Twisted intertwining operators and nonabelian orbifold theories

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Intertwining operators and chiral conformal field theories

- Intertwining operators (or more generally logarithmic intertwining operators) among modules for a vertex operator algebra (chiral algebra) are building blocks of chiral conformal field theories.
- The construction of chiral conformal field theories have been reduced to the proofs of associativity, commutativity, modular invariance for (logarithmic) intertwining operators and a suitable convergence for higher-genus correlation functions.

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Intertwining operators and chiral conformal field theories

- Let me first discuss the main theorems on the construction of rational conformal field theories using the representation theory of vertex operator algebras.
- We consider the following three conditions for a vertex operator algebra V:
  1. $V_{(n)} = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ and the contragredient V', as a V-module, is equivalent to V.
  2. Every grading-restricted generalized V-module is completely reducible.
  3. V is C_2-cofinite, that is, $\dim V/C_2(V) < \infty$, where $C_2(V)$ is the subspace of V spanned by the elements of the form $\mathrm{Res}_x x^{-2} Y(u, x)v$ for $u, v \in V$ and $Y : V \otimes V \to V[[x, x^{-1}]]$ is the vertex operator map for V.

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Intertwining operators and chiral conformal field theories

Theorem (H.)

Let V be a simple vertex operator algebra satisfying the three conditions above. Then we have

1. *The intertwining operators among V-modules satisfy associativity, communitivity (thus form an intertwining operator algebra) and the modular invariance property.*
2. *The category of V-modules has a natural structure of modular tensor category.*

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Intertwining operators and full conformal field theories

Theorem (H.-Kong)

Let V be a simple vertex operator algebra satisfying the three conditions above. Let $\{W_i\}_{i=1}^m$ be a complete set of all inequivalent irreducible V-modules. Then there is a natural full field algebra structure (equivalent to a genus-zero full conformal field theory) on the space $\otimes_{i=1}^m W_i \otimes W_i'$ satisfying the modular invariance property.

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Twisted intertwining operators and chiral orbifold conformal field theories

- **Problem**: Given a vertex operator algebra V and a group G of automorphisms of V, construct the conformal field theories corresponding to the fixed point vertex operator algebra V^G and vertex operator algebras containing V^G as a subalgebra.
- To study V^G-modules, we need to study twisted V-modules.

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Twisted modules

- Twisted modules associated to automorphisms of finite orders appeared first in the works of Frenkel-Lepowsky-Meurman and Lepowsky.
- They have been studied by Lepowsky, Frenkel-Lepowsky-Meurman, Dong, Dong-Lepowsky, Li, Dong-Li-Mason, Barron-Dong-Mason, Doyon-Lepowsky-Milas, Barron-H.-Lepowsky and many others.
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Definition of twisted modules (H., 2009)

A *g-twisted V-module* is a $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ (graded by *weights*) equipped with a linear map

$$\begin{aligned} Y_W^g : V \otimes W &\to W\{x\}[\log x], \\ v \otimes w &\mapsto Y_W^g(v, x)w \end{aligned}$$

satisfying the following conditions:

- The *equivariance property*: For $p \in \mathbb{Z}$, $z \in \mathbb{C}^\times$, $v \in V$ and $w \in W$,
$$(Y_W^g)^{p+1}(gv, z)w = (Y_W^g)^p(v, z)w,$$
where for $p \in \mathbb{Z}$, $(Y_W^g)^p(v, z)$ is the p-th analytic branch of $Y_W^g(v, x)$.
- The *identity property*: For $w \in W$, $Y^g(\mathbf{1}, x)w = w$.

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- The *duality property*: For any $u, v \in V$, $w \in W$ and $w' \in W'$, there exists a multivalued analytic function with preferred branch of the form

$$f(z_1, z_2) = \sum_{i,j,k,l=0}^{N} a_{ijkl} z_1^{m_i} z_2^{n_j} (\log z_1)^k (\log z_2)^l (z_1 - z_2)^{-t}$$

for $N \in \mathbb{N}$, $m_1, \ldots, m_N$, $n_1, \ldots, n_N \in \mathbb{C}$ and $t \in \mathbb{Z}_+$, such that the series

$$\langle w', (Y_W^g)^p(u, z_1)(Y_W^g)^p(v, z_2)w\rangle,$$
$$\langle w', (Y_W^g)^p(v, z_2)(Y_W^g)^p(u, z_1)w\rangle,$$
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are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, and their sums are equal to the branch

$$f^{p,p}(z_1, z_2) = \sum_{i,j,k,l=0}^{N} a_{ijkl} e^{m_i l_p(z_1)} e^{n_j l_p(z_2)} l_p(z_1)^k l_p(z_2)^l (z_1 - z_2)^{-t}$$

of $f(z_1, z_2)$ in the region $|z_1| > |z_2| > 0$, the region $|z_2| > |z_1| > 0$, the region given by $|z_2| > |z_1 - z_2| > 0$ and $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$, respectively.

- The $L(0)$*-grading condition* and g*-grading condition*: Let $L_W^g(0) = \mathrm{Res}_x x Y_W^g(\omega, x)$. Then for $n \in \mathbb{C}$ and $\alpha \in \mathbb{C}/\mathbb{Z}$, $w \in W_{[n]}^{[\alpha]}$, there exists $K, \Lambda \in \mathbb{Z}_+$ such that $(L_W^g(0) - n)^K w = (g - e^{2\pi\alpha i})^\Lambda w = 0$.
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Twisted intertwining operators

- To construct conformal field theories associated to V^G, we need to introduce and study twisted intertwining operators, that is, intertwining operators among twisted modules.
- In 1995, Xu introduced a notion of intertwining operators among twisted modules associated to commuting automorphisms of finite orders by generalizing the Jacobi identity defining intertwining operators among (untwisted) modules.
- But in general, an orbifold conformal field theory is associated to a nonabelian group of automorphisms. In particular, we have to introduce and study intertwining operators among twisted modules associated to not-necessarily-commuting automorphisms.

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- In general, the group might also not be finite. So we also have to introduce and study intertwining operators among twisted modules associated to automorphisms of infinite orders.
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- In a recent paper, I find a formulation of such a notion of twisted intertwining operators associated to not-necessarily-commuting automorphisms and proved their basic properties.
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Let g_1, g_2, g_3 be automorphisms of V and let W_1, W_2 and W_3 be g_1-, g_2- and g_3-twisted V-modules, respectively. A twisted intertwining operator of type $\binom{W_3}{W_1 W_2}$ is a linear map

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satisfying the following conditions:

- *The lower truncation property*: For $w_1 \in W_1$ and $w_2 \in W_2$, $n \in \mathbb{C}$ and $k = 0, \ldots, K$, $\mathcal{Y}_{n+l,k}(w_1)w_2 = 0$ for $l \in \mathbb{N}$ sufficiently large.

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\begin{aligned}
&f(z_1, z_2; u, w_1, w_2, w_3')\\
&\quad = \sum_{i,j,k,l,m,n=1}^{N} a_{ijklmn} z_1^{r_i} z_2^{s_j} (z_1 - z_2)^{t_k} \cdot \\
&\qquad\qquad \cdot (\log z_1)^l (\log z_2)^m (\log(z_1 - z_2))^n
\end{aligned}
$$

for $N \in \mathbb{N}$, $r_i, s_j, t_k, a_{ijklmn} \in \mathbb{C}$, such that for $p_1, p_2, p_{12} \in \mathbb{Z}$, the series

$$\langle w_3', (Y_{W_3}^{g_3})^{p_1}(u, z_1)\mathcal{Y}^{p_2}(w_1, z_2)w_2\rangle,$$
$$\langle w_3', \mathcal{Y}^{p_2}(w_1, z_2)(Y_{W_2}^{g_2})^{p_1}(u, z_1)w_2\rangle,$$
$$\langle w_3', \mathcal{Y}^{p_2}((Y_{W_1}^{g_1})^{p_{12}}(u, z_1 - z_2)v, z_2)w\rangle$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively. Moreover, their sums are equal to the branches

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& = \sum_{i,j,k,l,m,n=1}^{N} a_{ijklmn} z_1^{r_i} e^{r_i l_{p_1}(z_1)} e^{s_j l_{p_2}(z_2)} e^{t_k l_{p_1}(z_1 - z_2)} \cdot \\
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$$\cdot (l_{p_2}(z_1))^l (l_{p_2}(z_2))^m (l_{p_{12}}(z_1 - z_2))^n,$$

respectively, of $f(z_1, z_2; u, w_1, w_2, w_3')$ in the region given by $|z_1| > |z_2| > 0$ and $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$, the region given by $|z_2| > |z_1| > 0$ and $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$, the region given by $|z_2| > |z_1 - z_2| > 0$ and $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$, respectively.

- The $L(-1)$-*derivative property*:

$$\frac{d}{dx}\mathcal{Y}(w_1, x) = \mathcal{Y}(L(-1)w_1, x).$$

Outline

Compatibility with the group multiplication

Theorem (H., 2017)

Let g_1, g_2, g_3 be automorphisms of V and let W_1, W_2 and W_3 be g_1-, g_2- and g_3-twisted V-modules, respectively. Assume that the vertex operator map for W_3 given by $u \mapsto Y_{W_3}^{g_3}(u, x)$ is injective. If there exists a twisted intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$ such that the coefficients of the series $\mathcal{Y}(w_1, x)w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$ span W_3, then $g_3 = g_1 g_2$.

The proof of this result can be explained by three pictures:

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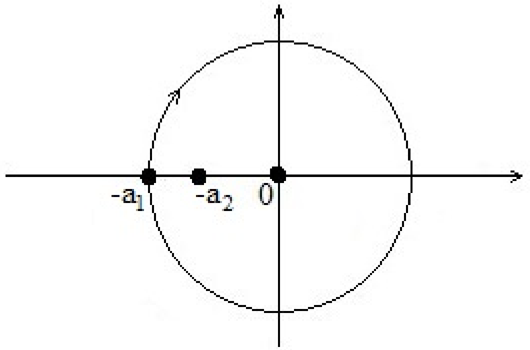

Figure: The loop Γ_1

Compatibility with the group multiplication

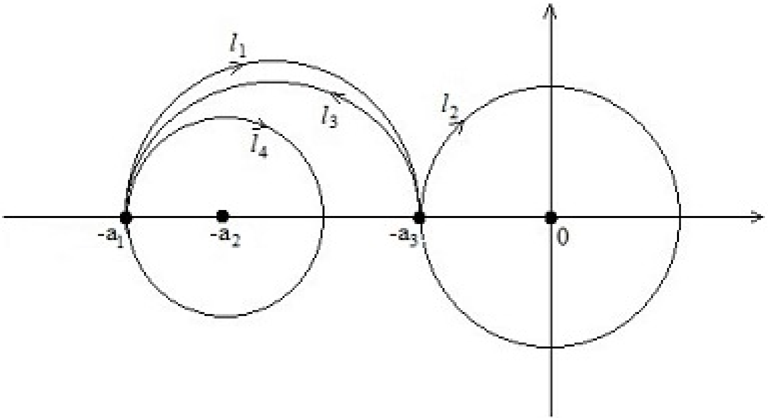

Figure: The loop Γ_2

Compatibility with the group multiplication

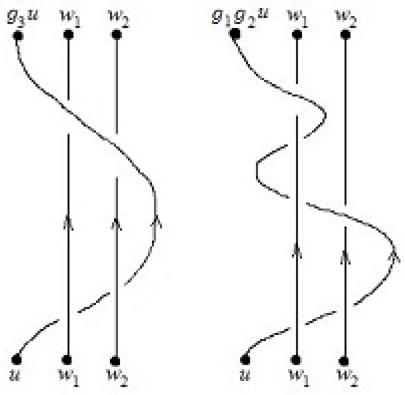

Figure: The braiding graphs corresponding to Γ_1 (left) and Γ_2 (right)

The skew symmetry isomorphisms

Let (W, Y_W^g) be a g-twisted V-module. Let h be an automorphism of V and let

$$\begin{aligned} \phi_h(Y^g) : V \times W &\to W\{x\}[\log x] \\ v \otimes w &\mapsto \phi_h(Y^g)(v, x)w \end{aligned}$$

be the linear map defined by

$$\phi_h(Y^g)(v, x)w = Y^g(h^{-1}v, x)w.$$

Proposition

The pair $(W, \phi_h(Y^g))$ is an hgh^{-1}-twisted V-module.

We shall denote the hgh^{-1}-twisted V-module in the proposition above by $\phi_h(W)$.

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$$\begin{aligned}\Omega_\pm(\mathcal{Y}) : W_2 \otimes W_1 &\to W_3\{x\}[\log x]\\ w_2 \otimes w_1 &\mapsto \Omega_\pm(\mathcal{Y})(w_2, x)w_1\end{aligned}$$

by

$$\Omega_\pm(\mathcal{Y})(w_2, x)w_1 = e^{xL(-1)}\mathcal{Y}(w_1, y)w_2\Big|_{y^n=e^{\pm n\pi \text{ß}}x^n,\ \log y=\log x\pm\pi \text{ß}}$$

for $w_1 \in W_1$ and $w_2 \in W_2$.

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Theorem (H., 2017)

The linear maps $\Omega_+(\mathcal{Y})$ and $\Omega_-(\mathcal{Y})$ are twisted intertwining operators of types $\binom{W_3}{W_2\,\phi_{g_2^{-1}}(W_1)}$ and $\binom{W_3}{\phi_{g_1}(W_2)\,W_1}$, respectively.

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The maps $\Omega_+ : \mathcal{V}^{W_3}_{W_1 W_2} \to \mathcal{V}^{W_3}_{W_2\phi_{g_2^{-1}}(W_1)}$ and $\Omega_- : \mathcal{V}^{W_3}_{W_1 W_2} \to \mathcal{V}^{W_3}_{\phi_{g_1}(W_2) W_1}$ are linear isomorphisms. In particular, $\mathcal{V}^{W_3}_{W_1 W_2}$, $\mathcal{V}^{W_3}_{\phi_{g_1}(W_2) W_1}$ and $\mathcal{V}^{W_3}_{W_2\phi_{g_2^{-1}}(W_1)}$ are linearly isomorphic.

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Let (W, Y_W^g) be a g-twisted V-module relative to G. Let W' be the graded dual of W. Define a linear map

$$\begin{aligned}(Y_W^g)' : V \otimes W' &\to W'\{x\}[\log x],\\ v \otimes w' &\mapsto (Y_W^g)'(v,x)w'\end{aligned}$$

by

$$\langle (Y_W^g)'(v,x)w', w\rangle = \langle w', Y_W^g(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w\rangle$$

for $v \in V$, $w \in W$ and $w' \in W'$.

Proposition

The pair $(W', (Y_W^g)')$ is a g^{-1}-twisted V-module.

We call $(W', (Y_W^g)')$ the contragredient twisted V-module of (W, Y_W^g).

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Outline

Orbifold theory conjecture

- **Conjecture (H.)**: Let V is a simple vertex operator algebra satisfying the three conditions given above, that is, (i) $V_{(0)} = \mathbb{C}\mathbf{1}$, $V_{(n)} = 0$ for $n < 0$ and the contragredient V', as a V-module, is equivalent to V, (ii) every grading-restricted generalized V-module is completely reducible, (iii) V is C_2-cofinite. Let G be a finite group of automorphisms of V. Then the twisted intertwining operators among the g-twisted V-modules for all $g \in G$ satisfy the associativity, commutativity and modular invariance properties.
- If this conjecture is proved, we obtain the genus-zero and genus-one parts of the chiral orbifold conformal field theory associated with the vertex operator algebra V and the group G of automorphisms of V.

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- **Conjecture (H.)**: Let V be a vertex operator satisfying the two conditions in the orbifold conjecture above and let G be a finite group of automorphisms of V. The the category of g-twisted V-modules for all $g \in G$ is a G-crossed (tensor) category in the sense of Turaev.

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- One of the main motivation to construct and study orbifold conformal field theories is the uniqueness conjecture of the moonshine module vertex operator algebra proposed by Frenkel, Lepowsky and Meurman.
- If these conjecture are proved, we will be able to use it to study the representation theory of and conformal field theories associated to the fixed-point vertex operator algebra V^G and vertex operator algebras containing V^G as a subalgebra.
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