^{[g]}}(x_1)(x_1 - x_2)^{-\mathcal{N}_g}\psi^a_{\widehat{M}_B^{[g]}}(x_2)v \\
&\quad = (-1)^{|u||w|}(-x_2 + x_1)^{\alpha^i + M_{i,a}}\psi^a_{\widehat{M}_B^{[g]}}(x_2)\cdot \\
&\qquad \cdot (-x_2 + x_1)^{\mathcal{N}_g}\phi^i(x_1)(-x_2 + x_1)^{-\mathcal{N}_g}v,
\end{aligned}
$$

$$
\begin{aligned}
&L_{\widehat{M}_B^{[g]}}(0)\psi^a_{\widehat{M}_B^{[g]}}(x)v - \psi^a_{\widehat{M}_B^{[g]}}(x)L_V(0)v \\
&\quad = x\frac{d}{dx}\psi^a_{\widehat{M}_B^{[g]}}(x) + (\mathrm{wt}\ w^a)\psi^a_{\widehat{M}_B^{[g]}}(x) + \psi^{L_M(0)_N a}_{\widehat{M}_B^{[g]}}(x),
\end{aligned}
$$

$$
L_{\widehat{M}_B^{[g]}}(-1)\psi^a_{\widehat{M}_B^{[g]}}(x)v - \psi^a_{\widehat{M}_B^{[g]}}(x)L_V(-1)v = \frac{d}{dx}\psi^a_{\widehat{M}_B^{[g]}}(x)v.
$$

The graded space

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$$(x_1 - x_2)^{\alpha^i + M_{i,a}}(x_1 - x_2)^{\mathcal{N}_g}\phi^i_{\widehat{M}_B^{[g]}}(x_1)(x_1 - x_2)^{-\mathcal{N}_g}\psi^a_{\widehat{M}_B^{[g]}}(x_2)v$$
$$= (-1)^{|u||w|}(-x_2 + x_1)^{\alpha^i + M_{i,a}}\psi^a_{\widehat{M}_B^{[g]}}(x_2)\cdot$$
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$$L_{\widehat{M}_B^{[g]}}(0)\psi^a_{\widehat{M}_B^{[g]}}(x)v - \psi^a_{\widehat{M}_B^{[g]}}(x)L_V(0)v$$
$$= x\frac{d}{dx}\psi^a_{\widehat{M}_B^{[g]}}(x) + (\mathrm{wt}\ w^a)\psi^a_{\widehat{M}_B^{[g]}}(x) + \psi^{L_M(0)_N a}_{\widehat{M}_B^{[g]}}(x),$$

$$L_{\widehat{M}_B^{[g]}}(-1)\psi^a_{\widehat{M}_B^{[g]}}(x)v - \psi^a_{\widehat{M}_B^{[g]}}(x)L_V(-1)v = \frac{d}{dx}\psi^a_{\widehat{M}_B^{[g]}}(x)v.$$

The graded space

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\begin{aligned}
&(x_1 - x_2)^{\alpha^i + M_{i,a}}(x_1 - x_2)^{\mathcal{N}_g}\phi^i_{\widehat{M}_B^{[g]}}(x_1)(x_1 - x_2)^{-\mathcal{N}_g}\psi^a_{\widehat{M}_B^{[g]}}(x_2)v\\
&\quad = (-1)^{|u||w|}(-x_2 + x_1)^{\alpha^i + M_{i,a}}\psi^a_{\widehat{M}_B^{[g]}}(x_2)\cdot\\
&\qquad \cdot(-x_2 + x_1)^{\mathcal{N}_g}\phi^i(x_1)(-x_2 + x_1)^{-\mathcal{N}_g}v,\\
&L_{\widehat{M}_B^{[g]}}(0)\psi^a_{\widehat{M}_B^{[g]}}(x)v - \psi^a_{\widehat{M}_B^{[g]}}(x)L_V(0)v\\
&\quad = x\frac{d}{dx}\psi^a_{\widehat{M}_B^{[g]}}(x) + (\mathrm{wt}\ w^a)\psi^a_{\widehat{M}_B^{[g]}}(x) + \psi^{L_M(0)_N a}_{\widehat{M}_B^{[g]}}(x),\\
&L_{\widehat{M}_B^{[g]}}(-1)\psi^a_{\widehat{M}_B^{[g]}}(x)v - \psi^a_{\widehat{M}_B^{[g]}}(x)L_V(-1)v = \frac{d}{dx}\psi^a_{\widehat{M}_B^{[g]}}(x)v.
\end{aligned}
$$

The graded space

- Let $\widehat{M}_B^{[g]} = (\widetilde{M}_\ell^{[g]}/J_B(\widetilde{M}^{[g]}))/J(\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]}))$.
- In $\widehat{M}_B^{[g]}$, we have

$$\begin{aligned}
&(x_1 - x_2)^{\alpha^i + M_{i,a}}(x_1 - x_2)^{\mathcal{N}_g}\phi^i_{\widehat{M}_B^{[g]}}(x_1)(x_1 - x_2)^{-\mathcal{N}_g}\psi^a_{\widehat{M}_B^{[g]}}(x_2)v\\
&\quad = (-1)^{|u||w|}(-x_2 + x_1)^{\alpha^i + M_{i,a}}\psi^a_{\widehat{M}_B^{[g]}}(x_2)\cdot\\
&\qquad \cdot (-x_2 + x_1)^{\mathcal{N}_g}\phi^i(x_1)(-x_2 + x_1)^{-\mathcal{N}_g}v,\\
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\end{aligned}$$

The lower-bounded generalized twisted module

Theorem

The twisted fields $\phi^i_{\widehat{M}_B^{[g]}}$ *for* $i \in I$ *generate a twisted vertex operator map*

$$Y^g_{\widehat{M}_B^{[g]}} : V \otimes \widehat{M}_B^{[g]} \to \widehat{M}_B^{[g]}\{x\}[\log x]$$

such that $(\widehat{M}_B^{[g]}, Y^g_{\widehat{M}_B^{[g]}})$ *is a lower-bounded generalized* g*-twisted* V*-module. Moreover, this is the unique generalized* g*-twisted* V*-module structure on* $\widehat{M}_B^{[g]}$ *generated by the coefficients of* $(\psi^a_{\widehat{M}_B^{[g]}})(x)v$ *for* $a \in A$ *and* $v \in V$ *such that* $Y^g_{\widehat{M}_B^{[g]}}(\phi^i_{-1}\mathbf{1}, z) = \phi^i_{\widehat{M}_B^{[g]}}(z)$ *for* $i \in I$.

Outline

The universal property $\widehat{M}_B^{[g]}$

Theorem

Let (W, Y_W^g) be a lower-bounded generalized g-twisted V-module and M_0 a $\mathbb{Z}_2$-graded subspace of W invariant under the actions of g, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)$, $L_W(0)_S$ and $L_W(0)_N$. Let $B \in \mathbb{R}$ such that $W_{[n]} = 0$ when $\Re(n) < B$. Assume that there is a linear map $f : M \to M_0$ preserving the $\mathbb{Z}_2$-fermion number grading and commuting with the actions of g, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)$ ($L_{\widehat{M}_B^{[g]}}(0)$), $L_W(0)_S$ ($L_{\widehat{M}_B^{[g]}}(0)_S$) and $L_W(0)_N$ ($L_{\widehat{M}_B^{[g]}}(0)_N$). Then there exists a unique module map $\tilde{f} : \widehat{M}_B^{[g]} \to W$ such that $\tilde{f}|_M = f$. If f is surjective and (W, Y_W^g) is generated by the coefficients of $(Y^g)_{WV}^W(w_0, x)v$ for $w_0 \in M_0$ and $v \in V$, where $(Y^g)_{WV}^W$ is the twist vertex operator map obtained from Y_W^g, then $\tilde{f}$ is surjective.

Lower-bounded generalized twisted modules as quotients

Corollary

Let (W, Y_W^g) be a lower-bounded generalized g-twisted V-module generated by the coefficients of $(Y^g)_{WV}^W(w, x)v$ for $w \in M$, where $(Y^g)_{WV}^W$ is the twist vertex operator map obtained from Y_W^g and M is a $\mathbb{Z}_2$-graded subspace of W invariant under the actions of g, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)$, $L_W(0)_S$ and $L_W(0)_N$. Let $B \in \mathbb{R}$ such that $W_{[n]} = 0$ when $\Re(n) < B$. Then there is a generalized g-twisted V-submodule J of $\widehat{M}_B^{[g]}$ such that W is equivalent as a lower-bounded generalized g-twisted V-module to the quotient module $\widehat{M}_B^{[g]}/J$.

The structure and existence of lower-bounded generalized twisted modules

Theorem

The lower-bounded generalized g-twisted V-module $\widehat{M}_B^{[g]}$ is in fact generated by $(\psi^a_{\widehat{M}_B^{[g]}})_{-k-1,0}\mathbf{1} = L_{\widehat{M}_B^{[g]}}(-1)^k(\psi^a_{\widehat{M}_B^{[g]}})_{-1,0}\mathbf{1}$ for $a \in A$ and $k \in \mathbb{N}$.

Theorem

The generators $(\psi^a_{\widehat{M}_B^{[g]}})_{-k-1,0}\mathbf{1}$ for $k \in \mathbb{N}$ and $a \in A$ of $\widehat{M}_B^{[g]}$ are linearly independent. In particular, $\widehat{M}_B^{[g]}$ is not 0.

- In the case that g is of finite order, the last theorem solves the open problem of more than 20 years on the existence of lower-bounded generalized twisted modules

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