Twist vertex operators for twisted modules

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Abstract

We introduce and study twist vertex operators for a (lower-bounded generalized) twisted modules for a grading-restricted vertex (super)algebra. We prove the duality property, weak associativity, a Jacobi identity, a generalized commutator formula, generalized weak commutativity, and convergence and commutativity for products of more than two operators involving twist vertex operators. These properties of twist vertex operators play an important role in the author's recent general, direct and explicit construction of (lower-bounded generalized) twisted modules.

1 Introduction

Frenkel, Lepowsky and Meurman introduced twisted modules associated to automorphisms of finite order of a vertex operator algebra in their construction [FLM1] [FLM2] [FLM3] of the moonshine module vertex operator algebra $V^\natural$. In [H2], the author introduced (generalized) twisted module associated a general automorphism g of a vertex operator algebra V. Twisted modules for vertex operator algebras have been studied extensively in mathematics and physics. See for example, [Le1], [FLM2], [Le2], [FLM3], [D], [DL], [DonLM1], [DonLM2], [Li], [BDM], [DoyLM1], [DoyLM2], [BHL], [H2], [B], [Y], and the references in these papers.

Recently, the author has found a general and direct construction of (lower-bounded generalized) twisted modules for a grading-restricted vertex (super)algebra V [H5]. In this construction, besides "twisted fields" or twisted vertex operators, one crucial ingredient in this construction is what we call the "twist fields" or "twist vertex operators." Such an operators is a generalization of the vertex operator Y_{WV}^{W} for a V-module W given by the skew-symmetry in Section 5.6 in [FHL] and is a special type of twisted intertwining operators obtained from twisted vertex operator for the (lower-bounded generalized) twisted module using the skew-symmetry isomorphism Ω_+ in [H4].

The reason why twist vertex operators are important for the construction of twisted modules (and in particular, modules) is quite simple. To construct twisted modules (or even modules), we usually start with a set of generating twisted fields corresponding to a set of generating fields of the algebra V. Usually it is not difficult to verify that these twisted fields satisfy weak commutativity or commutativity. But for twisted modules (or even modules), associativity is the main property to be verified and is not a consequence of the weak commutativity or commutativity. For a set of generating twisted fields, we cannot formulate the associativity as one of our assumptions on these generating twisted fields. But for a twisted module W, if we use the skew-symmetry isomorphism for intertwining operators to introduce an intertwining operator of type $\binom{W}{WV}$ called twist vertex operator map, then the associativity for twisted vertex operators can be rewritten as the commutativity involving twisted vertex operators, twist vertex operators and vertex operators for the algebra V. Thus by introducing additional twist fields and assuming that the commutativity holds for the generating twisted fields, twist fields and generating fields for V, we can construct twisted modules using the same method as in the first construction of grading-restricted vertex algebras in [H3].

In this paper, we study twist vertex operators for a (lower-bounded generalized) twisted modules for a grading-restricted vertex (super)algebra V associated to an automorphism g of V. We first prove that both the semisimple and unipotent parts of an automorphism g of V are also automorphisms of V and the unipotent part of g is in fact the exponential of a derivation of V. Similar results also hold for twisted modules. We also prove or reformulate some results for twisted vertex operators, including weak commutativity, a Jacobi identity, a commutator formula, and convergence and commutativity for products of more than two twisted vertex operators. We then prove the main results involving twist vertex operators, including duality, weak associativity, a Jacobi identity, a generalized commutator formula, generalized weak commutativity, and convergence and commutativity for products of more than two operators with one operator being a twist vertex operator.

In our formulations and proofs of the main results in this paper, we use both the complex variable approach and the formal variable approach. The formal variable approach can be used because we have explicit expressions of the correlation functions obtained by analytically extending the convergent products or iterates of twisted, untwisted and twist vertex operators. The convergence and commutativity are formulated and proved using complex variables approach while weak associativity, Jacobi identities, generalized commutator formulas and generalized weak commutativity are formulated using formal variable approach and proved using both approaches. The complex variable formulations often look simpler than the formal variable formulations because we can choose branches of multivalued correlation functions and assume that the reader is familiar with multivalued analytic functions and their branches. On the other hand, although the formal variable formulations look more complicated, they reveal explicitly the information about suitable branches of multivalued correlation functions in terms of purely algebraic expressions.

Our results are formulated and proved for lower-bounded generalized g-twisted V-modules. But these results hold for generalized g-twisted V-modules satisfying the lower-truncation properties in Remark 3.2.

This paper is organized as follows: In Section 2, we recall the notion of grading-restricted vertex (super)algebra and their variants. We also recall some basic and useful properties of automorphisms of grading-restricted vertex algebras. In Section 3, we recall the notion of generalized twisted modules and their variants. We also recall and prove some properties of twisted vertex operators for lower-bounded generalized twisted modules. In Section 4, we introduce twist vertex operators for a lower-bounded twisted module. The main properties for these vertex operators are proved in this section.

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2 Grading-restricted vertex (super)algebras and automorphisms

We recall the notions of grading-restricted vertex (super)algebra and (generalized) twisted module in this section. We also prove some properties of automorphisms of a grading-restricted vertex algebra and their actions on a (generalized) twisted module.

Before we recall the basic definitions, we first discuss the conventions used throughout the present paper. We shall use $z, z_1, z_2, \dots, \zeta, \zeta_1, \zeta_2, \xi, \xi_1, \xi_2, \dots$ to denote complex variables and $x, x_1, x_2, \dots, y, y_1, y_2, \dots$ to denote formal variables. Logarithms of formal variables are also formal variables.

For $z \in \mathbb{C}^\times$, we use $l_p(z)$ to denote $\log|z| + \arg z i + 2p\pi i$, where $0 \leq \arg z < 2\pi$. We shall always use expressions such as $(x_1 - x_2)^n$ for $n \in \mathbb{C}$ or $(\log(x_1 - x_2))^k$ for $k \in \mathbb{N}$ as power series in x_2. For a semisimple operator $\mathcal{S}$, $x^{\mathcal{S}}$ or $(x_1 - x_2)^{\mathcal{S}}$ and so on acting on the eigenspace for $\mathcal{S}$ with eigenvalue $\alpha \in \mathbb{C}$ are defined to be x^α or $(x_1 - x_2)^\alpha$ and so on. For a nilpotent operator $\mathcal{N}$, $x^{\mathcal{N}}$ or $(x_1 - x_2)^{\mathcal{N}}$ and so on are defined to be $e^{\mathcal{N} \log x}$ or $e^{\mathcal{N} \log(x_1 - x_2)}$ and so on.

We first recall the definitions of grading-restricted vertex (super)algebra, some variants and their automorphisms.

Definition 2.1 A *grading-restricted vertex superalgebra* is a $\frac{\mathbb{Z}}{2}$-graded vector space $V = \coprod_{n \in \frac{\mathbb{Z}}{2}} V_{(n)}$, equipped with a linear map

$$\begin{aligned} Y_V : V \otimes V &\to V[[x, x^{-1}]], \\ u \otimes v &\mapsto Y_V(u, x)v, \end{aligned}$$

or equivalently, an analytic map

$$\begin{aligned} Y_V : \mathbb{C}^\times &\to \mathrm{Hom}(V \otimes V, \overline{V}), \\ z &\mapsto Y_V(\cdot, z)\cdot : u \otimes v \mapsto Y_V(u, z)v \end{aligned}$$

called the *vertex operator map* and a *vacuum* $\mathbf{1} \in V_{(0)}$ satisfying the following axioms:

1. Axioms for the grading: (a) *Grading-restriction condition*: When n is sufficiently negative, $V_{(n)} = 0$ and $\dim V_{(n)} < \infty$ for $n \in \frac{\mathbb{Z}}{2}$. (b) *$L(0)$-commutator formula*: Let $L_V(0) : V \to V$ be defined by $L_V(0)v = nv$ for $v \in V_{(n)}$. Then

$$[L_V(0), Y_V(v, x)] = x \frac{d}{dx} Y_V(v, x) + Y_V(L_V(0)v, x)$$

for $v \in V$.

2. Axioms for the vacuum: (a) *Identity property*: Let 1_V be the identity operator on V. Then $Y_V(\mathbf{1}, x) = 1_V$. (b) *Creation property*: For $u \in V$, $\lim_{x\to 0} Y_V(u, x)\mathbf{1}$ exists and is equal to u.

3. $L(-1)$-*derivative property* and $L(-1)$-*commutator formula*: Let $L_V(-1) : V \to V$ be the operator given by

$$L_V(-1)v = \lim_{x\to 0} \frac{d}{dx} Y_V(v, x)\mathbf{1}$$

for $v \in V$. Then for $v \in V$,

$$\frac{d}{dx} Y_V(v, x) = Y_V(L_V(-1)v, x) = [L_V(-1), Y_V(v, x)].$$

4. *Duality*: For u_1, u_2 in either $\coprod_{n\in\mathbb{Z}} V_{(n)}$ or $\coprod_{n\in\mathbb{Z}+\frac{1}{2}} V_{(n)}$, $v \in V$ and $v' \in V'$, the series

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle, \tag{2.1}$$

$$(-1)^{|u_1||u_2|}\langle v', Y_V(u_2, z_2)Y_V(u_1, z_1)v\rangle, \tag{2.2}$$

$$\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle, \tag{2.3}$$

where $|u_1|$ (or $|u_2|$) is 0 if u_1 (or u_2) is in $\coprod_{n\in\mathbb{Z}} V_{(n)}$ and is 1 if u_1 (or u_2) is in $\coprod_{n\in\mathbb{Z}+\frac{1}{2}} V_{(n)}$, are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.

In the case that $\coprod_{n\in\mathbb{Z}+\frac{1}{2}} V_{(n)} = 0$, the grading-restricted vertex superalgebra just defined is called a *grading-restricted vertex algebra*.

For simplicity, in the formulations, calculations, proofs and discussions of the results in this paper, we always choose elements of V to be either in $\coprod_{n\in\mathbb{Z}} V_{(n)}$ or $\coprod_{n\in\mathbb{Z}+\frac{1}{2}} V_{(n)}$ and we shall use $|v|$ to denote 0 if $v \in \coprod_{n\in\mathbb{Z}} V_{(n)}$ and to denote 1 if $v \in \coprod_{n\in\mathbb{Z}+\frac{1}{2}} V_{(n)}$.

Remark 2.2 In Definition 2.1, the duality property can be stated separately as three axioms, that is, the *rationality* (the convergence of (2.1), (2.2) and (2.3) to rational functions in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively), the *commutativity* (the statement that the rational functions to which (2.1) and (2.2) converge are equal) and the *associativity* (the statement that the (2.1) and (2.3) are equal in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$). These axioms are not independent. In fact, the associativity follows from the rationality and commutativity (see [FHL]) and the commutativity also follows from the rationality and associativity (see [H1]).

Definition 2.3 A *quasi-vertex operator (super)algebra* or a *Möbius vertex algebra* is a grading-restricted vertex (super)algebra $(V, Y_V, \mathbf{1})$ together with an operator $L_V(1)$ of weight 1 on V satisfying

$$\begin{aligned}
[L_V(-1), L_V(1)] &= -2L_V(0),\\
[L_V(1), Y_V(v, x)] &= Y_V(L_V(1)v, x) + 2xY_V(L_V(0)v, x) + x^2Y_V(L_V(-1)v, x)
\end{aligned}$$

for $v \in V$.

Definition 2.4 Let V_1 and V_2 be grading-restricted vertex superalgebras. A homomorphism from V_1 to V_2 is a weight-preserving linear map $g : V_1 \to V_2$ such that $gY_{V_1}(u,x)v = Y_{V_2}(gu,x)gv$. An isomorphism from V_1 to V_2 is an invertible homomorphism from V_1 to V_2. When $V_1 = V_2 = V$, an isomorphism from V to V is called an automorphism of V.

In the rest of this paper, we fix a grading-restricted vertex superalgebra V and an automorphism g of V. We shall need the following linear algebra lemma:

Lemma 2.5 *There exist a semisimple operator $\mathcal{S}_g$ on V with eigenvalues of the form $e^{2\pi i\alpha}$ for some $\alpha \in \mathbb{C}$ and an operator $\mathcal{N}_g$ nilpotent on the homogeneous subspace $V_{(n)}$ of V for $n \in \mathbb{Z}$ such that $\mathcal{S}_g$ and $\mathcal{N}_g$ commute with each other and $g = e^{2\pi i(\mathcal{S}_g+\mathcal{N}_g)}$. In particular, there exists an operator $\mathcal{L}_g = \mathcal{S}_g + \mathcal{N}_g$ on V such that $g = e^{2\pi i\mathcal{L}_g}$ and $\mathcal{S}_g$ and $\mathcal{N}_g$ are the semisimple and nilpotent parts of $\mathcal{L}_g$.*

Proof. One proof can be found in Section 2 of [HY], where $\mathcal{S}_g$ and $\mathcal{N}_g$ are denoted by $\mathbf{A}_V$ and $\mathcal{N}$, respectively. ∎

Let P_V be set of $\alpha \in \mathbb{C}$ such that $0 \leq \Re(\alpha) < 1$ and $e^{2\pi i\alpha}$ is an eigenvalue of g. Then V can be decomposed as a direct sum

$$V = \coprod_{\alpha\in P_V} V^{[\alpha]},$$

where $V^{[\alpha]}$ for $\alpha \in P_V$ is the generalized eigenspace for g with eigenvalue $e^{2\pi i\alpha}$. In particular, $V^{[\alpha]}$ is the eigenspace for $e^{2\pi \mathcal{S}_g}$ with eigenvalue $e^{2\pi i\alpha}$.

Apply Lemma 2.4 in [HY] to V viewed as a 1_V-twisted V-module, we obtain:

Proposition 2.6 *The operators $\mathcal{S}_g$ and $\mathcal{N}_g$ have the following properties:*

1. *The operators $e^{2\pi i\mathcal{S}_g}$ and $e^{2\pi i\mathcal{N}_g}$ are also automorphisms of V.*
2. *The operator $\mathcal{N}_g$ is a derivation of V, that is,*
$$[\mathcal{N}_g, Y_V(u,x)] = Y_V(\mathcal{N}_g u, x).$$
3. *For any $t \in \mathbb{C}$, $e^{t\mathcal{N}_g}$ is an automorphism of V. For any formal variable y and $u, v \in V$,*
$$e^{y\mathcal{N}_g}Y_V(u,x)v = Y_V(e^{y\mathcal{N}_g}u,x)e^{y\mathcal{N}_g}v. \tag{2.4}$$
In particular, when $y = \log x_0$,
$$x_0^{\mathcal{N}_g}Y_V(u,x)v = Y_V(x_0^{\mathcal{N}_g}u,x)x_0^{\mathcal{N}_g}v. \tag{2.5}$$

Remark 2.7 It is important to note that in general $\mathcal{S}_g$ might not be a derivation of V. The reason is that in general $\log(1-(e^{2\pi i\mathcal{S}_g}-1))$ expanded in nonnegative powers of $e^{2\pi i\mathcal{S}_g}-1$ might not be well defined. One can also see this from examples. For example, the automorphism group of the moonshine module vertex operator algebra $V^\natural$ is the Monster (see [FLM3]). So every automorphism g of $V^\natural$ is of finite order and semisimple. In particular, $g = e^{2\pi i\mathcal{S}_g}$ for some operator $\mathcal{S}_g$. If $\mathcal{S}_g$ is a derivation of $V^\natural$, then for $t \in \mathbb{C}$, $e^{t\mathcal{S}_g}$ is also an automorphism of $V^\natural$ and thus there are infinitely many automorphisms of $V^\natural$. But the automorphism group of $V^\natural$ is the Monster, a finite simple group. Contradiction.

3 Twisted modules and twisted vertex operators

In this section, we first recall various notions of (generalized) g-twisted V-module. Then we recall and prove some basic properties of twisted vertex operators for a generalized g-twisted V-module W.

Definition 3.1 A *generalized g-twisted V-module* is a $\mathbb{C}\times\mathbb{Z}_2\times\mathbb{C}/\mathbb{Z}$-graded vector space

$$W=\coprod_{n\in\mathbb{C},s\in\mathbb{Z}_2,[\alpha]\in\mathbb{C}/\mathbb{Z}} W_{[n]}^{s;[\alpha]}=\coprod_{n\in\mathbb{C},s\in\mathbb{Z}_2,\alpha\in P_W} W_{[n]}^{s;[\alpha]}$$

(graded by *weights*, $\mathbb{Z}_2$-*fermion number* and *g-weights*, where P_W is the subset of the set $\{\alpha\in\mathbb{C}\mid \Re(\alpha)\in[0,1)\}$ such that $W_{[n]}^{s;[\alpha]}\neq 0$ for $\alpha\in P_W$, equipped with operators $L_W(0)$ and $L_W(-1)$ on W, a linear map

$$\begin{aligned}Y_W^g: V\otimes W &\to W\{x\}[\log x],\\ v\otimes w &\mapsto Y_W^g(v,x)w=\sum_{n\in\mathbb{C}}\sum_{k\in\mathbb{N}}(Y_W^g)_{n,k}x^n(\log x)^k\end{aligned}$$

called *twisted vertex operator map* and an action of g satisfying the following conditions:

1. The *equivariance property*: For $p\in\mathbb{Z}$, $z\in\mathbb{C}^\times$, $v\in V$ and $w\in W$,

   $$(Y_W^g)^{p+1}(gv,z)w=(Y_W^g)^p(v,z)w,$$

   where for $p\in\mathbb{Z}$, $(Y_W^g)^p(v,z)$ is the p-th analytic branch of $Y_W^g(v,x)$.

2. The *identity property*: For $w\in W$, $Y_W^g(\mathbf{1},x)w=w$.

3. The *duality property*: For $u,v\in V$ (recall our convention that we always choose elements of V to be either in $\coprod_{n\in\mathbb{Z}}V_{(n)}$ or $\coprod_{n\in\mathbb{Z}+\frac{1}{2}}V_{(n)}$ so that $|u|$ and $|v|$ are well defined), $w\in W$ and $w'\in W'$, there exists a multivalued analytic function with preferred branch of the form

   $$f(z_1,z_2)=\sum_{i,j,k,l=0}^{N}a_{ijkl}z_1^{m_i}z_2^{n_j}(\log z_1)^k(\log z_2)^l(z_1-z_2)^{-t} \tag{3.1}$$

   for $N\in\mathbb{N}$, $m_1,\dots,m_N$, $n_1,\dots,n_N\in\mathbb{C}$ and $t\in\mathbb{Z}_+$, such that the series

   $$\langle w',(Y_W^g)^p(u,z_1)(Y_W^g)^p(v,z_2)w\rangle, \tag{3.2}$$
   $$(-1)^{|u||v|}\langle w',(Y_W^g)^p(v,z_2)(Y_W^g)^p(u,z_1)w\rangle, \tag{3.3}$$
   $$\langle w',(Y_W^g)^p(Y_V(u,z_1-z_2)v,z_2)w\rangle \tag{3.4}$$

   are absolutely convergent in the regions $|z_1|>|z_2|>0$, $|z_2|>|z_1|>0$, $|z_2|>|z_1-z_2|>0$, respectively, and their sums are equal to the branch

   $$f^{p,p}(z_1,z_2)=\sum_{i,j,k,l=0}^{N}a_{ijkl}e^{m_il_p(z_1)}e^{n_jl_p(z_2)}l_p(z_1)^kl_p(z_2)^l(z_1-z_2)^{-t} \tag{3.5}$$

of $f(z_1, z_2)$ in the region $|z_1| > |z_2| > 0$, the region $|z_2| > |z_1| > 0$, the region given by $|z_2| > |z_1 - z_2| > 0$ and $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$, respectively.

4. Properties about the gradings: (a) The $L(0)$-*grading condition*: For $w \in W_{[n]}$, $n \in \mathbb{C}$, there exists $K \in \mathbb{Z}_+$ such that $(L_W(0) - n)^K w = 0$. (b) The $L(0)$-*commutator formula*:

$$[L_W(0), Y_W^g(v, z)] = z\frac{d}{dz}Y_W^g(v, z) + Y_W^g(L_V(0)v, z)$$

for $v \in V$. (c) The g-*grading condition*: For $\alpha \in P_W$, $w \in W[\alpha]$, there exist $\Lambda \in \mathbb{Z}_+$ such that $(g - e^{2\pi\alpha i})^\Lambda w = 0$. (d) The g-*compatibility condition* and $\mathbb{Z}_2$-*fermion number compatibility condition*: For $u \in V$ and $w \in W$, $gY_W^g(u, x)w = Y_W^g(gu, x)gw$ and $|Y_W^g(u, x)w| = |u| + |w|$.

5. The $L(-1)$-*derivative property* and $L(-1)$-*commutator formula*: For $v \in V$,

$$\frac{d}{dx}Y_W^g(v, x) = [L_W(-1), Y_W^g(v, x)].$$

A *lower-bounded generalized g-twisted V-module* is a generalized g-twisted V-module W such that $W_{[n]} = 0$ when $\Re(n) < B$ for some $B \in \mathbb{R}$. A *grading-restricted generalized g-twisted V-module* or simply a *g-twisted V-module* is a lower-bounded generalized g-twisted V-module W such that for each $n \in \mathbb{C}$, $\dim W_{[n]} < \infty$.

For $w \in \coprod_{n\in\mathbb{C},\alpha\in P_W} W_{[n]}^{s;[\alpha]}$, we shall use $|w|$ to denote s. We shall also use the convention that in the formulations, calculations, proofs and discussions of the results in this paper, we always choose elements of W to be in $\coprod_{n\in\mathbb{C},\alpha\in P_W} W_{[n]}^{s;[\alpha]}$ for either $s = 0$ or $s = 1$. Thus when we let $w \in W$, $|w|$ is always well defined.

We shall need the homogeneous subspaces $W_{[n]} = \coprod_{s=1,2,\alpha\in P_W} W_{[n]}^{s;[\alpha]}$ for $n \in \mathbb{C}$ and $W^{[\alpha]} = \coprod_{n\in\mathbb{C},s=1,2} W_{[n]}^{s;[\alpha]}$ for $\alpha \in P_W$ of W.

Remark 3.2 For a lower-bounded generalized g-twisted V-module W, by (2.7) in [HY] (or more explicitly, the two displayed formula after (2.7) in [HY]), for $u \in V^{[\alpha]}$,

$$Y_W^g(u, x) = \sum_{k=1}^{N} \sum_{n\in\alpha+\mathbb{Z}} (Y_W^g)_{n,k}(u)x^{-n-1}(\log x)^k. \tag{3.6}$$

Moreover, for $u \in V^{[\alpha]}$ and $w \in W$, $Y_W^g(u, x)w$ has only finitely many terms containing $x^{-\alpha+n}$ for $n \in -\mathbb{N}$ and for $u \in V^{[\alpha]}$ and $w' \in W'$, $\langle w', Y_W^g(u, x)\cdot\rangle$ has only finitely many terms containing $x^{-\alpha+n}$ for $n \in \mathbb{Z}_+$. In the rest of the present paper, we shall study only lower-bounded generalized g-twisted V-modules though all the results still hold for generalized g-twisted V-modules such that the twisted vertex operators have these properties remarked above.

Since (3.2), (3.3) and (3.4) all converges absolutely in the corresponding regions to the corresponding branches of the multivalued analytic function $f(z_1, z_2)$ with preferred branch, we shall denote $f(z_1, z_2)$ by

$$F(\langle w', Y_W^g(u, z_1)Y_W^g(v, z_2)w\rangle),$$
$$F((-1)^{|u||v|}\langle w', Y_W^g(v, z_2)Y_W^g(u, z_1)w\rangle)$$

or

$$F(\langle w', Y_W^g(Y_V(u, z_1 - z_2)v, z_2)w\rangle).$$

We shall also denote the branch $f^{p,p}(z_1, z_2)$ for $p \in \mathbb{Z}$ by

$$F^p(\langle w', Y_W^g(u, z_1)Y_W^g(v, z_2)w\rangle),$$
$$F^p((-1)^{|u||v|}\langle w', Y_W^g(v, z_2)Y_W^g(u, z_1)w\rangle)$$

or

$$F^p(\langle w', Y_W^g(Y_V(u, z_1 - z_2)v, z_2)w\rangle).$$

We shall use the similar notations to denote the multivalued analytic functions or branches to which the products or iterates of more than two twisted vertex operators and vertex operators for the algebra converge.

Remark 3.3 The duality property in Definition 3.1 can also be stated separately as three axioms, generalized rationality, commutativity and associativity. We do not state them explicitly here since they are similar to the corresponding properties for V (see Remark 2.2). But for W, associativity does not follow from the generalized rationality and commutativity while the commutativity does follow from the generalized rationality and associativity. Also, it was observed by Bin Gui that the duality property for W implies the $L(-1)$-derivative property for W: For $v \in V$,

$$\frac{d}{dx}Y_W^g(v, x) = Y_W^g(L_V(-1)v, x).$$

In fact, for $w \in W$ and $w' \in W'$,

$$\begin{aligned}\langle w', (Y_W^g)^p(u, z_1)w\rangle &= F^p(\langle w', Y_W^g(u, z_1)Y_W^g(\mathbf{1}, z_2)w\rangle)\\ &= F^p(\langle w', Y_W^g(Y_V(u, z_1 - z_2)\mathbf{1}, z_2)w\rangle)\end{aligned} \tag{3.7}$$

for $u \in V$. Taking derivative with respect to z_1 on both sides of (3.7), using the $L(-1)$-derivative property for V and using the duality property again, we obtain

$$\begin{aligned}\left\langle w', \frac{d}{dz_1}(Y_W^g)^p(u, z_1)w\right\rangle &= F^p\left(\left\langle w', Y_W^g\left(\frac{\partial}{\partial z_1}Y_V(u, z_1 - z_2)\mathbf{1}, z_2\right)w\right\rangle\right)\\ &= F^p(\langle w', Y_W^g(Y_V(L_V(-1)u, z_1 - z_2)\mathbf{1}, z_2)w\rangle)\\ &= \langle w', (Y_W^g)^p(L_V(-1)u, z_1)w\rangle.\end{aligned} \tag{3.8}$$

This is equivalent to the $L(-1)$-derivative property for W.

In the rest of this paper, we fix a lower-bounded g-twisted V-module W. It has been proved in Section 2 of [HY] that the action of g on W has the following properties similar to those of the action of g on V:

Lemma 3.4 *There exist a semisimple operator on W, denoted still by $\mathcal{S}_g$, such that for $\alpha \in P_W$, $W^{[\alpha]}$ is the eigenspace of $\mathcal{S}_g$ with the eigenvalue $e^{2\pi i\alpha}$ and an operator nilpotent on any element of W and preserving the gradings of W, denoted still by $\mathcal{N}_g$, such that $\mathcal{S}_g$ and $\mathcal{N}_g$ on W commute with each other and $g = e^{2\pi i(\mathcal{S}_g+\mathcal{N}_g)}$ on W. In particular, there exists an operator $\mathcal{L}_g = \mathcal{S}_g + \mathcal{N}_g$ on W such that $g = e^{2\pi i\mathcal{L}_g}$ and $\mathcal{S}_g$ and $\mathcal{N}_g$ are the semisimple and nilpotent parts of $\mathcal{L}_g$.*

Proof. Since W is a direct sum of generalized eigenspaces of the action of g on W, we have a simpler proof of the existence of $\mathcal{S}_g$, $\mathcal{N}_g$ and $\mathcal{L}_g$ than the one in [HY] (in [HY], $\mathcal{S}_g$ and $\mathcal{N}_g$ are denoted by $\mathbf{A}_W$ and $\mathcal{N}$). Here we give this proof.

For $w \in W^{[\alpha]}$, $\alpha \in P_W$, $(g-e^{2\pi\alpha})^\Lambda w = 0$ for some $\Lambda \in \mathbb{Z}_+$. Define $\mathcal{S}_g$ on W by $\mathcal{S}_g w = \alpha w$. Then $e^{2\pi i\mathcal{S}_g}w = e^{2\pi i\alpha}w$ and

$$\begin{aligned}(e^{-2\pi i\mathcal{S}_g}g - 1_W)^\Lambda w &= e^{-2\pi i\Lambda\mathcal{S}_g}(g - e^{2\pi\alpha})^\Lambda w\\ &= 0.\end{aligned}$$

So $e^{-2\pi i\mathcal{S}_g}g - 1_W$ is nilpotent on w. Define

$$\begin{aligned}\mathcal{N}_g w &= \frac{1}{2\pi i}\log(1_W + (e^{-2\pi i\mathcal{S}_g}g - 1_W))w\\ &= \sum_{j\in\mathbb{Z}_+}\frac{(-1)^{j+1}}{j}(e^{-2\pi i\mathcal{S}_g}g - 1_W)^j w \qquad (3.9)\end{aligned}$$

for $w \in W$. Then $e^{-2\pi i\mathcal{S}_g}gw = e^{2\pi i\mathcal{N}_g}w$ and thus $gw = e^{2\pi i(\mathcal{S}_g+\mathcal{N}_g)}w$ for $w \in W$. It is clear that $\mathcal{S}_g$ commutes with g and thus with $\mathcal{N}_g$. ∎

Proposition 3.5 *For $u \in V$, $w \in W$, $t \in \mathbb{C}$ and formal variables x, y and x_0, we have*

$$e^{2\pi i\mathcal{S}_g}Y_W^g(u,x)w = Y_W^g(e^{2\pi i\mathcal{S}_g}u,x)e^{2\pi i\mathcal{S}_g}w, \qquad (3.10)$$
$$e^{2\pi i\mathcal{N}_g}Y_W^g(u,x)w = Y_W^g(e^{2\pi i\mathcal{N}_g}u,x)e^{2\pi i\mathcal{N}_g}w, \qquad (3.11)$$
$$\mathcal{N}_gY_W^g(u,x)w = Y_W^g(\mathcal{N}_gu,x)w + Y_W^g(u,x)\mathcal{N}_gw, \qquad (3.12)$$
$$e^{t\mathcal{N}_g}Y_W^g(u,x)w = Y_W^g(e^{t\mathcal{N}_g}u,x)e^{t\mathcal{N}_g}w, \qquad (3.13)$$
$$e^{y\mathcal{N}_g}Y_W^g(u,x)w = Y_W^g(e^{y\mathcal{N}_g}u,x)e^{y\mathcal{N}_g}w, \qquad (3.14)$$
$$x_0^{\mathcal{N}_g}Y_W^g(u,x)w = Y_W^g(x_0^{\mathcal{N}_g}u,x)x_0^{\mathcal{N}_g}w. \qquad (3.15)$$

By Lemma 2.3 in [HY], we have

$$Y_W^g(u,x) = (Y_W^g)_0(x^{-\mathcal{N}_g}u,x), \qquad (3.16)$$

where $(Y_W^g)_0(v,x)$ is the constant term of $Y_W^g(u,x)$ viewed as a power series in $\log x$. First we need another explicit form of the expansion of twisted vertex operators in the powers of the logarithm of the variable.

Proposition 3.6 *For $u \in V$,*

$$Y_W^g(u,x) = x^{-\mathcal{N}_g}(Y_W^g)_0(u,x)x^{\mathcal{N}_g}$$
$$= \sum_{k\in\mathbb{N}} \frac{(-1)^k}{k!}(\log x)^k \overbrace{[\mathcal{N}_g, \cdots, [\mathcal{N}_g}^{k}, (Y_W^g)_0(u,x)]\cdots], \tag{3.17}$$

Proof. Using (3.15) with $x_0 = x$ and (3.16), we have

$$\begin{aligned} Y_W^g(u,x) &= Y_W^g(x^{-\mathcal{N}_g}x^{\mathcal{N}_g}u, x) \\ &= x^{-\mathcal{N}_g}Y_W^g(x^{\mathcal{N}_g}u,x)x^{\mathcal{N}_g} \\ &= x^{-\mathcal{N}_g}(Y_W^g)_0(u,x)x^{\mathcal{N}_g}. \end{aligned}$$

This is the first equality in (3.17). Expanding $x^{-\mathcal{N}_g}(Y_W^g)_0(u,x)x^{\mathcal{N}_g}$ as a power series in $\log x$, we obtain the second equality in (3.17). ∎

We have the following weak commutativity for twisted vertex operators:

Proposition 3.7 *For $u, v \in V$, let $M_{u,v} \in \mathbb{Z}_+$ such that $x^{M_{u,v}}Y_V(u,x)v \in V[[x]]$. Then*

$$(x_1 - x_2)^{M_{u,v}}Y_W^g(u,x_1)Y_W^g(v,x_2) = (x_1-x_2)^{M_{u,v}}(-1)^{|u||v|}Y_W^g(v,x_2)Y_W^g(u,x_1). \tag{3.18}$$

Proof. This weak commutativity can be derived easily from the Jacobi identity (3.24) below for twisted vertex operators. Here we directly derive it from the duality property so that the reader will be familiar with the complex variable approach which is necessary when we prove the convergence later.

Since $x^{M_{u,v}}Y_V(u,x)v \in V[[x]]$,

$$(z_1 - z_2)^{M_{u,v}}\langle w', (Y_W^g)^p(Y_V(u, z_1 - z_2)v, z_2)w\rangle \tag{3.19}$$

is a Laurent series in z_2 and $z_1 - z_2$ with only nonnegative powers of $z_1 - z_2$. By the duality property, (3.19) is the expansion of

$$\sum_{i,j,k,l=0}^{N} a_{ijkl}e^{m_i l_p(z_1)}e^{n_j l_p(z_2)}l_p(z_1)^k l_p(z_2)^l(z_1-z_2)^{M_{u,v}-t}. \tag{3.20}$$

in the region given by $|z_2| > |z_1 - z_2| > 0$ and $|\arg z_1 - \arg z_2| < \frac{1}{2}$ as a Laurent series of z_2 and $z_1 - z_2$. Since (3.19) has only nonnegative powers of $z_1 - z_2$, (3.20) cannot have a pole at $z_1 - z_2 = 0$ and therefore can be rewritten as

$$\sum_{i,j,k,l=0}^{N'} a'_{ijkl}e^{m_i l_p(z_1)}e^{n_j l_p(z_2)}l_p(z_1)^k l_p(z_2)^l \tag{3.21}$$

for some $N' \in \mathbb{N}$ and $a'_{ijkl} \in \mathbb{C}$. Then from the duality property, for $w \in W$ and $w' \in W'$,

$$(z_1 - z_2)^{M_{u,v}} \langle w', (Y_W^g)^p(u, z_1)(Y_W^g)^p(v, z_2)w\rangle \tag{3.22}$$

and

$$(z_1 - z_2)^{M_{u,v}} (-1)^{|u||v|} \langle w', (Y_W^g)^p(v, z_2)(Y_W^g)^p(u, z_1)w\rangle \tag{3.23}$$

converges absolutely in the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively, to (3.21). But the expansions of (3.21) in the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$ are just (3.21) itself. Therefore both (3.22) and (3.23) are equal to the finite sum (3.21) in the region given by $z_1, z_2 \neq 0$, $z_1 \neq z_2$. In particular, (3.22) and (3.23) are equal in this region.

Since w and w' are arbitrary and $M_{u.v}$ are independent of w and w', we obtain

$$(z_1 - z_2)^{M_{u.v}} (Y_W^g)^p(u, z_1)(Y_W^g)^p(v, z_2) = (z_1 - z_2)^{M_{u.v}} (-1)^{|u||v|} (Y_W^g)^p(v, z_2)(Y_W^g)^p(u, z_1)$$

as series of the form

$$\sum_{m,n\in\mathbb{C},k,l\in\mathbb{N}} c_{m,n,k,l} e^{ml_p(z_1)} e^{nl_p(z_2)} l_p(z_1)^k l_p(z_2)^l$$

for $c_{m,n,k,l} \in W$. This is equivalent to (3.18). ∎

For the map $(Y_W^g)_0$, there is a Jacobi identity obtained in [B] and proved in [HY] to be equivalent to the duality property for Y_W^g. Here we reformulate it to obtain a Jacobi identity for the twisted vertex operator map Y_W^g.

Theorem 3.8 *For $u, v \in V$,*

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W^g(u, x_1)Y_W^g(v, x_2) - (-1)^{|u||v|} x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) Y_W^g(v, x_2)Y_W^g(u, x_1) \\ = x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right) Y_W^g\left(Y_V\left(\left(\frac{x_2 + x_0}{x_1}\right)^{\mathcal{L}_g} u, x_0\right) v, x_2\right). \end{aligned} \tag{3.24}$$

Proof. In the case that $u \in V^{[\alpha]}$, we have the Jacobi identity

$$\begin{aligned} &x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) (Y_W^g)_0(u, x_1)(Y_W^g)_0(v, x_2) \\ &\quad - (-1)^{|u||v|} x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) (Y_W^g)_0(v, x_2)(Y_W^g)_0(u, x_1) \\ &\quad = x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right)\left(\frac{x_2 + x_0}{x_1}\right)^{\alpha} (Y_W^g)_0\left(Y_V\left(\left(1 + \frac{x_0}{x_2}\right)^{\mathcal{N}_g} u, x_0\right) v, x_2\right) \end{aligned} \tag{3.25}$$

for $(Y_W^g)_0$. See [HY] for the proof in the case that V is a grading-restricted vertex algebra; in the case that V is a grading-restricted vertex superalgebra, the proof is completely the same. By (3.25) and (3.16), we have

$$
\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_W^g(u,x_1)Y_W^g(v,x_2)w-(-1)^{|u||v|}x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)Y_W^g(v,x_2)Y_W^g(u,x_1)w\\
&\quad=x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\left(\frac{x_2+x_0}{x_1}\right)^{\alpha}(Y_W^g)_0\left(Y_V\left(\left(1+\frac{x_0}{x_2}\right)^{\mathcal{N}_g}x_1^{-\mathcal{N}_g}u,x_0\right)x_2^{-\mathcal{N}_g}v,x_2\right)w\\
&\quad=x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\left(\frac{x_2+x_0}{x_1}\right)^{\alpha}Y_W^g\left(Y_V\left(x_2^{\mathcal{N}_g}\left(1+\frac{x_0}{x_2}\right)^{\mathcal{N}_g}x_1^{-\mathcal{N}_g}u,x_0\right)v,x_2\right)w\\
&\quad=x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\left(\frac{x_2+x_0}{x_1}\right)^{\alpha}Y_W^g\left(Y_V\left(\left(\frac{x_2+x_0}{x_1}\right)^{\mathcal{N}_g}u,x_0\right)v,x_2\right)w. \qquad (3.26)
\end{aligned}
$$

The right-hand side of (3.26) can be rewritten as the right-hand side of (3.24). So (3.24) holds. ∎

Corollary 3.9 *For $u,v\in V$,*

$$
\begin{aligned}
&Y_W^g(u,x_1)Y_W^g(v,x_2)-(-1)^{|u||v|}Y_W^g(v,x_2)Y_W^g(u,x_1)\\
&\quad=\mathrm{Res}_{x_0}x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_W^g\left(Y_V\left(\left(\frac{x_2+x_0}{x_1}\right)^{\mathcal{L}_g}u,x_0\right)v,x_2\right). \qquad (3.27)
\end{aligned}
$$

Next we prove that the product of k twisted vertex operators is absolutely convergent to a multivalued analytic function of a certain form.

Theorem 3.10 *For $v_1\in V^{[\alpha_1]},\dots,v_k\in V^{[\alpha_k]}$, $w\in W$ and $w'\in W'$, the series*

$$\langle w',(Y_W^g)^p(v_1,z_1)\cdots(Y_W^g)^p(v_k,z_k)w\rangle \qquad (3.28)$$

is absolutely convergent in the region $|z_1|>\cdots>|z_k|>0$. Moreover, there exists a multivalued analytic function with preferred branch of the form

$$\sum_{n_1,\dots,n_k=0}^{N}f_{n_1\cdots n_{k+l}}(z_1,\dots,z_k)z_1^{-\alpha_1}\cdots z_k^{-\alpha_k}(\log z_1)^{n_1}\cdots(\log z_k)^{n_k},$$

denoted by

$$F(\langle w',Y_W^g(v_1,z_1)\cdots Y_W^g(v_k,z_k)w\rangle) \qquad (3.29)$$

using our notations introduced in the preceding section, where $N\in\mathbb{N}$ and $f_{i_1\cdots i_k n_1\cdots n_k}(z_1,\dots,z_k)$ for $i_1,\dots,i_k$, $n_1,\cdots,n_k=0,\dots,N$ are rational functions of $z_1,\dots,z_k$ with the only possible poles $z_i=0$ for $i=1,\dots,k$, $z_i-z_j=0$ for $i,j=1,\dots,k$, $i\neq j$, such that the sum of the series (3.28) is equal to the branch

$$
\begin{aligned}
&F^p(\langle w',Y_W^g(v_1,z_1)\cdots Y_W^g(v_k,z_k)w\rangle)\\
&\quad=\sum_{n_1,\dots,n_k=0}^{N}f_{n_1\cdots n_k}(z_1,\dots,z_k)e^{-\alpha_1 l_p(z_1)}\cdots e^{-\alpha_k l_p(z_k)}(l_p(z_1))^{n_1}\cdots(l_p(z_k))^{n_k}, \qquad (3.30)
\end{aligned}
$$

of (3.29) in the region given by $|z_1| > \cdots > |z_k| > 0$*. In addition, the orders of the pole* $z_i = 0$ *of the rational functions* $f_{n_1\cdots n_k}(z_1, \dots, z_k)$ *have a lower bound independent of* v_q *for* $q \neq i$ *and* w'*; the orders of the pole* $z_i = z_j$ *of the rational functions* $f_{i_1\cdots i_k n_1\cdots n_k}(z_1, \dots, z_k)$ *have a lower bound independent of* v_q *for* $q \neq i, j$*,* w *and* w'*.*

Proof. Consider the formal series

$$\prod_{i=l}^{k} x_l^{\alpha_l} \prod_{1\leq i<j\leq k} (x_i - x_j)^{M_{v_i,v_j}} \langle w', Y_W^g(v_1, x_1)\cdots Y_W^g(v_k, x_k)w\rangle. \tag{3.31}$$

For $1 \leq q \leq k$, using (3.18), the series (3.31) is equal to

$$\prod_{i=l}^{k} x_l^{\alpha_l} \prod_{1\leq i<j\leq k} (x_i - x_j)^{M_{v_i,v_j}} (-1)^{|v_q||v_{q+1}|+\cdots+|v_q||v_k|}\cdot$$
$$\cdot \langle w', Y_W^g(v_1, x_1)\cdots Y_W^g(v_{q-1}, x_{q-1})Y_W^g(v_{q+1}, x_{q+1})\cdots Y_W^g(v_k, x_k)Y_W^g(v_q, x_q)w\rangle. \tag{3.32}$$

Since by Remark 3.2, $x_q^{\alpha_q} Y_W^g(v_q, x_q)w$ is a Laurent series in x^q with only finitely many negative power terms, (3.32) is also a Laurent series in x_q with only finitely many negative power terms. So the same is true for (3.31). On the other hand, using (3.7) again, (3.31) is equal to

$$\prod_{i=l}^{k} x_l^{\alpha_l} \prod_{1\leq i<j\leq k} (x_i - x_j)^{M_{v_i,v_j}} (-1)^{|v_q||v_1|+\cdots+|v_q||v_{q-1}|}\cdot$$
$$\cdot \langle w', Y_W^g(v_q, x_q)Y_W^g(v_1, x_1)\cdots Y_W^g(v_{q-1}, x_{q-1})Y_W^g(v_{q+1}, x_{q+1})\cdots Y_W^g(v_k, x_k)w\rangle. \tag{3.33}$$

Since by Remark 3.2, $x_q^{\alpha_q}\langle w', Y_W^g(v_q, x_q)\cdot\rangle$ is a Laurent series in x_q with only finitely many positive power terms, (3.33) is also Laurent series in x_q with only finitely many positive power terms. So the same is true for (3.31). Thus (3.31) must be a Laurent polynomial in x_q with polynomials in $\log x_q$ as coefficients, or equivalently, a polynomial in $\log x_q$ with Laurent polynomials in x_q as coefficients. Since this is true for $q = 1, \dots, k$, (3.31) is a polynomial in $\log x_1, \dots, \log x_k$ with Laurent polynomials in $x_1, \dots, x_k$ as coefficients.

On the other hand, by Remark 3.2, we have

$$\prod_{i=l}^{k} x_l^{-\alpha_l} \langle w', Y_W^g(v_1, x_1)\cdots Y_W^g(v_k, x_k)w\rangle \in \mathbb{C}((x_1))\cdots((x_k)), \tag{3.34}$$

where as usual, for a ring R and a formal variable x, we use $R((x))$ to denote the ring of Laurent series in x with coefficients in R and finitely many negative power terms. Since $\mathbb{C}((x_1))\cdots((x_k))$ is a ring and $\prod_{1\leq i<j\leq k}(x_i - x_j)^{-M_{v_i,v_j}}$ is in fact in this ring, $\prod_{1\leq i<j\leq k}(x_i - x_j)^{M_{v_i,v_j}}$ is invertible in this ring with the inverse $\prod_{1\leq i<j\leq k}(x_i - x_j)^{-M_{v_i,v_j}}$. Since (3.31) is a polynomial in $\log x_1, \dots, \log x_k$ with Laurent polynomials in $x_1, \dots, x_k$ as coefficients, it is

also in this ring. Therefore (3.34) is equal to the product of $\prod_{1\le i<j\le k}(x_i-x_j)^{-M_{v_i,v_j}}$ and a polynomial in $\log x_1,\dots,\log x_k$ with Laurent polynomials in $x_1,\dots,x_k$ as coefficients. So

$$\langle w', Y_W^g(v_1,x_1)\cdots Y_W^g(v_k,x_k)w\rangle\in\mathbb{C}((x_1))\cdots((x_k))$$

is equal to the product of $\prod_{i=l}^k x_l^{-\alpha_l}\prod_{1\le i<j\le k}(x_i-x_j)^{-M_{v_i,v_j}}$ and a polynomial in $\log x_1,\dots,\log x_k$ with Laurent polynomials in $x_1,\dots,x_k$ as coefficients. Thus (3.28) is absolutely convergent in the region $|z_1|>\cdots>|z_k|>0$ to an analytic function of the form (3.30).

The properties of the function (3.28) follow from Remark 3.2 and the duality property. ∎

Corollary 3.11 *For $v_1,\dots,v_k\in V$, $w\in W$, $w'\in W'$, $p\in\mathbb{Z}$ and $\sigma\in S_k$,*

$$F^p(\langle w', Y_W^g(v_1,z_1)\cdots Y_W^g(v_k,z_k)w\rangle)=\pm F^p(\langle w', Y_W^g(v_{\sigma(1)},z_{\sigma(1)})\cdots Y_W^g(v_{\sigma(k)},z_{\sigma(k)})w\rangle),$$

where the sign $\pm$ is uniquely determined by $|v_1,\dots,|v_k|$ and σ.

Proof. This result follows immediately from Theorem 3.10 and the duality property. ∎

4 Twist vertex operators

We introduce twist vertex operators and prove the main results on twist vertex operators in this section.

Recall that W is a lower-bounded generalized g-twisted V-module. We first introduce twist vertex operators. Let

$$\begin{aligned}(Y^g)_{WV}^W: W\otimes V&\to W\{x\}[\log x],\\ w\otimes v&\mapsto (Y^g)_{WV}^W(w,x)v=\sum_{n\in\mathbb{C}}\sum_{k\in\mathbb{N}}((Y^g)_{WV}^W)_{n,k}x^n(\log x)^k\end{aligned}$$

be defined by

$$(Y^g)_{WV}^W(w,x)v=(-1)^{|v||w|}e^{xL_W(-1)}Y_W^g(v,y)w\Big|_{y^n=e^{\pi ni}x^n,\ \log y=\log x+\pi i}$$

for $v\in V$ and $w\in W$. In particular, for $p\in\mathbb{Z}$, we have

$$\begin{aligned}((Y^g)_{WV}^W)^p(w,z)v&=(-1)^{|v||w|}e^{zL_W(-1)}Y_W^g(v,y)w\Big|_{y^n=e^{n(l_p(z)+\pi i)},\ \log y=l_p(z)+\pi i}\\&=\begin{cases}(-1)^{|v||w|}e^{zL_W(-1)}(Y_W^g)^p(v,-z)w & 0\le\arg z<\pi,\\ (-1)^{|v||w|}e^{zL_W(-1)}(Y_W^g)^{p+1}(v,-z)w & \pi\le\arg z<2\pi.\end{cases}\end{aligned}\tag{4.1}$$

We shall call $(Y^g)_{WV}^W(w,x)$ a *twist vertex operator from V to W* and $(Y^g)_{WV}^W$ a *twist vertex operator map of type $\binom{W}{WV}$*.

Note that though the definitions and results in [H4] are given for a vertex operator algebras, they can be adapted to give definitions and results for grading-restricted vertex superalgebras by adding appropriate signs when two elements change their order. In particular, it is easy to see from the definition of generalized twisted module that the twisted vertex operator map Y_W^g is a twisted intertwining operator of type $\binom{W}{VW}$. By the definition of $(Y^g)_{WV}^W$ above and the definition of Ω_+ in Section 5 of [H4], we have

$$(Y^g)_{WV}^W = \Omega_+(Y_W^g).$$

In fact there is another twisted intertwining operator $\Omega_-(Y_W^g)$ of the same type. But we shall not use it in this paper. From the properties of Y_W^g and Section 5 of [H4], we obtain the following result:

Theorem 4.1 *The linear map $(Y^g)_{WV}^W$ is a twisted intertwining operator of type $\binom{W}{WV}$. In particular, we have the following properties of $(Y^g)_{WV}^W$:*

1. *The lower truncation property: For $w \in W$ and $v \in V$, $(Y^g)_{WV}^W(w,x)v$ has only finitely many terms involving x^n for $n \in \mathbb{C}$ with $\Re(n) < 0$ and $(\log x)^m$ for $m \in \mathbb{N}$.*

2. *The duality property: For $u, v \in V$, $w \in W$ and $w' \in W'$, the series*

$$\langle w', (Y_W^g)^{p_1}(u, z_1)((Y^g)_{WV}^W)^{p_2}(w, z_2)v\rangle, \tag{4.2}$$
$$(-1)^{|u||w|}\langle w', ((Y^g)_{WV}^W)^{p_2}(w, z_2)Y_V(u, z_1)v\rangle, \tag{4.3}$$
$$\langle w', ((Y^g)_{WV}^W)^{p_2}((Y_W^g)^{p_1}(u, z_1 - z_2)w, z_2)v\rangle\rangle \tag{4.4}$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively. Moreover, there exists a multivalued analytic function with preferred branch

$$f(z_1, z_2; u, w, v, w') = \sum_{j,k,m,n=0}^{N} a_{jkmn} z_1^r z_2^{s_j}(z_1 - z_2)^{t_k}(\log z_2)^m(\log(z_1 - z_2))^n \tag{4.5}$$

for $N \in \mathbb{N}$, $r \in -\mathbb{N}$, $s_j, t_k, a_{ijklmn} \in \mathbb{C}$ such that for $p_1, p_2 \in \mathbb{Z}$, their sums are equal to the branches

$$\begin{aligned} &f^{p_1,p_2,p_1}(z_1, z_2; u, w, v, w') \\ &\quad = \sum_{j,k,m,n=0}^{N} a_{jkmn} z_1^r e^{s_j l_{p_2}(z_2)} e^{t_k l_{p_1}(z_1 - z_2)}(l_{p_2}(z_2))^m(l_{p_1}(z_1 - z_2))^n, \end{aligned} \tag{4.6}$$

$$\begin{aligned} &f^{p_1,p_2,p_2}(z_1, z_2; u, w, v, w') \\ &\quad = \sum_{j,k,m,n=0}^{N} a_{jkmn} z_1^r e^{s_j l_{p_2}(z_2)} e^{t_k l_{p_2}(z_1 - z_2)}(l_{p_2}(z_2))^m(l_{p_2}(z_1 - z_2))^n, \end{aligned} \tag{4.7}$$

$$\begin{aligned} &f^{p_2,p_2,p_1}(z_1, z_2; u, w, v, w') \\ &\quad = \sum_{j,k,m,n=0}^{N} a_{jkmn} z_1^r e^{s_j l_{p_2}(z_2)} e^{t_k l_{p_1}(z_1 - z_2)}(l_{p_2}(z_2))^m(l_{p_1}(z_1 - z_2))^n, \end{aligned} \tag{4.8}$$

respectively, of $f(z_1,z_2;u,w_1,w_2,w_3')$ *in the region given by* $|z_1|>|z_2|>0$ *and* $|\arg(z_1-z_2)-\arg z_1|<\frac{\pi}{2}$*, the region given by* $|z_2|>|z_1|>0$ *and* $-\frac{3\pi}{2}<\arg(z_1-z_2)-\arg z_2<-\frac{\pi}{2}$*, the region given by* $|z_2|>|z_1-z_2|>0$*, respectively. In addition, when* $u\in V^{[\alpha]}$ *and* $v\in V^{[\beta]}$*, we can always take* $a_{ijkl}=0$ *for* $j,k\neq 0$*,* $s_0=-\beta$ *and* $t_0=-\alpha$*.*

3. *The* $L(-1)$*-derivative property and* $L(-1)$*-commutator formula:*

$$\begin{aligned}\frac{d}{dx}(Y^g)_{WV}^W(w,x) &= (Y^g)_{WV}^W(L_W(-1)w,x)\\ &= L_W(-1)(Y^g)_{WV}^W(w,x)-(Y^g)_{WV}^W(w,x)L_V(-1).\end{aligned}$$

Proof. Note that Theorem 5.1 in [H4] also holds for twisted intertwining operators for grading-restricted vertex superalgebras by adding appropriate signs. By this theorem, the map $(Y^g)_{WV}^W$ is a twisted intertwining operator of type $\binom{W}{WV}$. By the definition of twisted intertwining operator, $(Y^g)_{WV}^W$ have the lower truncation property, the $L(-1)$-derivative property and the and $L(-1)$-commutator formula.

By the duality property for twisted intertwining operators, (4.2), (4.3) and (4.4) are absolutely convergent in the regions $|z_1|>|z_2|>0$, $|z_2|>|z_1|>0$, $|z_2|>|z_1-z_2|>0$, respectively. Moreover there exists a multivalued analytic function

$$\begin{aligned}&f(z_1,z_2;u,w,v,w')\\ &\quad=\sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn}z_1^{r_i}z_2^{s_j}(z_1-z_2)^{t_k}(\log z_1)^l(\log z_2)^m(\log(z_1-z_2))^n\end{aligned}$$

for $N\in\mathbb{N}$, $r_i,s_j,t_k,a_{ijklmn}\in\mathbb{C}$, such that (4.2), (4.3) and (4.4) are equal to the branches

$$\begin{aligned}&f^{p_1,p_2,p_1}(z_1,z_2;u,w,v,w')\\ &\quad=\sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn}e^{r_il_{p_1}(z_1)}e^{s_jl_{p_2}(z_2)}e^{t_kl_{p_1}(z_1-z_2)}(l_{p_1}(z_1))^l(l_{p_2}(z_2))^m(l_{p_1}(z_1-z_2))^n,\\ &f^{p_1,p_2,p_2}(z_1,z_2;u,w,v,w')\\ &\quad=\sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn}e^{r_il_{p_1}(z_1)}e^{s_jl_{p_2}(z_2)}e^{t_kl_{p_2}(z_1-z_2)}(l_{p_1}(z_1))^l(l_{p_2}(z_2))^m(l_{p_2}(z_1-z_2))^n,\\ &f^{p_2,p_2,p_1}(z_1,z_2;u,w,v,w')\\ &\quad=\sum_{i,j,k,l,m,n=0}^{N} a_{ijklmn}e^{r_il_{p_2}(z_1)}e^{s_jl_{p_2}(z_2)}e^{t_kl_{p_1}(z_1-z_2)}(l_{p_2}(z_1))^l(l_{p_2}(z_2))^m(l_{p_1}(z_1-z_2))^n,\end{aligned}$$

respectively, of $f(z_1,z_2;u,w,v,w')$ in the region given by $|z_1|>|z_2|>0$ and $|\arg(z_1-z_2)-\arg z_1|<\frac{\pi}{2}$, the region given by $|z_2|>|z_1|>0$ and $-\frac{3\pi}{2}<\arg(z_1-z_2)-\arg z_2<-\frac{\pi}{2}$, the region given by $|z_2|>|z_1-z_2|>0$ and $|\arg z_1-\arg z_2|<\frac{\pi}{2}$, respectively. Since (4.3) contains only integral powers of z_1, $f^{p_1,p_2,p_2}(z_1,z_2;u,w,v,w')$ must be independent of p_1 and of the form of the right-hand side of (4.7). Thus $f(z_1, z_2; u, w, v, w')$ must be of the form of (4.5). In particular, (4.2) and (4.4) must be of the forms of (4.6) and (4.8), respectively.

When $u \in V^{[\alpha]}$, by (3.6),

$$(Y_W^g)^{p_1}(u, z_1 - z_2)w = \sum_{k=0}^{N} \sum_{n\in\alpha+\mathbb{Z}} (Y_W^g)_{n,k}(u)we^{(-n-1)l_{p_1}(z_1-z_2)}(l_{p_1}(z_1 - z_2))^k.$$

When $v \in V^{[\beta]}$, by the definition of $(Y^g)_{WV}^W$ and (3.6), we have

$$\begin{aligned}
&((Y^g)_{WV}^W)^{p_2}(w, z_2)v \\
&\quad = (-1)^{|v||w|}e^{z_2 L_W(-1)}Y_W^g(v, y)w\Big|_{y^n=e^{n(l_{p_2}(z_2)+\pi i)},\ \log y=l_{p_2}(z_2)+\pi i} \\
&\quad = (-1)^{|v||w|}e^{z_2 L_W(-1)} \sum_{l=0}^{M} \sum_{m\in\beta+\mathbb{Z}} (Y_W^g)_{m,l}(v)we^{(-m-1)(l_{p_2}(z_2)+\pi i)}(l_{p_2}(z_2) + \pi i)^l
\end{aligned}$$

Since (4.4) and (4.2) are convergent absolutely to $f^{p_2,p_2,p_1}(z_1, z_2; u, w, v, w')$ in the region given by $|z_2| > |z_1 - z_2| > 0$ and $|z_1| > |z_2| > 0$, respectively, we can always choose the coefficients a_{ijkl} for $i, j, k, l = 0, \ldots, N$ such that $a_{ijkl} = 0$ for $j, k \neq 0$, $s_0 = -\beta$ and $t_0 = -\alpha$. ∎

Remark 4.2 Using the definition of $(Y^g)_{WV}^W$ and Remark 3.2 we have the following stronger property than the lower-truncation property: For $w \in W$ and $v \in V^{[\alpha]}$, there exist $N \in \mathbb{N}$ and $M \in \mathbb{Z}$ such that

$$(Y^g)_{WV}^W(w, x)v = \sum_{k=1}^{N} \sum_{n\in\alpha+M-\mathbb{N}} ((Y^g)_{WV}^W)_{n,k}(w)vx^{-n-1}(\log x)^k.$$

For $w' \in W'$ and $w \in W$, $\langle w', (Y^g)_{WV}^W(w, x)\cdot\rangle$ has only finitely many terms involving x^n for $n \in \mathbb{C}$ with $\Re(n) > 0$ and $(\log x)^m$ for $m \in \mathbb{N}$.

We shall use our notations introduced in Section 2 to denote the multivalued analytic function $f(z_1, z_2; u, w, v, w')$ with preferred branch in Theorem 4.1 by

$$F(\langle w', Y_W^g(u, z_1)(Y^g)_{WV}^W(w, z_2)v\rangle)$$

or

$$F((-1)^{|u||w|}\langle w', (Y^g)_{WV}^W(w, z_2)Y_V(u, z_1)v\rangle)$$

or

$$F(\langle w', (Y^g)_{WV}^W(Y_W^g(u, z_1 - z_2)w, z_2)v\rangle).$$

We shall also denote the branches (4.6), (4.7), (4.8) of $f(z_1, z_2; u, w, v, w')$ when $p_1 = p_2 = p$ by

$$F^p(\langle w', Y_W^g(u, z_1)(Y^g)_{WV}^W(w, z_2)v\rangle),$$
$$F^p((-1)^{|u||w|}\langle w', (Y^g)_{WV}^W(w, z_2)Y_V(u, z_1)v\rangle),$$
$$F^p(\langle w', (Y^g)_{WV}^W(Y_W^g(u, z_1 - z_2)w, z_2)v\rangle),$$

respectively. Then in particular, we have the following commutativity for twisted and twist vertex operators:

Corollary 4.3 *For $u, v \in V$, $w \in W$ and $w' \in W'$, we have*

$$F^p(\langle w', Y_W^g(u, z_1)(Y^g)_{WV}^W(w, z_2)v\rangle) = (-1)^{|u||w|}F^p(\langle w', (Y^g)_{WV}^W(w, z_2)Y_V(u, z_1)v\rangle). \quad (4.9)$$

We also have the following formal weak associativity property involving twist vertex operators:

Proposition 4.4 *For $u, v \in V$,*

$$(x_0+x_2)^{M_{u,v}}Y_W^g(u, x_0+x_2)(Y^g)_{WV}^W(w, x_2)v = (x_0+x_2)^{M_{u,v}}(Y^g)_{WV}^W(Y_W^g(u, x_0)w, x_2)v, \quad (4.10)$$

where $M_{u,v}$ is the positive integer in Proposition 3.7

Proof. We take $u, v \in V$ and $w \in W$. Then by the definition of $(Y^g)_{WV}^W$, the $L(-1)$-commutator and $L(-1)$-derivative properties, Proposition 3.7 and the $\mathbb{Z}_2$-fermion number compatibility condition,

$$\begin{aligned}
&(x_0 + x_2)^{M_{u,v}}Y_W^g(u, x_0 + x_2)(Y^g)_{WV}^W(w, x_2)v\\
&\quad = (x_0 + x_2)^{M_{u,v}}Y_W^g(u, x_0 + x_2)(-1)^{|v||w|}Y_W^g(v, y)w\Big|_{y^n=e^{\pi ni}x_2^n,\ \log y=\log x_2+\pi i}\\
&\quad = (-1)^{|v||w|}e^{x_2L_W(-1)}(x_0 - y)^{M_{u,v}}Y_W^g(u, x_0)Y_W^g(v, y)w\Big|_{y^n=e^{\pi ni}x_2^n,\ \log y=\log x_2+\pi i}\\
&\quad = (-1)^{|v||w|}(-1)^{|v||u|}e^{x_2L_W(-1)}(x_0 - y)^{M_{u,v}}Y_W^g(v, y)Y_W^g(u, x_0)w\Big|_{y^n=e^{\pi ni}x_2^n,\ \log y=\log x_2+\pi i}\\
&\quad = (x_0 - y)^{M_{u,v}}(-1)^{|v||Y_W^g(u,x_0)w|}e^{x_2L_W(-1)}Y_W^g(v, y)Y_W^g(u, x_0)w\Big|_{y^n=e^{\pi ni}x_2^n,\ \log y=\log x_2+\pi i}\\
&\quad = (x_0 + x_2)^{M_{u,v}}(Y^g)_{WV}^W(Y_W^g(u, x_0)w, x_2)v.
\end{aligned}$$

∎

Recall (3.16) (see Lemma 2.3 in [HY]). We have a similar result for $(Y^g)_{WV}^W$. For $w \in W$, let

$$((Y^g)_{WV}^W)_0(w, x) = (Y^g)_{WV}^W(w, x)x^{\mathcal{N}_g}.$$

Then we have:

Lemma 4.5 *For $w \in W$, $((Y^g)^W_{WV})_0(w,x)v \in W\{x\}$ and*

$$(Y^g)^W_{WV}(w,x) = ((Y^g)^W_{WV})_0(w,x)x^{-\mathcal{N}_g}. \tag{4.11}$$

In particular, $((Y^g)^W_{WV})_0(w,x)$ is the constant term of $(Y^g)^W_{WV}(w,x)$ viewed as a power series in $\log x$.

Proof. By definition, for $v \in V$ and $w \in W$,

$$\begin{aligned}(Y^g)^W_{WV}(w,x)v &= (-1)^{|v||w|}e^{xL_W(-1)}Y^g_W(v,y)w\Big|_{y^n=e^{\pi ni}x^n,\ \log y=\log x+\pi i}\\ &= (-1)^{|v||w|}e^{xL_W(-1)}(Y^g_W)_0(y^{-\mathcal{N}_g}v,y)w\Big|_{y^n=e^{\pi ni}x^n,\ \log y=\log x+\pi i}\\ &= (-1)^{|v||w|}e^{xL_W(-1)}(Y^g_W)_0(e^{-\log y\mathcal{N}_g}v,y)w\Big|_{y^n=e^{\pi ni}x^n,\ \log y=\log x+\pi i}\\ &= (-1)^{|v||w|}e^{xL_W(-1)}(Y^g_W)_0(x^{-\mathcal{N}_g}e^{-\pi i\mathcal{N}_g}v,y)w\Big|_{y^n=e^{\pi ni}x^n}. \end{aligned} \tag{4.12}$$

Replacing v in (4.12) by $x^{\mathcal{N}_g}$, we obtain

$$(Y^g)^W_{WV}(w,x)x^{\mathcal{N}_g}v = (-1)^{|v||w|}e^{xL_W(-1)}(Y^g_W)_0(e^{-\pi i\mathcal{N}_g}v,y)w\Big|_{y^n=e^{\pi ni}x^n}. \tag{4.13}$$

From (4.13) and the fact that $(Y^g_W)_0(v,x)w \in W\{x\}$, we see that $((Y^g)^W_{WV})_0(w,x)v \in W\{x\}$ and (4.11) holds. ∎

We now prove a Jacobi identity involving $(Y^g)^W_{WV}$. In the results below, for simplicity, we shall always use the convention that for any operator or number A,

$$(-x_2+x_1)^A = (x_2-x_1)^A e^{\pi iA}. \tag{4.14}$$

Theorem 4.6 *For $u, v \in V$ and $w \in W$, we have*

$$\begin{aligned}&{x_0}^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y^g_W\left(\left(\frac{x_1-x_2}{x_0}\right)^{\mathcal{L}_g}u,x_1\right)(Y^g)^W_{WV}(w,x_2)v\\ &\quad -(-1)^{|u||w|}x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)(Y^g)^W_{WV}(w,x_2)Y_V\left(\left(\frac{-x_2+x_1}{x_0}\right)^{\mathcal{L}_g}u,x_1\right)v\\ &= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)(Y^g)^W_{WV}\left(Y^g_W\left(u,x_0\right)w,x_2\right)v. \end{aligned} \tag{4.15}$$

Proof. This Jacobi identity can be proved by directly using the duality property in Theorem 4.1. Here we give a proof using the Jacobi identity (3.24).

For $u \in V^{[\alpha]}$ and $v \in V$, (3.26) holds. Using the definition of $(Y^g)_{WV}^W$ and (3.26), we obtain

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1+x_2}{x_0}\right)\left(\frac{x_1+x_2}{x_0}\right)^{-\alpha} Y_W^g\left(\left(\frac{x_1+x_2}{x_0}\right)^{-\mathcal{N}_g} u, x_0\right)(Y^g)_{WV}^W(w,x_2)v \\
&\quad - x_0^{-1}\delta\left(\frac{x_2+x_1}{x_0}\right)(Y^g)_{WV}^W(Y_W^g(u,x_1)w,x_2)v \\
&\quad = x_0^{-1}\delta\left(\frac{x_1+x_2}{x_0}\right)\left(\frac{x_1+x_2}{x_0}\right)^{-\alpha}(Y_W^g)_0\left(x_0^{-\mathcal{N}_g}\left(\frac{x_1+x_2}{x_0}\right)^{-\mathcal{N}_g} u, x_0\right)(Y^g)_{WV}^W(w,x_2)v \\
&\quad\quad - x_0^{-1}\delta\left(\frac{x_2+x_1}{x_0}\right)(Y^g)_{WV}^W(Y_W^g(u,x_1)w,x_2)v \\
&\quad = x_0^{-1}\delta\left(\frac{x_1+x_2}{x_0}\right)(Y_W^g)_0((x_1+x_2)^{-\mathcal{N}_g}u, x_1+x_2)(Y^g)_{WV}^W(w,x_2)v \\
&\quad\quad - x_0^{-1}\delta\left(\frac{x_2+x_1}{x_0}\right)(Y^g)_{WV}^W(Y_W^g(u,x_1)w,x_2)v \\
&\quad = x_0^{-1}\delta\left(\frac{x_1+x_2}{x_0}\right)Y_W^g(u,x_1+x_2)(Y^g)_{WV}^W(w,x_2)v \\
&\quad\quad - x_0^{-1}\delta\left(\frac{x_2+x_1}{x_0}\right)(Y^g)_{WV}^W(Y_W^g(u,x_1)w,x_2)v \\
&\quad = (-1)^{|v||w|}x_0^{-1}\delta\left(\frac{x_1+x_2}{x_0}\right)Y_W^g(u,x_1+x_2)e^{x_2L_W^g(-1)}Y_W^g(v,y)w\Big|_{y^n=e^{\pi n i}x^n,\ \log y=\log x+\pi i} \\
&\quad\quad - (-1)^{(|u|+|w|)|v|}x_0^{-1}\delta\left(\frac{x_2+x_1}{x_0}\right)e^{x_2L_W^g(-1)}Y_W^g(v,y)Y_W^g(u,x_1)w\Big|_{y^n=e^{\pi n i}x^n,\ \log y=\log x+\pi i} \\
&\quad = (-1)^{|v||w|}e^{x_2L_W^g(-1)}\Bigg(x_0^{-1}\delta\left(\frac{x_1-y}{x_0}\right)Y_W^g(u,x_1)Y_W^g(v,y)w \\
&\quad\quad\quad -(-1)^{|u||v|}x_0^{-1}\delta\left(\frac{-y+x_1}{x_0}\right)Y_W^g(v,y)Y_W^g(u,x_1)w\Bigg)\Bigg|_{y^n=e^{\pi n i}x^n,\ \log y=\log x+\pi i} \\
&\quad = (-1)^{|v||w|}x_1^{-1}\delta\left(\frac{y+x_0}{x_1}\right)\left(\frac{y+x_0}{x_1}\right)^{\alpha}e^{x_2L_W^g(-1)}\cdot \\
&\quad\quad\quad \cdot Y_W^g\left(Y_V\left(\left(\frac{y+x_0}{x_1}\right)^{\mathcal{N}_g}u,x_0\right)v,y\right)w\Big|_{y^n=e^{\pi n i}x^n,\ \log y=\log x+\pi i} \\
&\quad = (-1)^{|u||w|}x_1^{-1}\delta\left(\frac{-x_2+x_0}{x_1}\right)\left(\frac{-x_2+x_0}{x_1}\right)^{\alpha}\cdot \\
&\quad\quad\quad \cdot (Y^g)_{WV}^W(w,x_2)Y_V\left(\left(\frac{-x_2+x_0}{x_1}\right)^{\mathcal{N}_g}u,x_0\right)v,
\end{aligned} \tag{4.16}$$

where we have used our convention (4.14).

From (4.16), we obtain

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\left(\frac{x_0+x_2}{x_1}\right)^{-\alpha}Y_W^g\left(\left(\frac{x_0+x_2}{x_1}\right)^{-\mathcal{N}_g}u,x_1\right)(Y^g)_{WV}^W(w,x_2)v$$
$$-(-1)^{|u||w|}x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\left(\frac{-x_2+x_1}{x_0}\right)^{\alpha}\cdot$$
$$\cdot(Y^g)_{WV}^W(w,x_2)Y_V\left(\left(\frac{-x_2+x_1}{x_0}\right)^{\mathcal{N}_g}u,x_1\right)v$$
$$=x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)(Y^g)_{WV}^W\left(Y_W^g\left(u,x_0\right)w,x_2\right)v. \tag{4.17}$$

Using the property of the formal δ-function, we see that the first term in the left-hand side of (4.17) is equal to

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)x_0^{-\alpha}\left(\frac{1+\frac{x_2}{x_1-x_2}}{x_1}\right)^{-\alpha}\cdot$$
$$\cdot Y_W^g\left(x_0^{-\mathcal{N}_g}\left(\frac{1+\frac{x_2}{x_1-x_2}}{x_1}\right)^{-\mathcal{N}_g}u,x_1\right)(Y^g)_{WV}^W(w,x_2)v. \tag{4.18}$$

For $a\in\mathbb{C}$, we have

$$\begin{aligned}\left(\frac{1+\frac{x_2}{x_1-x_2}}{x_1}\right)^{-\alpha}&=(x_1-x_2)^{\alpha}(x_1-x_2)^{-\alpha}\left(\frac{1+\frac{x_2}{x_1-x_2}}{x_1}\right)^{-\alpha}\\&=(x_1-x_2)^{\alpha}\left(1-\frac{x_2}{x_1}\right)^{-\alpha}\left(1+\frac{x_2}{x_1-x_2}\right)^{-\alpha}\\&=(x_1-x_2)^{\alpha}\left(\left(1-\frac{x_2}{x_1}\right)\left(1+\frac{x_2}{x_1}\left(1-\frac{x_2}{x_1}\right)^{-1}\right)\right)^{-\alpha}\\&=(x_1-x_2)^{\alpha}.\end{aligned} \tag{4.19}$$

Similarly, we have

$$\begin{aligned}\left(\frac{1+\frac{x_2}{x_1-x_2}}{x_1}\right)^{-\mathcal{N}_g}&=x^{\mathcal{N}_g}\left(1+\frac{x_2}{x_1-x_2}\right)^{-\mathcal{N}_g}\\&=x^{\mathcal{N}_g}\left(1-\frac{x_2}{x_1}\right)^{\mathcal{N}_g}\left(1-\frac{x_2}{x_1}\right)^{-\mathcal{N}_g}\left(1+\frac{x_2}{x_1}\left(1-\frac{x_2}{x_1}\right)^{-1}\right)^{-\mathcal{N}_g}\end{aligned}$$

$$= (x_1 - x_2)^{\mathcal{N}_g}\left(\left(1-\frac{x_2}{x_1}\right)\left(1+\frac{x_2}{x_1}\left(1-\frac{x_2}{x_1}\right)^{-1}\right)\right)^{-\mathcal{N}_g}$$
$$= (x_1 - x_2)^{\mathcal{N}_g}. \tag{4.20}$$

Using (4.18), (4.19) and (4.20), we see that (4.17) becomes

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\left(\frac{x_1-x_2}{x_0}\right)^{\alpha} Y_W^g\left(\left(\frac{x_1-x_2}{x_0}\right)^{\mathcal{N}_g} u, x_1\right)(Y^g)_{WV}^W(w,x_2)v$$
$$- (-1)^{|u||w|}x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\left(\frac{-x_2+x_1}{x_0}\right)^{\alpha}\cdot$$
$$\cdot (Y^g)_{WV}^W(w,x_2)Y_V\left(\left(\frac{-x_2+x_1}{x_0}\right)^{\mathcal{N}_g} u, x_1\right)v$$
$$= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)(Y^g)_{WV}^W(Y_W^g(u,x_0)w,x_2)v. \tag{4.21}$$

Using our convention (4.14), we see that the identity (4.21) is (4.15) in the case that $u \in V^{[\alpha]}$.
∎

There are many important and useful consequences of the Jacobi identity. For example, by taking Res_{x_1} on both sides of the Jacobi identity (4.15), we obtain an iterate formula for $(Y^g)_{WV}^W$. From this iterate formula, we obtain another generalized weak associativity formula involving the iterate $(Y^g)_{WV}^W((Y^g)_{WV}^W(w,x_0)u,x_2)v$. We shall not give all such consequences in this paper. We give only the following generalized commutator formula and generalized weak commutativity which will be used as the main assumptions on twist fields in our construction of lower-bounded generalized twisted modules in [H5] and will play an important role in our proof below of the convergence of products of more than one twisted vertex operators and vertex operators for V and one twist vertex operator:

Corollary 4.7 *For $u \in V, v \in V$ and $w \in W$, we have:*

1. *The generalized commutator formula:*

$$Y_W^g((x_1-x_2)^{\mathcal{L}_g}u,x_1)(Y^g)_{WV}^W(w,x_2)v$$
$$- (-1)^{|u||w|}(Y^g)_{WV}^W(w,x_2)Y_V((-x_2+x_1)^{\mathcal{L}_g}u,x_1)v$$
$$= \mathrm{Res}_{x_0} x_0^{\alpha} x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)(Y^g)_{WV}^W((Y_W^g)_0(u,x_0)w,x_2)v. \tag{4.22}$$

2. *The generalized weak commutativity: For $M_{u,w} \in \mathbb{Z}_+$ such that $x^{\alpha+M_{u,w}}(Y_W^g)_0(u,x)w \in W[[x]]$,*

$$(x_1-x_2)^{M_{u,w}}Y_W^g((x_1-x_2)^{\mathcal{S}_g+\mathcal{N}_g}u,x_1)(Y^g)_{WV}^W(w,x_2)v$$
$$= (-1)^{|u||w|}(x_1-x_2)^{M_{u,w}}(Y^g)_{WV}^W(w,x_2)Y_V((-x_2+x_1)^{\mathcal{S}_g+\mathcal{N}_g}u,x_1)v. \tag{4.23}$$

Proof. Let $u \in V^{[\alpha]}$. Then the Jacobi identity (4.15) becomes (4.21). Multiplying both sides of (4.21) by $x_0^{\alpha+m}$ for $m \in \mathbb{Z}_+$, we see that the two sides contains only integral powers of x_0. Then applying Res_{x_0} to both sides and using (3.16), we obtain

$$\begin{aligned}
&(x_1 - x_2)^{\alpha+m} Y_W^g \left(\left(\frac{x_1 - x_2}{x_0} \right)^{\mathcal{N}_g} u, x_1 \right) (Y^g)_{WV}^W (w, x_2) v \\
&\quad - (-1)^{|u||w|} (-x_2 + x_1)^{\alpha+m} (Y^g)_{WV}^W (w, x_2) Y_V \left(\left(\frac{-x_2 + x_1}{x_0} \right)^{\mathcal{N}_g} u, x_1 \right) v \\
&= \mathrm{Res}_{x_0} x_0^{\alpha+m} x_1^{-1} \delta \left(\frac{x_2 + x_0}{x_1} \right) (Y^g)_{WV}^W (Y_W^g (u, x_0) w, x_2) v \\
&= \mathrm{Res}_{x_0} x_0^{\alpha+m} x_1^{-1} \delta \left(\frac{x_2 + x_0}{x_1} \right) (Y^g)_{WV}^W ((Y_W^g)_0 (x_0^{-\mathcal{N}_g} u, x_0) w, x_2) v. \qquad (4.24)
\end{aligned}$$

Taking u in (4.24) to be $x_0^{\mathcal{N}_g} u$, we see that (4.24) becomes

$$\begin{aligned}
&(x_1 - x_2)^{\alpha+m} Y_W^g ((x_1 - x_2)^{\mathcal{N}_g} u, x_1) (Y^g)_{WV}^W (w, x_2) v \\
&\quad - (-1)^{|u||w|} (-x_2 + x_1)^{\alpha+m} (Y^g)_{WV}^W (w, x_2) Y_V ((-x_2 + x_1)^{\mathcal{N}_g} u, x_1) v \\
&= \mathrm{Res}_{x_0} x_0^{\alpha+m} x_1^{-1} \delta \left(\frac{x_2 + x_0}{x_1} \right) (Y^g)_{WV}^W ((Y_W^g)_0 (u, x_0) w, x_2) v. \qquad (4.25)
\end{aligned}$$

Taking $m = 0$ in (4.25), we obtain (4.22) in the case $u \in V^{[\alpha]}$.

Since $x_0^{\alpha+M_{u,w}} (Y_W^g)_0 (u, x_0) w$ is a power series in x_0, for $m = M_{u,w}$, the right-hand side of (4.25) is equal to 0. Thus the left-hand side of (4.25) is equal to 0 when $m = M_{u,w}$, proving (4.23). ∎

The generalized commutator formula and generalized weak commutativity above are formulated in terms of $Y_W^g((x_1 - x_2)^{\mathcal{L}_g} u, x_1)$ and $Y_V((-x_2 + x_1)^{\mathcal{L}_g} u, x_1)$. In the construction of lower-bounded generalized twisted modules in [H5], we need a generalized commutator formula and generalized weak commutativity expressed in terms of twisted fields and untwisted fields without $(x_1 - x_2)^{\mathcal{L}_g}$ and $(-x_2 + x_1)^{\mathcal{L}_g}$. Here we rewrite the generalized commutator formula and generalized weak commutativity above as follows:

Corollary 4.8 *For $u \in V^{[\alpha]}$, $v \in V$ and $w \in W$, the generalized commutator formula (4.22) and generalized weak commutativity (4.22) can be rewritten as*

$$\begin{aligned}
&(x_1 - x_2)^{\alpha} (x_1 - x_2)^{\mathcal{N}_g} Y_W^g (u, x_1) (x_1 - x_2)^{-\mathcal{N}_g} (Y^g)_{WV}^W (w, x_2) v \\
&\quad - (-1)^{|u||w|} (-x_2 + x_1)^{\alpha} (Y^g)_{WV}^W (w, x_2) (-x_2 + x_1)^{\mathcal{N}_g} Y_V (u, x_1) (-x_2 + x_1)^{-\mathcal{N}_g} v \\
&= \sum_{k=0}^{M_{u,w}-1} \frac{1}{k!} x_1^{-1} \frac{\partial^k}{\partial x_2^k} \delta \left(\frac{x_2}{x_1} \right) (Y^g)_{WV}^W ((Y_W^g)_{\alpha+k,0} (u) w, x_2) v. \qquad (4.26)
\end{aligned}$$

and

$$
\begin{aligned}
&(x_1-x_2)^{\alpha+M_{u,w}}(x_1-x_2)^{\mathcal{N}_g}Y_W^g(u,x_1)(x_1-x_2)^{-\mathcal{N}_g}(Y^g)_{WV}^W(w,x_2)v\\
&\quad=(-1)^{|u||w|}(-x_2+x_1)^{\alpha+M_{u,w}}(Y^g)_{WV}^W(w,x_2)\cdot\\
&\qquad\cdot(-x_2+x_1)^{\mathcal{N}_g}Y_V(u,x_1)(-x_2+x_1)^{-\mathcal{N}_g}v,
\end{aligned}
\tag{4.27}
$$

respectively. In particular, when g is semisimple (for example, when g is of finite order), we have

$$
\begin{aligned}
&(x_1-x_2)^{\alpha}Y_W^g(u,x_1)(Y^g)_{WV}^W(w,x_2)v\\
&\quad-(-1)^{|u||w|}(-x_2+x_1)^{\alpha}(Y^g)_{WV}^W(w,x_2)Y_V(u,x_1)v\\
&\quad=\sum_{k=0}^{M_{u,w}-1}\frac{1}{k!}x_1^{-1}\frac{\partial^k}{\partial x_2^k}\delta\left(\frac{x_2}{x_1}\right)(Y^g)_{WV}^W((Y_W^g)_{\alpha+k,0}(u)w,x_2)v.
\end{aligned}
\tag{4.28}
$$

and

$$
\begin{aligned}
&(x_1-x_2)^{\alpha+M_{u,w}}Y_W^g(u,x_1)(Y^g)_{WV}^W(w,x_2)v\\
&\quad=(-1)^{|u||w|}(-x_2+x_1)^{\alpha+M_{u,w}}(Y^g)_{WV}^W(w,x_2)Y_V(u,x_1)v,
\end{aligned}
\tag{4.29}
$$

Proof. In the case $u\in V^{[\alpha]}$,

$$
\begin{aligned}
(x_1-x_2)^{\mathcal{S}_g}u&=(x_1-x_2)^{\alpha},\\
(-x_2+x_1)^{\mathcal{S}_g}u&=(-x_2+x_1)^{\alpha}.
\end{aligned}
$$

Also

$$
\begin{aligned}
(x_1-x_2)^{\mathcal{N}_g}u&=\sum_{k\in\mathbb{N}}\frac{1}{k!}(\log(x_1-x_2))^k\mathcal{N}_g^k u,\\
(-x_2+x_1)^{\mathcal{N}_g}u&=\sum_{k\in\mathbb{N}}\frac{1}{k!}(\log(-x_2+x_1))^k\mathcal{N}_g^k u.
\end{aligned}
$$

Using these formulas, (2.4), (3.14), $(Y_W^g)_0(u,x_0)w=\sum_{n\in\alpha+\mathbb{Z}}(Y_W^g)_{n,0}(u)wx_0^{-n-1}$ and the definition of $M_{u,w}$, we see that (4.22) and (4.23) become (4.26) and (4.27), respectively. ∎

We now study products of more than one twisted vertex operators or vertex operators for V and one twist vertex operators:

Theorem 4.9 *For $w'\in W'$, $v_1\in V^{[\alpha_1]},\dots,v_{k+l}\in V^{[\alpha_{k+l}]},v\in V^{[\alpha]}$ and $w\in W$, the series*

$$
\langle w',(Y_W^g)^p(v_1,z_1)\cdots(Y_W^g)^p(v_k,z_k)((Y^g)_{WV}^W)^p(w,z)Y_V(v_{k+1},z_{k+1})\cdots Y_V(v_{k+l},z_{k+l})v\rangle \tag{4.30}
$$

is absolutely convergent in the region $|z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l}| > 0$. *Moreover, there exists a multivalued analytic function of the form*

$$\sum_{n_1,\ldots,n_{k+l},n=0}^{N} f_{n_1\cdots n_{k+l}n}(z_1,\ldots,z_{k+l},z)\cdot \\ \cdot (z_1-z)^{-\alpha_1}\cdots(z_{k+l}-z)^{-\alpha_{k+l}}z^{-\alpha}(\log(z_1-z))^{n_1}\cdots(\log(z_{k+l}-z))^{n_{k+l}}(\log z)^n, \tag{4.31}$$

denoted by

$$F(\langle w', Y_W^g(v_1,z_1)\cdots Y_W^g(v_k,z_k)(Y^g)_{WV}^W(w,z)Y_V(v_{k+1},z_{k+1})\cdots Y_V(v_{k+l},z_{k+l})v\rangle),$$

where $N\in\mathbb{N}$ *and* $f_{n_1\cdots n_{k+l}n}(z_1,\ldots,z_{k+l},z)$ *for* $n_1,\cdots,n_{k+l},n=0,\ldots,N$ *are rational functions of* $z_1,\ldots,z_{k+l},z$ *with the only possible poles* $z_i=0$ *for* $i=1,\ldots,k+l$, $z=0$, $z_i-z_j=0$ *for* $i,j=1,\ldots,k+l$, $i\neq j$, $z_i-z=0$ *for* $i=1,\ldots,k+l$, *such that the sum of (4.30) is equal to the branch*

$$F^p(\langle w', Y_W^g(v_1,z_1)\cdots Y_W^g(v_k,z_k)(Y^g)_{WV}^W(w,z)Y_V(v_{k+1},z_{k+1})\cdots Y_V(v_{k+l},z_{k+l})v\rangle) \\ = \sum_{n_1,\ldots,n_{k+l},n=0}^{N} f_{n_1\cdots n_{k+l}n}(z_1,\ldots,z_{k+l},z)\cdot \\ \cdot e^{-\alpha_1 l_p(z_1-z)}\cdots e^{-\alpha_{k+l}l_p(z_{k+l}-z)}e^{-\alpha l_p(z)}(l_p(z_1-z))^{n_1}\cdots(l_p(z_{k+l}-z))^{n_{k+l}}(l_p(z))^n, \tag{4.32}$$

of (4.31) in the region given by $|z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l}| > 0$, $|\arg(z_i-z)-\arg z_i| < \frac{\pi}{2}$ *for* $i=1,\ldots,k$ *and* $|\arg(z_i-z)-\arg z| < \frac{\pi}{2}$ *for* $i=k+1,\ldots,k+l$. *In addition, the orders of the pole* $z_j=0$ *of the rational functions* $f_{n_1\cdots n_{k+l}n}(z_1,\ldots,z_{k+l},z)$ *have a lower bound independent of* v_q *for* $q\neq j$, w *and* w'*; the orders of the pole* $z=0$ *of the rational functions* $f_{n_1\cdots n_{k+l}n}(z_1,\ldots,z_{k+l},z)$ *have a lower bound independent of* $v_1,\ldots,v_{k+l}$ *and* w'*; the orders of the pole* $z_j=z_m$ *of the rational functions* $f_{n_1\cdots n_{k+l}n}(z_1,\ldots,z_{k+l},z)$ *have a lower bound independent of* v_q *for* $q\neq j,m$, v, w *and* w'*; the orders of the pole* $z_j=z$ *of the rational functions* $f_{n_1\cdots n_{k+l}n}(z_1,\ldots,z_{k+l},z)$ *have a lower bound independent of* v_q *for* $q\neq j$, v *and* w'.

Proof. Let $L_W(-1)'$ be the adjoint of $L_W(-1)$ on W'. By Theorem 3.10,

$$\langle e^{zL_W(-1)'}w', (Y_W^g)^p(v_1,z_1-z)\cdots(Y_W^g)^p(v_k,z_k-z)\cdot \\ \cdot (Y_W^g)^p(v_{k+1},z_{k+1}-z)\cdots(Y_W^g)^p(v_{k+l},z_{k+l}-z)(Y_W^g)^p(v,-z)w\rangle \tag{4.33}$$

converges absolutely in the region $|z_1-z| > \cdots > |z_{k+l}-z| > |z| > 0$ to

$$\sum_{n_1,\ldots,n_k,n=0}^{N} g_{n_1\cdots n_{k+l}n}(z_1-z,\ldots,z_{k+l}-z,-z)\cdot \\ \cdot e^{-\alpha_1 l_p(z_1-z)}\cdots e^{-\alpha_{k+l}l_p(z_{k+l}-z)}e^{-\alpha l_p(-z)}(l_p(z_1-z))^{n_1}\cdots(l_p(z_{k+l}-z))^{n_{k+l}}(l_p(z))^n, \tag{4.34}$$

where $n_1, \ldots, n_{k+l} \in \mathbb{N}$ and $g_{n_1\cdots n_{k+l}n}(z_1 - z, \ldots, z_{k+l} - z, -z)$ are rational functions in $z_1 - z, \ldots, z_{k+l} - z, -z$ with the only possible poles at $z_i - z = 0$, $z = 0$ and $z_i - z_j = 0$ for $i \neq j$. Moreover, the orders of the pole $z_j = 0$ of the rational functions $g_{n_1\cdots n_{k+l}n}(z_1 - z, \ldots, z_{k+l} - z, -z)$ have a lower bound independent of v_q for $q \neq j$, w and w'; the orders of the pole $z = 0$ of the rational functions $g_{n_1\cdots n_{k+l}n}(z_1 - z, \ldots, z_{k+l} - z, -z)$ have a lower bound independent of $v_1, \ldots, v_{k+l}$ and w'; the orders of the pole $z_j = z_m$ of the rational functions $g_{n_1\cdots n_{k+l}n}(z_1 - z, \ldots, z_{k+l} - z, -z)$ have a lower bound independent of v_q for $q \neq j, m$, v, w and w'; the orders of the pole $z_j = z$ of the rational functions $g_{n_1\cdots n_{k+l}n}(z_1 - z, \ldots, z_{k+l} - z, -z)$ have a lower bound independent of v_q for $q \neq j$, v and w'. Let

$$f_{n_1\cdots n_{k+l}n}(z_1, \ldots, z_{k+l}, z) = g_{n_1\cdots n_{k+l}n}(z_1 - z, \ldots, z_{k+l} - z, -z)e^{-\alpha\pi i}.$$

Since $l_p(-z) = l_p(z) + \pi i$ when $0 \leq \arg z < \pi$, (4.34) is equal to a multivalued analytic function of the form (4.31) with branches of the form (4.32) satisfying all the properties when $0 \leq \arg z < \pi$. Using our notations, we denote this function by

$$F(\langle e^{zL_W(-1)'}w', Y_W^g(v_1, z_1 - z) \cdots Y_W^g(v_k, z_k - z)\cdot \\ \cdot Y_W^g(v_{k+1}, z_{k+1} - z) \cdots Y_W^g(v_{k+l}, z_{k+l} - z)Y_W^g(v, -z)w\rangle)$$

On the other hand, when $0 \leq \arg z < \pi$, using the $L(-1)$-commutator formula, the $L(-1)$-derivative property and the associativity for the twisted vertex operators, we have

$$\begin{aligned}
F^p(&\langle e^{zL_W(-1)'}w', Y_W^g(v_1, z_1 - z) \cdots Y_W^g(v_k, z_k - z)\cdot \\
&\quad \cdot Y_W^g(v_{k+1}, z_{k+1} - z) \cdots Y_W^g(v_{k+l}, z_{k+l} - z)Y_W^g(v, -z)w\rangle) \\
&= F^p(\langle w', Y_W^g(v_1, z_1) \cdots Y_W^g(v_k, z_k)\cdot \\
&\quad \cdot e^{zL_W(-1)}Y_W^g(v_{k+1}, z_{k+1} - z) \cdots Y_W^g(v_{k+l}, z_{k+l} - z)Y_W^g(v, -z)w\rangle) \\
&= F^p(\langle w', Y_W^g(v_1, z_1) \cdots Y_W^g(v_k, z_k)\cdot \\
&\quad \cdot e^{zL_W(-1)}Y_W^g(Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l}, z_{k+l})v, -z)w\rangle) \\
&= F^p(\langle w', Y_W^g(v_1, z_1) \cdots Y_W^g(v_k, z_k)(Y^g)_{WV}^W(w, z)Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l}, z_{k+l})v\rangle).
\end{aligned} \tag{4.35}$$

Then by analytic extensions, (4.35) holds without the condition $0 \leq \arg z < \pi$. But (4.30) is a series of the same form as the expansion of the left-hand side of (4.35) in the region given by $|z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l}| > 0$, $|\arg(z_i - z) - \arg z_i| < \frac{\pi}{2}$ for $i = 1, \ldots, k$ and $|\arg(z_i - z) - \arg z| < \frac{\pi}{2}$ for $i = k+1, \ldots, k+l$. By (4.35), (4.30) must be the expansion of the left-hand side of (4.35). Since we have proved that the left-hand side of (4.35) is of the form (4.32) satisfying all the properties, the result is proved. ∎

Corollary 4.10 *For $v_1, \ldots, v_{k-1} \in V$, $v \in V$, $w \in W$, $w' \in W'$, $p \in \mathbb{Z}$, $\tau \in S_k$ and fixed $1 \leq i \leq k$,*

$$F^p(\langle w', \varphi_1(z_1) \cdots \varphi_k(z_k)v\rangle) = \pm F^p(\langle w', \varphi_{\tau(1)}(z_{\tau(1)}) \cdots \varphi_{\tau(k)}(z_{\tau(k)})v\rangle),$$

where $\varphi_j(z_j) = Y_W^g(v_j, z_j)$ for $j \neq i$ and $\varphi_i = (Y^g)_{WV}^W(w, z_l)$ and the sign $\pm$ is uniquely determined by τ and $|v_1|, \ldots, |v_{k-1}|, |w|$.

Proof. This result follows immediately from Theorem 3.10, the duality property for Y_W^g and Corollary 4.3. ∎

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