Representation theory of vertex operator algebras, conformal field theories and tensor categories

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1. Vertex operator algebras (VOAs, chiral algebras)

Symmetry algebras for conformal field theories

Conformal field theory can be viewed as the representation theory of vertex operator algebras

Vertex operator algebras are analogues of both commutative associative algebras and Lie algeberas

Commutative associative algebras:

A vector space: C
Multiplication: $a, b \in C$, $ab \in C$
Identity: $1 \in C$
Associativity: $(ab)c = a(bc)$
Commutativity: $ab = ba$
Identity property: $1a = a1 = a$

Examples:
(i) The space of polynomials in $x_1, x_2, \cdots, x_n$

$\mathbb{C}[x_1, \cdots, x_n]$

(ii) The space of functions on a manifold, e.g., space-time.

Lie algebras:

A vector space: $\mathfrak{g}$
A Lie bracket: $a, b \in \mathfrak{g}$, $[a,b] \in \mathfrak{g}$
Skew-symmetry: $[a,b] = -[b,a]$
Jacobi identity: $[a,[b,c]] - [b,[a,c]] = [[a,b],c]$

Vertex operator algebras are analogues of both commutative associative algebras and Lie algebras.

A vector space which is a direct sum of finite-dimensional vector spaces: $V = \bigoplus_{n\in\mathbb{Z}} V_{(n)} = \coprod_{n\in\mathbb{Z}} V_{(n)}$. $V_{(n)}$: subspaces of conformal dimensional n.

Vextex operators: For $v \in V$, there is a vertex operator $Y(v, z) = v(z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}$ where v_n for $n \in \mathbb{Z}$ are operators on V.

A vacuum: $\mathbb{1}$ or $|0\rangle \in V_{(0)}$

A conformal element: $\omega \in V_{(2)}$ corresponding to the stress energy tensor, that is, $Y(\omega, z)$ or $\omega(z)$ is the stress energy tensor $T(z)$
satisfying the following:

Properties for the vacuum: $Y(|0\rangle, z) = 1_V$,
$\lim_{z\to 0} Y(u,z)|0\rangle = \lim_{z\to 0} u(z)|0\rangle = u$.

Associativity or (meromorphic) operator product expansion:

$V' = \bigoplus_{n\in\mathbb{Z}} V_{(n)}^*$. $\langle\ ,\ \rangle : V'\otimes V \to \mathbb{C}$, $v'\in V'$, $v\in V$

$\langle v', v\rangle \in \mathbb{C}$ (Notation in physics: $\langle v' | v\rangle$)

$\langle v', Y(u_1, z_1)Y(u_2, z_2)v\rangle = \langle v' | u_1(z_1)\, u_2(z_2) | v\rangle$

is absolutely convergent when $|z_1| > |z_2| > 0$ to a rational function with the only possible poles at $z_1, z_2 = 0$ and $z_1 - z_2 = 0$ and is equal to

$\langle v', Y(Y(u_1, z_1 - z_2)u_2, z_2)v\rangle = \langle v' | (u_1(z_1 - z_2)\, u_2)(z_2) | v\rangle$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

Since $Y(u_1, z_1 - z_2) = \sum_{n\in\mathbb{Z}} (u_1)_n (z_1 - z_2)^{-n-1}$

$\langle v', Y(Y(u_1, z_1 - z_2)u_2, z_2)v\rangle$

$= \langle v', \sum_{n\in\mathbb{Z}} Y((u_1)_n u_2, z_2)\,(z_1 - z_n)^{-n-1}\rangle$

Operator product expansion!

Commutativity:

$\langle v', Y(u_1, z_1)Y(u_2, z_2)v\rangle$

and

$\langle v', Y(u_2, z_2)Y(u_1, z_1)v\rangle$

are absolutely convergent to a common rational function but in different regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively.

Properties for the conformal element or the stress-energy tensor:

$$Y(\omega, z) = \omega(z) = T(z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$$

Then $[L(m), L(n)] = (m-n) L(m+n) + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$

$$\frac{d}{dz} Y(u, z) = \frac{d}{dz} u(z) = Y(L(-1)u, z) = (L(-1)u)(z).$$

$L(0)u = nu$ for $u \in V_{(n)}$.

Examples:

(i) Trivial examples: Commutative associative algebras

$C = C_{(0)}$, $Y(a, z)b = a(z)b = ab$, $|0\rangle = 1$, $\omega = 0$

(ii) Free boson:

Heisenberg algebra: $[a_m, a_n] = m\,\delta_{m+n,0}$ $m, n \in \mathbb{Z}$

Fock space V: Space spanned by elements of the form $a_{-n_1} \cdots a_{-n_\ell}|0\rangle$ $n_1, \dots, n_\ell > 0$

$V_{(n)}$ elements of conformal dimension n

$$V = \bigoplus_{n=0}^{\infty} V_{(n)}$$

$|0\rangle$

$$\omega = \frac{1}{2} a_{-1} a_{-1}|0\rangle$$

Vertex operators: $Y(|0\rangle, z) = 1_V$

$$Y(a_{-1}|0\rangle, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} = a(z)$$

$$Y(a_{-n_1}\cdots a_{-n_\ell}|0\rangle, z)$$
$$= {:}\frac{1}{(n_1-1)!}\frac{d^{n_1-1}}{dz^{n_1-1}}a(z)\cdots\frac{1}{(n_\ell-1)!}\frac{d^{n_\ell-1}}{dz^{n_\ell-1}}a(z){:}$$

(iii) Affine Lie algebras:

$\mathfrak{g}$ finite-dimensional Lie algebras

$\hat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K$ $\left(\mathbb{C}[t,t^{-1}]\ \text{space of polynomials in } t \text{ and } t^{-1}\right)$

$a(m) = a\otimes t^m$ for $a\in\mathfrak{g}$ and $m\in\mathbb{Z}$ Span $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$

$$[a\otimes t^m, b\otimes t^n] = [a,b]\otimes t^{m+n} + m(a,b)\delta_{m+n,0}K$$

$$[K, a\otimes t^n] = 0$$

V space spanned by elements of the form

$$a_1(-m_1)\cdots a_\ell(-m_\ell)|0\rangle,\quad a_1,\cdots,a_\ell\in\mathfrak{g},\ m_1\geq\cdots\geq m_\ell>0$$

$$V = \bigoplus_{n=0}^{\infty} V_{(n)}$$

$|0\rangle$

$$\omega = \frac{1}{2(k+h^\vee)}\sum_{i=1}^{\dim\mathfrak{g}} a_i(-1)a_i(-1)|0\rangle,\quad a_i \text{ form a basis of } \mathfrak{g}$$

(Segal-Sugawara construction)

Vertex operator: $Y(|0\rangle, z) = 1_V$

$$Y(a(-1)|0\rangle, z) = \sum_{n\in\mathbb{Z}} a(n)x^{-n-1} = a(z)$$

$$Y(a(-n)|0\rangle, z) = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}a(z)$$

$Y(a_1(-n_1)\cdots a_\ell(-n_\ell)|0\rangle, z)$ can be defined using a noncommutative version of normal ordering

V module for affine Lie algebra

There exists a maximal proper submodule J.

So V/J is irreducible

V/J is also a VOA

(iv) Virasoro algebra:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{c}{12}(m^3-m)\delta_{m+n,0}$$

$$[c, L(n)] = 0$$

V space spanned by elements of the form

$$L(-n_1)\cdots L(-n_\ell)|0\rangle \qquad n_1 \geq \cdots \geq n_\ell \geq 2$$

$$V = \bigoplus_{n=0}^{\infty} V_{(n)} \qquad V_{(1)} = 0$$

$|0\rangle$

$\omega = L(-2)|0\rangle$

$$Y(|0\rangle, z) = 1_V, \quad Y(L(-2)|0\rangle, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$$

There is a way similar to the affine Lie algebra case to define vertex operator $Y(L(-n_1)\cdots L(-n_\ell)|0\rangle, z)$

Jacobi identity for vertex operator algebras:

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n, \quad \delta\left(\frac{z_1 - z_2}{z_0}\right) = \sum_{n\in\mathbb{Z}} \left(\frac{z_1 - z_2}{z_0}\right)^n$$
$$= \sum_{n\in\mathbb{Z}} \sum_{k\geq 0} \binom{n}{k} z_1^{n-k} (-z_2)^k z_0^{-n}$$

Let V be a vertex operator algebra and $u_1, u_2 \in V$.

Then

$u_1(z_1) \quad u_2(z_2) \qquad\qquad u_2(z_2) \quad u_1(z_1)$

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u_1, z_1) Y(u_2, z_2) u_3 - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(u_2, z_2) Y(u_1, z_1) u_3$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u_1, z_0)u_2, z_2) u_3$$

$(u_1(z_0)u_2)(z_2)$

Special cases:

(i) Commutator formular:

$$[Y(u_1, z_1), Y(u_2, z_2)] = [u_1(z_1), u_2(z_2)]$$
$$= \mathrm{Res}_{z_0} z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u_1, z_0)u_2, z_2)$$
$$= \mathrm{Res}_{z_0} z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) (u(z_0)u_2)(z_2)$$

(ii) Associator formula:

$$Y(Y(u_1, z_0)u_2, z_2)$$
$$= Y(u_1, z_0 + z_2) Y(u_2, z_2) - \mathrm{Res}_{z_1} z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(u_2, z_2) Y(u_1, z_1)$$

2. Modules for vertex operator algebras:

Let V be a vertex operator algebra.

Module for V:

A vector space: $W = \bigoplus_{n \in \mathbb{C}} W_{(n)}$

Vertor operators: For $v \in V$, there exists a vertex operator

$Y_W(v, z) = v(z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$

satisfying all the properties for VOAs that still make sense, that is:

Identity property: $Y_W(|0\rangle, z) = 1_W$

Associativity: $W' = \bigoplus_{n \in \mathbb{Z}} W_{(n)}^*$. $w' \in W'$, $w \in W$, $u_1, u_2 \in V$

$\langle w', Y_W(u_1, z_1) Y_W(u_2, z_2) w \rangle$

is absolutely convergent to a rational function of the only possible poles $z_1, z_2 = 0$ and $z_1 - z_2 = 0$ when $|z_1| > |z_2| > 0$ and is equal to

$\langle w', Y_W(Y(u_1, z_1 - z_2) u_2, z_2) w \rangle$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

Commutativity:

$\langle w', Y_W(u_1, z_1) Y_W(u_2, z_2) w \rangle$

and

$\langle w', Y_W(u_2, z_2) Y_W(u_1, z_1) w \rangle$

are absolutely convergent to a common rational function in the region $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively.

Properties for the conformal element:

$$Y_W(\omega, z) = \sum_{n \in \mathbb{Z}} L_W(n) z^{-n-2}$$

$$[L_W(m), L_W(n)] = (m-n) L(m+n) + \frac{c}{12}(m^3 - m) \delta_{m+n,0}$$

$$\frac{d}{dz} Y_W(u, z) = Y_W(L(-1)u, z)$$

$$L_W(0) w = n w \quad \text{for } w \in W_{(n)}$$

Vectors in modules can have noninteger conformal dimensions but vertex operators for modues are still meromorphic.

Examples:

(i) Free boson: Let V be the vertex operator algebra for one free boson. Let $|k\rangle$ be a vector satisfying $a(0)|k\rangle = k|k\rangle$. Let W be the Fock space generated from this vector, that is, W is spanned by elements of the form

$$a_{-n_1} \cdots a_{-n_l} |k\rangle$$

Vertex operators:

$$Y_W(a_{-n_1} \cdots a_{-n_l} |0\rangle, z) = {}^{\circ}_{\circ} \frac{1}{(n_1-1)!} \frac{d^{n_1-1}}{dz^{n_1-1}} a(z) \cdots \frac{1}{(n_l-1)!} \frac{d^{n_l-1}}{dz^{n_l-1}} a(z) {}^{\circ}_{\circ}$$

where $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$ and a_n are operators on W.

(ii) Affine Lie algebra: Let V be the vertex operator algebra for affine Lie algebra. Let M be a finite-dimensional module for the finite-dimensional Lie algebra $\mathfrak{g}$.

Let W be the space spanned by elements of the form

$$a_1(-n_1) \cdots a_\ell(-n_\ell) w, \quad a_1, \dots, a_n \in \mathfrak{g}, \; n_1 \geq \cdots \geq n_\ell > 0, \; w \in M$$

Vertex operators:

$$Y_W(a(-n)|0\rangle, z) = \frac{1}{(n-1)} \frac{d^{n-1}}{dz^{n-1}} \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$$

where $a(n)$ are operators on W

For the vertex operator algebra V/J, only some particular modules M are allowed.

(iii) Virasoro algebra: Let V be the vertex operator algebra for the Virasoro algebra. Let $|h\rangle$ be a vector satisfying

$$L(0)|h\rangle = h|h\rangle \quad \text{and} \quad C|h\rangle = c|h\rangle.$$

Let W be the vector space spanned by elements of the form

$$L(-n_1) \cdots L(-n_\ell)|h\rangle, \quad n_1 \geq \cdots \geq n_\ell \geq 1$$

Vertex operators:

$$Y_W(L(-2)|0\rangle, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$$

3. Intertwining operators and intertwining operator algebras:

W_1, W_2, W_3 modules for V

An intertwining operator of type $\binom{W_3}{W_1 W_2}$ is a linear map for W_1 to the space of series of the form $\sum_{n\in\mathbb{C}} a_n z^n$ where a_n are linear maps from W_2 to W_3 satisfying the following conditions: Let $\mathcal{Y}$ be the intertwining operator and $\mathcal{Y}(w_1, z)$ is the image of $w_1 \in W_1$ under this map (the intertwining operator associated to w_1).

Associativity: For $u \in V$, $w_1 \in W_1$, $w_2 \in W_2$, $w_3' \in W_3'$,

$$\langle w_3', Y_{W_3}(u, z_1)\mathcal{Y}(w_1, z_2)w_2\rangle$$

is absolutely convergent when $|z_1| > |z_2| > 0$ to an analytic function in z_1, z_2 (possibly multivalued in z_2) with the only possible singularities or branch points $z_1, z_2 = 0$ or $z_1 - z_2 = 0$. and is equal to

$$\langle w', \mathcal{Y}(Y_{W_1}(u, z_1 - z_2)w_1, z_2)w_2\rangle$$

when $|z_1|>|z_2|>|z_1-z_2|>0$.

Commutativity:

$$\langle w_3', Y_{W_3}(u, z_1)\mathcal{Y}(w_1, z_2)w_2\rangle$$

and

$$\langle w_3', \mathcal{Y}(w_1, z_2)Y_{W_2}(u, z_1)w_2\rangle$$

are absolutely convergent when $|z_1|>|z_2|>0$ and $|z_2|>|z_1|>0$, respectively, and their sums, as analytic functions, are analytic extensions of each other.

L(-1)-derivative property

$$\frac{d}{dz}\mathcal{Y}(w_1, z) = \mathcal{Y}(L_{W_1}(-1)w_1, z)$$

Examples:

Free boson: Let W_1, W_2, W_3 be modules for the free boson vertex operator algebra V constructed from $|k_1\rangle$, $|k_2\rangle$, $|k_3\rangle$, respectively. Let $k_3 = k_1+k_2$. We define

$$\mathcal{Y}(|k_1\rangle, z)|k_2\rangle$$
$$= {:}e^{k_1\int a(z)dz}{:}|k_2\rangle = e^{-k_1\sum_{n<0}\frac{a_n}{n}z^{-n}}e^{-k_1\sum_{n>0}\frac{a_n}{n}z^{-n}}z^{k_1k_2}\underbrace{|k_3\rangle}_{k_1+k_2}$$

$$\mathcal{Y}(a_{-n_1}\cdots a_{-n_\ell}|k_1\rangle, z)$$
$$= {:}\frac{1}{(n_1-1)!}\frac{d^{n_1-1}}{dz^{n_1-1}}a(z)\cdots\frac{1}{(n_\ell-1)!}\frac{d^{n_\ell-1}}{dz^{n_\ell-1}}a(z)\, e^{k_1\int a(z)\,dz}{:}$$

Intertwining operators of type $\binom{W_3}{W_1 W_2}$ form a vector space.

Question: Do we have associativity or operator product expansion for the product of two arbitrary intertwining operators?

Yes, if the vertex operator algebra satisfies some additional conditions.

Modules: $W=\bigoplus_{n\in\mathbb{C}} W_{(n)}$, $\dim W_{(n)}<\infty$, $W_{(n)}=0$ when $\mathrm{Re}(n)$ is sufficiently negative.

Generalized modules: Do not require that these two conditions hold.

C_1-cofiniteness or quasi-rationality: Let W be a module for V. $C_1(W)$: Subspace of W spanned by elements of the form $u_{-1}w$ for $u\in V_+=\bigoplus_{n=1}^{\infty}V_{(n)}$, $w\in W$ (recall that $Y_W(u,z)w=\sum_{n\in\mathbb{Z}}u_n w z^{-n-1}$).

W is C_1-cofinite or quasi-rational if $W/C_1(W)$ is finite dimensional, that is, there is a finite-dimensional subspace X of W such that $W = X + C_1(W)$.

Theorem [H]: Let V be a vertex operator algebra satisfying the following conditions:

(i) There are only finitely many irreducible V-modules whose conformal dimensions are all real numbers.

(ii) Every generalized module for V is completely reducible.

(iii) Every irreducible module for V is C_1-cofinite.

Then the following associativity (operator product expansion) and commutativity (braiding relation) of intertwining operators hold:

For any intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of types $\binom{W_4}{W_1\, W_5}$ and $\binom{W_5}{W_2\, W_3}$, respectively, there exist a module W_6 and intertwining operators $\mathcal{Y}_3$ and $\mathcal{Y}_4$ of types $\binom{W_4}{W_6\, W_3}$ and $\binom{W_6}{W_1\, W_2}$, respectively, such that

$$\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle$$

is absolutely convergent when $|z_1| > |z_2| > 0$ to a (multivalued) analytic function with the only possible singularities or branch points at $z_1, z_2 = 0$ and $z_1 - z_2 = 0$ and is equal to

$$\langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3\rangle$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

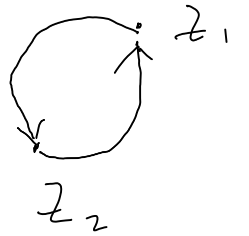
There also exist a module W_7 and intertwining operators $\mathcal{Y}_5$ and $\mathcal{Y}_6$ of types $\binom{W_4}{W_2\, W_7}$ and $\binom{W_7}{W_1\, W_3}$ such that

$$\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle$$

and

$$\langle w_4', \mathcal{Y}_5(w_2, z_2)\mathcal{Y}_6(w_1, z_1)w_3\rangle$$

are analytic extensions of each other along the path



Express the results using numbers: Let W^a, $a \in \overset{\text{finite set}}{I}$, be all inequivalent irreducible modules for V. Choose a basis $\{\mathcal{Y}^{a_3}_{a_1 a_2; i}\}$ of the space $\mathcal{V}^{a_3}_{a_1 a_2}$ of all intertwining operators of type $\binom{W^{a_3}}{W^{a_1} W^{a_2}}$. Then for $\mathcal{Y}_1 = \mathcal{Y}^{a_4}_{a_1 a_5; i}$, $\mathcal{Y}_2 = \mathcal{Y}^{a_5}_{a_2 a_3; j}$,

W_6 and W_7 can be decomposed into direct sums of W^a for $a \in I$. In addition, intertwining operators $\mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5, \mathcal{Y}_6$ of types $\binom{W^{a_4}}{W_6 W^{a_3}}$, $\binom{W_6}{W^{a_1} W^{a_2}}$, $\binom{W^{a_4}}{W^{a_2} W_7}$, $\binom{W_7}{W^{a_1} W^{a_3}}$ can be decomposed into sums of intertwining operators of types $\binom{W^{a_4}}{W^{a_6} W^{a_3}}$, $\binom{W^{a_6}}{W^{a_1} W^{a_2}}$, $\binom{W^{a_4}}{W^{a_2} W^{a_7}}$, $\binom{W^{a_7}}{W^{a_1} W^{a_3}}$. But these intertwining operators are linear combinations of basis elements we have chosen. So the associativity for intertwining operators can be written in terms of basis and numbers as:

$$\langle w'_{a_4}, \mathcal{Y}^{a_4}_{a_1 a_5; i}(w_{a_1}, z_1)\, \mathcal{Y}^{a_5}_{a_2 a_3; j}(w_{a_2}, z_2)\, w_{a_3}\rangle$$

$$= \sum_{a_6} \sum_{k} \sum_{l} F^{a_4; a_6; kl}_{a_1 a_2 a_3; a_5; ij}$$

$$\langle w'_{a_4}, \mathcal{Y}^{a_4}_{a_6 a_3; k}(\mathcal{Y}^{a_6}_{a_1 a_2; l}(w_{a_1}, z_1 - z_2)\, w_{a_2}, z_2)\, w_{a_3}\rangle$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$, where $F^{a_4; a_6; kl}_{a_1 a_2 a_3; a_5; ij} \in \mathbb{C}$.

$\left(F^{a_4; a_6; kl}_{a_1 a_2 a_3; a_5; ij}\right)$: Fusing matrices

The commutativity of intertwining operators can be written as: There exist $B^{a_4; a_7; pq}_{a_1 a_2 a_3; a_5; ij} \in \mathbb{C}$

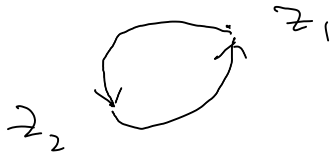
$$\langle w'_{a_4}, \mathcal{Y}^{a_4}_{a_1 a_5; i}(w_{a_1}, z_1)\, \mathcal{Y}^{a_5}_{a_2 a_3; j}(w_{a_2}, z_2)\, w_{a_3}\rangle$$

and

$$\sum_{a_7} \sum_{p} \sum_{q} B^{a_4; a_7; pq}_{a_1 a_2 a_3; a_5; ij}$$

$$\langle w'_{a_4}, \mathcal{Y}^{a_4}_{a_2 a_7; p}(w_{a_2}, z_2)\, \mathcal{Y}^{a_7}_{a_1 a_3; q}(w_{a_1}, z_1)\, w_{a_3}\rangle$$

are analytic extensions of each other along the path



$$\left(B^{a_4;a_7;pq}_{a_1a_2a_3;a_5;ij}\right)$$ Braiding matrices, giving a representation of the braid group

Differential equations of regular singular points: The proof of the theorem above is based on some differential equations satisfied by the products or iterates of intertwining operators. For affine Lie algebras, there are the famous KZ equations. For minimal models the Virasoro algebra case, there are BPZ equations. In general, the C_1-cofiniteness allows us to derive differential equations satified by the products and iterates. These equations have singular points but they are all regular singular points.

Examples:

$$\left(z\frac{d}{dz}\right)^n \varphi(z) + a_{n-1}(z)\left(z\frac{d}{dz}\right)^{n-1}\varphi(z) + \cdots + a_1(z)\, z\frac{d}{dz}\varphi(z) + a_0(z) = 0$$

Intertwining operator algebras:

$W = \bigoplus_{a\in I} W^a$, equipped with all intertwining operators, satisfying the associativity and commutativity of intertwining operators.

Intertwining operator algebras describe nonabelian anyons.

4. Fusion rules and tensor products:

Fusion rules: Let W_1, W_2, W_3 be modules for a vertex operator algebras and let $\mathcal{V}_{W_1W_2}^{W_3}$ be the space of intertwining operators of type $\binom{W_3}{W_1 W_2}$.

$$N_{W_1W_2}^{W_3} = \dim \mathcal{V}_{W_1W_2}^{W_3} \quad \text{fusion rule of type } \binom{W_3}{W_1 W_2}.$$

Fusion rules and tensor products for modules for Lie algebras:

Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra and $M_1, M_2, \cdots$ are all inequivalent finite-dimensional irreducible modules for $\mathfrak{g}$. Then $M_i \otimes M_j$ a module for $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, $M_i \otimes M_j \simeq \bigoplus_k N_{ij}^k M_k$. N_{ij}^k is the fusion rule of type $\binom{M_k}{M_i M_j}$.

Question: Is there a tensor product operation for moules for V such that the fusion rules can be calculated using the tensor products as in the Lie algebra case.

Answer: Yes, if V satisfies the three conditions in the theorem giving the operator product expansion of intertwining operators.

Let W_1 and W_2 be modules for V. The

Vector space tensor product $W_1 \otimes W_2$ does not have a structure of modules for V. We have to construct the tensor product of W_1 and W_2 using a completely different method.

For a module W_3, an element $w_3' \in W_3'$ and an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$, we obtain an element $\langle w_3', \mathcal{Y}(\cdot, 1)\cdot \rangle$ of $(W_1 \otimes W_2)^*$. Let $W_1 \boxdot W_2$ be the subspace of all such elements of $(W_1 \otimes W_2)^*$. Then $W_1 \boxdot W_2$ is a module for V. Given any module W, W' is also a module for V called contragredient module. We define $W_1 \boxtimes W_2 = (W_1 \boxdot W_2)'$.

Let W^a for $a \in I$ are all irreducible modules. Then $W^{a_1} \boxtimes W^{a_2} \cong \bigotimes_{a_3} N_{a_1 a_2}^{a_3} W^{a_3}$. $N_{a_1 a_2}^{a_3}$ is equal to the fusion rule $N_{W^{a_1} W^{a_2}}^{W^{a_3}} = \dim \mathcal{V}_{W^{a_1} W^{a_2}}^{W^{a_3}}$.

Theorem [H]: Let V be a vertex operator algebra satisfying the following conditions:

(i) There are only finitely many irreducible V-modules whose conformal dimensions are all real numbers.

(ii) Every generalized module for V is completely reducible.

(iii) Every irreducible module for V is C_1-cofinite.

Then the category of modules for V is a braided tensor category.

5. Modular invariance, the Verlinde conjecture and the rigidity

We need stronger conditions now.

Logarithmic modules: $W = \bigoplus_{n \in \mathbb{C}} W_{(n)}$, $(L(0)-n)^N w = 0$ for $w \in W_{(n)}$.

C_2-cofiniteness: $C_2(V)$ = the subspace of V spanned by elements of the form $u_{-2}v$ for $u, v \in V$ (recall that $Y(u,z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1}$). If $\dim V/C_2(V) < \infty$, that is, there exists a finite dimensional subspace X of V such that $V = X + C_2(V)$, we say that V

is C_2-cofinite.

Let W^a for $a \in I$ be all irreducible modules for V.

Define $Ch_{W^a}(\tau) = \sum_{n \in \mathbb{C}} (\dim W^a_{(n)}) e^{2\pi i \tau (n - \frac{c}{24})}$.

Theorem[Y. Zhu] Let V be a vertex operator algebra satisfying the following conditions:

(i) $V_{(n)} = 0$ for $n < 0$.

(ii) Every logarithmic generalized module for V is completely reducible.

(iii) Every irreducible module for V is C_2-cofinite.

Then there are only finitely many irreducible modules for V and the finite-dimensional vector space spanned by

$$Ch_{W^a}(\tau) \text{ for } a \in I$$

are invariant under the modular transformations

$$\tau \longmapsto \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \quad \text{where } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{Z})$$

that is, $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$, $\alpha\delta - \beta\gamma = 1$.

For $a \in I$, $Ch_{W^a}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right)$ is an element of the vector space spanned by $Ch_{W^a}(\tau)$ for $a \in I$. So there exist $A_a^b \in \mathbb{C}$ for $a, b \in I$ such that

$$Ch_{W^a}\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = \sum_{b\in I} A_a^b \, Ch_{W^b}(\tau).$$

In particular, for $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, that is, for the modular transformation $\tau \longrightarrow -\frac{1}{\tau}$, we have $S_a^b \in \mathbb{C}$ for $a, b \in I$ such that

$$Ch_{W^a}\left(\frac{-1}{\tau}\right) = \sum_{b\in I} S_a^b \, Ch_{W^b}(\tau)$$

Theorem[H] Let V be a vertex operator algebra satisfying the following conditions:

(i) $V_{(n)} = 0$ for $n < 0$, V as a module is irreducible and is equivalent to the contragredient module V'.

(ii) Every logarithmic generalized module for V is completely reducible.

(iii) Every irreducible module for V is C_2-cofinite.

Then the following Verlinde conjecture is true:

Let $N_a = (N_{ab}^c)$ for $a \in I$ where N_{ab}^c is the fusion rule of type $\binom{W^c}{W^a\, W^b}$. Then $S = (S_a^b)$ diagonalizes N_a for $a \in I$ simultaneously and the diagonal entries can be expressed using special entries of the braiding and fusing matrices.

6. Modular tensor category structures:

Using all the results above, we have the following main theorem:

Theorem[H] Under the assumption of the previous theorem, the category of modules for V is a modular tensor category.

I would like to mention that the material presented here are based on the works of many physicists and many mathematicians for more than 30 years, especially the works of Goddard, Belavin-polyakov-Zamolodchikov, Friedan-Shenker, Witten, Borcherds, Frenkel-Lepowsky-Meurman, Verlinde, Moore-Seiberg, Zhu and Lepowsky-Huang.