Intertwining operators among twisted modules associated to not-necessarily-commuting automorphisms

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Abstract

We introduce intertwining operators among twisted modules or twisted intertwining operators associated to not-necessarily-commuting automorphisms of a vertex operator algebra. Let V be a vertex operator algebra and let g_1 , g_2 and g_3 be automorphisms of V. We prove that for g_1 -, g_2 - and g_3 -twisted V-modules W_1 , W_2 and W_3 , respectively, such that the vertex operator map for W_3 is injective, if there exists a twisted intertwining operator of type $\binom{W_3}{W_1W_2}$ such that the images of its component operators span W_3 , then $g_3 = g_1g_2$. We also construct what we call the skew-symmetry and contragredient isomorphisms between spaces of twisted intertwining operators among twisted modules of suitable types. The proofs of these results involve careful analysis of the analytic extensions corresponding to the actions of the not-necessarily-commuting automorphisms of the vertex operator algebra.

1 Introduction

In the present paper, we initiate the study of intertwining operators among twisted modules associated to not-necessarily-commuting automorphisms of a vertex operator algebra. Here by twisted modules we mean (generalized or logarithmic) twisted modules introduced in [H7]. For simplicity and to avoid confusion, when twisted modules are not mentioned, we shall call such intertwining operators "twisted intertwining operators," although these intertwining operators are not twisted directly, but are twisted instead in a suitable sense through twisted modules.

Intertwining operators among (untwisted) modules for a vertex operator algebra were first introduced mathematically by Frenkel, Lepowsky and the author in [FHL] and correspond to chiral vertex operators in physics (see [MS]). They have been studied systematically in the papers [HL], [H1]–[H6], [HLZ1]–[HLZ4], [Y], [Ch1]–[Ch2] and [Fi1]–[Fi2]. Intertwining operators give chiral three-point correlation functions in two-dimensional conformal field theories and are the building blocks of multi-point correlation functions on Riemann surfaces of arbitrary genus. They are the main objects of interest in the representation theory of vertex operator algebras and two-dimensional conformal field theory. Almost all important results in these theories are in fact properties of intertwining operators.

Intertwining operators among twisted modules associated to commuting automorphisms of finite orders appeared implicitly in the work [FFR] of Feingold, Frenkel and Ries and were introduced explicitly by Xu in [X] in terms of a generalization of the Jacobi identity for twisted modules. However, there is still no definition of intertwining operators among twisted modules associated to noncommuting automorphisms in the literature. To construct orbifold conformal field theories associated to a noncommutative group of automorphisms of a vertex operator algebra, it is necessary to study these intertwining operators.

In [H8], the author formulated the following conjecture:

Conjecture 1.1 ([H8]) Assume that V is a simple vertex operator algebra satisfying the following conditions:

- 1. $V_{(0)} = \mathbb{C}\mathbf{1}$, $V_{(n)} = 0$ for n < 0 and the contragredient V', as a V-module, is equivalent to V.
- 2. Every grading-restricted generalized V-module is completely reducible
- 3. V is C₂-cofinite, that is, dim $V/C_2(V) < \infty$, where $C_2(V)$ is the subspace of V spanned by the elements of the form $\operatorname{Res}_x x^{-2}Y(u, x)v$ for $u, v \in V$ and $Y : V \otimes V \to V[[x, x^{-1}]]$ is the vertex operator map for V.

Let G be a finite group of automorphisms of V. Then the twisted intertwining operators among the g-twisted V-modules for all $g \in G$ satisfy the associativity, commutativity and modular invariance properties.

If this conjecture is proved, then we obtain the genus-zero and genus-one parts of the chiral orbifold conformal field theory associated with the vertex operator algebra V and the group G of automorphisms of V. One consequence of Conjecture 1.1 is that the category of g-twisted modules for all $g \in G$ has a natural structure of G-crossed braided tensor category satisfying additional properties.

To even formulate this conjecture precisely, we have to first introduce twisted intertwining operators or intertwining operators among twisted modules associated to general automorphisms and study their basic properties. In this paper, we give a definition of such twisted intertwining operators. Let V be a vertex operator algebra and let g_1 , g_2 and g_3 be automorphisms of V. We prove that for g_1 -, g_2 - and g_3 -twisted V-modules W_1 , W_2 and W_3 , respectively, such that the vertex operator map for W_3 is injective, if there exists a twisted intertwining operator of type $\binom{W_3}{W_1W_2}$ such that the images of its component operators span W_3 , then $g_3 = g_1g_2$. We also construct what we call the skew-symmetry and contragredient isomorphisms between spaces of twisted intertwining operators among twisted modules of suitable types. The proofs of these results are much more subtle and delicate than those for the corresponding results in [FHL], [HL], [X] and [HLZ1] because they involve careful analysis of the analytic extensions corresponding to the actions of the not-necessarily-commuting automorphisms of the vertex operator algebra.

The motivation for the study of twisted intertwining operators does not just come from orbifold conformal field theories and their potential applications in geometry and physics. It in fact also comes intrinsically from the study of the uniqueness conjecture of the moonshine module vertex operator algebra proposed by Frenkel, Lepowsky and Meurman [FLM]. This conjecture was obtained by using the analogy among the Golay code, the Leech lattice and the moonshine module vertex operator algebra. In Conway's proof of the uniqueness of the Leech lattice [Co], the 24-dimensional vector space \mathbb{R}^{24} plays a fundamental role. For the uniqueness conjecture for the moonshine module vertex operator algebra, the first difficulty is that there is no analogue of \mathbb{R}^{24} . But we still need a structure large enough such that all the works can be done in this structure.

If the vertex operator subalgebra fixed by the automorphism group of a vertex operator algebra satisfy suitable conditions (for example, the three conditions in Conjecture 1.1), then one possible choice of such a structure large enough for our purpose is the intertwining operator algebra formed by the modules and intertwining operators for the fixed-point vertex operator subalgebra. However, the assumption that these suitable conditions hold is in fact a main difficult conjecture that we have to prove first. Because of this, instead of assuming these conditions, we have to develop a theory that will lead us to a proof of these conditions and a construction of the intertwining operator algebra.

Since in this area, conjectures are sometimes claimed to have been proved in some books, papers, preprints or unpublished manuscripts without the evidence that the proofs indeed exist, the author would like to comment that the mathematics for obtaining conjectures and the mathematics for finding proofs are often very different. To obtain conjectures, one can assume what one believes to be true and derive the consequences. But to prove conjectures, the assumptions used to derive the conjectures are often themselves the most difficult parts of the conjectures. Thus one might need different mathematical approaches and theories to prove these assumptions first. The theory of twisted intertwining operators initiated in this paper is what the author believes to be needed in the proofs of the conjectures mentioned above, including Conjecture 1.1 and the conjecture that suitable conditions hold for the fixed-point vertex operator subalgebra. We expect that this theory will play an important role in the study of orbifold conformal field theory and, in particular, in the study of the uniqueness conjecture of the moonshine module vertex operator algebra.

In this paper, we use the approach based on multivalued analytic functions with preferred branches developed and used in [H1], [H3], [H6], [HLZ2]–[HLZ3] and [Ch1]–[Ch2]. Intertwining operators among (untwisted) modules can be defined using either such multivalued analytic functions or formal variables. But to study products and iterates of intertwining operators, it is necessary to use the approach based on such multivalued analytic functions. For twisted intertwining operators introduced and studied in this paper, even for the definition and the properties involving only one twisted intertwining operator, we need the approach based on such multivalued analytic functions, because the vertex operators for twisted modules contain nonintegral powers and logarithm of the variable and, more importantly, because the automorphisms associated to different twisted modules do not necessarily commute with each other.

In Section 2, we discuss the notations and conventions used in this paper, especially those involving multivalued analytic functions with preferred branches. In Section 3, we recall the notion of twisted module introduced in [H7]. We also discuss in this section the functors associated to automorphisms of the vertex operator algebra and the contragredient functor on the category of twisted modules. Twisted intertwining operators are introduced in Section 4. In the same section, we prove the result that for g_1 -, g_2 - and g_3 -twisted V-modules W_1 , W_2 and W_3 , respectively, such that the vertex operator map for W_3 is injective, if there exists a twisted intertwining operator of type $\binom{W_3}{W_1W_2}$ such that the images of its component operators span W_3 , then $g_3 = g_1g_2$. The skew-symmetry isomorphisms and contragredient isomorphisms are constructed in Section 5 and Section 6, respectively.

2 Notations and conventions

To study intertwining operators, we have to work with multivalued analytic functions with preferred branches. The approach that we use in this paper is the same as the one used in [H1], [H3], [H6], [HLZ2]–[HLZ3] and [Ch1]–[Ch2]. In this section, we recall and introduce some notations and conventions.

We shall use **i** to denote $\sqrt{-1}$. For $z \in \mathbb{C}^{\times}$, we choose the value $\arg z$ of the argument of z to be the one satisfying $0 \leq \arg z < 2\pi$. We shall not use $\log z$ to denote the particular value $\log |z| + (\arg z)\mathbf{i}$ of the logarithm of z as in [H1], [H3], [HLZ2]-[HLZ3] and [Ch1]-[Ch2]. Instead, we shall always use $l_p(z)$ to denote the value $\log |z| + (\arg z + 2p\pi)\mathbf{i}$ of the logarithm of z for $p \in \mathbb{Z}$.

Intertwining operators defined using formal variables in fact give multivalued analytic functions with preferred branches. We shall use $\log z$ to denote the multivalued logarithm function of the variable z with the preferred branch $l_0(z) = \log |z| + (\arg z)\mathbf{i}$. For $n \in \mathbb{C}$, we shall use z^n to denote the multivalued analytic function $e^{n\log z}$ with the preferred branch $e^{nl_0(z)}$. Multivalued analytic functions with preferred branch on a region form a commutative associative algebra and can also be divided by such functions on the same region to obtain such functions on possibly smaller regions. In particular,

$$f(z_1, z_2) = \sum_{i,j,k,l,m,n=1}^{N} a_{ijklmn} z_1^{r_i} z_2^{s_j} (z_1 - z_2)^{t_k} (\log z_1)^l (\log z_2)^m (\log(z_1 - z_2))^n$$

for $a_{ijklmn}, r_i, s_j, t_k \in \mathbb{C}$ is a multivalued analytic function with preferred branch on the region given by $z_1, z_2 \neq 0, z_1 \neq z_2$. For $p_1, p_2, p_{12} \in \mathbb{Z}$, we shall use $f^{p_1, p_2, p_{12}}(z_1, z_2)$ to denote the single-valued branch

$$\sum_{i,j,k,l,m,n=1}^{N} a_{ijklmn} e^{r_i l_{p_1}(z_1)} e^{s_j l_{p_2}(z_2)} e^{t_k l_{p_{12}}(z_1-z_2)} (l_{p_1}(z_1))^l (l_{p_2}(z_2))^m (l_{p_{12}}(z_1-z_2))^n$$

of $f(z_1, z_2)$.

For a \mathbb{C} -graded vector space $W = \coprod_{n \in \mathbb{C}} W_{[n]}$, let $W' = \coprod_{n \in \mathbb{C}} W_{[n]}^*$ be the graded dual of W and $\overline{W} = \prod_{n \in \mathbb{C}} W_{[n]}$ the algebraic completion of W. For $n \in \mathbb{C}$, we use π_n to denote the projection from W or \overline{W} to $W_{[n]}$.

Let W be a vector space and

$$X(x) = \sum_{k=0}^{K} \sum_{n \in \mathbb{C}} a_{n,k} x^{n} (\log x)^{k} \in (\text{End } W)\{x\} [\log x].$$

For $z \in \mathbb{C}^{\times}$, we shall use $X^p(z)$ to denote the series

$$X(x)\Big|_{x^n = e^{nl_p(z)}, \log x = l_p(z)} = \sum_{k=0}^K \sum_{n \in \mathbb{C}} a_{n,k} e^{nl_p(z)} (l_p(z))^k$$

with terms in End W. When $W = \coprod_{n \in \mathbb{C}} W_{[n]}$ is a \mathbb{C} -graded vector space and $a_{n,k}$ for different n are homogeneous operators of different degrees, $X^p(z) \in \text{Hom}(W, \overline{W})$. When z changes in \mathbb{C}^{\times} , $X^p(z)$ can be viewed as a function on \mathbb{C}^{\times} valued in the space of series in W. We call this function $X^p(z)$ the *p*-th analytic branch of X(x).

3 Twisted modules

In this paper, we fix a vertex operator algebra $(V, Y_V, \mathbf{1}_V, \omega_V)$. In fact, the results of the present paper hold for a grading-restricted Möbius vertex algebra, that is, a \mathbb{Z} -graded vertex algebra $V = \coprod_{n \in \mathbb{Z}} V_{(n)}$ equipped with operators $L_V(-1)$, $L_V(0)$ and $L_V(1)$ such that $V_{(n)} = 0$ when n is sufficiently negative, dim $V_{(n)} < \infty$ for $n \in \mathbb{Z}$, the usual $\mathfrak{sl}(2, \mathbb{C})$ commutator relations for $L_V(-1)$, $L_V(0)$ and $L_V(1)$ hold and the usual commutator relations between $L_V(-1)$, $L_V(0)$ and $L_V(1)$ and vertex operators hold. But for what we want to prove in the future, it is necessary for V to be a vertex operator algebra with the additional data of a conformal element satisfying some additional conditions.

Let g be an automorphism of V. We first recall the definition of generalized g-twisted V-module first introduced in [H7]. But for simplicity, we shall omit the word "generalized." In particular, in this paper, the vertex operator map for a g-twisted V-module in general contain the logarithm of the variable and the operator L(0) in general does not have to act semisimply.

Definition 3.1 A *g*-twisted *V*-module is a $\mathbb{C} \times \mathbb{C}/\mathbb{Z}$ -graded vector space $W = \coprod_{n \in \mathbb{C}, \alpha \in \mathbb{C}/\mathbb{Z}} W_{[n]}^{[\alpha]}$ (graded by weights and *g*-weights) equipped with a linear map

$$\begin{array}{rcl} Y^g_W : V \otimes W & \to & W\{x\}[\log x], \\ & v \otimes w & \mapsto & Y^g_W(v,x)w \end{array}$$

satisfying the following conditions:

1. The equivariance property: For $p \in \mathbb{Z}$, $z \in \mathbb{C}^{\times}$, $v \in V$ and $w \in W$,

$$(Y_W^g)^{p+1}(gv,z)w = (Y_W^g)^p(v,z)w,$$

where for $p \in \mathbb{Z}$, $(Y_W^g)^p(v, z)$ is the *p*-th analytic branch of $Y_W^g(v, x)$.

- 2. The *identity property*: For $w \in W$, $Y^g(\mathbf{1}, x)w = w$.
- 3. The duality property: For any $u, v \in V, w \in W$ and $w' \in W'$, there exists a multivalued analytic function with preferred branch of the form

$$f(z_1, z_2) = \sum_{i,j,k,l=0}^{N} a_{ijkl} z_1^{m_i} z_2^{n_j} (\log z_1)^k (\log z_2)^l (z_1 - z_2)^{-t}$$

for $N \in \mathbb{N}$, $m_1, \ldots, m_N, n_1, \ldots, n_N \in \mathbb{C}$ and $t \in \mathbb{Z}_+$, such that the series

$$\langle w', (Y_W^g)^p(u, z_1)(Y_W^g)^p(v, z_2)w \rangle = \sum_{n \in \mathbb{C}} \langle w', (Y_W^g)^p(u, z_1)\pi_n(Y_W^g)^p(v, z_2)w \rangle,$$

$$\langle w', (Y_W^g)^p(v, z_2)(Y_W^g)^p(u, z_1)w \rangle = \sum_{n \in \mathbb{C}} \langle w', (Y_W^g)^p(v, z_2)\pi_n(Y_W^g)^p(u, z_1)w \rangle,$$

$$\langle w', (Y_W^g)^p(Y_V(u, z_1 - z_2)v, z_2)w \rangle = \sum_{n \in \mathbb{C}} \langle w', (Y_W^g)^p(\pi_n Y_V(u, z_1 - z_2)v, z_2)w \rangle$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1-z_2| > 0$, respectively, and their sums are equal to the branch

$$f^{p,p}(z_1, z_2) = \sum_{i,j,k,l=0}^{N} a_{ijkl} e^{m_i l_p(z_1)} e^{n_j l_p(z_2)} l_p(z_1)^k l_p(z_2)^l (z_1 - z_2)^{-t}$$

of $f(z_1, z_2)$ in the region $|z_1| > |z_2| > 0$, the region $|z_2| > |z_1| > 0$, the region given by $|z_2| > |z_1 - z_2| > 0$ and $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$, respectively.

- 4. The L(0)-grading condition and g-grading condition: Let $L_W^g(0) = \operatorname{Res}_x X_W^g(\omega, x)$. Then for $n \in \mathbb{C}$ and $\alpha \in \mathbb{C}/\mathbb{Z}$, $w \in W_{[n]}^{[\alpha]}$, there exist $K, \Lambda \in \mathbb{Z}_+$ such that $(L_W^g(0) - n)^K w = (g - e^{2\pi\alpha i})^\Lambda w = 0$. Moreover, $gY_W^g(u, x)v = Y_W^g(gu, x)gv$.
- 5. The L(-1)-derivative property: For $v \in V$,

$$\frac{d}{dx}Y_W^g(v,x) = Y_W^g(L_V(-1)v,x).$$

A lower-bounded generalized g-twisted V-module is a g-twisted V-module W such that for each $n \in \mathbb{C}$, $W_{[n+l]} = 0$ for sufficiently negative real number l. A g-twisted V-module W is said to be grading-restricted if it is lower bounded and for each $n \in \mathbb{C}$, dim $W_{[n]} < \infty$. We shall denote the g-twisted V-module just defined by (W, Y_W^g) or simply by W when Y_W^g is clear.

Let (W, Y_W^g) be a g-twisted V-module. Using the notation introduced in Section 2, we have the p-th analytic branch $(Y_W^g)^p(\omega, z)$ of the formal series $Y_W^g(\omega, x)$ for $p \in \mathbb{Z}$. Since $g\omega = \omega$,

$$(Y_W^g)^{p+1}(\omega, z) = (Y_W^g)^{p+1}(g\omega, z) = (Y_W^g)^p(\omega, z)$$

for $p \in \mathbb{Z}$. Thus $Y_W^g(\omega, x)$ involves only integral powers of x. Let

$$Y_W^g(\omega, x) = \sum_{n \in \mathbb{Z}} L_W(n) x^{-n-2}.$$

Then the same argument deriving the Virasoro relations for (untwisted) modules from axioms other than those for the Virasoro operators give

$$[L_W(m), L_W(n)] = (m-n)L_W(m+n) + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

for $m, n \in \mathbb{Z}$, where c is the central charge of V.

Let (W, Y_W^g) be a g-twisted V-module. Let h be an automorphism of V and let

$$\begin{aligned} \phi_h(Y^g) &: V \times W \quad \to \quad W\{x\}[\log x] \\ v \otimes w \quad \mapsto \quad \phi_h(Y^g)(v, x)w \end{aligned}$$

be the linear map defined by

$$\phi_h(Y^g)(v,x)w = Y^g(h^{-1}v,x)w.$$

The following result can be proved by a straightforward use of the axioms:

Proposition 3.2 The pair $(W, \phi_h(Y^g))$ is an hgh⁻¹-twisted V-module.

We shall denote the hgh^{-1} -twisted V-module in the proposition above by $\phi_h(W)$.

We also need contragredient twisted V-modules. Let (W, Y_W^g) be a g-twisted V-module relative to G. Let W' be the graded dual of W. Define a linear map

$$(Y_W^g)': V \otimes W' \to W'\{x\}[\log x], v \otimes w' \mapsto (Y_W^g)'(v, x)w'$$

by

$$\langle (Y_W^g)'(v,x)w',w\rangle = \langle w', Y_W^g(e^{xL(1)}(-x^{-2})^{L(0)}v,x^{-1})w\rangle$$

for $v \in V$, $w \in W$ and $w' \in W'$.

Proposition 3.3 The pair $(W', (Y_W^g)')$ is a g^{-1} -twisted V-module.

The proof of this result is a special case of the proof of Theorem 6.1 in Section 6 with $W_1 = V$, $g_1 = 1_V$, $g_2 = g$ and $W_2 = W_3 = W$ (see also Remark 4.2 in Section 4 below). Since the proof of Theorem 6.1 uses only the definition of $(Y_W^g)'$, quoting the proof of Theorem 6.1 to give a proof of Proposition 3.3 does not constitute circular reasoning.

The g^{-1} -twisted V-module $(W', (Y_W^g)')$ is called the *contragredient twisted V-module of* (W, Y^g) .

4 Twisted intertwining operators

We introduce the notion of twisted intertwining operator or intertwining operator among twisted modules in this section. Twisted intertwining operators in this paper in general involve the logarithm of the variable. Such intertwining operators are usually called logarithmic intertwining operators. For simplicity, we omit the word "logarithm," unless there is a need to emphasize that the intertwining operator indeed involve the logarithm of the variable.

Definition 4.1 Let g_1, g_2, g_3 be automorphisms of V and let W_1, W_2 and W_3 be g_1 -, g_2 and g_3 -twisted V-modules, respectively. A twisted intertwining operator of type $\binom{W_3}{W_1W_2}$ is a linear map

$$\mathcal{Y}: W_1 \otimes W_2 \to W_3\{x\}[\log x]$$
$$w_1 \otimes w_2 \mapsto \mathcal{Y}(w_1, x)w_2 = \sum_{k=0}^K \sum_{n \in \mathbb{C}} \mathcal{Y}_{n,k}(w_1)w_2 x^{-n-1}(\log x)^k$$

satisfying the following conditions:

- 1. The lower truncation property: For $w_1 \in W_1$ and $w_2 \in W_2$, $n \in \mathbb{C}$ and $k = 0, \ldots, K$, $\mathcal{Y}_{n+l,k}(w_1)w_2 = 0$ for $l \in \mathbb{N}$ sufficiently large.
- 2. The duality property: For $u \in V$, $w_1 \in W_1$, $w_2 \in W_2$ and $w'_3 \in W'_3$, there exists a multivalued analytic function with preferred branch

$$f(z_1, z_2; u, w_1, w_2, w'_3) = \sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn} z_1^{r_i} z_2^{s_j} (z_1 - z_2)^{t_k} (\log z_1)^l (\log z_2)^m (\log(z_1 - z_2))^n$$

for $N \in \mathbb{N}$, $r_i, s_j, t_k, a_{ijklmn} \in \mathbb{C}$, such that for $p_1, p_2, p_{12} \in \mathbb{Z}$, the series

$$\langle w_3', (Y_{W_3}^{g_3})^{p_1}(u, z_1)\mathcal{Y}^{p_2}(w_1, z_2)w_2 \rangle = \sum_{n \in \mathbb{C}} \langle w_3', (Y_{W_3}^{g_3})^{p_1}(u, z_1)\pi_n \mathcal{Y}^{p_2}(w_1, z_2)w_2 \rangle,$$
(4.1)

$$\langle w_3', \mathcal{Y}^{p_2}(w_1, z_2)(Y_{W_2}^{g_2})^{p_1}(u, z_1)w_2 \rangle = \sum_{n \in \mathbb{C}} \langle w_3', \mathcal{Y}^{p_2}(w_1, z_2)\pi_n(Y_{W_2}^{g_2})^{p_1}(u, z_1)w_2 \rangle,$$
(4.2)

$$\langle w_3', \mathcal{Y}^{p_2}((Y_{W_1}^{g_1})^{p_{12}}(u, z_1 - z_2)w_1, z_2)w_2 \rangle = \sum_{n \in \mathbb{C}} \langle w_3', \mathcal{Y}^{p_2}(\pi_n(Y_{W_1}^{g_1})^{p_{12}}(u, z_1 - z_2)w_1, z_2)w_2 \rangle$$

$$(4.3)$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1-z_2| > 0$, respectively. Moreover, their sums are equal to the branches

$$\begin{split} f^{p_1,p_2,p_1}(z_1,z_2;u,w_1,w_2,w_3') \\ &= \sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn} e^{r_i l_{p_1}(z_1)} e^{s_j l_{p_2}(z_2)} e^{t_k l_{p_1}(z_1-z_2)} (l_{p_1}(z_1))^l (l_{p_2}(z_2))^m (l_{p_1}(z_1-z_2))^n, \\ f^{p_1,p_2,p_2}(z_1,z_2;u,w_1,w_2,w_3') \\ &= \sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn} e^{r_i l_{p_1}(z_1)} e^{s_j l_{p_2}(z_2)} e^{t_k l_{p_2}(z_1-z_2)} (l_{p_1}(z_1))^l (l_{p_2}(z_2))^m (l_{p_2}(z_1-z_2))^n, \\ f^{p_2,p_2,p_{12}}(z_1,z_2;u,w_1,w_2,w_3') \\ &= \sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn} e^{r_i l_{p_2}(z_1)} e^{s_j l_{p_2}(z_2)} e^{t_k l_{p_{12}}(z_1-z_2)} (l_{p_2}(z_1))^l (l_{p_2}(z_2))^m (l_{p_{12}}(z_1-z_2))^n, \end{split}$$

respectively, of $f(z_1, z_2; u, w_1, w_2, w'_3)$ (recall the notations and convention in Section 2) in the region given by $|z_1| > |z_2| > 0$ and $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$, the region given by $|z_2| > |z_1| > 0$ and $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$, the region given by $|z_2| > |z_1 - z_2| > 0$ and $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$, respectively.

3. The L(-1)-derivative property:

$$\frac{d}{dx}\mathcal{Y}(w_1,x) = \mathcal{Y}(L(-1)w_1,x).$$

Remark 4.2 Let (W, Y_W^g) be a *g*-twisted *V*-module. Then by definition, the vertex operator map Y_W^g is a twisted intertwining operator of type $\binom{W}{VW}$.

Remark 4.3 Let g_1, g_2 and g_3 be automorphisms of V, let $V^{\langle g_1, g_2, g_3 \rangle}$ be the vertex operator subalgebra of V consisting of elements of V fixed under the actions of g_1, g_2 and g_3 . Let W_1, W_2 and W_3 be g_1 -, g_2 - and g_3 -twisted V-modules and let \mathcal{Y} be a twisted intertwining operator of type $\binom{W_3}{W_1W_2}$. Then W_1, W_2 and W_3 are $V^{\langle g_1, g_2, g_3 \rangle}$ -modules and \mathcal{Y} is an (untwisted) intertwining operator of type $\binom{W_3}{W_1W_2}$ when W_1, W_2 and W_3 are viewed as $V^{\langle g_1, g_2, g_3 \rangle}$ -modules. But in general, an (untwisted) intertwining operator of type $\binom{W_3}{W_1W_2}$ when W_1, W_2 and W_3 are viewed as $V^{\langle g_1, g_2, g_3 \rangle}$ -modules might not be a twisted intertwining operator of type $\binom{W_3}{W_1W_2}$. In fact, an (untwisted) intertwining operator of type $\binom{W_3}{W_1W_2}$ when W_1, W_2 and W_3 are viewed as $V^{\langle g_1, g_2, g_3 \rangle}$ -modules is required to satisfy only the duality property for vertex operators associated to elements in $V^{\langle g_1, g_2, g_3 \rangle}$ while a twisted intertwining operator of type $\binom{W_3}{W_1W_2}$ must satisfy the more restrictive duality property in Definition 4.1. This is the reason why even when W_1, W_2 and W_3 are known to be twisted V-modules, we still want to add the word "twisted" in front of "intertwining operator" to call an intertwining operator of type $\binom{W_3}{W_1W_2}$ in Definition 4.1 a twisted intertwining operator. **Remark 4.4** In the duality property in the definition above, we require that the sum of (4.2) is equal to $f^{p_1,p_2,p_2}(z_1, z_2; u, w_1, w_2, w'_3)$ in the region given by $|z_2| > |z_1| > 0$ and $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$. The choice of this region in fact gives an order of W_1 and W_2 to be W_1 first and W_2 second. See Theorem 4.7 and especially its proof below. The other region is the region given by $|z_2| > |z_1| > 0$ and $\frac{\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < \frac{3\pi}{2}$. If we require the sum of (4.2) is equal to $f^{p_1,p_2,p_2}(z_1, z_2; u, w_1, w_2, w'_3)$ in this region, then the order of W_1 and W_2 is chosen to be W_2 first and W_1 second. We choose the more natural order.

We shall need the following two lemmas:

Lemma 4.5 In Definition 4.1, the requirement in the duality property that the sum of (4.2) be equal to $f^{p_1,p_2,p_2}(z_1, z_2; u, w_1, w_2, w'_3)$ in the region given by $|z_2| > |z_1| > 0$ and $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$ can be replaced by the requirement that the sum of (4.2) be equal to $f^{p_1,p_2,p_2-1}(z_1, z_2; u, w_1, w_2, w'_3)$ in the region given by $|z_2| > |z_1| > 0$ and $\frac{\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < \frac{3\pi}{2}$. In the same definition, the requirement in the duality property that the sum of (4.1) be equal to $f^{p_1,p_2,p_1-1}(z_1, z_2; u, w_1, w_2, w'_3)$ in the region given by $|z_2| > |z_1| > 0$ and $\frac{\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < \frac{3\pi}{2}$. In the same definition, the requirement in the duality property that the sum of (4.1) be equal to $f^{p_1,p_2,p_1}(z_1, z_2; u, w_1, w_2, w'_3)$ in the region given by $|z_2| > |z_1| > 0$ and $-2\pi < \arg(z_1 - z_2) - \arg z_2 < -\frac{3\pi}{2}$ and to $f^{p_1,p_2,p_1+1}(z_1, z_2; u, w_1, w_2, w'_3)$ in the region given by $|z_2| > |z_1| > 0$ and $-2\pi < \arg(z_1 - z_2) - \arg z_2 < -\frac{3\pi}{2}$ and to $f^{p_1,p_2,p_1+1}(z_1, z_2; u, w_1, w_2, w'_3)$ in the region given by $|z_2| > |z_1| > 0$ and $-2\pi < \arg(z_1 - z_2) - \arg z_2 < -\frac{3\pi}{2}$ and to $f^{p_1,p_2,p_1+1}(z_1, z_2; u, w_1, w_2, w'_3)$ in the region given by $|z_2| > |z_1| > 0$ and $-2\pi < \arg(z_1 - z_2) - \arg z_2 < -\frac{3\pi}{2}$ and to $f^{p_1,p_2,p_1+1}(z_1, z_2; u, w_1, w_2, w'_3)$ in the region given by $|z_2| > |z_1| > 0$ and $-2\pi < \arg(z_1 - z_2) - \arg z_2 < -\frac{3\pi}{2}$ and to $f^{p_1,p_2,p_1+1}(z_1, z_2; u, w_1, w_2, w'_3)$ in the region given by $|z_2| > |z_1| > 0$ and $-2\pi < \arg(z_1 - z_2) - \arg(z_2) - \arg(z_1 - z_2) - \arg(z_2) - \arg(z_2)$

Proof. We prove only the first part. The second part can be proved similarly.

Assume that \mathcal{Y} is a twisted intertwining operator satisfying Definition 4.1. We choose a path l in the region given by $|z_1| > |z_2| > 0$ from a point $(z_1^{(1)}, z_2^{(0)})$ in the subregion given by $|z_2| > |z_1| > 0$ and $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$ to a point $(z_1^{(2)}, z_2^{(0)})$ in the subregion given by $|z_2| > |z_1| > 0$ and $\frac{\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < \frac{3\pi}{2}$ by letting z_1 pass through the set given by $\arg(z_1 - z_2) = 0$ in the counter clockwise direction for the variable $z_1 - z_2$ but keeping $\arg z_1$ between 0 and 2π and fixing $z_2 = z_2^{(0)}$. See Figure 1. The sum of the series (4.2) is an analytic function of z_1 and z_2 and thus we can analytically



Figure 1: The path l

extend its value at $(z_1^{(1)}, z_2^{(0)})$ through the path *l* to its value at $(z_1^{(2)}, z_2^{(0)})$. At $(z_1^{(1)}, z_2^{(0)})$, its

value is given by $f^{p_1,p_2,p_2}(z_1^{(1)}, z_2^{(0)}; u, w_1, w_2, w_3')$. When the path l pass the point at which $\arg(z_1 - z_2) = 0$, there is a jump of $\arg(z_1 - z_2)$ from 0 to 2π . When $\arg(z_1 - z_2) = 0$, its value (at z_1, z_2) is still $f^{p_1,p_2,p_2}(z_1, z_2; u, w_1, w_2, w_3')$. But after the jump, since the sum is analytic and in particular is continuous, its value at (z_1, z_2) must be $f^{p_1,p_2,p_2-1}(z_1, z_2; u, w_1, w_2, w_3')$. In particular, its value at the arbitrary point $(z_1^{(2)}, z_2^{(0)})$ in the region given by $|z_2| > |z_1| > 0$ and $\frac{\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < \frac{3\pi}{2}$ must be $f^{p_1,p_2,p_2-1}(z_1^{(2)}, z_2^{(0)}; u, w_1, w_2, w_3')$.

If we assume that \mathcal{Y} satisfies the requirement that the sum of (4.2) be equal to

 $f^{p_1,p_2,p_2-1}(z_1,z_2;u,w_1,w_2,w_3)$

in the region given by $|z_2| > |z_1| > 0$ and $\frac{\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < \frac{3\pi}{2}$ and all the other axioms in Definition 4.1, a completely analogous argument shows that the sum of (4.2) be equal to $f^{p_1,p_2,p_2}(z_1, z_2; u, w_1, w_2, w'_3)$ in the region given by $|z_2| > |z_1| > 0$ and $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$.

Lemma 4.6 For $p_1, p_2, p_{12} \in \mathbb{Z}$, $u \in V$, $w_1 \in W_1$, $w_2 \in W_2$ and $w'_3 \in W'_3$, we have

$$f^{p_1,p_2,p_{12}+1}(z_1,z_2;g_1u,w_1,w_2,w_3') = f^{p_1,p_2,p_{12}}(z_1-z_2,z_2;u,w_1,w_2,w_3')$$
(4.4)

and

$$f^{p_1+1,p_2,p_{12}}(z_1, z_2; g_2u, w_1, w_2, w_3') = f^{p_1,p_2,p_{12}}(z_1, z_2; u, w_1, w_2, w_3'),$$
(4.5)

where $f^{p_1,p_2,p_{12}}(z_1, z_2; u, w_1, w_2, w'_3)$ for $p_1, p_2, p_{12} \in \mathbb{Z}$ is the branch given by p_1, p_2, p_{12} of the multivalued analytic function $f(z_1, z_2; u, w_1, w_2, w'_3)$ with preferred branch in Definition 4.1.

Proof. By the duality property for \mathcal{Y} , when $|z_2| > |z_1 - z_2| > 0$ and $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$,

$$\langle w_3', \mathcal{Y}^{p_2}((Y_{W_1}^{g_1})^{p_{12}+1}(g_1u, z_1 - z_2)w_1, z_2)w_2 \rangle$$

converges absolutely to $f^{p_2, p_2, p_{12}+1}(z_1, z_2; g_1u, w_1, w_2, w'_3)$ and

$$\langle w_3', \mathcal{Y}^{p_2}((Y_{W_1}^{g_1})^{p_{12}}(u, z_1 - z_2)w_1, z_2)w_2 \rangle$$

converges absolutely to $f^{p_2,p_2,p_{12}}(z_1, z_2; u, w_1, w_2, w'_3)$. But

$$\langle w_3', \mathcal{Y}^{p_2}((Y_{W_1}^{g_1})^{p_{12}+1}(g_1u, z_1 - z_2)w_1, z_2)w_2 \rangle = \langle w_3', \mathcal{Y}^{p_2}((Y_{W_1}^{g_1})^{p_{12}}(u, z_1 - z_2)w_1, z_2)w_2 \rangle$$

Thus we have

$$f^{p_2,p_2,p_{12}+1}(z_1, z_2; g_1u, w_1, w_2, w'_3) = f^{p_2,p_2,p_{12}}(z_1, z_2; u, w_1, w_2, w'_3)$$

For general $p_1, p_2, p_{12} \in \mathbb{Z}$, we obtain (4.4) by analytic extensions.

On the other hand, by the duality property for \mathcal{Y} , when $|z_2| > |z_1| > 0$ and $-\frac{3\pi}{2} < \arg z_1 - \arg(-z_2) < -\frac{\pi}{2}$,

$$\langle w_3', \mathcal{Y}^{p_2}(w_1, z_2)(Y_{W_1}^{g_2})^{p_1+1}(g_2 u, z_1)w_2 \rangle$$

converges absolutely to $f^{p_1+1,p_2,p_2}(z_1, z_2; g_2u, w_1, w_2, w'_3)$ and

$$\langle w_3', \mathcal{Y}^{p_2}(w_1, z_2)(Y^{g_2}_{W_1})^{p_1}(u, z_1)w_2 \rangle$$

converges absolutely to $f^{p_1, p_2, p_2}(z_1, z_2; u, w_1, w_2, w'_3)$. But

$$\langle w_3', \mathcal{Y}^{p_2}(w_1, z_2)(Y_{W_1}^{g_2})^{p_1+1}(g_2u, z_1)w_2 \rangle = \langle w_3', \mathcal{Y}^{p_2}(w_1, z_2)(Y_{W_1}^{g_2})^{p_1}(u, z_1)w_2 \rangle.$$

Thus we have

$$f^{p_1+1,p_2,p_2}(z_1, z_2; g_2u, w_1, w_2, w'_3) = f^{p_1,p_2,p_2}(z_1 - z_2, -z_2; u, w_1, w_2, w'_3).$$

For general $p_1, p_2, p_{12} \in \mathbb{Z}$, we obtain (4.5) by analytic extensions.

We now prove that under suitable minor conditions, $g_3 = g_1 g_2$ for the twisted intertwining operator defined in Definition 4.1.

Theorem 4.7 Let g_1, g_2, g_3 be automorphisms of V and let W_1, W_2 and W_3 be g_1 -, g_2 - and g_3 -twisted V-modules, respectively. Assume that the vertex operator map for W_3 given by $u \mapsto Y_{W_3}^{g_3}(u, x)$ is injective. If there exists a twisted intertwining operator \mathcal{Y} of type $\binom{W_3}{W_1W_2}$ such that the coefficients of the series $\mathcal{Y}(w_1, x)w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$ span W_3 , then $g_3 = g_1g_2$.

Proof. Let \mathcal{Y} be a twisted intertwining operator of type $\binom{W_3}{W_1W_2}$ such that the coefficients of $\mathcal{Y}(w_1, x)w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$ span W_3 . For $u \in V$, $w_1 \in W_1$, $w_2 \in W_2$ and $w'_3 \in W'_3$, consider the multivalued analytic function $f(z_1, z_2; u, w_1, w_2, w'_3)$ with preferred branch for the twisted intertwining operator \mathcal{Y} (see Definition 4.1). Fix z_2 to be a nonzero negative real number $-a_2$ where $a_2 \in \mathbb{R}_+$. Then for any $p \in \mathbb{Z}$, we have an analytic function

$$f_{-a_2}^p(z_1) = \sum_{i,j,k,l,m,n=0}^N a_{ijklmn} e^{r_i l_p(z_1)} e^{s_j l_p(-a_2)} e^{t_k l_p(z_1+a_2)} (l_p(z_1))^l (l_p(-a_2))^m (l_p(z_1+a_2))^n e^{s_j l_p(z_1)} e^{s_j l_p(z_1)} e^{s_j l_p(z_1+a_2)} e^{s_j l_p(z_1+a_2)} (l_p(z_1))^l (l_p(z_1))^{l_p(z_1+a_2)} e^{s_j l_p(z_1)} e^{s_j l_p(z_1+a_2)} e^{s_j l_p(z_1+a_2)}$$

of z_1 and can be analytically extended to a multivalued analytic function

$$f_{-a_2}(z_1) = \sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn} z_1^{r_i} e^{s_j l_p(-a_2)} (z_1 + a_2)^{t_k} (\log z_1)^l (l_p(-a_2))^m (\log(z_1 + a_2))^n (\log(z_1 + a_2))^{t_k} (\log z_1)^l (\log z_1)^l (\log z_1)^{t_k} (\log z_$$

of z_1 with preferred branch. Let $a_1 \in \mathbb{R}_+$ such that $a_1 > a_2 > a_1 - a_2$. Consider the loop Γ_1 in the z_1 plane based at $z_1 = -a_1$ in Figure 2. We consider the value $f_{-a_2}^p(-a_1)$ of the multivalued analytic function $f_{-a_2}(z_1)$ at $-a_1$. By the definition of twisted intertwining operator above,

$$f_{-a_2}^p(-a_1) = \langle w_3', (Y_{W_3}^{g_3})^p(u, -a_1)\mathcal{Y}^p(w_1, -a_2)w_2 \rangle.$$
(4.6)

But by definition, when z_1 goes around the loop above, the right-hand side of (4.6) changes to

$$\langle w'_3, (Y^{g_3}_{W_3})^{p-1}(u, -a_1)\mathcal{Y}^p(w_1, -a_2)w_2 \rangle.$$
 (4.7)



Figure 2: The loop Γ_1

By the equivariance property of the g_3 -twisted module W_3 , (4.7) is equal to

$$\langle w_3', (Y_{W_3}^{g_3})^p(g_3u, -a_1)\mathcal{Y}^p(w_1, -a_2)w_2 \rangle.$$
 (4.8)

We also consider another loop Γ_2 in the z_1 plane and based at $z_1 = -a_1$ given in the order l_1 first, l_2 second, l_3 third and l_4 last in Figure 3. The loop Γ_2 is in fact homotopy equivalent



Figure 3: The loop Γ_2

to the loop Γ_1 . Thus when z_1 goes around Γ_2 , the right-hand side of (4.6) also changes to (4.8).

On the other hand, we look at how the function values change when z_1 goes through l_1 , l_2 , l_3 and l_4 . When z_1 goes from $-a_1$ to $-a_3$ (see Figure 3) through l_1 , the right-hand side of (4.6) changes to

$$\sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn} e^{r_i l_p(-a_3)} e^{s_j l_p(-a_2)} e^{t_k l_p(-a_3+a_2)} (l_p(-a_3))^l (l_p(-a_2))^m (l_p(-a_3+a_2))^n$$
(4.9)

Note that $\arg(-a_2) = \pi$, $\arg(-a_3) = \pi$ and $\arg(-a_3 + a_2) = 0$. Hence $\arg(-a_3 + a_2) - \arg(-a_2) = -\pi$. Since $|-a_1| > |-a_3| > 0$ and $-\frac{3\pi}{2} < \arg(-a_3 + a_2) - \arg(-a_2) < -\frac{\pi}{2}$, by the duality property of the twisted intertwining operator \mathcal{Y} , (4.9) is equal to

$$\langle w'_3, \mathcal{Y}^p(w_1, -a_2)(Y^{g_2}_{W_2})^p(u, -a_3)w_2 \rangle.$$
 (4.10)

Next let z_1 go around the loop l_2 . Then (4.10) changes to

$$\langle w_3', \mathcal{Y}^p(w_1, -a_2)(Y_{W_2}^{g_2})^{p-1}(u, -a_3)w_2 \rangle.$$
 (4.11)

By the equivariance property of the g_2 -twisted module W_2 , (4.11) is equal to

$$\langle w_3', \mathcal{Y}^p(w_1, -a_2)(Y_{W_2}^{g_2})^p(g_2u, -a_3)w_2 \rangle.$$
 (4.12)

Now let z_1 go from $-a_3$ to $-a_1$ through l_3 . Then by reversing the argument above on the change of the values when z_1 goes through l_1 , we see that the values change from (4.12) to

$$\langle w'_3, (Y^{g_3}_{W_3})^p (g_2 u, -a_1) \mathcal{Y}^p (w_1, -a_2) w_2 \rangle.$$
 (4.13)

Since $|-a_1| > |-a_2| > |-a_1 - (-a_2)| > 0$, $|\arg(-a_1 - (a_2)) - \arg(-a_1)| = 0 < \frac{\pi}{2}$ and $|\arg(-a_1) - \arg(-a_2)| = 0 < \frac{\pi}{2}$, by the duality property of the twisted intertwining operator \mathcal{Y} , (4.13) is equal to

$$\langle w'_3, \mathcal{Y}^p((Y^{g_3}_{W_1})^p(g_2u, -a_1+a_2)w_1, -a_2)w_2 \rangle.$$
 (4.14)

Finally let z_1 go around the loop l_4 . The value changes from (4.14) to

$$\langle w'_3, \mathcal{Y}^p((Y^{g_3}_{W_1})^{p-1}(g_2u, -a_1+a_2)w_1, -a_2)w_2 \rangle.$$
 (4.15)

By the equivariance property of the g_1 -twisted module W_1 , (4.15) is equal to

$$\langle w_3', \mathcal{Y}^p((Y_{W_1}^{g_3})^p(g_1g_2u, -a_1+a_2)w_1, -a_2)w_2 \rangle.$$
 (4.16)

Again since $|-a_1| > |-a_2| > |-a_1 - (-a_2)| > 0$, $|\arg(-a_1 - (a_2)) - \arg(-a_1)| = 0 < \frac{\pi}{2}$ and $|\arg(-a_1) - \arg(-a_2)| = 0 < \frac{\pi}{2}$, by the duality property of the twisted intertwining operator \mathcal{Y} , (4.16) is equal to

$$\langle w_3', (Y_{W_3}^{g_3})^p (g_1 g_2 u, -a_1) \mathcal{Y}^p (w_1, -a_2) w_2 \rangle.$$
 (4.17)

From the discussions above, we see that when z_1 goes around Γ_2 , the right-hand side of (4.6) changes to (4.17). Thus we see that (4.8) and (4.17) are equal, that is

$$\langle w_3', (Y_{W_3}^{g_3})^p (g_3 u - g_1 g_2 u, -a_1) \mathcal{Y}^p (w_1, -a_2) w_2 \rangle = 0.$$
 (4.18)

Since w_1, w_2 and w'_3 are arbitrary and the coefficients of $\mathcal{Y}(w_1, x)w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$ span W_3 , we obtain from (4.18)

$$(Y_{W_3}^{g_3})^p(g_3u - g_1g_2u, -a_1) = 0. (4.19)$$

Replacing u in (4.19) by $L_V(-1)^m u$ for $m \in \mathbb{N}$, using the fact that $L_V(-1)$ commutes with g_1, g_2 and g_3 and then using the L(-1)-derivative property for the twisted vertex operator $Y_{W_3}^{g_3}$, we obtain

$$\frac{d^m}{dx^m} Y_{W_3}^{g_3}(g_3 u - g_1 g_2 u, x) \bigg|_{x^n = e^{nl_p(-a_1)}, \log x = l_p(-a_1)} = 0.$$

Using the Taylor series expansion, we obtain

$$Y_{W_3}^{g_3}(g_3u - g_1g_2u, x) \bigg|_{x^n = e^{nl_p(z)}, \log x = l_p(z)} = 0$$
(4.20)

for $z \in \mathbb{C}^{\times}$. Thus

$$Y_{W_3}^{g_3}(g_3u - g_1g_2u, x) = 0. (4.21)$$

Since the vertex operator map $u \mapsto Y_{W_3}^{g_3}(u, x)$ is injective, (4.21) implies $g_3u - g_1g_2u = 0$ or $g_3u = g_1g_2u$. Since u is also arbitrary, we obtain $g_3 = g_1g_2$.

Remark 4.8 The proof of Theorem 4.7 can be intuitively understood using two braiding graphs. See Figure 4. Since the two braiding graphs are topologically equivalent (isotopic),



Figure 4: The braiding graphs corresponding to Γ_1 (left) and Γ_2 (right)

the corresponding algebraic objects are equal and thus we have $g_3 u = g_1 g_2 u$. In fact, just like the theory of braided tensor categories, the correspondence between algebraic and analytic calculations and the braiding graphs can be made mathematically precise so that proofs such as the one for Theorem 4.7 can be given using such graphs. Note that in these graphs, we have suppressed the associativity for the twisted vertex operators and the twisted intertwining operator, just as what people usually do in the graphs for braided tensor categories. We also note that these braiding graphs explain only those results that are topological in nature. For analytic results such as those we shall prove in the next two sections, these graphs are not very useful.

Because of Theorem 4.7, in the rest of this paper, we shall discuss only twisted intertwining operators of type $\binom{W_3}{W_1W_2}$ with W_1 , W_2 and W_3 being g_1 -, g_2 - and g_1g_2 -twisted V-modules.

5 The skew symmetry isomorphisms

In this section, we construct what we call the skew-symmetry isomorphisms between the spaces of twisted intertwining operators of suitable types. These linear isomorphisms correspond to braidings in the still-to-be-constructed G-crossed braided tensor category structure on the category of g-twisted V-modules for all g in a group G of automorphisms of V.

Let g_1, g_2 be automorphisms of V, W_1, W_2 and $W_3 g_1$ -, g_2 - and g_1g_2 -twisted V-modules and \mathcal{Y} a twisted intertwining operator of type $\binom{W_3}{W_1W_2}$. We define linear maps

$$\Omega_{\pm}(\mathcal{Y}) : W_2 \otimes W_1 \to W_3\{x\}[\log x] \\ w_2 \otimes w_1 \mapsto \Omega_{\pm}(\mathcal{Y})(w_2, x)w_1$$

by

$$\Omega_{\pm}(\mathcal{Y})(w_2, x)w_1 = e^{xL(-1)}\mathcal{Y}(w_1, y)w_2 \bigg|_{y^n = e^{\pm n\pi \mathbf{i}}x^n, \log y = \log x \pm \pi \mathbf{i}}$$
(5.1)

for $w_1 \in W_1$ and $w_2 \in W_2$.

From the definition (5.1), for $p \in \mathbb{Z}$, $w_1 \in W_1$, $w_2 \in W_2$ and $z \in \mathbb{C}^{\times}$,

$$\Omega_{\pm}(\mathcal{Y})^{p}(w_{2}, z)w_{1} = \Omega_{\pm}(\mathcal{Y})(w_{2}, x)w_{1}\Big|_{x^{n}=e^{nl_{p}(z)}, \log x=l_{p}(z)}$$

$$= \left(e^{xL(-1)}\mathcal{Y}(w_{1}, y)w_{2}\Big|_{y^{n}=e^{\pm n\pi \mathbf{i}}x^{n}, \log y=\log x\pm \pi \mathbf{i}}\right)\Big|_{x^{n}=e^{nl_{p}(z)}, \log x=l_{p}(z)}$$

$$= e^{zL(-1)}\mathcal{Y}(w_{1}, y)w_{2}\Big|_{y^{n}=e^{n(l_{p}(z)\pm \pi \mathbf{i})}, \log y=l_{p}(z)\pm \pi \mathbf{i}}.$$

When $\arg z < \pi$ and $\arg z \ge \pi$, $\arg(-z) = \arg z + \pi$ and $\arg(-z) = \arg z - \pi$, respectively. Hence

$$e^{zL(-1)}\mathcal{Y}(w_1,y)w_2\Big|_{y^n=e^{n(l_p(z)+\pi\mathbf{i})},\ \log y=l_p(z)+\pi\mathbf{i}}=e^{zL(-1)}\mathcal{Y}^p(w_1,-z)w_2$$

when $\arg z < \pi$ and

$$e^{zL(-1)}\mathcal{Y}(w_1,y)w_2\Big|_{y^n=e^{n(l_p(z)-\pi\mathbf{i})},\ \log y=l_p(z)-\pi\mathbf{i}}=e^{zL(-1)}\mathcal{Y}^p(w_1,-z)w_2$$

when $\arg z \ge \pi$. In particular, for $w_1 \in W_1$, $w_2 \in W_2$ and $z \in \mathbb{C}^{\times}$ satisfying $\arg z < \pi$ and $\arg z \ge \pi$, we have

$$\Omega_{+}(\mathcal{Y})^{p}(w_{2},z)w_{1} = e^{zL(-1)}\mathcal{Y}^{p}(w_{1},-z)w_{2}$$
(5.2)

and

$$\Omega_{-}(\mathcal{Y})^{p}(w_{2}, z)w_{1} = e^{zL(-1)}\mathcal{Y}^{p}(w_{1}, -z)w_{2}, \qquad (5.3)$$

respectively.

Theorem 5.1 The linear maps $\Omega_+(\mathcal{Y})$ and $\Omega_-(\mathcal{Y})$ are twisted intertwining operators of types $\binom{W_3}{W_2\phi_{g_2^{-1}}(W_1)}$ and $\binom{W_3}{\phi_{g_1}(W_2)W_1}$, respectively (recall the definition of ϕ_g for an automorphism g of V in Section 3).

Proof. The lower-truncation property and the L(-1)-derivative property are easy to verify. We prove only the duality property.

Let $u \in V$, $w_1 \in W_1$, $w_2 \in W_2$ and $w'_3 \in W'_3$. We first need to give the multivalued analytic functions with preferred branches in the duality property. We shall denote these multivalued analytic functions for $\Omega_+(\mathcal{Y})$ and $\Omega_-(\mathcal{Y})$ by $g_+(z_1, z_2; u, w_1, w_2, w'_3)$ and $g_-(z_1, z_2; u, w_1, w_2, w'_3)$, respectively. Let $f(z_1, z_2; u, w_1, w_2, w'_3)$ be the multivalued analytic function with preferred branch in the duality property for the twisted intertwining operator \mathcal{Y} . Then we can write

$$f(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2 L'(1)} w'_3) = \sum_{i,j,k,l,m,n=0}^N a_{ijklmn} (z_1 - z_2)^{r_i} (-z_2)^{s_j} z_1^{t_k} (\log z_1)^l (\log z_2)^m (\log(z_1 - z_2))^n,$$

where L'(1) is the adjoint operator of L(-1) on W_3 and is equal to the coefficient of the x^{-3} term in $(Y_{W_3}^g)'(\omega, x)$. Define

$$g_{\pm}(z_1, z_2; u, w_2, w_1, w'_3) = \sum_{i,j,k,l,m,n=0}^{N} e^{\pm s_j \pi \mathbf{i}} a_{ijklmn} (z_1 - z_2)^{r_i} z_2^{s_j} z_1^{t_k} (\log(z_1 - z_2))^l (\log z_2 + \pi \mathbf{i})^m (\log z_1)^n.$$
(5.4)

When $|z_1| > |z_2| > 0$, $|z_1 - z_2| > |z_2| > 0$, $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ and $\arg z_2 < \pi$ (for Ω_+) or $\arg z_2 \ge \pi$ (for Ω_-), from (5.3), the L(-1)-derivative property for $Y_{W_3}^{g_3}$ and the duality property for \mathcal{Y} ,

$$\langle w_3', (Y_{W_3}^{g_3})^{p_1}(u, z_1)\Omega_{\pm}(\mathcal{Y})^{p_2}(w_2, z_2)w_1 \rangle = \langle w_3', (Y_{W_3}^{g_3})^{p_1}(u, z_1)e^{z_2L(-1)}\mathcal{Y}^{p_2}(w_1, -z_2)w_2 \rangle = \langle e^{z_2L'(1)}w_3', (Y_{W_3}^{g_3})^{p_1}(u, z_1 - z_2)\mathcal{Y}^{p_2}(w_1, -z_2)w_2 \rangle$$
(5.5)

converges absolutely to

$$f^{p_1, p_2, p_1}(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2 L'(1)} w'_3) = \sum_{i, j, k, l, m, n=0}^{N} a_{ijklmn} e^{r_i l_{p_1}(z_1 - z_2)} e^{s_j l_{p_2}(-z_2)} e^{t_k l_{p_1}(z_1)} (l_{p_1}(z_1 - z_2))^l (l_{p_2}(-z_2))^m (l_{p_1}(z_1))^n.$$
(5.6)

Since $\arg z_2 < \pi$ (for Ω_+) or $\arg z_2 \ge \pi$ (for Ω_-), $\arg(-z_2) = \arg z_2 \pm \pi$ (for Ω_{\pm}). Hence the right-hand side of (5.6) is equal to

$$\sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn} e^{r_i l_{p_1}(z_1-z_2)} e^{s_j (l_{p_2}(z_2)\pm\pi \mathbf{i})} e^{t_k l_{p_1}(z_1)} (l_{p_1}(z_1-z_2))^l (l_{p_2}(z_2)\pm\pi \mathbf{i})^m (l_{p_1}(z_1))^n$$
$$= g_{\pm}^{p_1,p_2,p_1} (z_1, z_2; u, w_2, w_1, w_3').$$
(5.7)

But the left-hand side of (5.7) can be expanded in the region $|z_1| > |z_2| > 0$ as a series in powers of $e^{l_{p_1}(z_1)}$ and $e^{l_{p_2}(z_2)}$ and in finitely many nonnegative integral powers of $l_{p_1}(z_1)$ and $l_{p_2}(z_2)$ such that the real parts of the powers of $e^{l_{p_1}(z_1)}$ is bounded from above and the real parts of the powers of $e^{l_{p_2}(z_2)}$ is bounded from below. So the left-hand side of (5.5) as a series of the same form that converges to the left-hand side of (5.7) in a smaller region must be convergent absolutely in the larger region given by $|z_1| > |z_2| > 0$ and its sum must be equal to (5.7) when $|z_1| > |z_2| > 0$, $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ and $\arg z_2 < \pi$ (for Ω_+) or $\arg z_2 \ge \pi$ (for Ω_-). Since both the left-hand side of (5.5) and the left-hand side of (5.7) are single-valued analytic function in z_1 and z_2 with cuts at $z_1 \in \mathbb{R}_+$ and $z_2 \in \mathbb{R}_+$, the fact that they are equal when $|z_1| > |z_2| > 0$, $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ and $\arg z_2 < \pi$ (for Ω_+) or $\arg z_2 \ge \pi$ (for Ω_-) means that they are equal when $|z_1| > |z_2| > 0$, and $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$. Thus we have proved that when $|z_1| > |z_2| > 0$ and $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$, the left-hand side of (5.5) is equal to $g_{\pm}^{p_1,p_2,p_1}(z_1,z_2;u,w_2,w_1,w'_3)$.

When $|z_2| > |z_1| > 0$ and $\arg z_2 \ge \pi$,

$$\langle w_3', \Omega_{-}(\mathcal{Y})^{p_2}(w_2, z_2)(Y_{W_1}^{g_1})^{p_1}(u, z_1)w_1 \rangle = \langle w_3', e^{z_2L(-1)}\mathcal{Y}^{p_2}((Y_{W_1}^{g_1})^{p_1}(u, z_1)w_1, -z_2)w_2 \rangle = \langle e^{z_2L'(1)}w_3', \mathcal{Y}^{p_2}((Y_{W_1}^{g_1})^{p_1}(u, z_1)w_1, -z_2)w_2 \rangle$$
(5.8)

converges absolutely and if in addition, $|\arg(z_1-z_2)-\arg(-z_2)|<\frac{\pi}{2}$, its sum is equal to

$$f^{p_{2},p_{2},p_{1}}(z_{1}-z_{2},-z_{2};u,w_{1},w_{2},e^{z_{2}L'(1)}w_{3}') = \sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn}e^{r_{i}l_{p_{2}}(z_{1}-z_{2})}e^{s_{j}l_{p_{2}}(-z_{2})}e^{t_{k}l_{p_{1}}(z_{1})}(l_{p_{2}}(z_{1}-z_{2}))^{l}(l_{p_{2}}(-z_{2}))^{m}(l_{p_{1}}(z_{1}))^{n}.$$
(5.9)

When $\arg z_2 \ge \pi$, $\arg(-z_2) = \arg z_2 - \pi$. Hence the right-hand side of (5.9) is equal to

$$\sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn} e^{r_i l_{p_2}(z_1-z_2)} e^{s_j (l_{p_2}(z_2)-\pi \mathbf{i})} e^{t_k l_{p_1}(z_1)} (l_{p_2}(z_1-z_2))^l (l_{p_2}(z_2)-\pi \mathbf{i})^m (l_{p_1}(z_1))^n$$

= $g_{-}^{p_1,p_2,p_2}(z_1,z_2;u,w_1,w_2,w_3').$ (5.10)

The same argument as above shows that the left-hand side of (5.8) converges absolutely when $|z_2| > |z_1| > 0$ and its sum is equal to (5.10) when $|z_2| > |z_1| > 0$, $|\arg(z_1-z_2)-\arg(-z_2)| < \frac{\pi}{2}$ and $\arg z_2 \ge \pi$. But when $\arg z_2 \ge \pi$, $\arg(-z_2) = \arg z_2 - \pi$. Hence in this case, the inequality $|\arg(z_1-z_2) - \arg(-z_2)| < \frac{\pi}{2}$ becomes $-\frac{3\pi}{2} < \arg(z_1-z_2) - \arg z_2 < -\frac{\pi}{2}$. Also both the left-hand side of (5.8) and the left-hand side of (5.10) are single valued analytic functions in z_1 and z_2 with cuts at $z_1 \in \mathbb{R}_+$ and $z_2 \in \mathbb{R}_+$. Thus the same argument as above shows that when $|z_2| > |z_1| > 0$ and $-\frac{3\pi}{2} < \arg(z_1-z_2) - \arg z_2 < -\frac{\pi}{2}$, the left-hand side of (5.8) is equal to $g_{-}^{p_1,p_2,p_1}(z_1,z_2;u,w_2,w_1,w_3')$.

Next we discuss the iterate of $\Omega_{-}(\mathcal{Y})$ and the twisted vertex operator map $\phi_{g_1}(Y_{W_2}^{g_2})$. By Lemma 4.5, when $|z_2| > |z_1 - z_2| > 0$ and $\arg z_2 \ge \pi$,

$$\langle w_3', \Omega_{-}(\mathcal{Y})^{p_2}(\phi_{g_1}(Y_{W_2}^{g_2})^{p_{12}}(u, z_1 - z_2)w_2, z_2)w_1 \rangle$$

$$= \langle w_3', \Omega_{-}(\mathcal{Y})^{p_2}((Y_{W_2}^{g_2})^{p_{12}}(g_1^{-1}u, z_1 - z_2)w_2, z_2)w_1 \rangle$$

$$= \langle w_3', e^{z_2L(-1)}\mathcal{Y}^{p_2}(w_1, -z_2)(Y_{W_2}^{g_2})^{p_{12}}(g_1^{-1}u, z_1 - z_2)w_2 \rangle$$

$$= \langle e^{z_2L'(1)}w_3', \mathcal{Y}^{p_2}(w_1, -z_2)(Y_{W_2}^{g_2})^{p_{12}}(g_1^{-1}u, z_1 - z_2)w_2 \rangle$$

$$(5.11)$$

converges absolutely and if in addition, $\frac{\pi}{2} < \arg z_1 - \arg(-z_2) < \frac{3\pi}{2}$, its sum is equal to $f^{p_{12},p_2,p_2-1}(z_1 - z_2, -z_2; g_1^{-1}u, w_1, w_2, e^{z_2L'(1)}w'_3)$. By (4.4), we have

$$f^{p_{12},p_2,p_2-1}(z_1 - z_2, -z_2; g_1^{-1}u, w_1, w_2, e^{z_2L'(1)}w'_3) = f^{p_{12},p_2,p_2}(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2L'(1)}w'_3) = \sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn} e^{r_i l_{p_{12}}(z_1 - z_2)} e^{s_j l_{p_2}(-z_2)} e^{t_k l_{p_2}(z_1)} (l_{p_{12}}(z_1 - z_2))^l (l_{p_2}(-z_2))^m (l_{p_2}(z_1))^n.$$
(5.12)

When $\arg z_1 \ge \pi$, $\arg(-z_2) = \arg z_2 - \pi$ and hence the right-hand side of (5.12) is equal to

$$\sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn} e^{r_i l_{p_{12}}(z_1 - z_2)} e^{s_j (l_{p_2}(z_2) - \pi \mathbf{i})} e^{t_k l_{p_2}(z_1)} (l_{p_{12}}(z_1 - z_2))^l (l_{p_2}(z_2) - \pi \mathbf{i})^m (l_{p_2}(z_1))^n$$
$$= g_{-}^{p_2,p_2,p_{12}} (z_1, z_2; u, w_2, w_1, w_3').$$
(5.13)

Thus the left-hand side of (5.11) converges absolutely when $|z_2| > |z_1 - z_2| > 0$ and its sum is equal to $g_{-}^{p_2,p_2,p_{12}}(z_1, z_2; u, w_2, w_1, w'_3)$ when $|z_2| > |z_1 - z_2| > 0$, $\frac{\pi}{2} < \arg z_1 - \arg(-z_2) < \frac{3\pi}{2}$ and $\arg z_2 \ge \pi$. But when $\arg z_2 \ge \pi$, $\arg(-z_2) = \arg z_2 - \pi$ and thus the inequality $\frac{\pi}{2} < \arg z_1 - \arg(-z_2) < \frac{3\pi}{2}$ is equivalent to $|\arg z_1 - \arg(z_2)| < \frac{\pi}{2}$. Then by the same

arguments as in the cases above, we see that when $|z_2| > |z_1-z_2| > 0$ and $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$, the sum of left-hand side of (5.11) is equal to $g_{-}^{p_2,p_2,p_{12}}(z_1,z_2;u,w_2,w_1,w_3')$.

We now come back to discuss $\Omega_+(\mathcal{Y})$. When $|z_2| > |z_1| > 0$ and $\arg z_2 < \pi$,

$$\langle w_3', \Omega_+(\mathcal{Y})^{p_2}(w_2, z_2)\phi_{g_2^{-1}}(Y_{W_1}^{g_1})^{p_1}(u, z_1)w_1 \rangle = \langle w_3', e^{z_2L(-1)}\mathcal{Y}^{p_2}(\phi_{g_2^{-1}}(Y_{W_1}^{g_1})^{p_1}(u, z_1)w_1, -z_2)w_2 \rangle = \langle e^{z_2L'(1)}w_3', \mathcal{Y}^{p_2}((Y_{W_1}^{g_1})^{p_1}(g_2u, z_1)w_1, -z_2)w_2 \rangle$$

$$(5.14)$$

converges absolutely and if in addition, $|\arg(z_1 - z_2) - \arg(-z_2)| < \frac{\pi}{2}$, its sum is equal to $f^{p_2, p_2, p_1}(z_1 - z_2, -z_2; g_2 u, w_1, w_2, e^{z_2 L'(1)} w'_3)$. By (4.5), we have

$$f^{p_{2},p_{2},p_{1}}(z_{1}-z_{2},-z_{2};g_{2}u,w_{1},w_{2},e^{z_{2}L'(1)}w_{3}')$$

$$=f^{p_{2}-1,p_{2},p_{1}}(z_{1}-z_{2},-z_{2};u,w_{1},w_{2},e^{z_{2}L'(1)}w_{3}')$$

$$=\sum_{i,j,k,l,m,n=0}^{N}a_{ijklmn}e^{r_{i}l_{p_{2}-1}(z_{1}-z_{2})}e^{s_{j}l_{p_{2}}(-z_{2})}e^{t_{k}l_{p_{1}}(z_{1})}(l_{p_{2}-1}(z_{1}-z_{2}))^{l}(l_{p_{2}}(-z_{2}))^{m}(l_{p_{1}}(z_{1}))^{n}.$$
(5.15)

When $\arg z_1 < \pi$, $\arg(-z_2) = \arg z_2 + \pi$ and hence the right-hand side of (5.15) is equal to

$$\sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn} e^{r_i l_{p_2-1}(z_1-z_2)} e^{s_j (l_{p_2}(z_2)+\pi \mathbf{i})} e^{t_k l_{p_1}(z_1)} (l_{p_2-1}(z_1-z_2))^l (l_{p_2}(z_2)+\pi \mathbf{i})^m (l_{p_1}(z_1))^n$$
$$= g_+^{p_1,p_2,p_2-1} (z_1, z_2; u, w_2, w_1, w_3').$$
(5.16)

Thus we see that the left-hand side of (5.14) converges absolutely when $|z_2| > |z_1| > 0$ and by (5.16), its sum is equal to $g_+^{p_1,p_2,p_2-1}(z_1, z_2; u, w_2, w_1, w_3')$ when $|z_2| > |z_1| > 0$, $|\arg(z_1 - z_2) - \arg(-z_2)| < \frac{\pi}{2}$ and $\arg z_2 < \pi$. But when $\arg z < \pi$, $\arg(-z_2) = \arg z_2 + \pi$. Hence in this case, the inequality $|\arg(z_1 - z_2) - \arg(-z_2)| < \frac{\pi}{2}$ becomes $\frac{\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < \frac{3\pi}{2}$. Also both the left-hand side of (5.14) and the left-hand side of (5.16) are single-valued analytic function in z_1 and z_2 with cuts at $z_1 \in \mathbb{R}_+$ and $z_2 \in \mathbb{R}_+$. Thus the same argument as above shows that when $|z_2| > |z_1| > 0$ and $\frac{\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < \frac{3\pi}{2}$, the left-hand side of (5.14) converges absolutely to $g_+^{p_1,p_2,p_2-1}(z_1, z_2; u, w_2, w_1, w_3')$. Then by Lemma 4.5, when $|z_2| > |z_1| > 0$ and $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$, the sum of left-hand side of (5.14) is equal to $g_+^{p_1,p_2,p_2}(z_1, z_2; u, w_2, w_1, w_3')$.

Finally, we discuss the iterate of $\Omega_+(\mathcal{Y})$ and the twisted vertex operator map $Y_{W_2}^{g_2}$. When $|z_2| > |z_1 - z_2| > 0$ and $\arg z_2 < \pi$,

$$\langle w_3', \Omega_+(\mathcal{Y})^{p_2} ((Y_{W_2}^{g_2})^{p_{12}} (u, z_1 - z_2) w_2, z_2) w_1 \rangle = \langle w_3', e^{z_2 L(-1)} \mathcal{Y}^{p_2} (w_1, -z_2) (Y_{W_2}^{g_2})^{p_{12}} (u, z_1 - z_2) w_2 \rangle = \langle e^{z_2 L'(1)} w_3', \mathcal{Y}^{p_2} (w_1, -z_2) (Y_{W_2}^{g_2})^{p_{12}} (u, z_1 - z_2) w_2 \rangle$$

$$(5.17)$$

converges absolutely and if in addition, $-\frac{3\pi}{2} < \arg z_1 - \arg(-z_2) < -\frac{\pi}{2}$, its sum is equal to

$$f^{p_{12},p_2,p_2}(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2 L'(1)} w'_3) = \sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn} e^{r_i l_{p_{12}}(z_1 - z_2)} e^{s_j l_{p_2}(-z_2)} e^{t_k l_{p_2}(z_1)} (l_{p_{12}}(z_1 - z_2))^l (l_{p_2}(-z_2))^m (l_{p_2}(z_1))^n.$$
(5.18)

When $\arg z_1 < \pi$, $\arg(-z_2) = \arg z_2 + \pi$. Hence the right-hand side of (5.18) is equal to

$$\sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn} e^{r_i l_{p_{12}}(z_1-z_2)} e^{s_j (l_{p_2}(z_2)+\pi \mathbf{i})} e^{t_k l_{p_2}(z_1)} (l_{p_{12}}(z_1-z_2))^l (l_{p_2}(z_2)+\pi \mathbf{i})^m (l_{p_2}(z_1))^n = g_+^{p_1,p_2,p_2} (z_1, z_2; u, w_2, w_1, w_3').$$
(5.19)

The same argument as above shows that the left-hand side of (5.17) converges absolutely when $|z_2| > |z_1 - z_2| > 0$ and its sum is equal to (5.19) when $|z_2| > |z_1 - z_2| > 0$, $-\frac{3\pi}{2} < \arg z_1 - \arg(-z_2) < -\frac{\pi}{2}$ and $\arg z_2 < \pi$. But when $\arg z < \pi$, $\arg(-z_2) = \arg z_2 + \pi$. Hence in this case, the inequality $-\frac{3\pi}{2} < \arg z_1 - \arg(-z_2) < -\frac{\pi}{2}$ becomes $|\arg(z_1 - z_2) - \arg(-z_2)| < \frac{\pi}{2}$. Also both the left-hand side of (5.17) and the left-hand side of (5.19) are single valued analytic function in z_1 and z_2 with cuts at $z_1 \in \mathbb{R}_+$ and $z_2 \in \mathbb{R}_+$. Thus the same argument as above shows that when $|z_2| > |z_1 - z_2| > 0$, $-\frac{3\pi}{2} < \arg z_1 - \arg(-z_2) < -\frac{\pi}{2}$, the sum of the left-hand side of (5.8) is equal to $g_+^{p_1, p_2, p_1}(z_1, z_2; u, w_2, w_1, w'_3)$.

Let $\mathcal{V}_{W_1W_2}^{W_3}$ be the space of twisted intertwining operators of type $\binom{W_3}{W_1W_2}$. Then we have: **Corollary 5.2** The maps $\Omega_+ : \mathcal{V}_{W_1W_2}^{W_3} \to \mathcal{V}_{W_2\phi_{g_2}^{-1}(W_1)}^{W_3}$ and $\Omega_- : \mathcal{V}_{W_1W_2}^{W_3} \to \mathcal{V}_{\phi_{g_1}(W_2)W_1}^{W_3}$ are linear isomorphisms. In particular, $\mathcal{V}_{W_1W_2}^{W_3}$, $\mathcal{V}_{\phi_{g_1}(W_2)W_1}^{W_3}$ and $\mathcal{V}_{W_2\phi_{g_2}^{-1}(W_1)}^{W_3}$ are linearly isomorphic.

Proof. It is clear that Ω_+ and Ω_- are inverse of each other.

The linear isomorphisms Ω_+ and Ω_- are called the *skew-symmetry isomorphisms*.

6 The contragredient isomorphisms

In this section, we construct what we call the contragredient isomorphisms between the spaces of twisted intertwining operators of suitable types. These linear isomorphisms will play an important role in the study of rigidity and other related properties of the still-to-be-constructed G-crossed braided tensor category structure on the category of g-twisted V-modules for all g in a group G of automorphisms of V.

Let g_1, g_2 be automorphisms of V, W_1, W_2 and $W_3 g_1$ -, g_2 - and g_1g_2 -twisted V-modules and \mathcal{Y} a twisted intertwining operator of type $\binom{W_3}{W_1W_2}$. We define linear maps

$$\begin{array}{rcl} A_{\pm}(\mathcal{Y}): W_1 \otimes W'_3 & \to & W'_2\{x\}[\log x] \\ & w_1 \otimes w'_3 & \mapsto & A_{\pm}(\mathcal{Y})(w_1, x)w'_3 \end{array}$$

 $\langle A_{\pm}(\mathcal{Y})(w_1, x)w_3', w_2 \rangle = \langle w_3', \mathcal{Y}(e^{xL(1)}e^{\pm\pi \mathbf{i}L(0)}(x^{-L(0)})^2w_1, x^{-1})w_2 \rangle$ (6.1)

for $w_1 \in W_1$ and $w_2 \in W_2$ and $w'_3 \in W'_3$.

Let (W, Y_W^g) be a *g*-twisted V-module. When $W_1 = V$, $W_2 = W_3 = W$ and $\mathcal{Y} = Y_W^g$, by definition, $A_+(Y_W^g) = A_-(Y_W^g) = (Y_W^g)'$ (see Section 3).

Let $L_{W_1}^s(0)$ be the semisimple part of $L_{W_1}(0)$. From the definition (6.1), for $p \in \mathbb{Z}$, $w_1 \in W_1, w_2 \in W_2, w'_3 \in W'_3$ and $z \in \mathbb{C}^{\times}$, we have

When $\arg z = 0$, $\arg z^{-1} = \arg z = 0$ and $-l_p(z) = l_{-p}(z^{-1})$. When $\arg z \neq 0$, $\arg z^{-1} = -\arg z + 2\pi$ and $-l_p(z) = l_{-p-1}(z^{-1})$. Hence when $\arg z = 0$, the right-hand side of (6.2) is equal to

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and when $\arg z \neq 0$, it is equal to

$$\langle w_3', \mathcal{Y}(e^{zL_{W_1}(1)}e^{\pm\pi \mathbf{i}L_{W_1}(0)}e^{2l_{-p-1}(z^{-1})L_{W_1}(0)}w_1, y)w_2\rangle \Big|_{y^n = e^{nl_{-p-1}(z^{-1})}, \log y = l_{-p-1}(z^{-1})}$$

= $\langle w_3', \mathcal{Y}^{-p-1}(e^{zL_{W_1}(1)}e^{\pm\pi \mathbf{i}L_{W_1}(0)}e^{2l_{-p-1}(z^{-1})L_{W_1}(0)}w_1, z^{-1})w_2\rangle.$ (6.4)

From (6.2)–(6.4), for $w_1 \in W_1$, $w_2 \in W_2$, $w'_3 \in W'_3$ and $z \in \mathbb{C}^{\times}$, we have

$$\langle A_{\pm}(\mathcal{Y})^{p}(w_{1},z)w_{3}',w_{2}\rangle = \langle w_{3}',\mathcal{Y}^{-p}(e^{zL_{W_{1}}(1)}e^{\pm\pi\mathbf{i}L_{W_{1}}(0)}e^{2l_{-p}(z^{-1})L_{W_{1}}(0)}w_{1},z^{-1})w_{2}\rangle$$
(6.5)

by

when $\arg z = 0$ and

$$\langle A_{\pm}(\mathcal{Y})^{p}(w_{1},z)w_{3}',w_{2}\rangle = \langle w_{3}',\mathcal{Y}^{-p-1}(e^{zL_{W_{1}}(1)}e^{\pm\pi \mathbf{i}L_{W_{1}}(0)}e^{2l_{-p-1}(z^{-1})L_{W_{1}}(0)}w_{1},z^{-1})w_{2}\rangle \quad (6.6)$$

when $\arg z \neq 0$.

Theorem 6.1 The linear maps $A_+(\mathcal{Y})$ and $A_-(\mathcal{Y})$ are twisted intertwining operators of types $\binom{\phi_{g_1}(W'_2)}{W_1W'_3}$ and $\binom{W'_2}{W_1\phi_{g_1^{-1}}(W'_3)}$, respectively.

Proof. Just as in the proof of Theorem 5.1, compared with the duality property, the lower-truncation property and the $L_{W_1}(-1)$ -derivative property can be verified straightforwardly. So here we prove only the duality property.

We first need to give the multivalued analytic functions with preferred branches in the duality property. We shall denote these multivalued analytic functions for $A_+(\mathcal{Y})$ and $A_-(\mathcal{Y})$ by $h_+(z_1, z_2; u, w_1, w_2, w'_3)$ and $h_-(z_1, z_2; u, w_1, w_2, w'_3)$, respectively. Let $f(z_1, z_2; u, w_1, w_2, w'_3)$ be the multivalued analytic function with preferred branch in the duality property for the twisted intertwining operator \mathcal{Y} . Then we can write

$$f(z_1, z_2; e^{z_1^{-1}L_V(1)}(-z_1^2)^{L_V(0)}u, e^{z_2^{-1}L_{W_1}(1)}e^{\pm\pi \mathbf{i}L_{W_1}(0)}e^{2\log z_2L_{W_1}(0)}w_1, w_2, w_3')$$

= $\sum_{i,j,k,l,m,n=0}^N a_{ijklmn}^{\pm} z_1^{r_i} z_2^{s_j} (z_1 - z_2)^{t^k} (\log z_1)^l (\log z_2)^m (\log(z_1 - z_2))^n.$

Define

$$h_{\pm}(z_{1}, z_{2}; u, w_{2}, w_{1}, w_{3}') = \sum_{i,j,k,l,m,n=0}^{N} e^{\pm t_{k}\pi \mathbf{i}} a_{ijklmn}^{\pm} z_{1}^{-(r_{i}+t_{k})} z_{2}^{-(s_{j}+t_{k})} (z_{1}-z_{2})^{t_{k}} \cdot (-\log z_{1})^{l} (-\log z_{2})^{m} (\log(z_{1}-z_{2}) - \log z_{1} - \log z_{2} \pm \pi \mathbf{i})^{n}.$$

$$(6.7)$$

Let $u \in V$, $w_1 \in W_1$, $w_2 \in W_2$ and $w'_3 \in W'_3$. We consider $z_1, z_2 \in \mathbb{C}$ satisfying $|z_2^{-1}| > |z_1^{-1}| > 0$ (or equivalently $|z_1| > |z_2| > 0$) and $\arg z_1, \arg z_2 \neq 0$. Since $|z_2^{-1}| > |z_1^{-1}| > 0$, from (6.6), $(Y_{W_2}^{g_2})' = A_+(Y_{W_2}^{g_2})$ and the duality property for \mathcal{Y} ,

$$\langle \phi_{g_1}((Y_{W_2}^{g_2})')^{p_1}(u, z_1)A_+(\mathcal{Y})^{p_2}(w_1, z_2)w'_3, w_2 \rangle = \langle ((Y_{W_2}^{g_2})')^{p_1}(g_1^{-1}u, z_1)A_+(\mathcal{Y})^{p_2}(w_1, z_2)w'_3, w_2 \rangle = \langle w'_3, \mathcal{Y}^{-p_2-1}(e^{z_2L_{W_1}(1)}e^{\pi \mathbf{i}L_{W_1}(0)}e^{2\log(z_2^{-1})L_{W_1}(0)}w_1, z_2^{-1}) \cdot \cdot (Y_{W_2}^{g_2})^{-p_1-1}(e^{z_1L_V(1)}(-z_1^{-2})^{L_V(0)}g_1^{-1}u, z_1^{-1})w_2 \rangle$$

$$(6.8)$$

converges absolutely and if in addition, $-\frac{3\pi}{2} < \arg(z_1^{-1} - z_2^{-1}) - \arg z_2^{-1} < -\frac{\pi}{2}$, its sum is equal to

$$f^{-p_{1}-1,-p_{2}-1,-p_{2}-1}(z_{1}^{-1},z_{2}^{-1}; e^{z_{1}L_{V}(1)}(-z_{1}^{-2})^{L_{V}(0)}g_{1}^{-1}u, e^{z_{2}L_{W_{1}}(1)}e^{\pi \mathbf{i}L_{W_{1}}(0)}e^{2\log(z_{2}^{-1})L_{W_{1}}(0)}w_{1}, w_{2}, w_{3}') = f^{-p_{1}-1,-p_{2}-1,-p_{2}-1}(z_{1}^{-1},z_{2}^{-1}; g_{1}^{-1}e^{z_{1}L_{V}(1)}(-z_{1}^{-2})^{L_{V}(0)}u, e^{z_{2}L_{W_{1}}(1)}e^{\pi \mathbf{i}L_{W_{1}}(0)}e^{2\log(z_{2}^{-1})L_{W_{1}}(0)}w_{1}, w_{2}, w_{3}').$$

$$(6.9)$$

By (4.4), (6.9) is equal to

$$f^{-p_{1}-1,-p_{2}-1,-p_{2}}(z_{1}^{-1},z_{2}^{-1};e^{z_{1}L_{V}(1)}(-z_{1}^{-2})^{L_{V}(0)}u,e^{z_{2}L_{W_{1}}(1)}e^{\pi iL_{W_{1}}(0)}e^{2\log(z_{2}^{-1})L_{W_{1}}(0)}w_{1},w_{2},w_{3}')$$

$$=\sum_{i,j,k,l,m,n=0}^{N}a_{ijklmn}^{+}e^{r_{i}l_{-p_{1}-1}(z_{1}^{-1})}e^{s_{j}l_{-p_{2}-1}(z_{2}^{-1})}e^{t_{k}l_{-p_{2}}(z_{1}^{-1}-z_{2}^{-1})}\cdot (l_{-p_{1}-1}(z_{1}^{-1}))^{l}(l_{-p_{2}-1}(z_{2}^{-1}))^{m}(l_{-p_{2}}(z_{1}^{-1}-z_{2}^{-1}))^{n}.$$
(6.10)

Since $\arg z_1, \arg z_2 \neq 0$, $\arg z_1^{-1} = -\arg z_1 + 2\pi$, $\arg z_2^{-1} = -\arg z_2 + 2\pi$, $l_{-p_1-1}(z_1^{-1}) = -l_{p_1}(z_1)$ and $l_{-p_2-1}(z_2^{-1}) = -l_{p_2}(z_2)$. Thus we also have

$$\arg(z_1^{-1} - z_2^{-1}) = \arg\left(\frac{z_1 - z_2}{z_1(-z_2)}\right).$$

Since $0 \leq \arg z < 2\pi$ for any $z \in \mathbb{C}^{\times}$ and

$$-\frac{3\pi}{2} < \arg(z_1^{-1} - z_2^{-1}) - \arg z_2^{-1} = \arg\left(\frac{z_1 - z_2}{z_1(-z_2)}\right) - \arg z_2^{-1} < -\frac{\pi}{2},$$

we have

$$\arg\left(\frac{z_1 - z_2}{z_1(-z_2)}\right) = \arg(z_1 - z_2) - \arg z_1 - \arg z_2 + (2q+1)\pi$$
(6.11)

with q = -1 when $-2\pi < \arg(z_1 - z_2) - \arg z_1 < -\frac{3\pi}{2}$, with q = 0 when $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ and with q = 1 when $\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_1 < 2\pi$. From (6.11), we obtain

$$l_{-p_{2}}(z_{1}^{-1} - z_{2}^{-1})$$

$$= l_{-p_{2}}\left(\frac{z_{1} - z_{2}}{z_{1}(-z_{2})}\right)$$

$$= \log|z_{1} - z_{2}| - \log|z_{1}| - \log|z_{2}| + \arg(z_{1} - z_{2})\mathbf{i} - \arg z_{1}\mathbf{i} - \arg z_{2}\mathbf{i}$$

$$+ (2q + 1)\pi\mathbf{i} + 2(-p_{2})\pi\mathbf{i}$$

$$= l_{p_{1}+q}(z_{1} - z_{2}) - l_{p_{1}}(z_{1}) - l_{p_{2}}(z_{2}) + \pi\mathbf{i}.$$
(6.12)

From $l_{-p_1-1}(z_1^{-1}) = -l_{p_1}(z_1)$, $l_{-p_2-1}(z_2^{-1}) = -l_{p_2}(z_2)$ and (6.12), the right-hand side of (6.10) is equal to

$$\sum_{i,j,k,l,m,n=0}^{N} e^{\pm t_k \pi \mathbf{i}} a_{ijklmn}^{\pm} e^{-r_i l_{p_1}(z_1)} e^{-s_j l_{p_2}(z_2)} e^{t_k (l_{p_1+q}(z_1-z_2)-l_{p_1}(z_1)-l_{p_2}(z_2)+\pi \mathbf{i})} \cdot (-l_{p_1}(z_1))^l (-l_{p_2}(z_2))^m (l_{p_1+q}(z_1-z_2)-l_{p_1}(z_1)-l_{p_2}(z_2)+\pi \mathbf{i})^n = h_+^{p_1,p_2,p_1+q}(z_1,z_2;u,w_2,w_1,w_3').$$

$$(6.13)$$

From (6.8)–(6.13), the left-hand side of (6.8) converges absolutely when $|z_1| > |z_2| > 0$ and $\arg z_1, \arg z_2 \neq 0$ and its sum is equal to the branch $h_+^{p_1, p_2, p_1+q}(z_1, z_2; u, w_2, w_1, w'_3)$ when $|z_1| > |z_2| > 0, -\frac{3\pi}{2} < \arg(z_1^{-1} - z_2^{-1}) - \arg z_2^{-1} < -\frac{\pi}{2}$ and $\arg z_1, \arg z_2 \neq 0$. In the case that $\arg z_1 = 0$ or $\arg z_2 = 0$, we can also prove similarly that when $|z_1| > |z_2| > 0$, the left-hand side of (6.8) converges absolutely and if in addition, $-\frac{3\pi}{2} < \arg(z_1^{-1} - z_2^{-1}) - \arg z_2^{-1} < -\frac{\pi}{2}$, its sum is equal to $h_+^{p_1, p_2, p_1 + q}(z_1, z_2; u, w_2, w_1, w'_3)$. The main difference, for example, in the case $\arg z_1 = 0$ is that $\arg z_1^{-1} = \arg z_1 = 0$ instead of $\arg z_1^{-1} = -\arg z_1 + 2\pi$.

When q = -1, since the sum of the left-hand side of (6.8) is equal to the branch $h_{+}^{p_1,p_2,p_1-1}(z_1, z_2; u, w_2, w_1, w'_3)$ when $|z_1| > |z_2| > 0$ and $-2\pi < \arg(z_1 - z_2) - \arg z_1 < -\frac{3\pi}{2}$, by Lemma 4.5, the left-hand side of (6.8) converges absolutely to $h_{+}^{p_1,p_2,p_1}(z_1, z_2; u, w_2, w_1, w'_3)$ when $|z_1| > |z_2| > 0$ and $0 < \arg(z_1 - z_2) - \arg z_1 < \frac{\pi}{2}$. When q = 0, the left-hand side of (6.8) converges absolutely to $h_{+}^{p_1,p_2,p_1}(z_1, z_2; u, w_2, w_1, w'_3)$ when $|z_1| > |z_2| > 0$ and $0 < \arg(z_1 - z_2) - \arg z_1 < \frac{\pi}{2}$. When q = 0, the left-hand side of (6.8) converges absolutely to $h_{+}^{p_1,p_2,p_1}(z_1, z_2; u, w_2, w_1, w'_3)$ when $|z_1| > |z_2| > 0$ and $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$. When q = 1, since the left-hand side of (6.8) converges absolutely to $h_{+}^{p_1,p_2,p_1+1}(z_1, z_2; u, w_2, w_1, w'_3)$ when $|z_1| > |z_2| > 0$ and $\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_1 < 2\pi$, by Lemma 4.5, the sum of the left-hand side of (6.8) is equal to $h_{+}^{p_1,p_2,p_1+1}(z_1, z_2; u, w_2, w_1, w'_3)$ when $|z_1| > |z_2| > 0$ and $-\frac{\pi}{2} < \arg(z_1 - z_2) - \arg z_1 < 0$. Thus we have proved that when $|z_1| > |z_2| > 0$, $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$, the sum of the left-hand side of (6.8) is always equal to $h_{+}^{p_1,p_2,p_1}(z_1, z_2; u, w_2, w_1, w'_3)$.

Next we consider the product of $A_+(\mathcal{Y})$ and the twisted vertex operator $(Y_{W_3}^{g_2})'$. Let u, w_1 , w_2 and w'_3 be the same as above. When $|z_1^{-1}| > |z_2^{-1}| > 0$ (or equivalently $|z_2| > |z_1| > 0$) and $\arg z_1$, $\arg z_2 \neq 0$,

$$\langle A_{+}(\mathcal{Y})^{p_{2}}(w_{1}, z_{2})((Y_{W_{2}}^{g_{2}})')^{p_{1}}(u, z_{1})w_{3}', w_{2} \rangle$$

$$= \langle w_{3}', (Y_{W_{2}}^{g_{2}})^{-p_{1}-1}(e^{z_{1}L_{V}(1)}(-z_{1}^{-2})^{L_{V}(0)}u, z_{1}^{-1}) \cdot$$

$$\cdot \mathcal{Y}^{-p_{2}-1}(e^{z_{2}L_{W_{1}}(1)}e^{\pi \mathbf{i}L_{W_{1}}(0)}e^{2l_{-p_{2}-1}(z_{2}^{-1})L_{W_{1}}(0)}w_{1}, z_{2}^{-1})w_{2} \rangle$$

$$(6.14)$$

converges absolutely and if in addition, $|\arg(z_1^{-1}-z_2^{-1}) - \arg z_1^{-1}| < \frac{\pi}{2}$, its sum is equal to

$$f^{-p_{1}-1,-p_{2}-1,-p_{1}-1}(z_{1}^{-1},z_{2}^{-1}; e^{z_{1}L_{V}(1)}(-z_{1}^{-2})^{L_{V}(0)}u, e^{z_{2}L_{W_{1}}(1)}e^{\pi iL_{W_{1}}(0)}e^{2l_{-p_{2}-1}(z_{2}^{-1})L_{W_{1}}(0)}w_{1}, w_{2}, w_{3}')$$

$$= \sum_{i,j,k,l,m,n=0}^{N} a^{+}_{ijklmn}e^{r_{i}l_{-p_{1}-1}(z_{1}^{-1})}e^{s_{j}l_{-p_{2}-1}(z_{2}^{-1})}e^{t_{k}l_{-p_{1}-1}(z_{1}^{-1}-z_{2}^{-1})}.$$

$$\cdot (l_{-p_{1}-1}(z_{1}^{-1}))^{l}(l_{-p_{2}-1}(z_{2}^{-1}))^{m}(l_{-p_{1}-1}(z_{1}^{-1}-z_{2}^{-1}))^{n}. \quad (6.15)$$

Again we have $\arg z_1^{-1} = -\arg z_1 + 2\pi$, $\arg z_2^{-1} = -\arg z_2 + 2\pi$, $l_{-p_1-1}(z_1^{-1}) = -l_{p_1}(z_1)$ and $l_{-p_2-1}(z_2^{-1}) = -l_{p_2}(z_2)$. Since $0 \le \arg z < 2\pi$ for any $z \in \mathbb{C}^{\times}$ and

$$-\frac{\pi}{2} < \arg(z_1^{-1} - z_2^{-1}) - \arg z_1^{-1} = \arg\left(\frac{z_1 - z_2}{z_1(-z_2)}\right) - \arg z_1^{-1} < \frac{\pi}{2},$$

we have

$$\arg(z_1^{-1} - z_2^{-1}) = \arg\left(\frac{z_1 - z_2}{z_1(-z_2)}\right) = \arg(z_1 - z_2) - \arg(z_1 - \arg(z_2) - \arg(z$$

with q = 0 when $\frac{\pi}{2} < \arg(z_1 - z_2) - \arg z_1 < \frac{3\pi}{2}$ and with q = 1 when $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_1 < -\frac{\pi}{2}$. From (6.16) and by the same calculation as in (6.12), we obtain

$$l_{-p_1-1}(z_1^{-1} - z_2^{-1}) = l_{p_2+q-1}(z_1 - z_2) - l_{p_1}(z_1) - l_{p_2}(z_2) + \pi \mathbf{i}.$$
 (6.17)

From $l_{-p_1-1}(z_1^{-1}) = -l_{p_1}(z_1)$, $l_{-p_2-1}(z_2^{-1}) = -l_{p_2}(z_2)$ and (6.17), the right-hand side of (6.15) is equal to

$$\sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn}^{+} e^{-r_i l_{p_1}(z_1)} e^{-s_j l_{p_2}(z_2)} e^{t_k (l_{p_2+q-1}(z_1-z_2)-l_{p_1}(z_1)-l_{p_2}(z_2)+\pi \mathbf{i})} \cdot (-l_{p_1}(z_1))^l (-l_{p_2}(z_2))^m (l_{p_2+q-1}(z_1-z_2)-l_{p_1}(z_1)-l_{p_2}(z_2)+\pi \mathbf{i})^n$$

$$= h_{+}^{p_1, p_2, p_2+q-1}(z_1, z_2; u, w_2, w_1, w'_3).$$
(6.18)

From (6.14)–(6.18), the left-hand side of (6.14) converges absolutely when $|z_2| > |z_1| > 0$ and $\arg z_1, \arg z_2 \neq 0$ and if in addition, $|\arg(z_1^{-1} - z_2^{-1}) - \arg z_1^{-1}| < \frac{\pi}{2}$, its sum is equal to $h_+^{p_1, p_2, p_2+q-1}(z_1, z_2; u, w_2, w_1, w'_3)$. In the case that $\arg z_1 = 0$ or $\arg z_2 = 0$, we can also prove similarly that when $|z_1| > |z_2| > 0$, the left-hand side of (6.14) converges absolutely and if in addition, $-\frac{3\pi}{2} < \arg(z_1^{-1} - z_2^{-1}) - \arg z_2^{-1} < -\frac{\pi}{2}$, its sum is equal to $h_+^{p_1, p_2, p_1+q}(z_1, z_2; u, w_2, w_1, w'_3)$.

When q = 0, since the sum of the left-hand side of (6.14) is equal to the branch $h_{+}^{p_1,p_2,p_2-1}(z_1, z_2; u, w_2, w_1, w'_3)$ when $|z_2| > |z_1| > 0$ and $\frac{\pi}{2} < \arg(z_1 - z_2) - \arg z_1 < \frac{3\pi}{2}$, by Lemma 4.5, the left-hand side of (6.14) converges absolutely to $h_{+}^{p_1,p_2,p_2}(z_1, z_2; u, w_2, w_1, w'_3)$ when $|z_2| > |z_1| > 0$ and $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_1 < -\frac{\pi}{2}$. When q = 1, the left-hand side of (6.14) converges absolutely to $h_{+}^{p_1,p_2,p_2}(z_1, z_2; u, w_2, w_1, w'_3)$ when $|z_2| > |z_1| > 0$ and $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_1 < -\frac{\pi}{2}$. When q = 1, the left-hand side of (6.14) converges absolutely to $h_{+}^{p_1,p_2,p_2}(z_1, z_2; u, w_2, w_1, w'_3)$ when $|z_2| > |z_1| > 0$ and $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_1 < -\frac{\pi}{2}$. Thus we have proved that when $|z_1| > |z_2| > 0$, the left-hand side of (6.14) converges absolutely and if in addition, $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_1 < -\frac{\pi}{2}$, its sum is equal to $h_{+}^{p_1,p_2,p_2}(z_1, z_2; u, w_2, w_1, w'_3)$.

Now we discuss $A_{-}(\mathcal{Y})$. When $|z_{2}^{-1}| > |z_{1}^{-1}| > 0$ (or equivalently $|z_{1}| > |z_{2}| > 0$) and $\arg z_{1}, \arg z_{2} \neq 0$,

$$\langle ((Y_{W_2}^{g_2})')^{p_1}(u, z_1) A_{-}(\mathcal{Y})^{p_2}(w_1, z_2) w'_3, w_2 \rangle = \langle w'_3, \mathcal{Y}^{-p_2 - 1}(e^{z_2 L_{W_1}(1)} e^{-\pi \mathbf{i} L_{W_1}(0)} e^{2l_{-p_2 - 1}(z_2^{-1}) L_{W_1}(0)} w_1, z_2^{-1}) \cdot \cdot (Y_{W_2}^{g_2})^{-p_1 - 1}(e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)} u, z_1^{-1}) w_2 \rangle$$

$$(6.19)$$

converges absolutely and if in addition, $-\frac{3\pi}{2} < \arg(z_1^{-1} - z_2^{-1}) - \arg z_2^{-1} < -\frac{\pi}{2}$, its sum is equal to

$$f^{-p_{1}-1,-p_{2}-1,-p_{2}-1}(z_{1}^{-1},z_{2}^{-1}; e^{z_{1}L_{V}(1)}(-z_{1}^{-2})^{L_{V}(0)}u, e^{z_{2}L_{W_{1}}(1)}e^{-\pi \mathbf{i}L_{W_{1}}(0)}e^{2l_{-p_{2}-1}(z_{2}^{-1})L_{W_{1}}(0)}w_{1}, w_{2}, w_{3}')$$

$$= \sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn}^{-}e^{r_{i}l_{-p_{1}-1}(z_{1}^{-1})}e^{s_{j}l_{-p_{2}-1}(z_{2}^{-1})}e^{t_{k}l_{-p_{2}-1}(z_{1}^{-1}-z_{2}^{-1})}.$$

$$\cdot (l_{-p_{1}-1}(z_{1}^{-1}))^{l}(l_{-p_{2}-1}(z_{2}^{-1}))^{m}(l_{-p_{2}-1}(z_{1}^{-1}-z_{2}^{-1}))^{n}.$$
(6.20)

As in the case for $A_+(\mathcal{Y})$ above, since $\arg z_1, \arg z_2 \neq 0$, $l_{-p_1-1}(z_1^{-1}) = -l_{p_1}(z_1)$ and $l_{-p_2-1}(z_2^{-1}) = -l_{p_2}(z_2)$. Also, the same argument as in that case gives

$$l_{-p_2-1}(z_1^{-1} - z_2^{-1}) = l_{p_1+q}(z_1 - z_2) - l_{p_1}(z_1) - l_{p_2}(z_2) - \pi \mathbf{i}$$
(6.21)

with q = -1 when $-2\pi < \arg(z_1 - z_2) - \arg z_1 < -\frac{3\pi}{2}$, with q = 0 when $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ and with q = 1 when $\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_1 < 2\pi$. From $l_{-p_1-1}(z_1^{-1}) = -l_{p_1}(z_1)$, $l_{-p_2-1}(z_2^{-1}) = -l_{p_2}(z_2)$ and (6.21), the right-hand side of (6.20) is equal to

$$\sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn}^{-} e^{-r_i l_{p_1}(z_1)} e^{-s_j l_{p_2}(z_2)} e^{t_k (l_{p_1+q}(z_1-z_2)-l_{p_1}(z_1)-l_{p_2}(z_2)-\pi \mathbf{i})} \cdot (-l_{p_1}(z_1))^l (-l_{p_2}(z_2))^m (l_{p_1+q}(z_1-z_2)-l_{p_1}(z_1)-l_{p_2}(z_2)-\pi \mathbf{i})^n = h_{-}^{p_1,p_2,p_1+q} (z_1, z_2; u, w_2, w_1, w_3').$$

$$(6.22)$$

From (6.19)–(6.22) and the same argument as in the case for $A_+(\mathcal{Y})$ above, the left-hand side of (6.19) converges absolutely when $|z_1| > |z_2| > 0$ and its sum is equal to the branch $h_-^{p_1,p_2,p_1+q}(z_1, z_2; u, w_2, w_1, w'_3)$ when $|z_1| > |z_2| > 0$ and $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$. Now we consider the product of $A_-(\mathcal{Y})$ and the twisted vertex operator $\phi_{g_1^{-1}}((Y_{W_3}^{g_2})')$.

Now we consider the product of $A_{-}(\mathcal{Y})$ and the twisted vertex operator $\phi_{g_{1}^{-1}}((Y_{W_{3}}^{g_{2}})')$. When $|z_{1}^{-1}| > |z_{2}^{-1}| > 0$ (or equivalently $|z_{2}| > |z_{1}| > 0$) and $\arg z_{1}, \arg z_{2} \neq 0$,

$$\langle A_{-}(\mathcal{Y})^{p_{2}}(w_{1}, z_{2})\phi_{g_{1}^{-1}}((Y_{W_{3}}^{g_{2}})')^{p_{1}}(u, z_{1})w_{3}', w_{2} \rangle$$

$$= \langle A_{-}(\mathcal{Y})^{p_{2}}(w_{1}, z_{2})(Y_{W_{3}}^{g_{2}})'^{p_{1}}(g_{1}u, z_{1})w_{3}', w_{2} \rangle$$

$$= \langle w_{3}', (Y_{W_{2}}^{g_{2}})^{-p_{1}-1}(e^{z_{1}L_{V}(1)}(-z_{1}^{-2})^{L_{V}(0)}g_{1}u, z_{1}^{-1}) \cdot$$

$$\cdot \mathcal{Y}^{-p_{2}-1}(e^{z_{2}L_{W_{1}}(1)}e^{-\pi \mathbf{i}L_{W_{1}}(0)}e^{2l_{-p_{2}-1}(z_{2}^{-1})L_{W_{1}}(0)}w_{1}, z_{2}^{-1})w_{2} \rangle$$

$$(6.23)$$

converges absolutely and if in addition, $|\arg(z_1^{-1}-z_2^{-1}) - \arg z_1^{-1}| < \frac{\pi}{2}$, its sum is equal to

$$f^{-p_{1}-1,-p_{2}-1,-p_{1}-1}(z_{1}^{-1},z_{2}^{-1};$$

$$e^{z_{1}L_{V}(1)}(-z_{1}^{-2})^{L_{V}(0)}g_{1}u, e^{z_{2}L_{W_{1}}(1)}e^{-\pi \mathbf{i}L_{W_{1}}(0)}e^{2l_{-p_{2}-1}(z_{2}^{-1})L_{W_{1}}(0)}w_{1}, w_{2}, w_{3}')$$

$$= f^{-p_{1}-1,-p_{2}-1,-p_{1}-1}(z_{1}^{-1},z_{2}^{-1};$$

$$g_{1}e^{z_{1}L_{V}(1)}(-z_{1}^{-2})^{L_{V}(0)}u, e^{z_{2}L_{W_{1}}(1)}e^{-\pi \mathbf{i}L_{W_{1}}(0)}e^{2l_{-p_{2}-1}(z_{2}^{-1})L_{W_{1}}(0)}w_{1}, w_{2}, w_{3}').$$
(6.24)

By (4.4), (6.24) is equal to

$$f^{-p_{1}-1,-p_{2}-1,-p_{1}-2}(z_{1}^{-1},z_{2}^{-1}; e^{z_{1}L_{V}(1)}(-z_{1}^{-2})^{L_{V}(0)}u, e^{z_{2}L_{W_{1}}(1)}e^{-\pi \mathbf{i}L_{W_{1}}(0)}e^{2l_{-p_{2}-1}(z_{2}^{-1})L_{W_{1}}(0)}w_{1}, w_{2}, w_{3}')$$

$$= \sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn}^{-}e^{r_{i}l_{-p_{1}-1}(z_{1}^{-1})}e^{s_{j}l_{-p_{2}-1}(z_{2}^{-1})}e^{t_{k}l_{-p_{1}-2}(z_{1}^{-1}-z_{2}^{-1})}.$$

$$\cdot (l_{-p_{1}-1}(z_{1}^{-1}))^{l}(l_{-p_{2}-1}(z_{2}^{-1}))^{m}(l_{-p_{1}-2}(z_{1}^{-1}-z_{2}^{-1}))^{n}. \quad (6.25)$$

As in the case for $A_+(\mathcal{Y})$ above, since $\arg z_1, \arg z_2 \neq 0$, $l_{-p_1-1}(z_1^{-1}) = -l_{p_1}(z_1)$ and $l_{-p_2-1}(z_2^{-1}) = -l_{p_2}(z_2)$. Also, the same argument as in that case gives

$$l_{-p_1-2}(z_1^{-1}-z_2^{-1}) = l_{p_2+q-1}(z_1-z_2) - l_{p_1}(z_1) - l_{p_2}(z_2) - \pi \mathbf{i}$$
(6.26)

with q = 0 when $\frac{\pi}{2} < \arg(z_1 - z_2) - \arg z_1 < \frac{3\pi}{2}$ and with q = 1 when $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_1 < -\frac{\pi}{2}$. From $l_{-p_1-1}(z_1^{-1}) = -l_{p_1}(z_1)$, $l_{-p_2-1}(z_2^{-1}) = -l_{p_2}(z_2)$ and (6.26), the right-hand side of (6.25) is equal to

$$\sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn}^{-} e^{-r_i l_{p_1}(z_1)} e^{-s_j l_{p_2}(z_2)} e^{t_k (l_{p_2+q-1}(z_1-z_2)-l_{p_1}(z_1)-l_{p_2}(z_2)-\pi \mathbf{i})} \cdot (-l_{p_1}(z_1))^l (-l_{p_2}(z_2))^m (l_{p_2+q-1}(z_1-z_2)-l_{p_1}(z_1)-l_{p_2}(z_2)-\pi \mathbf{i})^n = h_{-}^{p_1,p_2,p_2+q-1} (z_1, z_2; u, w_2, w_1, w_3').$$

$$(6.27)$$

From (6.23)–(6.27) and the same argument as in the case for $A_+(\mathcal{Y})$ above, the left-hand side of (6.23) converges absolutely when $|z_2| > |z_1| > 0$ and its sum is equal to the branch $h^{p_1,p_2,p_1+q}_-(z_1,z_2;u,w_2,w_1,w_3)$ when $|z_2| > |z_1| > 0$ and $-\frac{3\pi}{2} < \arg(z_1-z_2) - \arg z_1 < -\frac{\pi}{2}$.

 $h^{p_1,p_2,p_1+q}_{-}(z_1,z_2;u,w_2,w_1,w_3)$ when $|z_2| > |z_1| > 0$ and $-\frac{3\pi}{2} < \arg(z_1-z_2) - \arg z_1 < -\frac{\pi}{2}$. Finally we study the iterate of $A_{\pm}(\mathcal{Y})$ and the twisted vertex operator $Y^{g_1}_{W_1}$. When $\arg z_2 \neq 0$, from (6.6), we have

$$\langle A_{\pm}(\mathcal{Y})^{p_2}((Y_{W_1}^{g_1})^{p_{12}}(u, z_1 - z_2)w_1, z_2)w_3', w_2 \rangle$$

= $\langle w_3', \mathcal{Y}^{-p_2-1}(e^{z_2L_{W_1}(1)}e^{\pm\pi \mathbf{i}L_{W_1}(0)}e^{2l_{-p_2-1}(z_2^{-1})L_{W_1}(0)}(Y_{W_1}^{g_1})^{p_{12}}(u, z_1 - z_2)w_1, z_2^{-1})w_2 \rangle.$
(6.28)

Note that (3.61) and (3.62) in [HLZ1] with \mathcal{Y} replaced by $Y_{W_1}^{g_1}$ still holds since $Y_{W_1}^{g_1}$ is a (logarithmic) intertwining operator among $V^{\langle g_1, g_2, g_1 g_2 \rangle}$ -modules. Using these formulas, we obtain the formulas for $Y_{W_1}^{g_1}(u, x)$ $(u \in V)$ conjugated by $e^{x_2 L_{W_1}(1)}$ and $y^{L_{W_1}(0)}$. Using these formulas, we obtain

$$e^{x_{2}L_{W_{1}}(1)}y^{L_{W_{1}}(0)}Y_{W_{1}}^{g_{1}}(u,x_{0})$$

$$= e^{x_{2}L_{W_{1}}(1)}Y_{W_{1}}^{g_{1}}(y^{L_{V}(0)}u,yx_{0})y^{L_{W_{1}}(0)}$$

$$= Y_{W_{1}}^{g_{1}}(e^{x_{2}(1-x_{2}yx_{0})L_{V}(1)}(1-x_{2}yx_{0})^{-2L_{V}(0)}y^{L_{V}(0)}u,yx_{0}(1-x_{2}yx_{0})^{-1})e^{x_{2}L_{W_{1}}(1)}y^{L_{W_{1}}(0)}.$$
(6.29)

Substituting $e^{\pm n\pi \mathbf{i}}x_2^{-2n}$ and $\pm \pi \mathbf{i} - 2\log x_2$ for y^n and $\log y$, respectively, in (6.29), we obtain

$$e^{x_{2}L_{W_{1}}(1)}e^{\pm\pi \mathbf{i}L_{W_{1}}(0)}(x_{2}^{-L_{W_{1}}(0)})^{2}Y_{W_{1}}^{g_{1}}(u,x_{0})$$

$$=Y_{W_{1}}^{g_{1}}\left(e^{(x_{2}+x_{0})L_{V}(1)}(-(x_{2}+x_{0})^{-2})^{L_{V}(0)}u,\frac{xx_{0}}{(x_{2}+x_{0})x_{2}}\right)\Big|_{x^{n}=e^{\pm n\pi \mathbf{i}},\log x=\pm \pi \mathbf{i}}\cdot e^{x_{2}L_{W_{1}}(1)}e^{\pm\pi \mathbf{i}L_{W_{1}}(0)}(x_{2}^{-L_{W_{1}}(0)})^{2}$$

$$(6.30)$$

Substituting $e^{nl_{p_12}(z_1-z_2)}$, $l_{p_{12}}(z_1-z_2)$, $e^{nl_{p_2}(z_2)}$ and $l_{p_2}(z_2)$ for x^n , $\log x_0$, x_2^n and $\log x_2$, respectively, in (6.30) and then applying the resulting equality to w_1 , we obtain in the region $|z_2| > |z_1 - z_2| > 0$

$$e^{z_{2}L_{W_{1}}(1)}e^{\pm\pi \mathbf{i}L_{W_{1}}(0)}e^{2l_{-p_{2}-1}(z_{2}^{-1})L_{W_{1}}(0)}Y_{W_{1}}^{g_{1},p_{12}}(u,z_{1}-z_{2})w_{1}$$

$$=Y_{W_{1}}^{g_{1}}\left(e^{z_{1}L_{V}(1)}(-z_{1}^{-2})^{L_{V}(0)}u,\frac{xx_{0}}{(x_{2}+x_{0})x_{2}}\right)\Big|_{x_{0}^{n}=e^{nl_{p_{12}}(z_{1}-z_{2})},\log x_{0}=l_{p_{12}}(z_{1}-z_{2}),x_{2}^{n}=e^{nl_{p_{2}}(z_{2})}}{\log x_{2}=l_{p_{2}}(z_{2}),x^{n}=e^{\pm n\pi \mathbf{i}},\log x=\pm \pi \mathbf{i}}\cdot e^{z_{2}L_{W_{1}}(1)}e^{\pm\pi \mathbf{i}L_{W_{1}}(0)}e^{2l_{-p_{2}-1}(z_{2}^{-1})L_{W_{1}}(0)}w_{1}.$$
(6.31)

In the region $|z_2| > |z_1 - z_2| > 0$, either $\arg(1 + \frac{z_1 - z_2}{z_2}) < \frac{\pi}{2}$ or $\frac{3\pi}{2} < \arg(1 + \frac{z_1 - z_2}{z_2})$. Hence when $|z_2| > |z_1 - z_2| > 0$, the expansion of $(1 + \frac{z_1 - z_2}{z_2})^m$ for $m \in \mathbb{C}$ as a power series in $\frac{z_1 - z_2}{z_2}$ is absolutely convergent to

$$e^{m\log|1+\frac{z_1-z_2}{z_2}|}e^{m(\arg(1+\frac{z_1-z_2}{z_2})\mathbf{i}+2q\pi\mathbf{i})} = e^{m\log|z_1|}e^{-m\log|z_2|}e^{m(\arg(1+\frac{z_1-z_2}{z_2})\mathbf{i}+2q\pi\mathbf{i})},$$
(6.32)

where q = 0 when $\arg(1 + \frac{z_1 - z_2}{z_2}) < \frac{\pi}{2}$ and q = -1 when $\frac{3\pi}{2} < \arg(1 + \frac{z_1 - z_2}{z_2})$. Also, when $|z_2| > |z_1 - z_2| > 0$ and $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$,

$$\arg z_1 = \arg z_2 + \arg \left(1 + \frac{z_1 - z_2}{z_2}\right) + 2q\pi,$$
 (6.33)

where q = 0 when $\arg(1 + \frac{z_1 - z_2}{z_2}) < \frac{\pi}{2}$ and q = -1 when $\frac{3\pi}{2} < \arg(1 + \frac{z_1 - z_2}{z_2})$. By (6.32) and (6.33), we obtain

$$\begin{aligned} x_{2}^{m} \left(1 + \frac{x_{0}}{x_{2}}\right)^{m} \Big|_{x_{0}^{n} = e^{nl_{p_{12}}(z_{1} - z_{2})}, x_{2}^{n} = e^{nl_{p_{2}}(z_{2})}} \\ &= e^{ml_{p_{2}}(z_{2})} e^{m\log|z_{1}|} e^{-m\log|z_{2}|} e^{m((\arg(1 + \frac{z_{1} - z_{2}}{z_{2}}))\mathbf{i} + 2q\pi\mathbf{i})} \\ &= e^{m((\arg z_{2})\mathbf{i} + 2p_{2}\pi\mathbf{i})} e^{m\log|z_{1}|} e^{m((\arg(1 + \frac{z_{1} - z_{2}}{z_{2}})\mathbf{i}) + 2q\pi\mathbf{i})} \\ &= e^{m((\arg z_{1})\mathbf{i} - (\arg(1 + \frac{z_{1} - z_{2}}{z_{2}}))\mathbf{i} - 2q\pi\mathbf{i} + 2p_{2}\pi\mathbf{i})} e^{m\log|z_{1}|} e^{m((\arg(1 + \frac{z_{1} - z_{2}}{z_{2}}))\mathbf{i} + 2q\pi\mathbf{i})} \\ &= e^{ml_{p_{2}}(z_{1})} \end{aligned}$$
(6.34)

for $m \in \mathbb{C}$. Using (6.34), we obtain

$$\left(\frac{xx_{0}}{(x_{2}+x_{0})x_{2}}\right)^{m}\Big|_{x_{0}^{n}=e^{nl_{p_{12}}(z_{1}-z_{2})}, x_{2}^{n}=e^{nl_{p_{2}}(z_{2})}, x^{n}=e^{\pm n\pi \mathbf{i}}} \\
= \left(\frac{e^{\pm m\pi \mathbf{i}}x_{0}^{m}}{\left(x_{2}^{m}\left(1+\frac{x_{0}}{x_{2}}\right)^{m}\right)x_{2}^{m}}\right)\Big|_{x_{0}^{n}=e^{nl_{p_{12}}(z_{1}-z_{2})}, x_{2}^{n}=e^{nl_{p_{2}}(z_{2})}} \\
= e^{\pm m\pi \mathbf{i}}e^{ml_{p_{12}}(z_{1}-z_{2})}e^{-ml_{p_{2}}(z_{1})}e^{-ml_{p_{2}}(z_{2})} \\
= e^{m(l_{p_{12}}(z_{1}-z_{2})-l_{p_{2}}(z_{1})-l_{p_{2}}(z_{2})\pm\pi \mathbf{i}}.$$
(6.35)

Similarly, when $|z_2| > |z_1 - z_2| > 0$, the expansion of $\log(1 + \frac{z_1 - z_2}{z_2})$ as a power series in $\frac{z_1 - z_2}{z_2}$ is absolutely convergent to

$$\log \left| 1 + \frac{z_1 - z_2}{z_2} \right| + \arg \left(1 + \frac{z_1 - z_2}{z_2} \right) \mathbf{i} + 2q\pi \mathbf{i} = \log |z_1| - \log |z_2| + \arg \left(1 + \frac{z_1 - z_2}{z_2} \right) \mathbf{i} + 2q\pi \mathbf{i},$$
(6.36)

where q = 0 when $\arg(1 + \frac{z_1 - z_2}{z_2}) < \frac{\pi}{2}$ and q = -1 when $\frac{3\pi}{2} < \arg(1 + \frac{z_1 - z_2}{z_2})$. By (6.33) and (6.36), we obtain

$$\log\left(\frac{xx_0}{(x_2+x_0)x_2}\right)\Big|_{\log x_0 = l_{p_{12}}(z_1-z_2), \log x_2 = l_{p_2}(z_2), \log x = \pm \pi \mathbf{i}} \\
= \left(\log x + \log x_0 - \log x_2 - \log\left(1 + \frac{x_0}{x_2}\right) - \log x_2\right)\Big|_{\log x_0 = l_{p_{12}}(z_1-z_2), \log x_2 = l_{p_2}(z_2), \log x = \pm \pi \mathbf{i}} \\
= l_{p_{12}}(z_1-z_2) - l_{p_2}(z_1) - l_{p_2}(z_2) \pm \pi \mathbf{i}.$$
(6.37)

On the other hand, for any $z_1, z_2 \in \mathbb{C}^{\times}$ such that $z_1 \neq z_2$, there exists $m \in \mathbb{Z}$ such that

$$l_{p_{12}}(z_1 - z_2) - l_{p_2}(z_1) - l_{p_2}(z_2) \pm \pi \mathbf{i}$$

$$= \log |z_1 - z_2| + (\arg(z_1 - z_2))\mathbf{i} - \log |z_1| - (\arg z_1)\mathbf{i}$$

$$- \log |z_2| - (\arg z_2)\mathbf{i} + 2(p_{12} - 2p_2)\pi\mathbf{i} \pm \pi\mathbf{i}$$

$$= \log \left|\frac{z_1 - z_2}{z_1(-z_2)}\right| + \left(\arg\left(\frac{z_1 - z_2}{z_1(-z_2)}\right)\right)\mathbf{i} + 2(p_{12} - 2p_2)\pi\mathbf{i} + (2m + 1)\pi\mathbf{i} \pm \pi\mathbf{i}$$

$$= \log |z_1^{-1} - z_2^{-1}| + (\arg(z_1^{-1} - z_2^{-1}))\mathbf{i} + 2\left(p_{12} - 2p_2 + m + \frac{1 \pm 1}{2}\right)\pi\mathbf{i}$$

$$= l_{p_{12} - 2p_2 + m + \frac{1 \pm 1}{2}}(z_1^{-1} - z_2^{-1}).$$
(6.38)

From (6.31) and (6.35)–(6.38), we see that when $|z_2| > |z_1 - z_2| > 0$ and $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$, the right-hand side of (6.28) is equal to

$$\langle w_{3}', \mathcal{Y}^{-p_{2}-1}((Y_{W_{1}}^{g_{1}})^{p_{12}-2p_{2}+m+\frac{1+1}{2}}(e^{z_{1}L_{V}(1)}(-z_{1}^{-2})^{L_{V}(0)}u, z_{1}^{-1}-z_{2}^{-1}) \cdot e^{z_{2}L_{W_{1}}(1)}e^{\pm\pi \mathbf{i}L_{W_{1}}(0)}e^{2l_{-p_{2}-1}(z_{2}^{-1})L_{W_{1}}(0)}w_{1}, z_{2}^{-1})w_{2} \rangle, \qquad (6.39)$$

which, by the duality property for \mathcal{Y} , converges absolutely to

i

$$f^{-p_{2}-1,-p_{2}-1,p_{12}-2p_{2}+m+\frac{1\pm 1}{2}}(z_{1}^{-1},z_{2}^{-1};e^{z_{1}L_{V}(1)}(-z_{1}^{-2})^{L_{V}(0)}u,$$

$$e^{z_{2}L_{W_{1}}(1)}e^{\pm\pi iL_{W_{1}}(0)}e^{2l_{-p_{2}-1}(z_{2}^{-1})L_{W_{1}}(0)}w_{1},w_{2},w_{3}')$$

$$=\sum_{i,j,k,l,m,n=0}^{N}a_{ijklmn}^{\pm}e^{r_{i}l_{-p_{2}-1}(z_{1}^{-1})}e^{s_{j}l_{-p_{2}-1}(z_{2}^{-1})}e^{t_{k}l_{p_{12}-2p_{2}+m+\frac{1\pm 1}{2}}(z_{1}^{-1}-z_{2}^{-1})}.$$

$$\cdot(l_{-p_{2}-1}(z_{1}^{-1}))^{l}(l_{-p_{2}-1}(z_{2}^{-1}))^{m}(l_{p_{12}-2p_{2}+m+\frac{1\pm 1}{2}}(z_{1}^{-1}-z_{2}^{-1}))^{n} \quad (6.40)$$

when $|z_2^{-1}| > |z_1^{-1} - z_2^{-1}| > 0$ and $|\arg z_1^{-1} - \arg z_2^{-1}| < \frac{\pi}{2}$. In the case that $\arg z_1, \arg z_2 \neq 0$, $\arg z_1^{-1} = -\arg z_1 + 2\pi$, $\arg z_2^{-1} = -\arg z_2 + 2\pi$, $l_{-p_2-1}(z_1^{-1}) = -l_{p_2}(z_1)$ and $l_{-p_2-1}(z_2^{-1}) = -l_{p_2}(z_2)$. Using these and (6.38), we see that the right-hand side of (6.40) is equal to

$$\sum_{j,k,l,m,n=0}^{N} a_{ijklmn}^{\pm} e^{-r_i l_{p_2}(z_1)} e^{-s_j l_{p_2}(z_2)} e^{t_k (l_{p_{12}}(z_1-z_2)-l_{p_2}(z_1)-l_{p_2}(z_2)\pm\pi\mathbf{i})} \cdot (-l_{p_2}(z_1))^l (-l_{p_2}(z_2))^m (l_{p_{12}}(z_1-z_2)-l_{p_2}(z_1)-l_{p_2}(z_2)\pm\pi\mathbf{i})^n = h_{\pm}^{p_2,p_2,p_{12}} (z_1, z_2; u, w_2, w_1, w_3').$$

$$(6.41)$$

Note that $|z_2^{-1}| > |z_1^{-1} - z_2^{-1}| > 0$ is equivalent to $|z_1| > |z_1 - z_2| > 0$ and $|\arg z_1^{-1} - \arg z_2^{-1}| < \frac{\pi}{2}$ is equivalent to $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$. Thus we have proved that the right-hand side of (6.28) is absolutely convergent to $h_{\pm}^{p_2, p_2, p_{12}}(z_1, z_2; u, w_2, w_1, w_3')$ in the region given by $|z_1|, |z_2| > |z_1 - z_2| > 0$, $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$ and $\arg z_1, \arg z_2 \neq 0$ ($|z_1| > |z_1 - z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$ are both needed in the proof above). Note that the left-hand side of (6.28) is a series in (complex) powers of $e^{l_{p_12}(z_1 - z_2)}$ and $e^{l_{p_2}(z_2)}$ and in nonnegative integral powers of $l_{p_{12}}(z_1 - z_2)$ and $l_{p_2}(z_2)$ with finitely many negative powers in $e^{l_{p_{12}(z_1 - z_2)}}$ and finitely many positive powers of $e^{l_{p_2(z_2)}}$. Since $h_{\pm}^{p_2, p_2, p_{12}}(z_1, z_2; u, w_2, w_1, w_3')$ can be expanded uniquely as such a series in the region $|z_2| > |z_1 - z_2| > 0$ and if in addition, $|\arg z_1 - \arg z_1| < \frac{\pi}{2}$, its sum is equal to $h_{\pm}^{p_2, p_2, p_{12}}(z_1, z_2; u, w_2, w_1, w_3')$.

Just as in the skew-symmetry case, we have the following immediate consequence:

Corollary 6.2 The maps $A_+ : \mathcal{V}_{W_1W_2}^{W_3} \to \mathcal{V}_{W_1W_3'}^{\phi_{g_1}(W_2')}$ and $A_- : \mathcal{V}_{W_1W_2}^{W_3} \to \mathcal{V}_{W_1\phi_{g_1}^{-1}(W_3')}^{W_2}$ are linear isomorphisms. In particular, $\mathcal{V}_{W_1W_2}^{W_3}$, $\mathcal{V}_{W_1W_3'}^{\phi_{g_1}(W_2')}$ and $\mathcal{V}_{W_1\phi_{g_1}^{-1}(W_3')}^{W_2}$ are linearly isomorphic.

Proof. It is clear that A_+ and A_- are inverse of each other.

The linear isomorphisms A_+ and A_- are called the *contragredient isomorphisms*.

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