

# Weight-one elements of vertex operator algebras and automorphisms of categories of generalized twisted modules

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## Abstract

Given a weight-one element  $u$  of a vertex operator algebra  $V$ , we construct an automorphism of the category of generalized  $g$ -twisted modules for automorphisms  $g$  of  $V$  fixing  $u$ . We apply this construction to the case that  $V$  is an affine vertex operator algebra to obtain explicit results on these automorphisms of categories. In particular, we give explicit constructions of certain generalized twisted modules from generalized twisted modules associated to diagram automorphisms of finite-dimensional simple Lie algebras and generalized (untwisted) modules.

## 1 Introduction

Orbifold conformal field theories are conformal field theories obtained from known conformal field theories and their automorphisms. One conjecture in orbifold conformal field theory is that a category of suitable generalized twisted modules for a suitable vertex operator algebra has a natural structure of a crossed tensor category in the sense of Turaev [T], where it is called a crossed group-category. See [H3] and [H8] for the precise conjectures and open problems.

To study orbifold conformal field theories and to construct crossed tensor category structures, we first need to study categories of generalized twisted modules satisfying suitable conditions. In [H5] and [H6], lower-bounded generalized twisted modules for a grading-restricted vertex algebra are constructed and studied. In [H7], the constructions and results in [H5] and [H6] are applied to affine vertex (operator) algebras to give explicit constructions of various types of lower-bounded and grading-restricted generalized twisted modules.

In this paper, for each weight-one element  $u$  of a vertex operator algebra  $V$ , we first construct a transformation sending a generalized  $g$ -twisted module for an automorphism  $g$  of  $V$  fixing  $u$  to another such generalized twisted module by generalizing the construction of examples of generalized twisted modules in [H1]. In the special case that the automorphisms of  $V$  involved (including the automorphisms obtained from the zero modes of the vertex operators of weight-one elements) are of finite order, such transformations for weak twisted modules (without gradings) were constructed by Li [L]. We then show that such a

transformation is in fact an automorphism of the category of generalized  $g$ -twisted modules for automorphisms  $g$  fixing  $u$ . Such automorphisms of categories seem to be independent of the conjectured crossed tensor category structure of suitable generalized twisted modules. These automorphisms should be a special feature of the conjectured crossed tensor category structure of suitable generalized twisted  $V$ -modules and are expected to be a necessary part in the tensor categorical study of orbifold conformal field theories, especially in the study of problems related to “crossed vertex tensor categories” (the twisted generalizations of vertex tensor categories introduced in [HL]) and the reconstruction problem.

We apply our construction mentioned above to the case of affine vertex operator algebras to obtain explicit results on these automorphisms of categories. In particular, we give explicit constructions of certain generalized twisted modules from generalized twisted modules associated to diagram automorphisms of finite-dimensional simple Lie algebras and generalized (untwisted) modules.

This paper is organized as follows: In Section 2, we review some basic facts on automorphisms of  $V$  and recall a formal series  $\Delta_V^{(u)}$  of operators on  $V$  associated to a weight-one element  $u$  of  $V$  introduced in [H1]. In Section 3, we recall the notions of variants of generalized twisted module for a vertex operator algebra  $V$ . We construct in Section 4 an automorphism of the category of generalized  $g$ -twisted  $V$ -modules for all automorphisms  $g$  of  $V$  fixing  $u$ . We recall some basic facts about affine vertex (operator) algebras  $M(\ell, 0)$  and  $L(\ell, 0)$  and their automorphisms in Section 5. In Sections 6 and 7, we discuss the applications of the automorphisms of the categories of generalized twisted modules to the case of affine vertex operator algebras  $M(\ell, 0)$  and  $L(\ell, 0)$ . In Section 6, we construct explicitly generalized  $g$ -twisted  $M(\ell, 0)$ - and  $L(\ell, 0)$ -modules for semisimple automorphisms  $g$  of  $M(\ell, 0)$ - and  $L(\ell, 0)$  from generalized  $\mu$ -twisted modules for diagram automorphisms  $\mu$  of the finite-dimensional simple Lie algebra. In particular, we construct generalized  $g$ -twisted modules for semisimple automorphisms  $g$  of  $M(\ell, 0)$  and  $L(\ell, 0)$  from generalized (untwisted)  $V$ -modules. In Section 7, we construct explicitly generalized  $g$ -twisted  $M(\ell, 0)$ - and  $L(\ell, 0)$ -modules for general (not-necessarily semisimple) automorphisms  $g$  of  $M(\ell, 0)$  and  $L(\ell, 0)$  from generalized  $\mu$ -twisted modules for diagram automorphisms  $\mu$  of the finite-dimensional simple Lie algebra. In particular, we construct generalized  $g$ -twisted modules for general automorphisms  $g$  of  $M(\ell, 0)$  and  $L(\ell, 0)$  from generalized (untwisted)  $V$ -modules.

## 2 Automorphisms of $V$ and formal series $\Delta_V^{(u)}$ of operators

In this section, we review some basic facts about automorphisms of a vertex operator algebra  $V$  and recall a formal series  $\Delta_V^{(u)}$  of operators on  $V$  associated to a weight-one element  $u$  of  $V$  introduced in [H1].

Let  $g$  be an automorphism of  $V$ , that is, a linear isomorphism  $g : V \rightarrow V$  such that  $g(Y_V(u, x)v) = Y_V(g(u), x)g(v)$ ,  $g(\mathbf{1}) = \mathbf{1}$  and  $g(\omega) = \omega$ . Using the multiplicative Jordan-Chevalley decomposition, there are unique commuting operators  $\sigma$  and  $g_u$  such that  $g = \sigma g_u$ ,

where  $\sigma$  is a semisimple automorphism of  $V$  and  $g_{\mathcal{U}}$  is a locally unipotent operator on  $V$ , in the sense that  $g_{\mathcal{U}} - 1_V$  is locally nilpotent on  $V$ . We have that  $V = \coprod_{\alpha \in P_V^g} V^{[\alpha]}$ , where

$$P_V^g = \{\alpha \in [0, 1) + i\mathbb{R} \mid e^{2\pi i\alpha} \text{ is an eigenvalue of } g\}.$$

Following [HY], we write  $\sigma = e^{2\pi i\mathcal{S}_g}$ , where  $\mathcal{S}_g$  is the operator defined by  $\mathcal{S}_g v = \alpha v$  for  $v \in V^{[\alpha]}$  for each  $\alpha \in P_V^g$  and extended linearly, and we write  $g_{\mathcal{U}} = e^{2\pi i\mathcal{N}_g}$ , where

$$\mathcal{N}_g = \sum_{j \in \mathbb{Z}_+} \frac{(-1)^j (g_{\mathcal{U}} - 1_V)^j}{2\pi i j}.$$

We note that the generalized eigenspaces of  $g$  are the eigenspaces  $V^{[\alpha]}$  of  $\sigma$  and  $\mathcal{S}_g$  for  $\alpha \in P_V^g$ , and that  $\mathcal{N}_g$  is locally nilpotent since  $g_{\mathcal{U}} - 1_V$  is locally nilpotent. Also, we note that since  $\sigma$  and  $g_{\mathcal{U}}$  commute, they preserve each other's generalized eigenspaces. In particular, if  $v \in V$  is an eigenvector of  $\sigma$  with eigenvalue  $e^{2\pi i\alpha}$ , then so is  $g_{\mathcal{U}}v$ . Since the eigenspaces of  $\sigma$  are the same as the eigenspaces of  $\mathcal{S}_g$ , we have that if  $v$  is an eigenvector of  $\mathcal{S}_g$  with eigenvalue  $\alpha$ , then so is  $g_{\mathcal{U}}v$ . Hence, by the definition of  $\mathcal{N}_g$ , we see that  $\mathcal{N}_g v$  is also an eigenvector of  $\mathcal{S}_g$  with eigenvalue  $\alpha$ . Thus, on an eigenvector  $v$  of  $\mathcal{S}_g$  with eigenvalue  $\alpha$ ,

$$\mathcal{S}_g \mathcal{N}_g v = \alpha \mathcal{N}_g v = \mathcal{N}_g(\alpha v) = \mathcal{N}_g \mathcal{S}_g v$$

and so  $[\mathcal{S}_g, \mathcal{N}_g] = 0$  on  $V$ . Hence, we have  $g = e^{2\pi i\mathcal{L}_g}$  where  $\mathcal{L}_g = \mathcal{S}_g + \mathcal{N}_g$ .

For  $u \in V$ , we write  $Y(u, x) = \sum_{n \in \mathbb{Z}} Y_n(u) x^{-n-1}$ . Recall from Proposition 5.1 in [H1], for  $u \in V_{(1)}$  satisfying  $L(1)u = 0$ ,  $g_u = e^{2\pi i Y_0(u)}$  is an automorphism of the vertex operator algebra  $V$ . Using the notation in [H5], we may decompose  $g_u$  into its semisimple and unipotent parts, which we denote  $e^{2\pi i Y_0(u)_S}$  and  $e^{2\pi i Y_0(u)_N}$ , respectively. Here  $Y_0(u)_S$  is the semisimple part of  $Y_0(u)$  and  $Y_0(u)_N$  is the nilpotent part of  $Y_0(u)$  so that

$$Y_0(u) = Y_0(u)_S + Y_0(u)_N.$$

We also recall from [H1] the formal series  $\Delta_V^{(u)}(x)$  for  $u \in V_{(1)}$  defined by

$$\Delta_V^{(u)}(x) = x^{-Y_0(u)} e^{-\int_0^{-x} Y^{\leq -2}(u, y)} = x^{-Y_0(u)_S} e^{-2\pi i(Y_0(u)_N) \log x} e^{-\int_0^{-x} Y^{\leq -2}(u, y)}$$

where the linear map

$$\int_0^x : V[[x]] + x^{-2}V[[x^{-1}]] \rightarrow xV[[x]] + x^{-1}V[[x^{-1}]]$$

is defined by

$$\int_0^x \sum_{n \in \mathbb{Z} \setminus \{-1\}} v_n y^n = \sum_{n \in \mathbb{Z} \setminus \{-1\}} \frac{v_n}{n+1} x^{n+1}$$

and where  $Y^{\leq -2}(u, x) = \sum_{m \in \mathbb{Z}_+} Y_m(u) x^{-m-1}$ . The properties of  $\Delta_V^{(u)}(x)$  for  $u \in V_{(1)}$  in the following theorem are given in Theorem 5.2 in [H1] except for (2.3):

**Theorem 2.1** For  $v \in V$ , there exist  $m_1, \dots, m_l \in \mathbb{R}$  such that

$$\Delta_V^{(u)}(x)v \in x^{m_1}V[x^{-1}][\log x] + \dots + x^{m_l}V[x^{-1}][\log x]. \quad (2.1)$$

Moreover, the series  $\Delta_V^{(u)}(x)$  satisfies

$$\Delta_V^{(u)}(x)Y(v, x_2) = Y(\Delta_V^{(u)}(x + x_2)v, x_2)\Delta_V^{(u)}(x), \quad (2.2)$$

$$[L(0), \Delta_V^{(u)}(x)] = x \frac{d}{dx} \Delta_V^{(u)}(x) + Y_0(u)\Delta_V^{(u)}(x), \quad (2.3)$$

$$[L(-1), \Delta_V^{(u)}(x)] = -\frac{d}{dx} \Delta_V^{(u)}(x). \quad (2.4)$$

*Proof.* We need only prove (2.3). Using the fact that  $u \in V_{(1)}$ , we have by the commutator formula that

$$[L(0), Y_m(u)] = -mY_m(u) \quad (2.5)$$

for  $u \in V_{(1)}$ . Thus, we have that, for  $v \in V$ ,

$$\begin{aligned} [L(0), \Delta_V^{(u)}(x)] &= \left[ L(0), x^{-Y_0(u)} \exp \left( - \sum_{m \geq 1} \frac{Y_m(u)}{-m} (-x)^{-m} \right) \right] \\ &= x^{-Y_0(u)} \left[ L(0), \sum_{k \geq 0} \frac{1}{k!} \left( - \sum_{m \geq 0} \frac{(-1)^m Y_m(u)}{-m} x^{-m} \right)^k \right] \\ &= x^{-Y_0(u)} \left[ L(0), \sum_{k \geq 0} \frac{1}{k!} \left( - \sum_{\substack{n_1, \dots, n_k \in \mathbb{Z}_+ \\ n_1 + \dots + n_k = l}} \frac{(-1)^l Y_{n_1}(u) \cdots Y_{n_k}(u)}{(-n_1) \cdots (-n_k)} x^{-l} \right) \right] \\ &= x^{-Y_0(u)} \sum_{k \geq 0} \frac{1}{k!} \left( - \sum_{\substack{n_1, \dots, n_k \in \mathbb{Z}_+ \\ n_1 + \dots + n_k = l}} (-n_1 - \dots - n_k) \frac{(-1)^l Y_{n_1}(u) \cdots Y_{n_k}(u)}{(-n_1) \cdots (-n_k)} x^{-l} \right) \\ &= x^{-Y_0(u)} x \frac{d}{dx} \sum_{k \geq 0} \frac{1}{k!} \left( - \sum_{\substack{n_1, \dots, n_k \in \mathbb{Z}_+ \\ n_1 + \dots + n_k = l}} \frac{(-1)^l Y_{n_1}(u) \cdots Y_{n_k}(u)}{(-n_1) \cdots (-n_k)} x^{-l} \right) \\ &= x^{-Y_0(u)} x \frac{d}{dx} \exp \left( - \sum_{m \geq 1} \frac{Y_m(u)}{-m} (-x)^{-m} \right) \\ &= x \frac{d}{dx} \Delta_V^{(u)}(x) + Y_0(u)\Delta_V^{(u)}(x). \end{aligned}$$

■

We also note the following properties of  $\Delta_V^{(u)}(x)$ :

**Proposition 2.2** Let  $\Delta_V^{(u)}(x)$  for  $u \in V_{(1)}$  be the series defined above.

1.  $\Delta_V^{(0)}(x) = 1_V$

2. If  $u_1, u_2 \in V$  and  $[Y(u_1, x_1), Y(u_2, x_2)] = 0$ , then

$$\Delta_V^{(u_1)}(x)\Delta_V^{(u_2)}(x) = \Delta_V^{(u_1+u_2)}(x)$$

3.  $\Delta_V^{(u)}(x)$  is invertible and  $\Delta_V^{(u)}(x)^{-1} = \Delta_V^{(-u)}(x)$ .

*Proof.* Property 1 is obvious. Property 2 follow from properties of the exponential function. Property 3 is a consequence of Properties 1 and 2. ■

In the proof of the main theorem in Section 4, we also need a lemma.

**Lemma 2.3** Let  $g$  be an automorphism of  $V$  and suppose  $u \in V_{(1)}$  such that  $g(u) = u$ . Then, we have that

$$[g, \Delta_V^{(u)}(x)] = 0.$$

*Proof.* First, we note that since  $g$  is an automorphism of  $V$ , we have that

$$gY(u, x) = Y(g(u), x)g = Y(u, x)g$$

so that

$$[g, Y(u, x)] = 0$$

and so  $[g, Y_n(u)] = 0$  for all  $n \in \mathbb{Z}$ . In particular, we have that  $[g, Y_0(u)] = 0$  and so  $[g, g_u] = 0$ . Next, we show that

$$[g, Y_0(u)_S] = [g, Y_0(u)_N] = 0.$$

Since  $[g, Y_0(u)] = 0$ , we have that  $g(Y_0(u) - \lambda I)^k = (Y_0(u) - \lambda I)^k g$  for any  $k \in \mathbb{N}$  and  $\lambda \in \mathbb{C}$ , so that  $g$  preserves the generalized eigenspaces of  $Y_0(u)$ . Thus, if  $v$  is a generalized eigenvector of  $Y_0(u)$  with eigenvalue  $\lambda \in \mathbb{C}$ , we have that  $gY_0(u)_S v = g\lambda v$  and  $Y_0(u)_S g v = \lambda g v$  and so  $[g, Y_0(u)_S] = 0$ . From this we immediately have  $[g, Y_0(u)_N] = 0$ . Our lemma now follows. ■

Using the fact that  $[g, Y_n(u)] = 0$  for each  $n \in \mathbb{Z}$ , a similar proof also shows that  $Y_n(u)$  preserves the generalized eigenspaces  $V^{[\alpha]}$  of  $g$  for  $\alpha \in P_V^g$ . Hence,  $Y_n(u)$  preserves the eigenspaces of  $\mathcal{S}_g$  and  $e^{2\pi i \mathcal{S}_g}$ , and so we have that  $[\mathcal{S}_g, Y_n(u)] = [e^{2\pi i \mathcal{S}_g}, Y_n(u)] = 0$  on  $V^{[\alpha]}$  and hence on  $V$ . From this, it follows that  $[e^{-2\pi i \mathcal{S}_g} g, Y_n(u)] = [g u, Y_n(u)] = 0$ . Using the fact that  $\mathcal{N}_g = \sum_{j \in \mathbb{Z}_+} \frac{(-1)^j (g u - 1_V)^j}{2\pi i j}$ , we have  $[\mathcal{N}_g, Y_n(u)] = 0$ . Hence, we have  $[\mathcal{L}_g, Y_n(u)] = 0$  for all  $n \in \mathbb{Z}$ .

### 3 Generalized twisted modules

We recall the notions of variants of generalized twisted module for a vertex operator algebra in this section. Our definition uses the Jacobi identity for the twisted vertex operators formulated in [H4] as the main axiom. See [H1] for the original definition in terms of duality properties, [B] for the component form of the Jacobi identity and [HY] for a derivation of the Jacobi identity from the duality properties.

**Definition 3.1** Let  $V$  be a vertex operator algebra and  $g$  an automorphism of  $V$ . A  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $g$ -twisted  $V$ -module is a  $\mathbb{C} \times \mathbb{C}/\mathbb{Z}$ -graded vector space  $W = \coprod_{n \in \mathbb{C}, \alpha \in \mathbb{C}/\mathbb{Z}} W_{[n]}^{[\alpha]}$  equipped with an action of  $g$  and a linear map

$$\begin{aligned} Y_W^g : V \otimes W &\rightarrow W\{x\}[\log x], \\ v \otimes w &\mapsto Y_W^g(v, x)w \end{aligned}$$

satisfying:

1. The *equivariance property*: For  $p \in \mathbb{Z}$ ,  $z \in \mathbb{C}^\times$ ,  $v \in V$  and  $w \in W$ , we have

$$Y_W^{g;p+1}(gv, z)w = Y_W^{g;p}(v, z)w$$

where

$$Y_W^{g;p}(v, z)w = Y_W^g(v, x)w \Big|_{x^n = e^{nl_p(z)}, \log x = l_p(z)}$$

is the  $p$ -th analytic branch of  $Y^g$ .

2. The *identity property*: For  $w \in W$ ,  $Y_W^g(\mathbf{1}, x)w = w$ .
3. The *lower truncation property*: For all  $v \in V$  and  $w \in W$ ,  $Y_W^g(v, x)w$  is lower truncated, that is,  $(Y_W^g)_n(v)w = 0$  when  $\Re(n)$  is sufficiently negative.
4. The *Jacobi identity*:

$$\begin{aligned} x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_W^g(u, x_1) Y_W^g(v, x_2) - x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) Y_W^g(v, x_2) Y_W^g(u, x_1) \\ = x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y_W^g \left( Y \left( \left( \frac{x_2 + x_0}{x_1} \right)^{\mathcal{L}_g} u, x_0 \right) v, x_2 \right) \end{aligned} \quad (3.1)$$

5. Properties about the gradings: Let  $L_W^g(0) = \text{Res}_x x Y_W^g(\omega, x)$ . Then we have:

- (a) The  *$L(0)$ -grading condition*: For each  $w \in W_{[n]} = \coprod_{\alpha \in \mathbb{C}/\mathbb{Z}} W_{[n]}^{[\alpha]}$ , there exists  $K \in \mathbb{Z}_+$  such that

$$(L_W^g(0) - n)^K w = 0.$$

- (b) The *g-grading condition*: For each  $w \in W^{[\alpha]} = \coprod_{n \in \mathbb{C}} W_{[n]}^{[\alpha]}$ , there exists  $\Lambda \in \mathbb{Z}_+$  such that

$$(g - e^{2\pi i \alpha})^\Lambda w = 0$$

where  $Y_W^g(\omega, x) = \sum_{n \in \mathbb{Z}} L_W^g(n) x^{-n-2}$ .

6. The *L(-1)-derivative property*: For  $v \in V$ ,

$$\frac{d}{dx} Y_W^g(v, x) = Y_W^g(L_V(-1)v, x).$$

We denote the  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $g$ -twisted  $V$ -module defined above by  $(W, Y_W^g)$  or simply by  $W$ .

**Definition 3.2** We call a generalized  $g$ -twisted  $V$ -module *lower-bounded* if  $W_{[n]} = 0$  for  $n \in \mathbb{C}$  such that  $\Re(n)$  is sufficiently negative. We call a lower bounded generalized  $g$ -twisted  $V$ -module *grading-restricted* (or simply a *g-twisted V-module*) if for each  $n \in \mathbb{C}$  we have  $\dim W_{[n]} < \infty$ .

Throughout the work, we will write

$$W = \coprod_{n \in \mathbb{C}, \alpha \in P_W^g} W_{[n]}^{[\alpha]}$$

where

$$P_W^g = \{\Re(\alpha) \in [0, 1) | e^{2\pi i \alpha} \text{ is an eigenvalue of } g \text{ acting on } W\}.$$

**Definition 3.3** A vertex operator algebra  $V$  is said to have a (*strongly*)  $\mathbb{C}$ -graded vertex operator algebra structure compatible with  $g$  if  $V$  has an additional  $\mathbb{C}$ -grading  $V = \coprod_{\alpha \in \mathbb{C}} V^{[\alpha]} = \coprod_{n \in \mathbb{Z}, \alpha \in \mathbb{C}} V_{(n)}^{[\alpha]}$  such that for  $V^{[\alpha]}$  for  $\alpha \in \mathbb{C}$  is the generalized eigenspace of  $g$  with eigenvalue  $e^{2\pi i \alpha}$ . For such a vertex operator algebra, a  $\mathbb{C}$ -graded generalized  $g$ -twisted  $V$ -module is a  $\mathbb{C} \times \mathbb{C}$ -graded vector space  $W = \coprod_{n, \alpha \in \mathbb{C}} W_{[n]}^{[\alpha]}$  equipped with an action of  $g$  and a linear map

$$\begin{aligned} Y_W^g : V \otimes W &\rightarrow W\{x\}[\log x], \\ v \otimes w &\mapsto Y_W^g(v, x)w \end{aligned}$$

satisfying the same axioms as in Definition 3.1 except that  $\mathbb{C}/\mathbb{Z}$  is replaced by  $\mathbb{C}$  and the *grading compatibility condition* holds: For  $\alpha, \beta \in \mathbb{C}$ ,  $v \in V^{[\alpha]}$ , and  $w \in W^{[\beta]}$  we have

$$Y_W^g(v, x)w \in W^{[\alpha+\beta]}[\log x].$$

**Definition 3.4** A generalized  $\mathbb{C}/\mathbb{Z}$ -graded (or  $\mathbb{C}$ -graded)  $g$ -twisted  $V$ -module  $W$  is said to be *strongly*  $\mathbb{C}/\mathbb{Z}$ -graded (or *strongly*  $\mathbb{C}$ -graded) if it satisfies the *grading restriction condition*: For each  $n \in \mathbb{C}$  and  $\alpha \in \mathbb{C}/\mathbb{Z}$  (or  $\alpha \in \mathbb{C}$ )  $\dim W_{[n]}^{[\alpha]} < \infty$  and  $W_{[n+l]}^{[\alpha]} = 0$  for all sufficiently negative real numbers  $l$ .

The original definition of lower-bounded generalized  $V$ -module in [H1] is in terms of the duality property. More precisely, for a lower-bounded generalized  $g$ -twisted  $V$ -module  $W$ , it was shown in [HY] that the lower truncation property for the twisted vertex operator map and Jacobi identity is equivalent to the duality property: Let  $W' = \coprod_{n \in \mathbb{C}, \alpha \in \mathbb{C}/\mathbb{Z}} \left(W_{[n]}^{[\alpha]}\right)^*$  and, for  $n \in \mathbb{C}$ ,  $\pi_n : W \rightarrow W_{[n]} = \coprod_{\alpha \in \mathbb{C}/\mathbb{Z}} W_{[n]}^{[\alpha]}$  be the projection. For any  $u, v \in V$ ,  $w \in W$  and  $w' \in W'$ , there exists a multivalued analytic function of the form

$$f(z_1, z_2) = \sum_{i,j,k,l=0}^N a_{ijkl} z_1^{m_i} z_2^{n_j} (\log z_1)^k (\log z_2)^l (z_1 - z_2)^{-t}$$

for  $N \in \mathbb{N}$ ,  $m_1, \dots, m_N, n_1, \dots, n_N \in \mathbb{C}$  and  $t \in \mathbb{Z}_+$ , such that the series

$$\langle w', Y^{g;p}(u, z_1) Y^{g;p}(v, z_2) w \rangle = \sum_{n \in \mathbb{C}} \langle w', Y^{g;p}(u, z_1) \pi_n Y^{g;p}(v, z_2) w \rangle,$$

$$\langle w', Y^{g;p}(v, z_2) Y^{g;p}(u, z_1) w \rangle = \sum_{n \in \mathbb{C}} \langle w', Y^{g;p}(v, z_2) \pi_n Y^{g;p}(u, z_1) w \rangle,$$

$$\langle w', Y^{g;p}(Y(u, z_1 - z_2)v, z_2) w \rangle = \sum_{n \in \mathbb{C}} \langle w', Y^{g;p}(\pi_n Y(u, z_1 - z_2)v, z_2) w \rangle$$

are absolutely convergent in the regions  $|z_1| > |z_2| > 0$ ,  $|z_2| > |z_1| > 0$ ,  $|z_2| > |z_1 - z_2| > 0$ , respectively, and are convergent to the branch

$$\sum_{i,j,k,l=0}^N a_{ijkl} e^{m_i l_p(z_1)} e^{n_j l_p(z_2)} l_p(z_1)^k l_p(z_2)^l (z_1 - z_2)^{-t}$$

of  $f(z_1, z_2)$  when  $\arg z_1$  and  $\arg z_2$  are sufficiently close (more precisely, when  $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$ ).

## 4 Automorphisms of categories of generalized twisted modules

Let  $V$  be a vertex operator algebra. In this section, given  $u \in V_{(1)}$ , we generalize the construction of generalized twisted  $V$ -modules from (untwisted) generalized  $V$ -modules in [H1] to obtain a transformation which sends generalized  $g$ -twisted  $V$ -modules for automorphisms  $g$  of  $V$  fixing  $u$  to generalized twisted  $V$ -modules of the same type. We then show that this transformation is in fact an automorphism of the category of generalized  $g$ -twisted  $V$ -modules for automorphisms  $g$  of  $V$  fixing  $u$ .

We first prove a result on the properties of a modified vertex operator map from a  $\mathbb{C}/\mathbb{Z}$ -graded (or  $\mathbb{C}$ -graded) generalized  $g$ -twisted  $V$ -module using  $\Delta_V^{(u)}(x)$ .



**Theorem 4.1** *Let  $V$  be a vertex operator algebra satisfying  $V_{(0)} = \mathbb{C}\mathbf{1}$ ,  $V_{(n)} = 0$  for  $n < 0$  and  $L(1)u = 0$ ,  $g$  an automorphism of  $V$  and  $u \in V_{(1)}$  satisfying  $g(u) = u$ . Let  $(W, Y_W^g)$  be a  $\mathbb{C}/\mathbb{Z}$ -graded (or  $\mathbb{C}$ -graded) generalized  $g$ -twisted  $V$ -module. Then the map*

$$Y_W^{gg_u} : V \otimes W \rightarrow W\{x\}[\log x]$$

$$v \otimes w \mapsto Y_W^g(v, x)w$$

defined by

$$Y_W^{gg_u}(v, x) = Y_W^g(\Delta_V^{(u)}(x)v, x)$$

for  $v \in V$  satisfies the identity property, the lower truncation property, the  $L(-1)$ -derivative property, the equivariance property and the Jacobi identity.

*Proof.* The proof of the identity property and  $L(-1)$ -derivative property are identical to the proof in [H1]. The lower truncation property of  $Y_W^{gg_u}$  follows from Theorem 2.1 and the lower truncation property of  $Y_W^g$ .

We now prove that the equivariance property holds, that is, we show that

$$Y^{gg_u; p+1}(gg_u v, z)w = Y^{gg_u; p}(v, z)w$$

for  $v \in V$  and  $w \in W$ . Recall from the preceding section that since  $g(u) = u$ , we have that  $[g, Y_n(u)] = 0$  for all  $n \in \mathbb{Z}$ . Let  $v \in V$  be a generalized eigenvector of  $Y_0(u)$  with eigenvalue  $\lambda$ . Then, we have

$$\begin{aligned} & \Delta_V^{(u)}(x)gg_u v \Big|_{x^n=e^{nl_{p+1}(z)}, \log x=l_{p+1}(z)} \\ &= g\Delta_V^{(u)}(x)e^{2\pi i Y_0(u)}v \Big|_{x^n=e^{nl_{p+1}(z)}, \log x=l_{p+1}(z)} \\ &= ge^{-\int_0^{-x} Y^{\leq -2}(u, y)}x^{-\lambda}e^{(-Y_0(u)_N)\log x}e^{2\pi i \lambda}e^{2\pi i(Y_0(u)_N)}v \Big|_{x^n=e^{nl_{p+1}(z)}, \log x=l_{p+1}(z)} \\ &= ge^{-\int_0^{-x} Y^{\leq -2}(u, y)}x^{-\lambda}e^{(-Y_0(u)_N)\log x}v \Big|_{x^n=e^{nl_p(z)}, \log x=l_p(z)} \\ &= g\Delta_V^{(u)}(x)v \Big|_{x^n=e^{nl_p(z)}, \log x=l_p(z)}. \end{aligned}$$

Thus, we have

$$\begin{aligned} Y_W^{gg_u; p+1}(gg_u v, z)w &= Y_W^g(\Delta_V^{(u)}(x)gg_u v, x)w \Big|_{x^n=e^{nl_{p+1}(z)}, \log x=l_{p+1}(z)} \\ &= Y_W^g\left(g\Delta_V^{(u)}(x)v \Big|_{x^n=e^{nl_p(z)}, \log x=l_p(z)}, x\right)w \Big|_{x^n=e^{nl_{p+1}(z)}, \log x=l_{p+1}(z)} \\ &= Y_W^g(\Delta_V^{(u)}(x)v, x)w \Big|_{x^n=e^{nl_p(z)}, \log x=l_p(z)} \\ &= Y^{gg_u; p}(v, z)w. \end{aligned}$$

Finally, we show that the Jacobi identity holds. Using the Jacobi identity for  $Y_W^g$ , we have for,  $v_1, v_2 \in V$ ,

$$\begin{aligned}
& x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_W^{gg_u}(v_1, x_1) Y_W^{gg_u}(v_2, x_2) - x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) Y_W^{gg_u}(v_2, x_2) Y_W^{gg_u}(v_1, x_1) \\
&= x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_W^g(\Delta_V^{(u)}(x_1) v_1, x_1) Y_W^g(\Delta_V^{(u)}(x_2) v_2, x_2) \\
&\quad - x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) Y_W^g(\Delta_V^{(u)}(x_2) v_2, x_2) Y_W^g(\Delta_V^{(u)}(x_1) v_1, x_1) \\
&= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y_W^g \left( Y \left( \left( \frac{x_2 + x_0}{x_1} \right)^{\mathcal{L}_g} \Delta_V^{(u)}(x_1) v_1, x_0 \right) \Delta_V^{(u)}(x_2) v_2, x_2 \right).
\end{aligned}$$

Our desired Jacobi identity is:

$$\begin{aligned}
& x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_W^{gg_u}(v_1, x_1) Y_W^{gg_u}(v_2, x_2) - x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) Y_W^{gg_u}(v_2, x_2) Y_W^{gg_u}(v_1, x_1) \\
&= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y_W^{gg_u} \left( Y \left( \left( \frac{x_2 + x_0}{x_1} \right)^{\mathcal{L}_{gg_u}} v_1, x_0 \right) v_2, x_2 \right) \\
&= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y_W^g \left( \Delta_V^{(u)}(x_2) Y \left( \left( \frac{x_2 + x_0}{x_1} \right)^{\mathcal{L}_g + Y_0(u)} v_1, x_0 \right) v_2, x_2 \right)
\end{aligned}$$

Thus, it suffices to prove that

$$\begin{aligned}
& x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y \left( \left( \frac{x_2 + x_0}{x_1} \right)^{\mathcal{L}_g} \Delta_V^{(u)}(x_1) v_1, x_0 \right) \Delta_V^{(u)}(x_2) \\
&= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) \Delta_V^{(u)}(x_2) Y \left( \left( \frac{x_2 + x_0}{x_1} \right)^{\mathcal{L}_g + Y_0(u)} v_1, x_0 \right)
\end{aligned}$$

or equivalently

$$\begin{aligned}
& x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y \left( \left( \frac{x_2 + x_0}{x_1} \right)^{\mathcal{L}_g} \Delta_V^{(u)}(x_1) v_1, x_0 \right) \\
&= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) \Delta_V^{(u)}(x_2) Y \left( \left( \frac{x_2 + x_0}{x_1} \right)^{\mathcal{L}_g + Y_0(u)} v_1, x_0 \right) \Delta_V^{(u)}(x_2)^{-1}.
\end{aligned}$$

By Theorem 2.1 we have

$$\begin{aligned} & x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) \Delta_V^{(u)}(x_2) Y \left( \left( \frac{x_2 + x_0}{x_1} \right)^{\mathcal{L}_g + Y_0(u)} v_1, x_0 \right) \Delta_V^{(u)}(x_2)^{-1} \\ &= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y \left( \Delta_V^{(u)}(x_2 + x_0) \left( \frac{x_2 + x_0}{x_1} \right)^{\mathcal{L}_g + Y_0(u)} v_1, x_0 \right). \end{aligned}$$

So it suffices to prove

$$\begin{aligned} & x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) \left( \frac{x_2 + x_0}{x_1} \right)^{\mathcal{L}_g} \Delta_V^{(u)}(x_1) v_1 \\ &= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) \Delta_V^{(u)}(x_2 + x_0) \left( \frac{x_2 + x_0}{x_1} \right)^{\mathcal{L}_g + Y_0(u)} v_1 \end{aligned}$$

In fact,

$$\begin{aligned} & x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) \Delta_V^{(u)}(x_2 + x_0) \left( \frac{x_2 + x_0}{x_1} \right)^{\mathcal{L}_g + Y_0(u)} v_1 \\ &= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) (x_2 + x_0)^{-Y_0(u)} e^{-\int_0^{-(x_2+x_0)} Y^{\leq -2}(u,y)} \left( \frac{x_2 + x_0}{x_1} \right)^{Y_0(u)} \left( \frac{x_2 + x_0}{x_1} \right)^{\mathcal{L}_g} v_1 \\ &= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) x_1^{-Y_0(u)} e^{-\int_0^{-x_1} Y^{\leq -2}(u,y)} \left( \frac{x_2 + x_0}{x_1} \right)^{\mathcal{L}_g} v_1 \\ &= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) \left( \frac{x_2 + x_0}{x_1} \right)^{\mathcal{L}_g} x_1^{-Y_0(u)} e^{-\int_0^{-x_1} Y^{\leq -2}(u,y)} v_1 \\ &= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) \left( \frac{x_2 + x_0}{x_1} \right)^{\mathcal{L}_g} \Delta_V^{(u)}(x_1) v_1. \end{aligned}$$

■

**Remark 4.2** Here the authors would like to correct a sign mistake in [H1]. In [H1],  $\Delta_V^{(u)}(x)$  is defined to be  $x^{Y_0(u)} e^{\int_0^x Y^{\leq 2}(u,y)}$ . But the correct definition is

$$\Delta_V^{(u)}(x) = x^{-Y_0(u)} e^{-\int_0^{-x} Y^{\leq -2}(u,y)}.$$

In fact, using the definition with the wrong signs, in the proof of Theorem 5.5 in [H1], the author proved

$$(Y_W^{(u)})^{e^{2\pi i Y_0(u)}; p} (e^{2\pi i Y_0(u)} v, z) w = (Y_W^{(u)})^{e^{2\pi i Y_0(u)}; p+1} (v, z) w. \quad (4.1)$$

But equivariance property is formulated as

$$(Y_W^{(u)})^{e^{2\pi i Y_0(u)}; p+1} (e^{2\pi i Y_0(u)} v, z) w = (Y_W^{(u)})^{e^{2\pi i Y_0(u)}; p} (v, z) w$$

in the definition of twisted modules (see Condition 2 in Definition 3.1 in [H1]). After the sign mistake is corrected, we obtain the equivariance property in the definition. See the proof of Theorem 4.1 above. Certainly, if we had used (4.1) as the equivariance property in the definition of generalized twisted module, then  $\Delta_V^{(u)}(x)$  given in [H1] would be correct.

With additional conditions on  $W$ , we have the following consequence of Theorem 4.1:

**Theorem 4.3** *Let  $V$ ,  $g$  and  $u$  be the same as in Theorem 4.1. Let  $(W, Y_W^g)$  be a  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $g$ -twisted  $V$ -module. Assume that  $W$  is a direct sum of generalized eigenspaces of  $(Y_W)_0(u)$ . Then  $(W, Y_W^{gg_u})$  is a  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $gg_u$ -twisted  $V$ -module.*

*Proof.* First, we note that, by the commutator formula, we have  $Y_1(u)\omega = u$  (we note here that [H1] has a sign mistake in the proof of this fact and states  $Y_1(u)\omega = -u$ , but it should be  $Y_1(u)\omega = u$  where  $u \in V_{(1)}$  and  $L(1)u = 0$ ). Using a proof identical to Lemma 5.3 in [H1], we have that the vertex operator  $Y_W^{gg_u}(\omega, x)$  is in  $\text{End } W[[x, x^{-1}]]$  and if we write

$$Y_W^{gg_u}(\omega, x) = \sum_{n \in \mathbb{Z}} L_W^{gg_u}(n) x^{-n-2},$$

then

$$L_W^{gg_u}(0) = L_W^g(0) - (Y_W^g)_0(u) + \frac{1}{2}\mu \quad (4.2)$$

where  $L_W^g(0)$  is the corresponding Virasoro operator for the generalized  $g$ -twisted  $V$ -module  $W$  and  $\mu \in \mathbb{C}$  is given by  $Y_1(u)u = \mu \mathbf{1}$ . The eigenvalues of  $e^{2\pi i(Y_W^g)_0(u)}$  are of the form  $e^{2\pi i\beta}$  where  $\beta \in P_W^{gu}$ . Also, a generalized eigenvector of  $e^{2\pi i(Y_W^g)_0(u)}$  with eigenvalue  $e^{2\pi i\beta}$  is also a generalized eigenvector of  $(Y_W^g)_0(u)$  with eigenvalue  $\beta + p$  for some  $p \in \mathbb{Z}$ . We note here that, by the commutator formula,  $[L_W^g(0), (Y_W^g)_0(u)] = 0$ . By assumption, we have

$$W = \coprod_{n \in \mathbb{C}, \alpha \in P_W^g} W_{[n]}^{[\alpha]},$$

where  $W_{[n]}^{[\alpha]}$  is the homogeneous subspace of  $W$  of generalized eigenvectors of  $g$  with eigenvalue  $e^{2\pi i\alpha}$  and of  $L_W^g(0)$  of eigenvalue  $n$ . By assumption, we also have

$$W_{[n]}^{[\alpha]} = \coprod_{\tilde{\beta} \in P_W^{gu} + \mathbb{Z}} W_{[n]}^{[\alpha], [\tilde{\beta}]}$$

for each  $\alpha \in P_W^g$  and  $n \in \mathbb{C}$ . In fact all three gradings are compatible since all the relevant operators commute. Let

$$W_{\langle n \rangle}^{[\alpha], [\beta]} = \coprod_{\tilde{\beta} \in \beta + \mathbb{Z}} W_{[n + \tilde{\beta} - \frac{1}{2}\mu]}^{[\alpha], [\tilde{\beta}]}$$

for  $n \in \mathbb{C}, \alpha \in P_W^g, \beta \in P_W^{gu}$ . The elements of  $W_{\langle n \rangle}^{[\alpha], [\beta]}$  are generalized eigenvectors for  $L_W^{gg_u}(0)$  with eigenvalue  $n$  (by (4.2)), generalized eigenvectors for  $g$  with eigenvalue  $e^{2\pi i\alpha}$

and generalized eigenvectors of  $g_u$  with eigenvalue  $e^{2\pi i\beta}$ . In particular, they are generalized eigenvectors of  $gg_u$  with eigenvalue  $e^{2\pi i(\alpha+\beta)}$ . We have

$$W = \coprod_{n \in \mathbb{C}, \alpha \in P_W^g, \beta \in P_W^{gu}} W_{\langle n \rangle}^{[\alpha], [\beta]}.$$

Since all operators commute, we have

$$P_W^{gg_u} = \{\gamma \in \mathbb{C} | \Re(\gamma) \in [0, 1), \gamma \in \alpha + \beta + \mathbb{Z}, \text{ for } \alpha \in P_W^g, \beta \in P_W^{gu}\}.$$

Define

$$W_{\langle n \rangle}^{\langle \gamma \rangle} = \coprod_{\substack{\alpha \in P_W^g, \beta \in P_W^{gu} \\ \alpha + \beta \in \gamma + \mathbb{Z}}} W_{\langle n \rangle}^{[\alpha], [\beta]}$$

Then  $W$  is doubly graded by the eigenvalues of  $L_W^{gg_u}(0)$  and  $gg_u$ , that is,  $W_{\langle n \rangle}^{\langle \gamma \rangle}$  is the intersection of the generalized eigenspace of  $L_W^{gg_u}(0)$  with eigenvalue  $n$  and the generalized eigenspace of  $gg_u$  with eigenvalue  $e^{2\pi i\gamma}$ , and

$$W = \coprod_{n \in \mathbb{C}, \gamma \in P_W^{gg_u}} W_{\langle n \rangle}^{\langle \gamma \rangle}.$$

Together with Theorem 4.1, we see that  $W$  is a  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $gg_u$ -twisted module. ■

**Remark 4.4** In this paper,  $V$  is always a vertex operator algebra. In particular,  $V$  has a conformal vector  $\omega$  such that the gradings on  $V$  and on generalized twisted  $V$ -modules are given by the coefficients of  $x^{-2}$  of the vertex operators of the conformal element  $\omega$ . But Theorem 4.1 also holds in the case that  $V$  is a grading-restricted vertex algebra (that is, a vertex algebra with grading-restricted grading given by  $L(0)$ ) since we do not use anything involving the conformal element. Theorem 4.3 can be generalized to the case that  $V$  is a grading-restricted vertex algebra (likely satisfying some additional conditions) by defining

$$\begin{aligned} Y_W^{gg_u}(v, x) &= Y_W^g(\Delta_V^{(u)}(x)v, x), \\ L_W^{gg_u}(0) &= L_W^g(0) - (Y_W^g)_0(u) + \frac{1}{2}\mu, \\ L_W^{gg_u}(-1) &= L_W^g(-1) - (Y_W^g)_{-1}(u) \end{aligned}$$

for  $v \in V$ , where  $\mu \in \mathbb{C}$  is given by  $(Y_V)_1(u)u = \mu \mathbf{1}$ .

The generalized  $gg_u$ -twisted  $V$ -module in Theorem 4.1 is  $\mathbb{C}/\mathbb{Z}$ -graded. We now discuss  $\mathbb{C}$ -graded generalized  $gg_u$ -twisted modules. Note that  $g = e^{2\pi i(\mathcal{S}_g + \mathcal{N}_g)}$  and  $g_u = e^{2\pi i Y_0(u)}$ . If there is another semisimple operator  $\tilde{\mathcal{S}}_g$  on  $V$  such that  $g = e^{2\pi i(\tilde{\mathcal{S}}_g + \mathcal{N}_g)}$ , then the set of eigenvalues of  $\tilde{\mathcal{S}}_g$  must be contained in  $P_V^g + \mathbb{Z}$ . In particular, we have

$$V = \coprod_{\tilde{\alpha} \in P_V^g + \mathbb{Z}} V^{[\tilde{\alpha}]} = \coprod_{\tilde{\alpha} \in P_V^g + \mathbb{Z}, n \in \mathbb{Z}} V_{\langle n \rangle}^{[\tilde{\alpha}]},$$

where  $V_{(n)}^{[\tilde{\alpha}]}$  for  $\tilde{\alpha} \in P_V^g + \mathbb{Z}$  and  $n \in \mathbb{Z}$  is the eigenspace of  $\tilde{\mathcal{S}}_g$  restricted  $V_{(n)}$  with eigenvalue  $\tilde{\alpha}$  and

$$V^{[\tilde{\alpha}]} = \coprod_{n \in \mathbb{Z}} V_{(n)}^{[\tilde{\alpha}]}$$

for  $\tilde{\alpha} \in P_V^g + \mathbb{Z}$  is the eigenspace of  $\tilde{\mathcal{S}}_g$  with eigenvalue  $\tilde{\alpha}$ .

We now assume that there is a semisimple operator  $\tilde{\mathcal{S}}_g$  on  $V$  such that  $g = e^{2\pi i(\tilde{\mathcal{S}}_g + \mathcal{N}_g)}$  and  $Y_V(u, x)v \in V^{[\tilde{\alpha}_1 + \tilde{\alpha}_2]}$  for  $u \in V^{[\tilde{\alpha}_1]}$  and  $v \in V^{[\tilde{\alpha}_2]}$ . In this case,

$$gg_u = e^{2\pi i(\tilde{\mathcal{S}}_g + \mathcal{N}_g)} e^{2\pi i Y_0(u)} = e^{2\pi i(\tilde{\mathcal{S}}_g + Y_0(u)_S)} e^{2\pi i \mathcal{N}_g} e^{2\pi i(Y_0(u)_N)}.$$

In particular,  $e^{2\pi i(\tilde{\mathcal{S}}_g + Y_0(u)_S)}$  is the semisimple part of  $gg_u$ . We denote the operator  $\tilde{\mathcal{S}}_g + Y_0(u)_S$  by  $\tilde{\mathcal{S}}_{gg_u}$ .

Let  $W$  be a generalized  $g$ -twisted  $V$ -module with a semisimple action of  $\tilde{\mathcal{S}}_g$  such that the actions of  $e^{2\pi i \mathcal{S}_g}$  and  $e^{2\pi i \tilde{\mathcal{S}}_g}$  on  $W$  are equal. Then  $W$  is a direct sum of eigenspaces of  $\tilde{\mathcal{S}}_g$  and the set of eigenvalues of  $\tilde{\mathcal{S}}_g$  on  $W$  must be contained in  $P_W^g + \mathbb{Z}$ . In particular, we have a grading

$$W = \coprod_{\tilde{\alpha} \in P_W^g + \mathbb{Z}} W^{[\tilde{\alpha}]} = \coprod_{\tilde{\alpha} \in P_W^g + \mathbb{Z}, n \in \mathbb{C}} W_{[n]}^{[\tilde{\alpha}]},$$

where  $W_{[n]}^{[\tilde{\alpha}]}$  for  $\tilde{\alpha} \in P_W^g + \mathbb{Z}$  and  $n \in \mathbb{C}$  is the eigenspace of  $\tilde{\mathcal{S}}_g$  restricted  $W_{[n]}$  with eigenvalue  $\tilde{\alpha}$  and

$$W^{[\tilde{\alpha}]} = \coprod_{n \in \mathbb{C}} W_{[n]}^{[\tilde{\alpha}]}$$

for  $\tilde{\alpha} \in P_W^g + \mathbb{Z}$  is the eigenspace of  $\tilde{\mathcal{S}}_g$  with eigenvalue  $\tilde{\alpha}$ . We further assume that this grading is compatible with the twisted module structure, that is,  $Y_W^g(v, x)w \in W^{[\tilde{\alpha}_1 + \tilde{\alpha}_2]}$  for  $u \in V^{[\tilde{\alpha}_1]}$  and  $w \in W^{[\tilde{\alpha}_2]}$ .

Since  $V$  is a vertex operator algebra,  $V_{(n)}^{[\tilde{\alpha}]}$  for  $\tilde{\alpha} \in P_W^g + \mathbb{Z}$  and  $n \in \mathbb{Z}$  is finite dimensional. In particular, it can be decomposed into generalized  $Y_0(u)$ -eigenspace

$$V_{(n)}^{[\tilde{\alpha}]} = \coprod_{\tilde{\beta} \in P_V^{gu} + \mathbb{Z}} V_{(n)}^{[\tilde{\alpha}], [\tilde{\beta}]},$$

where  $V_{(n)}^{[\tilde{\alpha}], [\tilde{\beta}]}$  is the generalized eigenspace of  $Y_0(u)$  restricted to  $V_{(n)}^{[\tilde{\alpha}]}$  with eigenvalue  $\tilde{\beta}$ . By assumption,  $W_{[n]}^{[\tilde{\alpha}]}$  can be decomposed into generalized  $(Y_W^g)_0(u)$ -eigenspaces, that is,

$$W_{[n]}^{[\tilde{\alpha}]} = \coprod_{\tilde{\beta} \in P_W^{gu} + \mathbb{Z}} W_{[n]}^{[\tilde{\alpha}], [\tilde{\beta}]},$$

where  $W_{[n]}^{[\tilde{\alpha}], [\tilde{\beta}]}$  is the generalized eigenspace of  $(Y_W^g)_0(u)$  restricted to  $W_{[n]}^{[\tilde{\alpha}]}$  with eigenvalue  $\tilde{\beta}$ .

On  $V_{(n)}^{[\tilde{\alpha}], [\tilde{\beta}]}$ ,  $\tilde{\mathcal{S}}_{gg_u}$  acts as  $\tilde{\alpha} + \tilde{\beta}$  so that  $V_{(n)}^{[\tilde{\alpha}], [\tilde{\beta}]}$  is in the generalized eigenspace of  $\tilde{\mathcal{S}}_{gg_u}$  with eigenvalue  $\tilde{\alpha} + \tilde{\beta}$ . For  $\tilde{\gamma} \in P_V^{gg_u} + \mathbb{Z}$  and  $n \in \mathbb{Z}$ , let

$$V_{(n)}^{(\tilde{\gamma})} = \coprod_{\substack{\tilde{\alpha} \in P_V^g + \mathbb{Z}, \tilde{\beta} \in P_V^{gu} + \mathbb{Z} \\ \tilde{\alpha} + \tilde{\beta} = \tilde{\gamma}}} V_{(n)}^{[\tilde{\alpha}], [\tilde{\beta}]}.$$

Then  $V_{(n)}^{(\tilde{\gamma})}$  is the generalized eigenspace of  $gg_h$  restricted to  $V_{(n)}$  with eigenvalue  $\tilde{\gamma}$ . Similarly, on  $W_{[n]}^{[\tilde{\alpha}], [\tilde{\beta}]}$ ,  $\tilde{\mathcal{S}}_{gg_u}$  acts as  $\tilde{\alpha} + \tilde{\beta}$  so that  $W_{[n]}^{[\tilde{\alpha}], [\tilde{\beta}]}$  is in the generalized eigenspace of  $\tilde{\mathcal{S}}_{gg_u}$  with eigenvalue  $\tilde{\alpha} + \tilde{\beta}$ . For  $\tilde{\gamma} \in P_W^{gg_u} + \mathbb{Z}$  and  $n \in \mathbb{C}$ , let

$$W_{\langle n \rangle}^{(\tilde{\gamma})} = \coprod_{\substack{\tilde{\alpha} \in P_W^g + \mathbb{Z}, \tilde{\beta} \in P_W^{gu} + \mathbb{Z} \\ \tilde{\alpha} + \tilde{\beta} = \tilde{\gamma}}} W_{[n + \tilde{\beta} - \frac{1}{2}\mu]}^{[\tilde{\alpha}], [\tilde{\beta}]}.$$

Then  $W_{\langle n \rangle}^{(\tilde{\gamma})}$  is the generalized eigenspace of  $gg_u$  restricted to

$$W_{\langle n \rangle} = \coprod_{\tilde{\gamma} \in P_W^{gg_u} + \mathbb{Z}} W_{\langle n \rangle}^{(\tilde{\gamma})} \quad (4.3)$$

with eigenvalue  $\tilde{\gamma}$ . Then these subspaces give new gradings to  $V$  and  $W$ , that is,

$$V = \coprod_{n \in \mathbb{Z}, \tilde{\gamma} \in P_V^{gg_u} + \mathbb{Z}} V_{(n)}^{(\tilde{\gamma})}$$

and

$$W = \coprod_{n \in \mathbb{C}, \tilde{\gamma} \in P_W^{gg_u} + \mathbb{Z}} W_{\langle n \rangle}^{(\tilde{\gamma})}.$$

**Theorem 4.5** *Let  $V$ ,  $g$  and  $u$  be the same as in Theorem 4.1. Let  $W$  be a generalized  $g$ -twisted  $V$ -module. Assuming that there is a semisimple operator  $\tilde{\mathcal{S}}_g$  on  $V$  such that  $g = e^{2\pi i(\tilde{\mathcal{S}}_g + \mathcal{N}_g)}$  and  $Y_V(u, x)v \in V^{[\tilde{\alpha}_1 + \tilde{\alpha}_2]}$  for  $u \in V^{[\tilde{\alpha}_1]}$  and  $v \in V^{[\tilde{\alpha}_2]}$ , where for  $\alpha \in P_V^g$ ,  $V^{[\alpha]}$  is the eigenspace of  $\tilde{\mathcal{S}}_g$  with eigenvalue  $\tilde{\alpha}$ . Assume also that  $\tilde{\mathcal{S}}_g$  acts on  $W$  semisimply, the actions of  $e^{2\pi i\mathcal{S}_g}$  and  $e^{2\pi i\tilde{\mathcal{S}}_g}$  on  $W$  are equal and  $Y_W^g(v, x)w \in W^{[\tilde{\alpha}_1 + \tilde{\alpha}_2]}$  for  $u \in V^{[\tilde{\alpha}_1]}$  and  $w \in W^{[\tilde{\alpha}_2]}$ . Then  $(W, Y_W^{gg_u})$  with the gradings of  $W$  given by (4.3) is a  $\mathbb{C}$ -graded generalized  $gg_u$ -twisted module. In particular, if we define a  $\mathbb{C}/\mathbb{Z}$ -grading of  $W$  by*

$$W = \coprod_{\alpha \in P_W^g} W^{(\alpha)},$$

where for  $\alpha \in P_W^g$ ,

$$W^{(\alpha)} = \coprod_{k \in \mathbb{Z}} W^{[\alpha + k]},$$

then  $(W, Y_W^{gg_u})$  with this  $\mathbb{C}/\mathbb{Z}$ -grading is a  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $gg_u$ -twisted module.

*Proof.* We need only show that the new gradings on  $V$  and  $W$  are compatible with the twisted vertex operator map. Indeed, suppose  $v \in V_{(n)}^{[\tilde{\alpha}], [\tilde{\beta}]} \subset V_{(n)}^{(\tilde{\gamma})}$ , where  $\tilde{\alpha} \in P_V^g + \mathbb{Z}$ ,  $\tilde{\beta} \in P_V^{g_u} + \mathbb{Z}$  and  $\tilde{\alpha} + \tilde{\beta} = \tilde{\gamma}$ , and  $w \in W_{[n+\tilde{\beta}'-\frac{1}{2}\mu]}^{[\tilde{\alpha}'], [\tilde{\beta}']} \subset W_{(n)}^{(\tilde{\gamma})}$ , where  $\tilde{\alpha}' \in P_W^g + \mathbb{Z}$ ,  $\tilde{\beta}' \in P_W^{g_u} + \mathbb{Z}$  and  $\tilde{\alpha}' + \tilde{\beta}' = \tilde{\gamma}'$ . Then

$$\begin{aligned}
& \tilde{\mathcal{S}}_{gg_h} Y_W^{gg_u}(v, x)w \\
&= (\tilde{\mathcal{S}}_g + Y_0(u)_S) Y_W^g(\Delta_V^{(u)}(x)v, x)w \\
&= \tilde{\mathcal{S}}_g Y_W^g(\Delta_V^{(u)}(x)v, x)w + Y_0(u)_S Y_W^g(\Delta_V^{(u)}(x)v, x)w \\
&= (\tilde{\alpha} + \tilde{\alpha}') Y_W^g(\Delta_V^{(u)}(x)v, x)w + Y_W^g(Y_0(u)_S \Delta_V^{(u)}(x)v, x)w + Y_W^g(\Delta_V^{(u)}(x)v, x) Y_0(u)_S w \\
&= (\tilde{\alpha} + \tilde{\alpha}') Y_W^g(\Delta_V^{(u)}(x)v, x)w + Y_W^g(\Delta_V^{(u)}(x) Y_0(u)_S v, x)w + Y_W^g(\Delta_V^{(u)}(x)v, x) \tilde{\beta}' w \\
&= (\tilde{\alpha} + \tilde{\alpha}' + \tilde{\beta} + \tilde{\beta}') Y_W^{gg_u}(v, x)w \\
&= (\tilde{\gamma} + \tilde{\gamma}') Y_W^{gg_u}(v, x)w.
\end{aligned}$$

This means

$$Y_W^{gg_u}(v, x)w \in W^{(\tilde{\gamma}+\tilde{\gamma}')} \{x\} [\log x],$$

proving that the new gradings on  $V$  and  $W$  are indeed compatible with the twisted vertex operator map.  $\blacksquare$

We now discuss the case of strongly  $\mathbb{C}$ -graded generalized  $g$ -twisted  $V$ -modules. Let  $W$  be a  $\mathbb{C}$ -graded generalized  $g$ -twisted  $V$ -module such that the assumptions in Theorem 4.5 hold. Then  $W$  with the  $\mathbb{C}/\mathbb{Z}$ -grading

$$W = \coprod_{\alpha \in P_W^g} W^{(\alpha)} = \coprod_{\alpha \in P_W^g, k \in \mathbb{Z}} W^{[\alpha+k]}$$

is a  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $g$ -twisted  $V$ -module. For each  $\alpha \in P_W^g$  and  $n \in \mathbb{C}$ , we denote the set consisting of those  $\tilde{\alpha} \in \alpha + \mathbb{Z}$  such that  $W_{[n]}^{[\tilde{\alpha}]}$  is not 0 by  $Q_n^\alpha$ . Let  $Q_n = \bigcup_{\alpha \in P_W^g} Q_n^\alpha$  for  $n \in \mathbb{C}$  and  $Q = \bigcup_{n \in \mathbb{C}} Q_n$ .

If  $W$  is strongly  $\mathbb{C}/\mathbb{Z}$ -graded, then

$$W_{[n]}^{(\alpha)} = \coprod_{\alpha \in P_W^g, \tilde{\alpha} \in \alpha + \mathbb{Z}} W_{[n]}^{[\tilde{\alpha}]}$$

for  $\alpha \in P_W^g$  and  $n \in \mathbb{C}$  are finite dimensional and for each  $\alpha \in P_W^g$  and  $n \in \mathbb{C}$ , there are only finitely many  $\tilde{\alpha} \in \alpha + \mathbb{Z}$  such that  $W_{[n]}^{[\tilde{\alpha}]}$  is not 0. In particular, for each  $\alpha \in P_W^g$  and  $n \in \mathbb{C}$ ,  $Q_n^\alpha$  is a finite set in this case.

**Theorem 4.6** *Let  $V$ ,  $g$  and  $u$  be the same as in Theorem 4.1. Let  $W$  be a  $\mathbb{C}$ -graded generalized  $g$ -twisted  $V$ -module such that the assumptions in Theorem 4.5 hold. Assume in addition that the corresponding  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $g$ -twisted  $V$ -module is strongly  $\mathbb{C}/\mathbb{Z}$ -graded,  $Q$  is a finite set, then the  $\mathbb{C}$ -graded generalized  $gg_u$ -twisted module  $(W, Y_W^{gg_u})$  is strongly  $\mathbb{C}$ -graded.*



*Proof.* For  $\tilde{\gamma} \in P_W^{gg_u}$  and  $n \in \mathbb{C}$ ,

$$W_{\langle n \rangle}^{\langle \tilde{\gamma} \rangle} = \coprod_{\substack{\tilde{\alpha} \in P_W^g + \mathbb{Z}, \tilde{\beta} \in P_W^{gu} + \mathbb{Z} \\ \tilde{\alpha} + \tilde{\beta} = \tilde{\gamma}}} W_{[n + \tilde{\beta} - \frac{1}{2}\mu]}^{[\tilde{\alpha}], [\tilde{\beta}]} = \coprod_{\substack{\tilde{\alpha} \in Q_{n + \tilde{\beta} - \frac{1}{2}\mu}, \tilde{\beta} \in P_W^{gu} + \mathbb{Z} \\ \tilde{\alpha} + \tilde{\beta} = \tilde{\gamma}}} W_{[n + \tilde{\beta} - \frac{1}{2}\mu]}^{[\tilde{\alpha}], [\tilde{\beta}]} = \coprod_{\substack{\tilde{\alpha} \in Q, \tilde{\beta} \in P_W^{gu} + \mathbb{Z} \\ \tilde{\alpha} + \tilde{\beta} = \tilde{\gamma}}} W_{[n + \tilde{\beta} - \frac{1}{2}\mu]}^{[\tilde{\alpha}], [\tilde{\beta}]}.$$
(4.4)

Since  $Q$  is a finite set, the right hand side of (4.4) is a finite direct sum. Since in this case,  $W_{[n + \tilde{\beta} - \frac{1}{2}\mu]}^{[\tilde{\alpha}], [\tilde{\beta}]}$  are all finite dimensional, we see that  $W_{\langle n \rangle}^{\langle \tilde{\gamma} \rangle}$  are all finite dimensional.

Since  $W$  is strongly  $\mathbb{C}/\mathbb{Z}$ -graded, for each  $n \in \mathbb{C}$ ,  $\tilde{\alpha} \in P_W^g + \mathbb{Z}$ ,  $\tilde{\beta} \in P_W^{gu}$  and  $\tilde{\gamma} \in P_W^{gg_u}$ , there exists  $L_{n, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}} \in \mathbb{Z}$  such that  $W_{[n - l + \tilde{\beta} - \frac{1}{2}\mu]}^{[\tilde{\alpha}], [\tilde{\beta}]} = 0$  when  $l \in \mathbb{Z}$  is larger than  $L_{n, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}}$ . Let  $L$  be an integer larger than  $L_{n, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}}$  for  $n \in \mathbb{C}$ ,  $\tilde{\alpha} \in P_W^g + \mathbb{Z}$ ,  $\tilde{\beta} \in P_W^{gu}$  and  $\tilde{\gamma} \in P_W^{gg_u}$  satisfying  $\tilde{\alpha} + \tilde{\beta} = \tilde{\gamma}$ . Then

$$W_{\langle n-l \rangle}^{\langle \tilde{\gamma} \rangle} = \coprod_{\substack{\tilde{\alpha} \in Q, \tilde{\beta} \in P_W^{gu} + \mathbb{Z} \\ \tilde{\alpha} + \tilde{\beta} = \tilde{\gamma}}} W_{[n-l + \tilde{\beta} - \frac{1}{2}\mu]}^{[\tilde{\alpha}], [\tilde{\beta}]} = 0.$$

■

Since  $\Delta_V^{(u)}(x)$  is invertible, we immediately have:

**Corollary 4.7** *Let  $V$ ,  $g$  and  $u$  be the same as in Theorem 4.1. If  $(W, Y_W^g)$  is an irreducible generalized  $g$ -twisted  $V$ -module, then  $(W, Y_W^{gg_u})$  is an irreducible generalized  $gg_u$ -twisted  $V$ -module.*

These results in fact give us functors from suitable subcategories of generalized twisted  $V$ -modules to themselves. Let  $\mathcal{C}$  be the category of all generalized twisted  $V$ -modules. Let  $u \in V_{(1)}$  and  $G^u$  the subgroup of the automorphism group of  $V$  consisting the automorphisms  $g$  fixing  $u$  (that is,  $g(u) = u$ ). Consider the subcategory  $\mathcal{C}^u$  of  $\mathcal{C}$  consisting of generalized  $g$ -twisted  $V$ -modules for  $g \in G^u$ . For  $g \in G^u$ , we also have  $gg_u \in G^u$ . Then we have a functor

$$\Delta^u : \mathcal{C}^u \rightarrow \mathcal{C}^u$$

defined as follows: For an object  $(W, Y_W^g)$  of  $\mathcal{C}^u$  (a generalized  $g$ -twisted  $V$ -module for  $g \in G^u$ ),

$$\Delta^u(W, Y_W^g) = (W, Y_W^{gg_u}).$$

For a morphism (a module map)  $f$  from an object  $(W_1, Y_{W_1}^g)$  of  $\mathcal{C}^u$  to another object  $(W_2, Y_{W_2}^g)$  of  $\mathcal{C}^u$ , we have

$$\begin{aligned} f(Y_{W_1}^{gg_u}(v, x)w_1) &= f(Y_{W_1}^g(\Delta_V^{(u)}(x)v, x)w_1) \\ &= Y_{W_2}^g(\Delta_V^{(u)}(x)v, x)f(w_1) \\ &= Y_{W_2}^{gg_u}(v, x)f(w_1). \end{aligned}$$

Hence  $f$  is also a module map from  $(W_1, Y_{W_1}^{gg_u})$  to  $(W_2, Y_{W_2}^{gg_u})$ . We define the image of  $f$  under  $\Delta^u$  to be  $f$  viewed as a module map from  $(W_1, Y_{W_1}^{gg_u})$  to  $(W_2, Y_{W_2}^{gg_u})$ .

We have the following result:

**Theorem 4.8** *For  $u \in V_{(1)}$ ,  $\Delta^u$  is a functor from  $\mathcal{C}^u$  to itself. Moreover,  $\Delta^u$  is in fact an automorphism of the category  $\mathcal{C}^u$ .*

*Proof.* It is clear that  $\Delta^u$  is a functor. Note that  $\mathcal{C}^{-u} = \mathcal{C}^u$ . Then we also have a functor  $\Delta^{-u}$  from  $\mathcal{C}^u$  to itself. By Proposition 2.2, we see that

$$\Delta^u \circ \Delta^{-u} = \Delta^{-u} \circ \Delta^u = 1_u,$$

where  $1_u$  is the identity functor on  $\mathcal{C}^u$ . ■

**Remark 4.9** Since the underlying vector space of  $(W, Y_W^g)$  and  $\Delta^u(W, Y_W^g) = (W, Y_W^{gg_u})$  are the same,  $\Delta^u$  is indeed an automorphism of the category  $\mathcal{C}^u$  (that is, an isomorphism from  $\mathcal{C}^u$  to itself), not just an equivalence from  $\mathcal{C}^u$  to itself.

## 5 Affine vertex (operator) algebras and their automorphisms

In the next two sections, we shall apply the automorphisms of categories obtained in the preceding section to construct certain particular generalized twisted modules for affine vertex operator algebras. In this section, we recall some basic facts on affine vertex operator algebras and their automorphisms.

Let  $\mathfrak{g}$  be a finite-dimensional simple Lie algebra and  $(\cdot, \cdot)$  its normalized Killing form. Let  $g$  be an automorphism of  $\mathfrak{g}$ . Assume also that  $(\cdot, \cdot)$  is invariant under  $g$ . Then  $g = e^{2\pi i S_g} e^{2\pi i N_g}$ , where  $e^{2\pi i S_g}$  and  $e^{2\pi i N_g}$  are the semisimple and unipotent parts of  $g$ . Let

$$P_g^g = \{\alpha \in \mathbb{C} \mid \Re(\alpha) \in [0, 1), e^{2\pi i \alpha} \text{ is an eigenvalue of } g\}.$$

Then

$$\mathfrak{g} = \coprod_{\alpha \in P_g^g} \mathfrak{g}^{[\alpha]},$$

where for  $\alpha \in P_g^g$ ,  $\mathfrak{g}^{[\alpha]}$  is the generalized eigenspace of  $g$  (or the eigenspace of  $e^{2\pi i S_g}$ ) with the eigenvalue  $e^{2\pi i \alpha}$ .

By Proposition 5.3 in [H7],  $e^{2\pi i S_g} = \sigma = \tau_\sigma \mu e^{2\pi i \text{ad}_h} \tau_\sigma^{-1}$  where  $h \in \mathfrak{h}$ ,  $\mu$  is a diagram automorphism of  $\mathfrak{g}$  such that  $\mu(h) = h$  and  $\tau_\sigma$  is an automorphism of  $\mathfrak{g}$ . By Proposition 5.4 in [H7],  $\mathcal{N}_g = \text{ad}_{a_{\mathcal{N}_g}}$ , where  $a_{\mathcal{N}_g} \in \mathfrak{g}^{[0]}$ . Then

$$\tau_\sigma^{-1} \mathcal{N}_g \tau_\sigma = \tau_\sigma^{-1} \text{ad}_{a_{\mathcal{N}_g}} \tau_\sigma = \text{ad}_{\tau_\sigma^{-1} a_{\mathcal{N}_g}}.$$

So

$$\begin{aligned} g &= e^{2\pi i S_g} e^{2\pi i N_g} = \tau_\sigma \mu e^{2\pi i \text{ad}_h} \tau_\sigma^{-1} e^{2\pi i N_g} \\ &= \tau_\sigma \mu e^{2\pi i \text{ad}_h} e^{2\pi i \tau_\sigma^{-1} N_g \tau_\sigma} \tau_\sigma^{-1} = \tau_\sigma \mu e^{2\pi i \text{ad}_h} e^{2\pi i \text{ad}_{\tau_\sigma^{-1} a_{N_g}}} \tau_\sigma^{-1}. \end{aligned}$$

Let  $g_\sigma = \mu e^{2\pi i \text{ad}_h} e^{2\pi i \text{ad}_{\tau_\sigma^{-1} a_{N_g}}}$ . Then  $\mu e^{2\pi i \text{ad}_h}$  and  $e^{2\pi i \text{ad}_{\tau_\sigma^{-1} a_{N_g}}}$  are the semisimple and unipotent, respectively, parts of  $g_\sigma$ . We shall denote the semisimple part  $\mu e^{2\pi i \text{ad}_h}$  of  $g_\sigma$  by  $g_{\sigma,s}$ .

Let  $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$  equipped with the bracket operation

$$\begin{aligned} [a \otimes t^m, b \otimes t^n] &= [a, b] \otimes t^{m+n} + (a, b) m \delta_{m+n, 0} \mathbf{k}, \\ [a \otimes t^m, \mathbf{k}] &= 0 \end{aligned}$$

for  $a, b \in \mathfrak{g}$  and  $m, n \in \mathbb{Z}$ . Let  $\widehat{\mathfrak{g}}_\pm = \mathfrak{g} \otimes t^{\pm 1} \mathbb{C}[t^{\pm 1}]$  so that

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_- \oplus \mathfrak{g} \oplus \mathbb{C}\mathbf{k} \oplus \widehat{\mathfrak{g}}_+.$$

Consider  $\mathbb{C}$  as a 1-dimensional  $\mathfrak{g} \oplus \mathbb{C}\mathbf{k} \oplus \widehat{\mathfrak{g}}_+$ -module where  $\mathfrak{g} \oplus \widehat{\mathfrak{g}}_+$  acts trivially,  $\mathbf{k}$  acts as  $\ell \in \mathbb{C}$ , and let

$$M(\ell, 0) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g} \oplus \mathbb{C}\mathbf{k} \oplus \widehat{\mathfrak{g}}_+)} \mathbb{C}$$

be the induced  $\widehat{\mathfrak{g}}$ -module. Let  $J(\ell, 0)$  be the maximal proper  $\widehat{\mathfrak{g}}$ -submodule of  $M(\ell, 0)$  and let  $L(\ell, 0) = M(\ell, 0)/J(\ell, 0)$ . It is well known ([FZ]) that  $M(\ell, 0)$  and  $L(\ell, 0)$  have the structure of a vertex operator algebra when  $\ell \neq -h^\vee$ , where  $h^\vee$  is the dual Coxeter number of  $\mathfrak{g}$ .

The automorphisms  $\mu$ ,  $e^{2\pi i \text{ad}_h}$ ,  $e^{2\pi i \text{ad}_{\tau_\sigma^{-1} a_{N_g}}}$ ,  $g_\sigma$  and  $g_{\sigma,s}$  also give automorphisms of  $M(\ell, 0)$  and  $L(\ell, 0)$ . But on  $M(\ell, 0)$  and  $L(\ell, 0)$ ,  $\text{ad}_h$  and  $\text{ad}_{\tau_\sigma^{-1} a_{N_g}}$  act as  $h(0)$  and  $(\tau_\sigma^{-1} a_{N_g})(0)$ . So the corresponding automorphisms of  $L(\ell, 0)$  are actually  $e^{2\pi i h(0)}$  and  $e^{2\pi i (\tau_\sigma^{-1} a_{N_g})(0)}$ .

We will also study inner automorphisms of  $\mathfrak{g}$  below. Fix an element  $a \in \mathfrak{g}$ . The element  $a$  has a Chevalley-Jordan decomposition given by  $a = s + n$  where  $\text{ad}_s$  is semisimple on  $\mathfrak{g}$ ,  $\text{ad}_n$  is nilpotent on  $\mathfrak{g}$ , and  $[s, n] = 0$ . In particular, if  $\text{ad}_s$  is semisimple, then  $s$  belongs to some Cartan subalgebra of  $\mathfrak{g}$ . Without loss of generality, we take  $s \in \mathfrak{h}$ . Let  $a_1, \dots, a_n \in \mathfrak{g}$  be eigenvectors of  $\text{ad}_s$  with eigenvalues  $\lambda_1, \dots, \lambda_n$ , respectively. That is, we have

$$\text{ad}_s(a_i) = [s, a_i] = \lambda_i a_i$$

for  $1 \leq i \leq n$ . We note here that if  $\lambda_i \neq 0$ , we have that

$$(s, a_i) = \frac{1}{\lambda_i} (s, [s, a_i]) = \frac{1}{\lambda_i} ([s, s], a_i) = 0. \quad (5.5)$$

Hence,  $(s, a_i) \neq 0$  is only possible when  $\lambda_i = 0$ .

We now use  $V$  to denote  $M(\ell, 0)$  or  $L(\ell, 0)$ . Then we have  $Y_V(a(-1)\mathbf{1}, x) = \sum_{n \in \mathbb{Z}} a(n) x^{-n-1}$ . The action  $\text{ad}_a$  on  $\mathfrak{g}$  becomes the action of  $a(0)$  on  $V$ . In particular, we have that  $a(0)$  is a derivation of  $V$  since by the commutator formula for a vertex operator algebra we have

$$Y(a(0)v, x) = [a(0), Y(v, x)] \quad (5.6)$$

for all  $v \in V$ . We note that, similarly,  $s(0)$  and  $n(0)$  are also derivations of  $V$ . We have that  $V$  is spanned by elements of the form

$$a_{i_1}(m_1) \cdots a_{i_k}(m_k) \mathbf{1} \quad (5.7)$$

where  $k \geq 0$ ,  $i_1, \dots, i_k \in \{1, \dots, n\}$  and  $m_1, \dots, m_k \in \mathbb{Z}$ . In general, elements of the form (5.7) are eigenvectors of  $s(0)$  and generalized eigenvectors of  $a(0)$ . That is, we have

$$s(0)a_{i_1}(m_1) \cdots a_{i_k}(m_k) \mathbf{1} = (\lambda_{i_1} + \cdots + \lambda_{i_k})a_{i_1}(m_1) \cdots a_{i_k}(m_k) \mathbf{1}, \quad (5.8)$$

We have that  $g_a = e^{2\pi i a(0)}$ , an automorphism of  $V$ , and its semisimple and unipotent parts are  $g_s = e^{2\pi i s(0)}$  and  $g_u = e^{2\pi i n(0)}$ , respectively. We have that (5.7) is an eigenvector for the automorphism  $e^{2\pi i s(0)}$  with eigenvalue  $e^{2\pi i (\lambda_{i_1} + \cdots + \lambda_{i_k})}$ . Thus, the actions of  $a(0)$  and  $s(0)$  give us a natural  $\mathbb{C}$ -grading on  $V$ , given by the eigenspaces of  $s(0)$ , which are also generalized eigenspaces of  $a(0)$ :

$$V = \coprod_{\alpha \in P_V^g + \mathbb{Z}} V^{[\alpha]} \quad (5.9)$$

where

$$V^{[\alpha]} = \text{span}\{a_{i_1}(m_1) \cdots a_{i_k}(m_k) \mathbf{1} \mid k \geq 0, m_1, \dots, m_k \in \mathbb{Z}, \lambda_{i_1} + \cdots + \lambda_{i_k} = \alpha\}. \quad (5.10)$$

Moreover, it is easy to see that the grading compatibility condition is satisfied by  $V$ : Let  $v_1 \in V^{[\alpha]}$  and  $v_2 \in V^{[\beta]}$  for some  $\alpha, \beta \in \mathbb{C}$ . Then, we have that

$$s(0)Y(v_1, x)v_2 = Y(s(0)v_1, x)v_2 + Y(v_1, x)s(0)v_2 = (\alpha + \beta)Y(v_1, x)v_2. \quad (5.11)$$

## 6 Generalized twisted modules for affine vertex operator algebras associated to semisimple automorphisms

In this section, let  $V$  be  $M(\ell, 0)$  or  $L(\ell, 0)$  for  $\ell \neq -h^\vee$  (in fact,  $V$  can be any quotient vertex operator algebra of  $M(\ell, 0)$ ). Recall the automorphisms  $g_\sigma = \mu e^{2\pi i \text{ad}_h} e^{2\pi i \text{ad}_{\tau_\sigma^{-1} a_{\mathcal{N}_g}}}$  and  $g_{\sigma, s} = \mu e^{2\pi i \text{ad}_h}$  of  $V$  associated to an automorphism  $g$  of  $\mathfrak{g}$  in the preceding section. These automorphisms of  $\mathfrak{g}$  give automorphisms of  $V$  and are still denoted using the same notations  $g_\sigma$  and  $g_{\sigma, s}$ .

From Proposition 3.2 in [H2], we have an invertible functor  $\phi_{\tau_\sigma}$  from the category of (lower-bounded or grading-restricted) generalized  $g_\sigma$ -twisted  $V$ -modules to the category of (lower-bounded or grading-restricted) generalized  $g$ -twisted  $V$ -modules. Thus these two categories are isomorphic (stronger than equivalence since the underlying vector spaces of  $W$  and  $\phi_{\tau_\sigma}(W)$  are the same). In particular, to construct (lower-bounded or grading-restricted) generalized  $g$ -twisted  $V$ -modules, we need only construct (lower-bounded or grading-restricted) generalized  $g_\sigma$ -twisted  $V$ -modules and then apply the functor  $\phi_{\tau_\sigma}$ .

To construct explicitly generalized  $g_\sigma$ -twisted  $V$ -modules, in this section, we first construct explicitly  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $g_{\sigma,s}$ -twisted  $V$ -modules from  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $\mu$ -twisted  $V$ -modules. Note that many  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $\mu$ -twisted  $V$ -modules have been constructed and studied extensively since they are in fact modules for the corresponding twisted affine Lie algebras associated to the diagram automorphism  $\mu$  of  $\mathfrak{g}$ . Then we discuss the construction of  $\mathbb{C}$ -graded generalized  $g_s$ -twisted  $V$ -modules ( $g_s = e^{2\pi i s(0)}$ , see Section 5) from generalized (untwisted)  $V$ -modules. An explicit construction of generalized  $g_\sigma$ -twisted  $V$ -modules will be given in the next section using the explicit construction in this section.

Let  $(W, Y_W^\mu)$  be a  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $\mu$ -twisted  $V$ -module. We assume that  $h_W(0) = \text{Res}_x x^{-1} Y_W^\mu(h(-1)\mathbf{1}, x)$  acts on  $W$  semisimply because  $g_{\sigma,s}$  acts on  $L(\ell, 0)$  semisimply and we always require that the semisimple part of an automorphism of a vertex operator algebra which acts on a twisted module is also the semisimple part of the action of the automorphism. Our construction in Theorem 4.1 is more general and only requires that  $W$  can be decomposed as a direct sum of generalized eigenspace of  $h_W(0)$ .

Let the order of  $\mu$  be  $r$ . Then eigenvalues of  $\mu$  are of the form  $e^{2\pi i \frac{q}{r}}$  for  $q = 0, \dots, r-1$ . Since  $W$  is  $\mathbb{C}/\mathbb{Z}$ -graded,  $W = \coprod_{n \in \mathbb{C}} \coprod_{q=0}^{r-1} W_{[n]}^{[\frac{q}{r}]}$ , where  $W_{[n]}^{[\frac{q}{r}]}$  is the homogeneous subspace of  $W$  with eigenvalue of  $\mu$  being  $e^{2\pi i \frac{q}{r}}$  and of conformal weight  $n$ . We define a new vertex operator map

$$\begin{aligned} Y_W^{g_{\sigma,s}} : V \otimes W &\rightarrow W\{x\}[\log x] \\ v \otimes w &\mapsto Y_W^{g_{\sigma,s}}(v, x)w \end{aligned}$$

by

$$Y_W^{g_{\sigma,s}}(v, x)w = Y_W^\mu(\Delta_V^{(h)}(x)v, x). \quad (6.1)$$

The eigenvalues of  $e^{2\pi i h_W(0)}$  are of the form  $e^{2\pi i \alpha}$ , where  $\alpha \in \mathbb{C}$  such that  $\Re(\alpha) \in [0, 1)$ . We shall denote the set of such  $\alpha$  by  $P_W^h$ . Also an eigenvector of  $e^{2\pi i h_W(0)}$  with eigenvalue  $e^{2\pi i \alpha}$  is also an eigenvector of  $h_W(0)$  with eigenvalue  $\alpha + p$  for some  $p \in \mathbb{Z}$ . Since  $h$  is fixed by  $\mu$ ,  $h_W(0)$  commutes with  $\mu$ . Therefore for  $q = 0, \dots, r-1$  and  $n \in \mathbb{C}$ ,  $W_{[n]}^{[\frac{q}{r}]}$  can be further decomposed into direct sums of eigenspaces  $W_{[n]}^{[\frac{q}{r}], [m]}$  with eigenvalue  $m \in P_W^h + \mathbb{Z}$ . Then we have

$$W = \coprod_{n \in \mathbb{C}} \coprod_{q=0}^{r-1} \coprod_{m \in P_W^h + \mathbb{Z}} W_{[n]}^{[\frac{q}{r}], [m]}.$$

By the proof of Theorem 4.3 we also have  $Y_W^{g_{\sigma,s}}(\omega, x) \in \text{End } W[[x, x^{-1}]]$  and if we write

$$Y_W^{g_{\sigma,s}}(\omega, x) = \sum_{n \in \mathbb{Z}} L_W^g(n) x^{-n-2},$$

then by (4.2) we have

$$L_W^{g_{\sigma,s}}(0) = L_W^\mu(0) - h_W(0) + \frac{1}{2}(h, h)\ell,$$

where  $L_W^\mu(0)$  is the corresponding Virasoro operator for the grading-restricted generalized  $\mu$ -twisted  $L(\ell, 0)$ -module  $W$ . Thus we have that nonzero elements of  $W_{[n-m+\frac{1}{2}(h,h)\ell]}^{[\frac{q}{r}], [m]}$  are generalized eigenvectors of  $L_W^{g_{\sigma,s}}(0)$  with eigenvalue  $n$  and are also eigenvectors of  $g_{\sigma,s} = \mu e^{2\pi i h_W(0)}$  with eigenvalue  $e^{2\pi i(\frac{q}{r}+m)}$ . Let

$$P_W^{g_{\sigma,s}} = \{\beta \in \mathbb{C} \mid \Re(\beta) \in [0, 1), \beta = \frac{q}{r} + \alpha \pmod{\mathbb{Z}}, \text{ for } q \in \{0, \dots, r-1\}, \alpha \in P_W^h\}.$$

Let

$$W_{\langle n \rangle}^{(\beta)} = \coprod_{\substack{q \in \{0, \dots, r-1\}, m \in P_W^h + \mathbb{Z} \\ \frac{q}{r} + m = \beta \pmod{\mathbb{Z}}}} W_{[n-m+\frac{1}{2}(h,h)\ell]}^{[\frac{q}{r}], [m]}$$

for  $n \in \mathbb{C}$  and  $\beta \in P_W^{g_{\sigma,s}}$ . Then we have that the space  $W$  is doubly graded by the eigenvalues of  $L_W^{g_{\sigma,s}}(0)$  and  $g_{\sigma,s}$ , that is,  $W_{\langle n \rangle}^{(\beta)}$  is the intersection of the generalized eigenspace of  $L_W^{g_{\sigma,s}}(0)$  with the eigenvalues  $n$  and the eigenspace of  $g_{\sigma,s}$  with eigenvalue  $e^{2\pi i \beta}$  and

$$W = \coprod_{n \in \mathbb{C}} \coprod_{\beta \in P_W^{g_{\sigma,s}}} W_{\langle n \rangle}^{(\beta)}. \quad (6.2)$$

Conversely, let  $(W, Y_W^{g_{\sigma,s}})$  be a generalized  $g_{\sigma,s}$ -twisted  $V$ -module. We define

$$Y_W^\mu(v, x) = Y_W^g(\Delta_V^h(x)^{-1}v, x). \quad (6.3)$$

In this case,

$$W = \coprod_{n \in \mathbb{C}} \coprod_{\beta \in P_W^{g_{\sigma,s}}} W_{[n]}^{[\beta]}.$$

Since  $\mu$  and  $g_{\sigma,s}$  commute,  $W_{[n]}^{[\beta]}$  can be decomposed into a direct sum of eigenspaces  $W_{[n]}^{[\beta], [\frac{q}{r}]}$  of  $\mu$  for  $q = 0, \dots, r-1$  and we have

$$W = \coprod_{n \in \mathbb{C}} \coprod_{q=0}^{r-1} \coprod_{\beta \in P_W^{g_{\sigma,s}}} W_{[n]}^{[\beta], [\frac{q}{r}]}$$

Let

$$W_{\langle n \rangle}^{\langle \frac{q}{r} \rangle} = \coprod_{\beta \in P_W^{g_{\sigma,s}}} W_{[n-m+\frac{1}{2}(h,h)\ell]}^{[\beta], [\frac{q}{r}]}$$

for  $n \in \mathbb{C}$  and  $q = 0, \dots, r-1$ . Then

$$W = \coprod_{n \in \mathbb{C}} \coprod_{q=0}^{r-1} W_{\langle n \rangle}^{\langle \frac{q}{r} \rangle}. \quad (6.4)$$

Now, applying Theorem 4.3 to the discussion above, we have the following result:

**Theorem 6.1** *Let  $(W, Y_W^\mu)$  be a  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $\mu$ -twisted  $V$ -module. Assume that  $h_W(0) = \text{Res}_x x^{-1} Y_W^\mu(h(-1)\mathbf{1}, x)$  acts on  $W$  semisimply. The pair  $(W, Y_W^{g_{\sigma,s}})$ , with  $W$  equipped with the new double gradings given by (6.2) and  $Y_W^{g_{\sigma,s}}$  defined by (6.1), is a  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $g_{\sigma,s}$ -twisted  $V$ -module. Conversely, let  $(W, Y_W^{g_{\sigma,s}})$  be a  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $g_{\sigma,s}$ -twisted  $V$ -module. Then the pair  $(W, Y_W^\mu)$ , with  $W$  equipped with the new double gradings given by (6.4) and  $Y_W^{g_{\sigma,s}}$  defined by (6.3), is a  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $\mu$ -twisted  $V$ -module.*

We now give explicit examples of the vertex operators  $Y_W^{g_{\sigma,s}}(v, x)$  for  $v$  in a set of generators of  $V$ . Let  $a_1, \dots, a_n \in \mathfrak{g}$  be eigenvectors of  $\text{ad}_h$  with eigenvalues  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ , respectively. That is, we have

$$\text{ad}_h(a_i) = [h, a_i] = \lambda_i a_i$$

for  $1 \leq i \leq n$ . We note here that if  $\lambda_i \neq 0$ , we have that

$$(h, a_i) = \frac{1}{\lambda_i} (h, [h, a_i]) = \frac{1}{\lambda_i} ([h, h], a_i) = 0. \quad (6.5)$$

Hence,  $(h, a_i) \neq 0$  is only possible when  $\lambda_i = 0$ . For any element  $b \in \mathfrak{g}$  with  $\mu(b) = e^{\frac{2\pi i j}{k}} b$  we will write

$$Y_W^\mu(b(-1)\mathbf{1}, x) = \sum_{n \in \frac{j}{k} + \mathbb{Z}} b_W^\mu(n) x^{-n-1} \quad (6.6)$$

and

$$Y_W^{g_{\sigma,s}}(b(-1)\mathbf{1}, x) = \sum_{n \in \mathbb{C}} b_W^{g_{\sigma,s}}(n) x^{-n-1}. \quad (6.7)$$

We note that since  $\mu(h) = h$ ,  $b$  can be decomposed into a sum of eigenvectors of  $\text{ad}_h$ , so without loss of generality we assume that  $b$  is an eigenvector of  $\text{ad}_h$  with eigenvalue  $\lambda \in \{\lambda_1, \dots, \lambda_n\}$ . We begin by explicitly computing  $\Delta_V^{(h)}(x)$ . We have that

$$-\int_0^{-x} Y^{\leq -2}(h(-1)\mathbf{1}, x) = -\int_0^{-x} \sum_{n \in \mathbb{Z}_+} h(n) y^{-n-1} \quad (6.8)$$

$$= \sum_{n \in \mathbb{Z}_+} \frac{h(n)}{n} (-x)^{-n} \quad (6.9)$$

and so we have that

$$\Delta_V^{(h)}(x) = x^{-h(0)} \exp \left( \sum_{n \in \mathbb{Z}_+} \frac{h(n)}{n} (-x)^{-n} \right). \quad (6.10)$$

For each element  $a_i(-1)\mathbf{1}$  with  $1 \leq i \leq n$ , we have that

$$\Delta_V^{(h)}(x) b(-1)\mathbf{1} = x^{-h(0)} \exp \left( \sum_{n \in \mathbb{Z}_+} \frac{h(n)}{n} (-x)^{-n} \right) b(-1)\mathbf{1} \quad (6.11)$$

$$= x^{-\lambda} (1 + h(1)(-x)^{-1}) b(-1)\mathbf{1} \quad (6.12)$$

$$= b(-1)\mathbf{1} x^{-\lambda} - (h, b) \ell \mathbf{1} x^{-\lambda-1} \quad (6.13)$$

From this, we immediately obtain

$$Y_W^{g_{\sigma_s}}(b(-1)\mathbf{1}, x) = Y_W^\mu(\Delta_V^{(h)}(x)b(-1)\mathbf{1}, x) = Y_W^\mu(b(-1)\mathbf{1}, x)x^{-\lambda} - (h, b)\ell x^{-\lambda-1} \quad (6.14)$$

In particular, since  $(h, b) = 0$  if  $\lambda \neq 0$ , we have

$$Y_W^{g_{\sigma_s}}(b(-1)\mathbf{1}, x) = \begin{cases} Y_W^\mu(b(-1)\mathbf{1}, x)x^{-\lambda} & \text{if } \lambda \neq 0 \\ Y_W^\mu(b(-1)\mathbf{1}, x) - (h, b)\ell x^{-1} & \text{if } \lambda = 0. \end{cases} \quad (6.15)$$

Thus we have that, for  $n \in \lambda + \frac{j}{k} + \mathbb{Z}$ ,

$$b_W^{g_{\sigma_s}}(n) = \begin{cases} b_W^\mu(n - \lambda) & \text{if } \lambda \neq 0 \\ b_W^\mu(n) & \text{if } \lambda = 0 \text{ and } n \neq 0 \\ b_W^\mu(0) - (h, b)\ell & \text{if } \lambda = 0 \text{ and } n = 0 \end{cases} \quad (6.16)$$

In particular, we have  $Y_W^{g_{\sigma_s}}(b(-1)\mathbf{1}, x) = \sum_{n \in \frac{j}{k} + \lambda + \mathbb{Z}} b_W^{g_{\sigma_s}}(n)x^{-n-1}$ .

Instead of direct computation using the definition of  $\Delta_V^{(h)}(x)$ , we use the  $L(-1)$ -derivative property to compute  $Y_W^{g_{\sigma_s}}(b(-n-1)\mathbf{1}, x)$  for  $n \in \mathbb{Z}$ ,  $n > 1$ . We recall that

$$[L(m), b(n)] = -nb(m+n) \quad (6.17)$$

for all  $b \in \mathfrak{g}$  and that  $L(-1)\mathbf{1} = 0$  (cf. [LL]). From this, it immediately follows that

$$b(-n-1)\mathbf{1} = \frac{1}{n!}L(-1)^nb(-1)\mathbf{1} \quad (6.18)$$

We now have

$$Y_W^{g_{\sigma_s}}(b(-n-1)\mathbf{1}, x) = \frac{1}{n!} \frac{\partial^n}{\partial x^n} Y_W^{g_{\sigma_s}}(b(-1)\mathbf{1}, x) \quad (6.19)$$

$$= \begin{cases} \frac{1}{n!} \left( \frac{\partial}{\partial x} \right)^n (Y_W^\mu(b(-1)\mathbf{1}, x)x^{-\lambda}) & \text{if } \lambda \neq 0 \\ \frac{1}{n!} \left( \frac{\partial}{\partial x} \right)^n (Y_W^\mu(b(-1)\mathbf{1}, x) - (h, b)\ell x^{-1}) & \text{if } \lambda = 0 \end{cases} \quad (6.20)$$

In particular, if  $\lambda \neq 0$  we have

$$Y_W^{g_{\sigma_s}}(b(-n-1)\mathbf{1}, x) = \frac{1}{n!} \sum_{k=0}^n \binom{-\lambda}{k} k! \left( \left( \frac{\partial}{\partial x} \right)^{n-k} Y_W^\mu(b(-1)\mathbf{1}, x) \right) x^{-\lambda-k} \quad (6.21)$$

and if  $\lambda = 0$  we have:

$$Y_W^{g_{\sigma_s}}(b(-n-1)\mathbf{1}, x) = \left( \frac{1}{n!} \left( \frac{\partial}{\partial x} \right)^n Y_W^\mu(b(-1)\mathbf{1}, x) \right) - (-1)^n (h, b)\ell x^{-n-1} \quad (6.22)$$

Next, we discuss the construction of  $\mathbb{C}$ -graded generalized  $g_s$ -twisted modules (recalling  $g_s = e^{2\pi i s(0)}$  from the preceding section) from generalized (untwisted)  $V$ -modules.



Let  $(W, Y_W)$  be a generalized  $V$ -module. Then

$$Y_W(b(-1)\mathbf{1}, x) = \sum_{n \in \mathbb{Z}} b_W(n) x^{-n-1}$$

for an arbitrary  $b \in \mathfrak{g}$ , where  $b_W(n)$  for  $n \in \mathbb{Z}$  are actions of  $b(n)$  on  $W$ . We assume that  $s_W(0)$  acts on  $W$  semisimply. We note here that,  $W$  is a  $1_V$ -twisted module and the additional grading given by  $1_V$  is trivial. That is, we have that  $W^{[0]} = W$  and  $W_{[n]}^{[0]} = W_{[n]}$ , and in particular,  $W$  is a  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $1_V$ -twisted  $V$ -module. In this case,  $\mathcal{S}_g = \tilde{\mathcal{S}}_g = \mathcal{N}_g = 0$  since  $g = e^{2\pi i(\mathcal{S}_g + \mathcal{N}_g)} = 1_V$ .

Let

$$Y_W^{gs}(v, x) = Y_W(\Delta_V^{(s)}(x)v, x) \quad (6.23)$$

for  $v \in V$ . We also need an additional  $\mathbb{C}$ -grading to check that the conditions of Theorems 4.5 and 4.6 are satisfied. In this case, since  $L_W(0)$  and  $s_W(0)$  commute on  $W$ , we have

$$W = W^{[0]} = \coprod_{n \in \mathbb{C}} \coprod_{\alpha \in P_W^{gs} + \mathbb{Z}} W_{[n]}^{[0], [\alpha]}$$

where

$$W_{[n]}^{[0], [\alpha]} = \{w \in W \mid L_W(0)w = nw, s_W(0)w = \alpha w\}.$$

The we have an additional  $\mathbb{C}$ -grading on  $W$  given by

$$W = \coprod_{\alpha \in P_W^{gs} + \mathbb{Z}} W^{[0], [\alpha]},$$

where

$$W^{[0], [\alpha]} = \coprod_{n \in \mathbb{C}} W_{[n]}^{[0], [\alpha]}.$$

We define

$$W_{\langle n \rangle}^{(\alpha)} = W_{[n + \alpha - \frac{1}{2}(s, s)\ell]}^{[0], [\alpha]}$$

so that

$$W = \coprod_{n \in \mathbb{C}, \alpha \in P_W^{gs} + \mathbb{Z}} W_{\langle n \rangle}^{(\alpha)} \quad (6.24)$$

By Theorems 4.5 and 4.6, we have:

**Theorem 6.2** *Let  $(W, Y_W)$  be a generalized  $V$ -module. Assume that  $s_W(0)$  acts on  $W$  semisimply. Then the pair  $(W, Y_W^{gs})$ , with  $W$  equipped with the new double gradings given by (6.24) and  $Y_W^{gs}$  defined by (6.23), is a strongly  $\mathbb{C}$ -graded generalized  $g_s$ -twisted  $V$ -module. If  $(W, Y_W)$  is grading restricted, then  $(W, Y_W^{gs})$  is strongly  $\mathbb{C}$ -graded (grading restricted).*

*Proof.* We need only check that the conditions in Theorems 4.1, 4.5 and 4.6 are satisfied. In this case, we have  $g = 1_V$  and  $h = s(-1)\mathbf{1}$  in Theorem 4.1. So it is clear that  $g(h) = h$  i.e.  $1_V(s(-1)\mathbf{1}) = s(-1)\mathbf{1}$ . Also,  $s(-1)\mathbf{1} \in V_{(1)}$  and we have, by the commutator formula, that

$$L(1)s(-1)\mathbf{1} = s(-1)L(1)\mathbf{1} + s(0)\mathbf{1} = 0.$$

Since  $\tilde{\mathcal{S}}_g = 0$ , the conditions of Theorem 4.5 are trivially met. Moreover, using the notation  $Q$  and  $Q_n^\alpha$  from Theorem 4.6, we have that  $Q = \{0\}$ , and thus is a finite set, so that  $(W, Y_W^{g_s})$  is strongly  $\mathbb{C}$ -graded.  $\blacksquare$

**Example 6.3** If  $V = L(\ell, 0)$ , let  $W = L(\ell, \lambda)$  be the irreducible quotient of

$$\text{Ind}_{\mathfrak{g}}^{\hat{\mathfrak{g}}}(L(\lambda)) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g} \oplus \mathbf{Ck} \oplus \hat{\mathfrak{g}}_+)} L(\lambda),$$

where  $L(\lambda)$  is an irreducible  $\mathfrak{g}$ -module with highest weight  $\lambda$  and  $\mathbf{k}$  and  $\hat{\mathfrak{g}}_+$  act trivially. In fact,  $s(0)$  has a natural action on  $L(\lambda)$  given by the action of  $s$ , and thus  $L(\lambda)$  can be decomposed into  $s$ -eigenspaces. Moreover, the maximal proper submodule  $J(\ell, \lambda)$  of  $M(\ell, \lambda)$  is a  $\hat{\mathfrak{g}}$ -submodule of  $\text{Ind}_{\mathfrak{g}}^{\hat{\mathfrak{g}}}(L(\lambda))$ , and thus is preserved by  $s(0)$ . Hence,  $L(\ell, \lambda)$  can be decomposed into  $s(0)$ -eigenspaces. In fact, since  $[L_W(0), s(0)] = 0$ , we have that the  $s(0)$ -grading is compatible with the conformal weight grading, and we thus have

$$L(\ell, \lambda) = \coprod_{m \in \mathbb{C}, \alpha \in \mathbb{C}} L(\ell, \lambda)_{\langle m \rangle}^{(\alpha)}. \quad (6.25)$$

We conclude this section with an example to show that for  $s \in \mathfrak{g}$  such that  $\text{ad}_s$  is semisimple on  $\mathfrak{g}$ ,  $\Delta_W^{s(-1)\mathbf{1}}(x)$  need not map a grading-restricted twisted module to another grading-restricted twisted module.

**Example 6.4** Let  $\mathfrak{g}$  be a finite dimensional simple Lie algebra, and consider an  $\mathfrak{sl}(2)$ -triple  $\{e_\alpha, f_\alpha, h_\alpha\}$  where

$$[e_\alpha, f_\alpha] = h_\alpha, \quad [h_\alpha, e_\alpha] = 2e_\alpha, \quad [h_\alpha, f_\alpha] = -2f_\alpha$$

Let  $s = \frac{1}{2}h_\alpha$  so that  $[s, e_\alpha] = e_\alpha$  and consider the automorphism  $g_s = e^{2\pi i s(0)}$  of  $V = M(\ell, 0)$ . Let  $W = M(\ell, 0)$ . We note that  $W$  trivially satisfies the grading-restriction condition. We have that

$$L_W^{g_s}(0) = L_W(0) - s(0) + \frac{1}{2}(s, s)\ell.$$

We note that  $s(0)e_\alpha(-1)^k\mathbf{1} = ke_\alpha(-1)^k\mathbf{1}$  and that  $L(0)e_\alpha(-1)^k\mathbf{1} = ke_\alpha(-1)^k\mathbf{1}$ , and so

$$L_W^{g_s}(0)x_\alpha(-1)^k\mathbf{1} = \frac{1}{2}(s, s)\ell x_\alpha(-1)^k\mathbf{1}$$

for all  $k \geq 0$ . Thus, we have that  $\{e_\alpha(-1)^k\mathbf{1} | k \geq 0\}$  is a linear independent subset of  $W_{\langle \frac{1}{2}(s, s)\ell \rangle}$  and so the module  $(Y_W^{g_s}, W)$  does not satisfy the grading restriction condition.

## 7 Twisted modules for affine vertex operator algebras associated to general automorphisms

In this section,  $V$  is still  $M(\ell, 0)$  or  $L(\ell, 0)$  (or a quotient vertex algebra of  $M(\ell, 0)$ ) for  $\ell \neq -h^\vee$  as in the preceding section. We use the automorphisms of subcategories of generalized twisted modules in Section 4 to obtain  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $g_\sigma$ -twisted  $V$ -modules from generalized  $g_{\sigma,s}$ -twisted  $V$ -modules in this section (see Section 5 for the automorphisms  $g_\sigma$  and  $g_{\sigma,s}$  of  $V$ ). We then discuss the construction of  $\mathbb{C}$ -graded generalized  $g_a$ -twisted modules, where  $g_a$  is an inner automorphism  $g_a = e^{2\pi i a(0)}$  (see Section 5) which in general might not be semisimple.

Let  $(W, Y_W^{g_{\sigma,s}})$  be a  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $g_{\sigma,s}$ -twisted  $V$ -module with

$$W = \coprod_{n \in \mathbb{C}} \coprod_{\gamma \in P_W^{g_{\sigma,s}}} W_{[n]}^{[\gamma]}.$$

Assume that  $g_{\sigma,s}$  acts on  $W$  semisimply and  $(\tau_\sigma^{-1} a_{\mathcal{N}_g})_W(0)$  on  $W$  is locally nilpotent. Since  $\tau_\sigma^{-1} a_{\mathcal{N}_g} \in \mathfrak{g}$ ,  $(\tau_\sigma^{-1} a_{\mathcal{N}_g})(-1)\mathbf{1} \in V_{(1)}$ . Let  $u = (\tau_\sigma^{-1} a_{\mathcal{N}_g})(-1)\mathbf{1}$ . We define a new vertex operator map

$$\begin{aligned} Y_W^{g_\sigma} : V \otimes W &\rightarrow W\{x\}[\log x] \\ v \otimes w &\mapsto Y_W^{g_\sigma}(v, x)w \end{aligned}$$

by

$$Y_W^{g_\sigma}(v, x)w = Y_W^{g_{\sigma,s}}(\Delta_V^{(u)}(x)v, x)w \quad (7.1)$$

for  $v \in V$ .

Since  $g_{\sigma,s}$  and  $e^{\tau_\sigma^{-1} a_{\mathcal{N}_g}}$  commute and  $(\tau_\sigma^{-1} a_{\mathcal{N}_g})_W(0)$  is locally nilpotent, elements of  $W_{[n]}^{[\gamma]}$  for  $n \in \mathbb{C}$  and  $\gamma \in P_W^{g_{\sigma,s}}$  are also generalized eigenvectors of  $(\tau_\sigma^{-1} a_{\mathcal{N}_g})_W(0)$  with eigenvalue 0. Then  $P_W^{e^{\tau_\sigma^{-1} a_{\mathcal{N}_g}}} = \{0\}$  and

$$W_{[n]}^{[\gamma], [0]} = W_{[n]}^{[\gamma]}$$

for  $\gamma \in P_W^{g_{\sigma,s}}$  and  $n \in \mathbb{C}$ . From (4.2), we have

$$L_W^{g_\sigma}(0) = L_W^{g_{\sigma,s}}(0) - (\tau_\sigma^{-1} a_{\mathcal{N}_g})(0) + \frac{1}{2}(\tau_\sigma^{-1} a_{\mathcal{N}_g}, \tau_\sigma^{-1} a_{\mathcal{N}_g})\ell.$$

Since  $(\tau_\sigma^{-1} a_{\mathcal{N}_g})_W(0)$  is locally nilpotent, elements of  $W_{[n]}^{[\gamma]}$  are generalized eigenvectors of  $g_\sigma$  with eigenvalue  $\gamma$  and generalized eigenvectors of  $L_W^{g_\sigma}(0)$  with eigenvalue  $n + \frac{1}{2}(\tau_\sigma^{-1} a_{\mathcal{N}_g}, \tau_\sigma^{-1} a_{\mathcal{N}_g})\ell$ . For  $\gamma \in P_W^{g_{\sigma,s}}$ , we thus define

$$W_{\langle n \rangle}^{(\gamma)} = W_{[n - \frac{1}{2}(\tau_\sigma^{-1} a_{\mathcal{N}_g}, \tau_\sigma^{-1} a_{\mathcal{N}_g})\ell]}^{[\gamma]}$$

so that

$$W = \coprod_{n \in \mathbb{C}} \coprod_{\gamma \in P_W^{g_{\sigma,s}}} W_{\langle n \rangle}^{(\gamma)}. \quad (7.2)$$

Now, as a application of Theorem 4.3, we obtain:

**Theorem 7.1** *Let  $(W, Y_W^{g_{\sigma,s}})$  be a  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $g_{\sigma,s}$ -twisted  $V$ -module. Assume that  $g_{\sigma,s}$  acts on  $W$  semisimply and  $(\tau_\sigma^{-1}a_{\mathcal{N}_g})_W(0)$  on  $W$  is locally nilpotent. Then the pair  $(W, Y_W^{g_\sigma})$ , with the grading of  $W$  given (7.2) and with  $Y_W^{g_\sigma}$  defined by (7.1), is a  $\mathbb{C}/\mathbb{Z}$ -graded generalized  $g_\sigma$ -twisted  $V$ -module.*

*Proof.* To apply Theorem 4.3, it only remains to check that  $g_{\sigma,s}(\tau_\sigma^{-1}a_{\mathcal{N}_g}) = \tau_\sigma^{-1}a_{\mathcal{N}_g}$ . Indeed, we have

$$\begin{aligned} g_\sigma(\tau_\sigma^{-1}a_{\mathcal{N}_g}) &= \tau_\sigma^{-1}g\tau_\sigma(\tau_\sigma^{-1}a_{\mathcal{N}_g}) \\ &= \tau_\sigma^{-1}a_{\mathcal{N}_g}. \end{aligned}$$

■

We now explore the structure of the twisted module  $(W, Y_W^{g_\sigma})$ . As in the previous section, consider an element  $b \in \mathfrak{g}$  which is an eigenvector of  $\mu$  with eigenvalue  $e^{2\pi i \frac{j}{k}}$  and an eigenvector of  $\text{ad}_h$  with eigenvalue  $\lambda$ . Importantly, we have that  $b$  is a generalized eigenvector of  $g_\sigma$  and  $e^{2\pi i(\tau_\sigma^{-1}a_{\mathcal{N}_g})(0)}$ . Since  $b$  is a generalized eigenvector of  $g_\sigma$ ,  $\text{ad}_{\tau_\sigma^{-1}a_{\mathcal{N}_g}}$  on  $b$  is nilpotent. Then there exists  $M \in \mathbb{Z}_+$  such that  $(\text{ad}_{\tau_\sigma^{-1}a_{\mathcal{N}_g}})^M(b) = 0$ . We write

$$Y_W^{g_\sigma}(b(-1)\mathbf{1}, x) = \sum_{k \geq 0} \sum_{n \in \mathbb{C}} b^{g_\sigma}(n, k) x^{-n-1} (\log x)^k$$

We begin by explicitly computing  $\Delta_V^{(u)}(x) = \Delta_V^{(\tau_\sigma^{-1}a_{\mathcal{N}_g})(0)}(x)$ . As above, we have

$$-\int_0^{-x} Y^{\leq -2}((\tau_\sigma^{-1}a_{\mathcal{N}_g})(-1)\mathbf{1}, x) = \sum_{m \in \mathbb{Z}_+} \frac{(\tau_\sigma^{-1}a_{\mathcal{N}_g})(m)}{m} (-x)^{-m}$$

and so we have that

$$\Delta_V^{(\tau_\sigma^{-1}a_{\mathcal{N}_g})}(x) = \exp(-(\tau_\sigma^{-1}a_{\mathcal{N}_g})(0) \log x) \exp \left( \sum_{m \in \mathbb{Z}_+} \frac{(\tau_\sigma^{-1}a_{\mathcal{N}_g})(m)}{m} (-x)^{-m} \right).$$

Thus we have that

$$\begin{aligned} &\Delta_V^{((\tau_\sigma^{-1}a_{\mathcal{N}_g})(0))}(x) b(-1)\mathbf{1} \\ &= \exp(-(\tau_\sigma^{-1}a_{\mathcal{N}_g})(0) \log x) \exp \left( \sum_{m \in \mathbb{Z}_+} \frac{(\tau_\sigma^{-1}a_{\mathcal{N}_g})(m)}{m} (-x)^{-m} \right) b(-1)\mathbf{1} \\ &= \exp(-(\tau_\sigma^{-1}a_{\mathcal{N}_g})(0) \log x) (b(-1)\mathbf{1} - ((\tau_\sigma^{-1}a_{\mathcal{N}_g}), b)\ell \mathbf{1} x^{-1}) \\ &= \sum_{l \geq 0} \frac{1}{l!} (-1)^l ((\tau_\sigma^{-1}a_{\mathcal{N}_g})(0) \log x)^l b(-1)\mathbf{1} - ((\tau_\sigma^{-1}a_{\mathcal{N}_g}), b)\ell \mathbf{1} x^{-1} \\ &= \sum_{l=0}^M \frac{1}{l!} (-1)^l (\text{ad}_{\tau_\sigma^{-1}a_{\mathcal{N}_g}}^l(b))(-1)\mathbf{1} (\log x)^l - ((\tau_\sigma^{-1}a_{\mathcal{N}_g}), b)\ell \mathbf{1} x^{-1}. \end{aligned}$$

We write

$$Y_W^{g_{\sigma_s}}((\text{ad}_{(\tau_{\sigma}^{-1}a_{\mathcal{N}_g})}^l(b))(-1)\mathbf{1}, x) = \sum_{m \in \lambda + \frac{j}{k} + \mathbb{Z}} (\text{ad}_{(\tau_{\sigma}^{-1}a_{\mathcal{N}_g})}^l(b))_W^{g_{\sigma_s}}(m) x^{-m-1}.$$

Note that there is no logarithm of  $x$  in  $Y_W^{g_{\sigma_s}}((\text{ad}_{(\tau_{\sigma}^{-1}a_{\mathcal{N}_g})}^l(b))(-1)\mathbf{1}, x)$  since the automorphism  $g_{\sigma_s}$  of  $V$  is semisimple. Thus, we have that

$$\begin{aligned} Y_W^{g_{\sigma}}(b(-1)\mathbf{1}, x) &= Y_W^{g_{\sigma_s}}(\Delta_V^{(\tau_{\sigma}^{-1}a_{\mathcal{N}_g})}(x)b(-1)\mathbf{1}, x) \\ &= \sum_{l=0}^M \frac{(-1)^l}{l!} Y_W^{g_{\sigma_s}}((\text{ad}_{(\tau_{\sigma}^{-1}a_{\mathcal{N}_g})}^l(b))(-1)\mathbf{1}, x) (\log x)^l - (\tau_{\sigma}^{-1}a_{\mathcal{N}_g}, b) Y^{g_s}(\ell\mathbf{1}, x) x^{-1} \\ &= \sum_{l=0}^M \sum_{m \in \lambda + \frac{j}{k} + \mathbb{Z}} \frac{(-1)^l}{l!} (\text{ad}_{(\tau_{\sigma}^{-1}a_{\mathcal{N}_g})}^l(b))_W^{g_{\sigma_s}}(m) x^{-m-1} (\log x)^l - (\tau_{\sigma}^{-1}a_{\mathcal{N}_g}, b) \ell x^{-1} \end{aligned}$$

In particular, we have that

$$b^{g_{\sigma}}(m, l) = \begin{cases} \frac{(-1)^l}{l!} (\text{ad}_{(\tau_{\sigma}^{-1}a_{\mathcal{N}_g})}^l(b))_W^{g_{\sigma_s}}(m) & \text{if } m \in \lambda + \frac{j}{k} + \mathbb{Z} \text{ and } 0 \leq l \leq M \\ \frac{(-1)^l}{l!} (\text{ad}_{(\tau_{\sigma}^{-1}a_{\mathcal{N}_g})}^l(b))_W^{g_{\sigma_s}}(0) - (\tau_{\sigma}^{-1}a_{\mathcal{N}_g}, b) \ell & \text{if } m = 0 \text{ and } 0 \leq l \leq M \\ 0 & \text{if } m \neq 0 \text{ and } m \notin \lambda + \frac{j}{k} + \mathbb{Z}. \end{cases}$$

As in the preceding section, instead of direct computation using the definition of  $\Delta_V^{(\tau_{\sigma}^{-1}a_{\mathcal{N}_g})}(x)$ , we use the  $L(-1)$ -derivative property to compute  $Y_W^{g_{\sigma}}(b(-n-1)\mathbf{1}, x)$  for  $n \in \mathbb{Z}$ ,  $n > 1$ . Here, the calculation is straightforward, though the expression becomes complicated:

$$\begin{aligned} &Y_W^{g_{\sigma}}(b(-n-1)\mathbf{1}, x) \\ &= \frac{1}{n!} Y_W^{g_{\sigma}}(L(-1)^n b(-1)\mathbf{1}, x) \\ &= \frac{1}{n!} \left( \frac{\partial}{\partial x} \right)^n Y_W^{g_{\sigma}}(b(-1)\mathbf{1}, x) \\ &= \frac{1}{n!} \left( \frac{\partial}{\partial x} \right)^n \left( \sum_{l=0}^M \sum_{m \in \lambda + \frac{j}{k} + \mathbb{Z}} \frac{(-1)^l}{l!} (\text{ad}_{(\tau_{\sigma}^{-1}a_{\mathcal{N}_g})}^l(b))_W^{g_{\sigma_s}}(m) x^{-m-1} (\log x)^j - (\tau_{\sigma}^{-1}a_{\mathcal{N}_g}, b) \ell x^{-1} \right) \\ &= \sum_{l=0}^M \sum_{m \in \lambda + \frac{j}{k} + \mathbb{Z}} \sum_{r=0}^n \frac{1}{n!} \binom{n}{r} \frac{(-1)^l}{l!} (\text{ad}_{(\tau_{\sigma}^{-1}a_{\mathcal{N}_g})}^l(b))_W^{g_{\sigma_s}}(m) \left( \left( \frac{\partial}{\partial x} \right)^{n-r} x^{-m-1} \right) \left( \left( \frac{\partial}{\partial x} \right)^r (\log x)^l \right) \\ &\quad - (-1)^n (\tau_{\sigma}^{-1}a_{\mathcal{N}_g}, b) \ell x^{-n-1}. \end{aligned}$$

Finally, we discuss the construction of  $\mathbb{C}$ -graded generalized  $g_a$ -twisted modules from  $\mathbb{C}$ -graded generalized  $g_s$ -twisted  $V$ -modules, where  $g_a = e^{2\pi i a(0)}$  is an inner automorphism of  $V$  and  $g_s = e^{2\pi i s(0)}$  is its semisimple part (See Section 5).

Suppose that  $(W, Y_W^{g_s})$  is a  $\mathbb{C}$ -graded generalized  $g_s$ -twisted  $V$ -module. We assume that the actions  $s_W(0)$  and  $n_W(0)$  of  $s(0)$  and  $n(0)$  on  $W$  are the semisimple and nilpotent parts of the action  $a_W(0)$  of  $a(0)$  since  $s$  and  $n$  are the semisimple and nilpotent part of the derivation  $a$  of  $\mathfrak{g}$ . In particular,

$$W = \coprod_{n \in \mathbb{C}, \alpha \in P_W^{g_s} + \mathbb{Z}} W_{[n]}^{[\alpha]}. \quad (7.3)$$

We now to apply the results in Section 4 to the case that  $V$  is  $M(\ell, 0)$  or  $L(\ell, 0)$ ,  $g = g_s$  and  $u = n(-1)\mathbf{1}$ . In this case,  $g_u = e^{2\pi i n(0)}$  and  $gg_u = e^{2\pi i a(0)}$ .

We define

$$Y_W^{g_a}(v, x) = Y_W^{g_s}(\Delta_V^{(n(-1)\mathbf{1})}(x)v, x) \quad (7.4)$$

for  $v \in V$ .

Since by assumption,  $n_W(0)$  is locally nilpotent, every element of  $W$  is a generalized eigenvector of  $n_W(0)$  with eigenvalue 0. Then the  $\mathbb{C}$ -grading on  $W$  given by the generalized eigenspace of  $n_W(0)$  is trivial, that is,

$$W_{[n]}^{[\alpha], [0]} = W_{[n]}^{[\alpha]}$$

for  $n \in \mathbb{C}$  and  $\alpha \in P_W^{g_s} + \mathbb{Z}$ . We note also that

$$L_W^{g_a}(0) = L_W^{g_s}(0) - n(0) + \frac{1}{2}(n, n)\ell$$

so that the space  $W_{[n]}$  is made up of generalized eigenvectors of  $L^{g_a}(0)$  with eigenvalue  $n - 0 + \frac{1}{2}(n, n)\ell$ . We define

$$W_{\langle n \rangle}^{(\alpha)} = W_{[n - \frac{1}{2}(n, n)\ell]}^{[\alpha]}. \quad (7.5)$$

Moreover, if  $W$  is grading restricted, we have that each  $W_{\langle n \rangle}^{(\alpha)}$  is clearly finite dimensional since each  $W_{[n - \frac{1}{2}(n, n)\ell]}^{[\alpha]}$  is finite dimensional, and that  $W_{\langle n \rangle}^{(\alpha)} = 0$  when the real part of  $n$  is sufficiently negative since  $W_{[n - \frac{1}{2}(n, n)\ell]}^{[\alpha]} = 0$  when the real part of  $n$  is sufficiently negative. Thus  $W$  is also grading restricted with the grading given by (7.5).

From Theorem 4.3, we obtain the following result:

**Theorem 7.2** *Let  $(W, Y_W^{g_s})$  be a  $\mathbb{C}$ -graded generalized  $g_s$ -twisted  $V$ -module. such that  $s_W(0)$  and  $n_W(0)$  are the semisimple and nilpotent parts of  $a_W(0)$ . Then the pair  $(W, Y_W^{g_a})$  with the grading of  $W$  given by (7.5) and with  $Y_W^{g_a}$  defined by (7.4) is a  $\mathbb{C}$ -graded generalized  $g_a$ -twisted  $V$ -module. If  $(W, Y_W^{g_s})$  is grading restricted, then  $(W, Y_W^{g_a})$  is also grading restricted.*

*Proof.* We need only check the conditions in Theorems 4.1 and 4.3. Since  $[s, n] = 0$ , we have that  $s(0)n(-1)\mathbf{1} = n(-1)s(0)\mathbf{1} = 0$ . Since  $g_s = e^{2\pi i s(0)}$ , we have that

$$g_s(n(-1)\mathbf{1}) = \sum_{k \geq 0} \frac{(2\pi i s(0))^k}{k!} n(-1)\mathbf{1} = n(-1)\mathbf{1}.$$

By the commutator formula, we also have

$$L(1)n(-1)\mathbf{1} = n(-1)L(1)\mathbf{1} + n(0)\mathbf{1} = 0.$$

Hence, all the conditions of Theorems 4.1 and 4.3 are satisfied and so our theorem follows. ■

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