Twisted intertwining operators and tensor products of (generalized) twisted modules

Jishen Du and Yi-Zhi Huang

Abstract

We study the general twisted intertwining operators (intertwining operators among twisted modules) for a vertex operator algebra V. We give the skew-symmetry and contragredient isomorphisms between spaces of twisted intertwining operators and also prove some other properties of twisted intertwining operators. Using twisted intertwining operators, we introduce a notion of $P(z)$-tensor product of two objects for $z \in \mathbb{C}^{\times}$ in a category of suitable g-twisted V-modules for g in a group of automorphisms of V and give a construction of such a $P(z)$-tensor product under suitable assumptions. We also construct G-crossed commutativity isomorphisms and G-crossed braiding isomorphisms. We formulate a $P(z)$-compatibility condition and a $P(z)$-grading-restriction condition and use these conditions to give another construction of the $P(z)$-tensor product.

1 Introduction

Modular tensor categories associated to conformal field theories were discovered first in physics by Moore and Seiberg [MS]. In [T1], Turaev formulated a precise notion of modular tensor category based on his joint work [RT] with Reshetikhin on the construction of quantum invariants of three manifolds using representations of quantum groups. In [H6], the second author proved the following theorem:

Theorem 1.1 *Let V be a simple vertex operator algebra staisfying the following conditions:*

1. *For $n < 0$, $V_{(n)} = 0$ and $V_{(0)} = \mathbb{C}\mathbf{1}$ and as a V-module, V is equivalent to its contragredient V-module V' (or equivalently, there exists a nondegenerate invariant bilinear form on V).*
2. *Every lower-bounded (generalized) V-module is completely reducible.*
3. *V is C_2-cofinite.*

Then the category of V-modules has a natural structure of modular tensor category in the sense of Turaev [T1].

The proof of Theorem 1.1 was based on the results obtained by Lepowsky and the second author in [HL2], [HL3], [HL4] and the results obtained by the second author in [H1], [H3], [H4], [H5].

It is natural to expect that Theorem 1.1 has generalizations in two-dimensional orbifold conformal field theory. Two-dimensional orbifold conformal field theories are two-dimensional conformal field theories constructed from known theories and their automorphisms. The first example of two-dimensional orbifold conformal field theories is the the moonshine module constructed by Frenkel, Lepowsky and Meurman [FLM1] [FLM2] [FLM3] in mathematics. In string theory, the systematic study of two-dimensional orbifold conformal field theories was started by Dixon, Harvey, Vafa and Witten [DHVW1] [DHVW2]. See [H14] for an exposition on general results, conjectures and open problems in the construction of two-dimensional orbifold conformal field theories using the approach of the representation theory of vertex operator algebras.

In [K3], Kirillov Jr. stated that the category of g-twisted modules for a vertex operator algebra V for g in a finite subgroup G of the automorphism group of V is a G-equivariant fusion category (G-crossed braided (tensor) category in the sense of Turaev [T2]). For general V, this is certainly not true. The vertex operator algebra V must satisfy certain conditions. Here is a precise conjecture formulated by the second author in [H9]:

Conjecture 1.2 *Let V be a vertex operator satisfying the three conditions in Theorem 1.1 and let G be a finite group of automorphisms of V. Then the category of g-twisted V-modules for all $g \in G$ is a G-crossed braided tensor category.*

We also conjecture that the category of g-twisted V-modules for all $g \in G$ is a G-crossed modular tensor category in a suiable sense. Since the definitions of G-crossed modular tensor category in [K3] and [T2] are different, more work needs to be done to find out which definition is the correct one for the category of twisted modules for a vertex operator algebra. But we do believe that this stronger G-crossed modular tensor category conjecture should be true in a suitable sense.

In the case that G is trivial (the group containing only the identity), Conjecture 1.2 and even the stronger G-crossed modular tensor category conjecture is true by Theorem 1.1. Thus the G-crossed modular tensor category conjecture is a natural generalization of Theorem 1.1 to the category of category of g-twisted V-modules for $g \in G$.

In the case that the fixed point subalgebra V^G of V under G satisfies the conditions in Theorem 1.1 above, the category of V^G-modules is a modular tensor category. In this case, Conjecture 1.2 can be proved using the modular tensor category structure on the category of V^G-modules and the results on tensor categories by Kirillov Jr. [K1] [K2] [K3] and Müger [Mü1] [Mü2]. In the special case that G is a finite cyclic group and V satisfies the conditions in Theorem 1.1, Carnahan-Miyamoto [CM] proved that V^G also satisfies the conditions in Theorem 1.1. In the case that G is a finite cyclic group and V is in addition a holomorphic vertex operator algebra (meaning that the only irreducible V-module is V itself), Conjecture 1.2 can be obtained as a consequence of the results of van Ekeren-Möller-Scheithauer [EMS] and Möller [Mö] on the modular tensor category of V^G-modules. Assuming that G is a finite group containing the parity involution and that the category of grading-restricted V^G-modules has a natrual structure of vertex tensor category structure in the sense of [HL1], McRae [Mc] constructed a nonsemisimple G-crossed braided tensor category structure on the category of grading-restricted (generalized) g-twisted V-modules.

For general finite group G, the conjecture that the fixed point subalgebra V^G of V under G also satisfies the conditions in Theorem 1.1 is still open and seems to be a difficult problem. On the other hand, using twisted modules and twisted intertwining operators to construct G-crossed braided tensor categories seems to be a more conceptual and direct approach. If this approach works, we expect that the category of V^G-modules can also be studied using the G-crossed braided tensor category structure on the category of twisted V-modules.

In the case that the vertex operator algebra V does not satisfy the three conditions in Theorem 1.1 and/or the group G is not finite, it is not even clear what should be the precise conjecture. This was proposed as an open problem in [H9].

In the present paper, we prove some initial results in a long term program to prove the conjecture and to solve the open problem above. We introduce a more general notion of twisted intertwining operator than the one introduced by the second author in [H8]. In [H8], the correlation functions obtained from the products and iterates of a twisted intertwining operators and twisted vertex operators are required to be of a special explicit form. But for a twisted intertwining operator in this paper, such correlation functions are not required to have such an explicit form.

As in [H8], we prove some basic properties and construct the skew-symmetry and contragredient isomorphsims for our general twisted intertwining operators. Using such general twisted intertwining operators, we introduce a notion of $P(z)$-tensor product of two twisted modules for $z \in \mathbb{C}^\times$ and give a construction of such a $P(z)$-tensor product under suitable assumptions. We also prove a result showing that under suitable conditions, these assumptions are satisfied.

We need $P(z)$-tensor products for $z \in \mathbb{C}^\times$ because we would like to construct G-crossed vertex tensor categories in the future, not just G-crossed braided tensor categories. Also note that to give the correct notion of $P(z)$-tensor product of twisted modules, we need to use the most general twisted intertwining operators. If we use only certain special set of twisted intertwining operators as in [H8] to define and construct the $P(z)$-tensor products, we would obtain submodules of the correct $P(z)$-tensor products.

We note that in the untwisted case, a $P(z)$-compatibility condition and a $P(z)$-grading-restriction condition (see [HL4] and [HLZ3]) play an important role in the proof of associativity (operator product expansion) of intertwining operators and in the construction of the associativity isomorphisms for the vertex tensor category structure (see [H1] and [HLZ5]). In this paper, we also formulate a $P(z)$-compatibility condition and a $P(z)$-grading-restriction condition and use these conditions to give another construction of the $P(z)$-tensor product. In the untwisted case, the $P(z)$-compatibility condition is formulated using a formula obtained from the Jacobi identity in the definition of intertwining operators (see [HL4] and [HLZ3]). But since in general we do not have a Jacobi identity that can be used as the main axioms in the definition of twisted intertwining operators, our formulation of this condition and the construction of tensor products using this condition are complex analytic and are very different from the the formulation and construction in [HL4] and [HLZ3]. We expect that these two conditions will play the same important role in the future proof of the conjectured associativity of twisted intertwining operators formulated in [H14] (where twisted intertwining operators should be replaced by the most general twisted intertwining operators introduced in this paper).

This paper is organized as follows: In Section 2, we recall the definitions of (generalized) twisted module, lower-bounded (generalized) twisted module and grading-restricted (generalized) twisted module. We then introduce the general notion of twisted intertwining operator mentioned above. In Section 3, we give the skew-symmetry and contragredient isomorphisms for these general twisted intertwining operators. We introduce the notion of $P(z)$-tensor product and give a construction under suitable assumptions in Section 4. We prove a result showing that under suitable conditions, these assumptions are satisfied. We also construct G-crossed commutativity isomorphisms and G-crossed braiding isomorphisms in this section. We give the $P(z)$-compatibility condition and $P(z)$-grading-restriction condition and give another construction of the $P(z)$-tensor product in Section 5.

2 Twisted modules and twisted intertwining operators

We first recall in this section the notion of (generalized) twisted module from [H7]. We then introduce a notion of twisted intertwining operators more general than the one in [H8]. We also give some basic results on such twisted intertwining operators.

For $z \in \mathbb{C}^\times$ and $p \in \mathbb{Z}$, we shall use the notation $l_p(z) = \log|z| + i \arg z + 2\pi p i$, where $0 \le \arg z < 2\pi$. We shall also use the notation $\log z = l_0(z) = \log|z| + i \arg z$. For a vector space U, $p \in \mathbb{Z}$ and a formal series

$$f(x) = \sum_{k=0}^{K} \sum_{n \in \mathbb{C}} a_{n,k} x^n (\log x)^k,$$

where $a_{n,k} \in U$, the series

$$f^p(z) = \sum_{k=0}^{K} \sum_{n \in \mathbb{C}} a_{n,k} e^{n l_p(z)} (l_p(z))^k$$

is called the p-th analytic branch of $f(x)$. We also denote $f^0(z)$ simply by $f(z)$.

Let g be an automorphism of V. We recall the definition of generalized g-twisted V-module first introduced in [H7]. For simplicity, we shall omit the word "generalized" as in [H8]. In particular, in this paper, the vertex operator map for a g-twisted V-module in general contain the logarithm of the variable and the operator $L(0)$ in general does not have to act semisimply.

Definition 2.1 A *g-twisted V-module* is a $\mathbb{C}\times\mathbb{C}/\mathbb{Z}$-graded vector space $W = \coprod_{n\in\mathbb{C},\alpha\in\mathbb{C}/\mathbb{Z}} W_{[n]}^{[\alpha]}$ (graded by *weights* and *g-weights*) equipped with a linear map

$$\begin{aligned} Y_W^g : V \otimes W &\to W\{x\}[\log x], \\ v \otimes w &\mapsto Y_W^g(v, x)w \end{aligned}$$

satisfying the following conditions:

1. The *equivariance property*: For $p \in \mathbb{Z}$, $z \in \mathbb{C}^\times$, $v \in V$ and $w \in W$,
$$(Y_W^g)^{p+1}(gv, z)w = (Y_W^g)^p(v, z)w,$$
where for $p \in \mathbb{Z}$, $(Y_W^g)^p(v, z)$ is the p-th analytic branch of $Y_W^g(v, x)$.

2. The *identity property*: For $w \in W$, $Y_W^g(\mathbf{1}, x)w = w$.

3. The *duality property*: For any $u, v \in V$, $w \in W$ and $w' \in W'$, there exists a maximally-extended multivalued analytic function with preferred branch of the form
$$f(z_1, z_2) = \sum_{i,j,k,l=0}^{N} a_{ijkl} z_1^{m_i} z_2^{n_j} (\log z_1)^k (\log z_2)^l (z_1 - z_2)^{-t}$$
for $N \in \mathbb{N}$, $m_1, \dots, m_N$, $n_1, \dots, n_N \in \mathbb{C}$ and $t \in \mathbb{Z}_+$, such that the series
$$\langle w', (Y_W^g)^p(u, z_1)(Y_W^g)^p(v, z_2)w\rangle = \sum_{n\in\mathbb{C}} \langle w', (Y_W^g)^p(u, z_1)\pi_n(Y_W^g)^p(v, z_2)w\rangle,$$
$$\langle w', (Y_W^g)^p(v, z_2)(Y_W^g)^p(u, z_1)w\rangle = \sum_{n\in\mathbb{C}} \langle w', (Y_W^g)^p(v, z_2)\pi_n(Y_W^g)^p(u, z_1)w\rangle,$$
$$\langle w', (Y_W^g)^p(Y_V(u, z_1 - z_2)v, z_2)w\rangle = \sum_{n\in\mathbb{C}} \langle w', (Y_W^g)^p(\pi_n Y_V(u, z_1 - z_2)v, z_2)w\rangle$$
are absolutely convergent on the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, and their sums are equal to the branch
$$f^{p,p}(z_1, z_2) = \sum_{i,j,k,l=0}^{N} a_{ijkl} e^{m_i l_p(z_1)} e^{n_j l_p(z_2)} l_p(z_1)^k l_p(z_2)^l (z_1 - z_2)^{-t}$$
of $f(z_1, z_2)$ on the region $|z_1| > |z_2| > 0$, the region $|z_2| > |z_1| > 0$, the region given by $|z_2| > |z_1 - z_2| > 0$ and $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$, respectively.

4. The *$L(0)$-grading condition* and *g-grading condition*: Let $L_W^g(0) = \mathrm{Res}_x x Y_W^g(\omega, x)$. Then for $n \in \mathbb{C}$ and $\alpha \in \mathbb{C}/\mathbb{Z}$, $w \in W_{[n]}^{[\alpha]}$, there exist $K, \Lambda \in \mathbb{Z}_+$ such that $(L_W^g(0) - n)^K w = (g - e^{2\pi\alpha i})^\Lambda w = 0$. Moreover, $gY_W^g(u, x)w = Y_W^g(gu, x)gw$ for $u \in V$ and $w \in W$.

5. The $L(-1)$-*derivative property*: For $v \in V$,

$$\frac{d}{dx} Y_W^g(v, x) = Y_W^g(L_V(-1)v, x).$$

A *lower-bounded g-twisted V-module* is a g-twisted V-module W such that for each $n \in \mathbb{C}$, $W_{[n+l]} = 0$ for sufficiently negative real number l. A g-twisted V-module W is said to be *grading-restricted* if it is lower bounded and for each $n \in \mathbb{C}$, $\dim W_{[n]} < \infty$.

For simplicity, we shall sometimes omit the subscript W to denote the twisted vertex operator map Y_W^g by Y^g.

Let (W, Y_W^g) be a g-twisted V-module. Let h be an automorphism of V. We recall the hgh^{-1}-twisted V-module $(W, \phi_h(Y_W^g))$ (see for example [H8]). Let

$$\begin{array}{rcl} \phi_h(Y_W^g) : V \times W & \to & W\{x\}[\log x] \\ v \otimes w & \mapsto & \phi_h(Y^g)(v, x)w \end{array}$$

be the linear map defined by

$$\phi_h(Y_W^g)(v, x)w = Y_W^g(h^{-1}v, x)w.$$

Then the pair $(W, \phi_h(Y_W^g))$ is an hgh^{-1}-twisted V-module. We shall denote the hgh^{-1}-twisted V-module in the proposition above by $\phi_h(W)$.

Note that when $h = g$, we obtain a g-twisted module $\phi_g(W)$ for which the twisted vertex operator is given by

$$\phi_g(Y_W^g)(v, x)w = Y_W^g(g^{-1}v, x)w.$$

But this g-twisted V-module is equivalent to the original g-twisted module W. The equivalence is given by $g^{-1} : W \to W$ since we have

$$g^{-1} Y_W^g(v, x) g = Y_W^g(g^{-1}v, x)w$$

for $v \in V$. By the equivariance property, we have $(Y_W^g)^p(g^{-1}v, x) = (Y_W^g)^{p+1}(v, x)$ and, if

$$Y_W^g(v, x)w = \sum_{k=0}^{K} \sum_{n \in \mathbb{C}} (Y_W^g)_{n,k}(v) x^{-n-1} (\log x)^k$$

for $v \in V$ and $w \in W$, we have

$$Y_W^g(g^{-1}v, x)w = \sum_{k=0}^{K} \sum_{n \in \mathbb{C}} (Y_W^g)_{n,k}(v) e^{2\pi i n} x^{-n-1} (\log x + 2\pi i)^k.$$

We also recall contragredient twisted V-modules (see for example [H8]). Let (W, Y_W^g) be a g-twisted V-module relative to G. Let W' be the graded dual of W. Define a linear map

$$(Y_W^g)' : V \otimes W' \quad \to \quad W'\{x\}[\log x],$$

$$v \otimes w' \mapsto (Y_W^g)'(v,x)w'$$

by

$$\langle (Y_W^g)'(v,x)w', w\rangle = \langle w', Y_W^g(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w\rangle$$

for $v \in V$, $w \in W$ and $w' \in W'$. Then the pair $(W', (Y_W^g)')$ is a g^{-1}-twisted V-module.

Let $M^2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \neq 0,\ z_2 \neq 0,\ z_1 \neq z_2\}$. Let $f(z_1, z_2)$ be a maximally extended multivalued analytic function on M^2 with a preferred single-valued branch $f^e(z_1, z_2)$ on the simply-connected region M_0^2 given by cutting M^2 along the positive real lines in the z_1-, z_2- and $(z_1 - z_2)$-planes, that is, the sets

$$\begin{aligned}
&\{(z_1, z_2) \in M^2 \mid z_1 \in \mathbb{R}_+\},\\
&\{(z_1, z_2) \in M^2 \mid z_2 \in \mathbb{R}_+\},\\
&\{(z_1, z_2) \in M^2 \mid z_1 - z_2 \in \mathbb{R}_+\},
\end{aligned}$$

with these sets attached to the upper half z_1-, z_2- and $(z_1 - z_2)$-planes. Note that given any point $(z_1^0, z_2^0) \in M^2$ and any loop γ based at (z_1^0, z_2^0), we obtain from the preferred branch of $f(z_1, z_2)$ another single-valued branch by going around γ. The resulting single-valued branch depends only on the homotopy class of the loop and is independent of the choices of the base point. Thus we obtain a right action of the fundamental group of M^2 on the set of single-valued branches of $f(z_1, z_2)$. Note that M^2 is homotopically equivalent to the configuration space

$$F_3(\mathbb{C}) = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_i \neq z_j,\ i \neq j\}.$$

So the fundamental group of M^2 is in fact the pure braid group PB_3 and has three generators b_{12}, b_{13} and b_{23}, which are given as follows: Choose the base point to be $(-3, -2)$. Then the generator b_{12} is the homotopy class of the loop given by letting z_1 go counterclockwise around the circle of radius 1 centered at -2 (see Figure 1). The generator b_{23} is the homotopy class

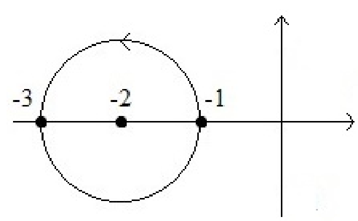

Figure 1: The loop for b_{12}

of the loop given by letting z_2 go counterclockwise around the circle of radius 2 centered at 0 (see Figure 2). The generator b_{13} is the homotopy class of the loop given by letting z_1 go around first the lower half circle of radius 3 centered at 0 counterclockwise, then the upper half circle of radius 2 centered at 1 counterclockwise and finally the lower half circle of radius 1 centered at -2 clockwise (see Figure 3). We know that the pure braid group PB_3 is isomorphic to the group generated by b_{12}, b_{13}, b_{23} with the relations

$$b_{13}b_{12}b_{23} = b_{12}b_{23}b_{13} = b_{23}b_{13}b_{12}.$$

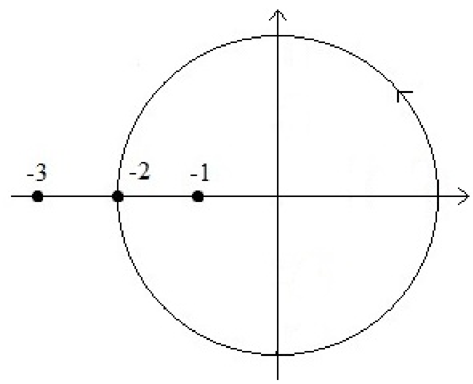

Figure 2: The loop for b_{23}

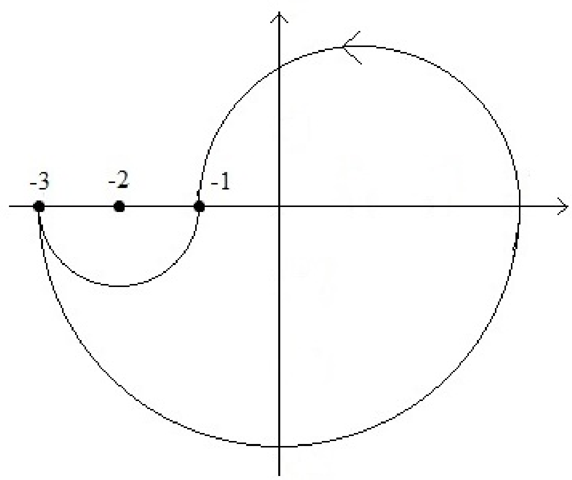

Figure 3: The loop for b_{13}

See [D] for more detailed discussions on M^2, its fundamental group, the pure braid groups and multivalued functions on M^2.

For $a \in \mathrm{PB}_3$, we denote the single-valued branch of $f(z_1, z_2)$ obtained by applying a to $f^e(z_1, z_2)$ by $f^a(z_1, z_2)$.

Let $\delta = (\delta_1, \ldots, \delta_n)$, where $\delta_i \in \{0, \infty\}$, $i = 1, \ldots, n$. Suppose $f(z_1, \ldots, z_n)$ is a multivalued analytic function defined on an open region Ω of $\mathbb{C}^n$. We say $(z_1, \ldots, z_n) = \delta$ is a *component-isolated singularity of* $f(z_1, \ldots, z_n)$ if there exists $r \in \mathbb{R}_+^n$ such that $\Delta^\times(\delta, r) \subset \Omega$. Let $A \in \mathrm{GL}(n, \mathbb{C})$ and $\beta \in \mathbb{C}^n$ (written as a row vector). Then $\zeta_1, \ldots, \zeta_n$ given by

$$(\zeta_1, \ldots, \zeta_n) = (z_1, \ldots, z_n)A - \beta$$

are also independent variables. Define

$$g(\zeta_1, \ldots, \zeta_n) = f\left((\zeta_1, \ldots, \zeta_n)A^{-1} + \beta A^{-1}\right). \tag{2.1}$$

For $\delta \in (\mathbb{C} \cup \{\infty\})^n$, we say that $(\zeta_1, \ldots, \zeta_n) = \delta$ is a *component-isolated sigularity of* $f(z_1, \ldots, z_n)$ if $(\zeta_1, \ldots, \zeta_n) = \delta$ is a component-isolated singularity of $g(\zeta_1, \ldots, \zeta_n)$.

Remark 2.2 Notice that $(\zeta_1, \ldots, \zeta_n)(= (z_1, \ldots, z_n)A - \beta) = \delta$ being a component-isolated singularity of a function is not equivalent to $(z_1, \ldots, z_n) = \delta A^{-1} + \beta A^{-1}$ being a component-isolated singularity of the same function. This is because we have different sets of independent variables. For example, consider $f(z_1, z_2) = \frac{1}{z_1 - z_2}$. Since $(\zeta_1, \zeta_2) = (0, 0)$ is a component-isolated singularity of the function $g(\zeta_1, \zeta_2) = 1/\zeta_1$, we also say that $(z_1 - z_2, z_2) = (0, 0)$ is a component-isolated singularity of $f(z_1, z_2)$. In this case, $z_1 - z_2$ and z_2 are independent variables. However, $(z_1, z_2) = (0, 0)$ is clearly not a component-isolated singularity of $f(z_1, z_2)$. In this case, the independent variables are z_1 and z_2.

Definition 2.3 Let $f(z_1, \ldots, z_n)$ be a multi-valued analytic function defined on an open region of $\mathbb{C}^n$. Let $\delta = (\delta, \ldots, \delta_n) \in \{0, \infty\}^n$. Suppose $(z_1, \ldots, z_n) = \delta$ is a component-isolated singularity of $f(z_1, \ldots, z_n)$. Let $\{f^b(z_1, \ldots, z_n)\}_{b \in B}$ be the set of all single valued branches of $f(z_1, \ldots, z_n)$ near δ with cuts at $z_i \in \mathbb{R}_+$. Then for each $b \in B$, there exists $r_b \in \mathbb{R}_+^n$ such that $f^b(z_1, \ldots, z_n)$ is analytic on $\Delta^\times(\delta, r_b)$. We say that $(z_1, \ldots, z_n) = \delta$ is a *regular singularity* of $f(z_1, \ldots, z_n)$ if there exists $K \in \mathbb{N}$, $D_i = \cup_{j=1}^{N_i} r_j^{(i)} + \mathbb{N}$ (or $D_i = \cup_{j=1}^{N_i} r_j^{(i)} - \mathbb{N}$) where $r_1^{(i)}, \ldots, r_{N_i}^{(i)} \in \mathbb{C}$ for $\delta_i = 0$ (or $\delta_i = \infty$), and $\alpha^{(b)}_{a_1, j_1; \ldots; a_n, j_n} \in \mathbb{C}$, such that on the region $\Delta^\times(\delta, r_b)$, the right-hand-side of the following equation is absolutely convergent, and

$$f^b(z_1, \ldots, z_n) = \sum_{i=1}^{n} \sum_{a_i \in D_i} \sum_{j_1, \ldots, j_n = 0}^{K} \alpha^{(b)}_{a_1, j_2; \ldots; a_n, j_n} z_1^{a_1} (\log z_1)^{j_1} \cdots z_n^{a_n} (\log z_n)^{j_n}. \tag{2.2}$$

If $K = 0$ and $D_i = -n^{(i)} + \mathbb{N}$ in the case $\delta_i = 0$ and $D_i = n^{(i)} - \mathbb{N}$ in the case $\delta_i = \infty$ for $i = 1, \ldots, n$ in ((2.2)), where $n^{(i)} \in \mathbb{Z}_+$ for $i \in I \subset \{1, \ldots, n\}$ and $n^{(i)} = 0$ for $i \neq I$, we say that $z_i = \delta_i$ for $i \in I$ are *poles of* $f(z_1, \ldots, z_n)$. If $n^{(i)} \in \mathbb{Z}_+$ for $i \in I \subset \{1, \ldots, n\}$, $\delta_i = 0$ are the smallest and such that (2.2) holds, we call n_i the orders of the poles $z_i = \delta_i$, respectively, for $i \in \mathbb{I}$. Let $A \in GL(n, \mathbb{C})$ and $\beta \in \mathbb{C}^n$ be the same as above. We say that $(\zeta_1, \ldots, \zeta_n) = \delta$ is a *regular singularity* of $f(z_1, \ldots, z_n)$ if $(\zeta_1, \ldots, \zeta_n) = \delta$ is a regular singularity of $g(\zeta_1, \ldots, \zeta_n)$, where $g(\zeta_1, \ldots, \zeta_n)$ is give by (2.1). We say that $\zeta_i = \delta_i$ for $i \in I$ are poles of $f(z_1, \ldots, z_n)$ with orders n_i, respectively, if $\zeta_i = \delta_i$ for $i \in I$ are poles of $g(\zeta_1, \ldots, \zeta_n)$ with orders n_i, respectively.

Remark 2.4 Let $f(z_1, \ldots, z_n)$ and $g(z_1, \ldots, z_n)$ be multivalued analytic functions with preferred branches defined on an open region. Then $\lambda f(z_1, \ldots, z_n) + \mu g(z_1, \ldots, z_n)$ for $\lambda, \mu \in \mathbb{C}$ and $f(z_1, \ldots, z_n) g(z_1, \ldots, z_n)$ are well defined using the preferred branches and are also multivalued analytic functions on the same region with preferred branches. If $(\zeta_1, \ldots, \zeta_n) = ((z_1, \ldots, z_n)A - \beta =)\delta$ is a regular singular point of both $f(z_1, \ldots, z_n)$ and $g(z_1, \ldots, z_n)$, then it is also a regular singular point for $\lambda f(z_1, \ldots, z_n) + \mu g(z_1, \ldots, z_n)$ and $f(z_1, \ldots, z_n) g(z_1, \ldots, z_n)$. Therefore, the set of multivalued analytic functions on the same region with a preferred branch such that $(\zeta_1, \ldots, \zeta_n) = \delta$ is a regular singular point form a commutative associative algebra over $\mathbb{C}$.

We also need the region

$$M^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq 0,\ z_i \neq z_j,\ i \neq j\}$$

for $n \in \mathbb{Z}_+$.

Definition 2.5 Let g_1, g_2, g_3 be automorphisms of V and let W_1, W_2 and W_3 be g_1-, g_2- and g_3-twisted V-modules, respectively. A *twisted intertwining operator of type* $\binom{W_3}{W_1 W_2}$ is a linear map

$$\begin{aligned} \mathcal{Y} : W_1 \otimes W_2 &\to W_3\{x\}[\log x] \\ w_1 \otimes w_2 &\mapsto \mathcal{Y}(w_1, x)w_2 = \sum_{k=0}^{K} \sum_{n \in \mathbb{C}} \mathcal{Y}_{n,k}(w_1)w_2 x^{-n-1}(\log x)^k \end{aligned}$$

satisfying the following conditions:

1. *The lower truncation property*: For $w_1 \in W_1$ and $w_2 \in W_2$, $n \in \mathbb{C}$ and $k = 0, \dots, K$, $\mathcal{Y}_{n+l,k}(w_1)w_2 = 0$ for $l \in \mathbb{N}$ sufficiently large.

2. The *duality property*: For $u \in V$, $w_1 \in W_1$, $w_2 \in W_2$ and $w_3' \in W_3'$, there exists a maximally extended multivalued analytic function $f(z_1, z_2; u, w_1, w_2, w_3')$ on M^2 with a preferred single-valued branch $f^e(z_1, z_2; u, w_1, w_2, w_3')$ on M_0^2 such that the series

$$\langle w_3', Y_{W_3}^{g_3}(u, z_1)\mathcal{Y}(w_1, z_2)w_2\rangle = \sum_{n \in \mathbb{C}} \langle w_3', Y_{W_3}^{g_3}(u, z_1)\pi_n \mathcal{Y}(w_1, z_2)w_2\rangle, \tag{2.3}$$

$$\langle w_3', \mathcal{Y}(w_1, z_2)Y_{W_2}^{g_2}(u, z_1)w_2\rangle = \sum_{n \in \mathbb{C}} \langle w_3', \mathcal{Y}(w_1, z_2)\pi_n Y_{W_2}^{g_2}(u, z_1)w_2\rangle, \tag{2.4}$$

$$\langle w_3', \mathcal{Y}(Y_{W_1}^{g_1}(u, z_1 - z_2)w_1, z_2)w_2\rangle = \sum_{n \in \mathbb{C}} \langle w_3', \mathcal{Y}(\pi_n Y_{W_1}^{g_1}(u, z_1 - z_2)w_1, z_2)w_2\rangle \tag{2.5}$$

are absolutely convergent on the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively. Moreover, their sums are equal to $f^e(z_1, z_2; u, w_1, w_2, w_3')$ on the region given by $|z_1| > |z_2| > 0$ and $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$, the region given by $|z_2| > |z_1| > 0$ and $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$, the region given by $|z_2| > |z_1 - z_2| > 0$ and $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$, respectively.

3. The *convergence and analytic extension for products with more than one twisted vertex operators*: For $k \in \mathbb{N} + 3$, $u_1, \dots, u_{k-1} \in V$, $w_1 \in W_1$, $w_2 \in W_2$ and $w_3' \in W_3'$, the series

$$\begin{aligned} &\langle w_3', Y_{W_3}^{g_3}(u_1, z_1) \cdots Y_{W_3}^{g_3}(u_{k-1}, z_{k-1})\mathcal{Y}(w_1, z_k)w_2\rangle \\ &\quad = \sum_{n_1, \dots, n_{k-1} \in \mathbb{C}} \langle w_3', Y_{W_3}^{g_3}(u_1, z_1)\pi_{n_1} \cdots \pi_{n_{k-2}} Y_{W_3}^{g_3}(u_{k-1}, z_{k-1})\pi_{k-1}\mathcal{Y}(w_1, z_2)w_2\rangle \end{aligned}$$

is absolutely convergent on the region $|z_1| > \cdots > |z_k| > 0$ and can be maximally extended to a multivalued analytic function on the region M^k such that all the component-isolated singularities of this function are regular.

4. The $L(-1)$-*derivative property*:

$$\frac{d}{dx}\mathcal{Y}(w_1, x) = \mathcal{Y}(L(-1)w_1, x).$$

Remark 2.6 For simplicity, in the duality property in Definition 2.5, we use only the preferred branches of the twisted intertwining operator and the twisted vertex operators. One can derive what the products and iterates of other branches of a twisted intertwining operator converge to using the actions of the elements of PB_3 on the single-valued branch $f^e(z_1, z_2; u, w_1, w_2, w_3')$ of the multivalued function $f(z_1, z_2; u, w_1, w_2, w_3')$ in the definition. Let $\mathcal{Y}$ be a twisted intertwining operator of type $\binom{W_3}{W_1W_2}$. For any $p_1, p_2, p_{12} \in \mathbb{Z}$, the series

$$\langle w_3', (Y_{W_3}^{g_3})^{p_1}(u, z_1)\mathcal{Y}^{p_2}(w_1, z_2)w_2\rangle = \sum_{n\in\mathbb{C}}\langle w_3', (Y_{W_3}^{g_3})^{p_1}(u, z_1)\pi_n\mathcal{Y}^{p_2}(w_1, z_2)w_2\rangle, \quad (2.6)$$

$$\langle w_3', \mathcal{Y}^{p_2}(w_1, z_2)(Y_{W_2}^{g_2})^{p_1}(u, z_1)w_2\rangle = \sum_{n\in\mathbb{C}}\langle w_3', \mathcal{Y}^{p_2}(w_1, z_2)\pi_n(Y_{W_2}^{g_2})^{p_1}(u, z_1)w_2\rangle, \quad (2.7)$$

$$\langle w_3', \mathcal{Y}^{p_2}((Y_{W_1}^{g_1})^{p_{12}}(u, z_1 - z_2)w_1, z_2)w_2\rangle = \sum_{n\in\mathbb{C}}\langle w_3', \mathcal{Y}^{p_2}(\pi_n(Y_{W_1}^{g_1})^{p_{12}}(u, z_1 - z_2)w_1, z_2)w_2\rangle \quad (2.8)$$

are absolutely convergent on the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively. Moreover, their sums are equal to the branches

$$f^{(b_{13}b_{12})^{p_1}b_{23}^{p_2}}(z_1, z_2; u, w_1, w_2, w_3'),$$
$$f^{(b_{12}b_{23})^{p_2}b_{13}^{p_1}}(z_1, z_2; u, w_1, w_2, w_3'),$$
$$f^{(b_{23}b_{13})^{p_2}b_{12}^{p_{12}}}(z_1, z_2; u, w_1, w_2, w_3'),$$

respectively, of $f(z_1, z_2; u, w_1, w_2, w_3')$ on the region given by $|z_1| > |z_2| > 0$ and $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$, the region given by $|z_2| > |z_1| > 0$ and $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$, the region given by $|z_2| > |z_1 - z_2| > 0$ and $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$, respectively. See [D] for more details.

Proposition 2.7 *Let g_1, g_2, g_3 be automorphisms of V, W_1, W_2, W_3 g_1-, g_2-, g_3-twisted generalized V-modules and $\mathcal{Y}$ a twisted intertwining operator of type $\binom{W_3}{W_1W_2}$. Assume that the map $u \mapsto Y_{W_3}^{g_3}(u, x)$ is injective and $\mathcal{Y}$ is surjective in the sense that the coefficients of the series $\mathcal{Y}(w_1, x)w_2$ for $w_1 \in W_1$, $w_2 \in W_2$ span W_3. Then $g_3 = g_1g_2$.*

Proof. By the definition of twisted intertwining operator, for $u \in V$, $w_1 \in W_1$, $w_2 \in W_2$ and $w_3' \in W_3'$, there exists a multivalued analytic function $f(z_1, z_2; u, w_1, w_2, w_3')$ on M^2 with a preferred single-valued branch $f^e(z_1, z_2; u, w_1, w_2, w_3')$ on M_0^2 such that (2.3), (2.4) and (2.5) are absolutely convergent to $f^e(z_1, z_2; u, w_1, w_2, w_3')$ on the corresponding regions in given in Defintion 2.5. In particular, on the region $|z_1| > |z_2| > 0$, $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ the sum of the series $\langle w_3', Y_{W_3}^{g_3}(g_3u, z_1)\mathcal{Y}(w_1, z_2)w_2\rangle$ is equal to $f^e(z_1, z_2; g_3u, w_1, w_2, w_3')$. By the equivariance property for W_3,

$$\langle w_3', Y_{W_3}^{g_3}(g_3u, z_1)\mathcal{Y}(w_1, z_2)w_2\rangle = \langle w_3', (Y_{W_3}^{g_3})^{-1}(u, z_1)\mathcal{Y}(w_1, z_2)w_2\rangle, \quad (2.9)$$

where as above $(Y_{W_3}^{g_3})^{-1}$ is the (-1)-th branch of the twisted vertex operator map $Y_{W_3}^{g_3}$. But the right-hand side of (2.9) can be obtained from $\langle w_3', Y_{W_3}^{g_3}(u, z_1)\mathcal{Y}(w_1, z_2)w_2\rangle$ on the region $|z_1| > |z_2| > 0$, $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ by letting z_1 go around clockwise a circle of radius larger than $|z_2|$. The homotopy class of such a circle is equal to $(b_{13}b_{12})^{-1}$. So (2.9) gives

$$f^e(z_1, z_2; g_3u, w_1, w_2, w_3') = f^{(b_{13}b_{12})^{-1}}(z_1, z_2; u, w_1, w_2, w_3')$$

or equivalently

$$f^{b_{13}b_{12}}(z_1, z_2; g_3u, w_1, w_2, w_3') = f^e(z_1, z_2; u, w_1, w_2, w_3'). \tag{2.10}$$

Similarly, on the region $|z_2| > |z_1| > 0$, $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$, the sum of the series $\langle w_3', \mathcal{Y}(w_1, z_2)Y_{W_3}^{g_2}(g_2u, z_1)w_2\rangle$ is equal to $f^e(z_1, z_2; g_3u, w_1, w_2, w_3')$. By the equivariance property for W_2,

$$\langle w_3', \mathcal{Y}(w_1, z_2)Y_{W_3}^{g_2}(g_2u, z_1)w_2\rangle = \langle w_3', \mathcal{Y}(w_1, z_2)(Y_{W_2}^{g_2})^{-1}(u, z_1)w_2\rangle, \tag{2.11}$$

where $(Y_{W_2}^{g_2})^{-1}$ is the (-1)-th branch of the twisted vertex operator map $Y_{W_2}^{g_2}$. The right-hand side of (2.11) can be obtained from $\langle w_3', \mathcal{Y}(w_1, z_2)Y_{W_3}^{g_2}(u, z_1)w_2\rangle$ on the region $|z_2| > |z_1| > 0$, $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$ by letting z_1 go around clockwise a circle of radius less than $|z_2|$. Such a circle as a loop must have a base point in the region $|z_2| > |z_1| > 0$, $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$. But there is a canonical isomorphism between the fundamental group of M^2 with such a base point and PB_3 which has a base point $(-3, -2)$. To see how the loop given by the circle above acts on the single-valued branches of $f(z_1, z_2; u, w_1, w_2, w_3')$, we need to find the element of PB_3 corresponding to this loop. We choose the following loop γ based at $(-3, -2)$: The first part γ_1 of γ is the lower half circle centered at -1 with radius 1 from -3 to -1 for z_1 and trivial ofor z_2 (meaning z_2 is always equal to -2). The second part γ_2 is the counterclockwise circle centered at 0 with radius 1 based at -1 for z_1 and trivial for z_2. The third part $\gamma_3 = \gamma_1^{-1}$ is also the the lower half circle centered at -1 with radius 1 but from -1 to -3 for z_1 and trivial for z_2. It is clear that γ is homotopically equivalent to the loop given in Figure 3. Then we have $b_{13} = [\gamma] = [\gamma_1][\gamma_2][\gamma_1]^{-1}$, where $[\gamma]$ for a path γ means its homotopy class. Equivalently, we have $[\gamma_2] = [\gamma_1]^{-1}b_{13}[\gamma_1]$. When we let z_1 go from -1 to -3 along γ_1^{-1}, since γ_1^{-1} passes the cut along the positive real line in the $z_1 - z_2$-plane, the single-valued branch $f^e(z_1, z_2; u, w_1, w_2, w_3')$ is changed to the single-valued branch $f^{b_{12}^{-1}}(z_1, z_2; u, w_1, w_2, w_3')$. Similarly, when we let z_1 go from -1 to -3 along γ_1, $f^b(z_1, z_2; u, w_1, w_2, w_3')$ is changed to the single-valued branch $f^{bb_{12}}(z_1, z_2; u, w_1, w_2, w_3')$ for any $b \in \mathrm{PB}_3$. Thus when we let z_1 go around the loop γ_2, $f^e(z_1, z_2; u, w_1, w_2, w_3')$ is changed to $f^{b_{12}^{-1}b_{13}b_{12}}(z_1, z_2; u, w_1, w_2, w_3')$, that is,

$$f^{[\gamma_2]}(z_1, z_2; u, w_1, w_2, w_3') = f^{b_{12}^{-1}b_{13}b_{12}}(z_1, z_2; u, w_1, w_2, w_3').$$

Note that the circle γ_2^{-1} is exactly the circle we use to obtain the right-hand side of (2.11) from $\langle w_3', \mathcal{Y}(w_1, z_2)Y_{W_3}^{g_2}(u, z_1)w_2\rangle$ on the region $|z_2| > |z_1| > 0$, $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$. So the right-hand side of (2.11) is equal to

$$f^{[\gamma_2^{-1}]}(z_1, z_2; u, w_1, w_2, w_3') = f^{b_{12}^{-1}b_{13}^{-1}b_{12}}(z_1, z_2; u, w_1, w_2, w_3').$$

But the sum of the left-hand side of (2.11) is equal to $f^e(z_1,z_2;g_2u,w_1,w_2,w_3')$. So we obtain

$$f^e(z_1,z_2;g_2u,w_1,w_2,w_3') = f^{b_{12}^{-1}b_{13}^{-1}b_{12}}(z_1,z_2;u,w_1,w_2,w_3')$$

or equivalently

$$f^{b_{12}^{-1}b_{13}b_{12}}(z_1,z_2;g_2u,w_1,w_2,w_3') = f^e(z_1,z_2;u,w_1,w_2,w_3'). \tag{2.12}$$

Similarly we also have

$$f^{b_{12}}(z_1,z_2;g_1u,w_1,w_2,w_3') = f^e(z_1,z_2;u,w_1,w_2,w_3'). \tag{2.13}$$

Using the right action of PB_3 on the set of single-valued branches of the multivalued analytic function $f(z_1,z_2;u,w_1,w_2,w_3')$, for $b\in \mathrm{PB}_3$, we obtain from (2.13)

$$f^{b_{12}b}(z_1,z_2;g_2u,w_1,w_2,w_3') = f^b(z_1,z_2;u,w_1,w_2,w_3'). \tag{2.14}$$

From (2.10), (2.12) and (2.14), we have

$$\begin{aligned} f^{b_{13}b_{12}}(z_1,z_2;g_3u,w_1,w_2,w_3') &= f^e(z_1,z_2;u,w_1,w_2,w_3')\\ &= f^{b_{12}^{-1}b_{13}b_{12}}(z_1,z_2;g_2u,w_1,w_2,w_3')\\ &= f^{b_{12}(b_{12}^{-1}b_{13}b_{12})}(z_1,z_2;g_1g_2u,w_1,w_2,w_3')\\ &= f^{b_{13}b_{12}}(z_1,z_2;g_1g_2u,w_1,w_2,w_3') \end{aligned} \tag{2.15}$$

When $|z_1|>|z_2|>0$ and $|\arg(z_1-z_2)-\arg z_1|<\frac{\pi}{2}$, the left- and right-hand sides of (2.15) are equal to the sum of $\langle w_3',(Y_{W_3}^{g_3})^1(g_3u,z_1)\mathcal{Y}(w_1,z_2)w_2\rangle$ and $\langle w_3',(Y_{W_3}^{g_3})^1(g_1g_2u,z_1)\mathcal{Y}(w_1,z_2)w_2\rangle$, respectively. Therefore, we obtain

$$\langle w_3',(Y_{W_3}^{g_3})^1(g_1g_2u-g_3u,z_1)\mathcal{Y}(w_1,z_2)w_2\rangle = 0 \tag{2.16}$$

for $w_1\in W_1, w_2\in W_2, w_3\in W_3'$. Since $\mathcal{Y}$ is surjective in the sense above and w_3' is arbitrary, (2.16) implies $(Y_{W_3}^{g_3})^1(g_1g_2u-g_3u,z_1)=0$. But the map given by $v\to Y_{W_3}^{g_3}(v,x)$ is injective, we obtain $g_1g_2u-g_3u=0$ for $u\in V$. So we have $g_3=g_1g_2$. ∎

3 Skew-symmetry and contragredient isomorphisms

Let g_1,g_2 be automorphisms of V, W_1, W_2 and W_3 g_1-, g_2- and g_1g_2-twisted V-modules and $\mathcal{Y}$ a twisted intertwining operator of type $\binom{W_3}{W_1W_2}$. We define linear maps

$$\begin{aligned} \Omega_\pm(\mathcal{Y}): W_2\otimes W_1 &\to W_3\{x\}[\log x]\\ w_2\otimes w_1 &\mapsto \Omega_\pm(\mathcal{Y})(w_2,x)w_1 \end{aligned}$$

by

$$\Omega_\pm(\mathcal{Y})(w_2,x)w_1 = e^{xL(-1)}\mathcal{Y}(w_1,y)w_2\Big|_{y^n=e^{\pm n\pi \mathbf{i}}x^n,\ \log y=\log x\pm\pi\mathbf{i}} \tag{3.1}$$

for $w_1 \in W_1$ and $w_2 \in W_2$. Note that we can also define Ω_p for $p \in \mathbb{Z}$ by changing $\pm$ in the right-hand side of (3.1) to $+p$. But we will not discuss Ω_p in this paper.

From the definition (3.1), for $p \in \mathbb{Z}$, $w_1 \in W_1$, $w_2 \in W_2$ and $z \in \mathbb{C}^\times$,

$$\begin{aligned}
\Omega_\pm(\mathcal{Y})^p(w_2, z)w_1 &= \Omega_\pm(\mathcal{Y})(w_2, x)w_1\Big|_{x^n = e^{nl_p(z)},\ \log x = l_p(z)} \\
&= \left(e^{xL(-1)}\mathcal{Y}(w_1, y)w_2\Big|_{y^n = e^{\pm n\pi \mathbf{i}}x^n,\ \log y = \log x \pm \pi \mathbf{i}}\right)\Bigg|_{x^n = e^{nl_p(z)},\ \log x = l_p(z)} \\
&= e^{zL(-1)}\mathcal{Y}(w_1, y)w_2\Big|_{y^n = e^{n(l_p(z) \pm \pi \mathbf{i})},\ \log y = l_p(z) \pm \pi \mathbf{i}}.
\end{aligned}$$

When $\arg z < \pi$ and $\arg z \geq \pi$, $\arg(-z) = \arg z + \pi$ and $\arg(-z) = \arg z - \pi$, respectively. Hence

$$e^{zL(-1)}\mathcal{Y}(w_1, y)w_2\Big|_{y^n = e^{n(l_p(z) + \pi \mathbf{i})},\ \log y = l_p(z) + \pi \mathbf{i}} = e^{zL(-1)}\mathcal{Y}^p(w_1, -z)w_2$$

when $\arg z < \pi$ and

$$e^{zL(-1)}\mathcal{Y}(w_1, y)w_2\Big|_{y^n = e^{n(l_p(z) - \pi \mathbf{i})},\ \log y = l_p(z) - \pi \mathbf{i}} = e^{zL(-1)}\mathcal{Y}^p(w_1, -z)w_2$$

when $\arg z \geq \pi$. In particular, for $w_1 \in W_1$, $w_2 \in W_2$ and $z \in \mathbb{C}^\times$ satisfying $\arg z < \pi$ and $\arg z \geq \pi$, we have

$$\Omega_+(\mathcal{Y})^p(w_2, z)w_1 = e^{zL(-1)}\mathcal{Y}^p(w_1, -z)w_2 \tag{3.2}$$

and

$$\Omega_-(\mathcal{Y})^p(w_2, z)w_1 = e^{zL(-1)}\mathcal{Y}^p(w_1, -z)w_2, \tag{3.3}$$

respectively.

Theorem 3.1 *The linear maps $\Omega_+(\mathcal{Y})$ and $\Omega_-(\mathcal{Y})$ are twisted intertwining operators of types $\binom{W_3}{W_2\phi_{g_2^{-1}}(W_1)}$ and $\binom{W_3}{\phi_{g_1}(W_2)W_1}$, respectively (recall the definition of ϕ_g for an automorphism g of V in Section 3).*

Proof. The main difference between the proof here and the proof of Theorem 5.1 in [H8] is that here we cannot use the explicit form of the correlation functions in [H8] obtained from the products and iterates of a twisted intertwining operators and twisted vertex operators. So our proof here is much more complicated and involves some technical convergence and analytic extension results, even though the idea is the same as in the proof of Theorem 5.1 in [H8].

Let $u \in V$, $w_1 \in W_1$, $w_2 \in W_2$ and $w_3' \in W_3'$. As in the proof of Theorem 5.1 in [H8], we use $f(z_1, z_2; u, w_1, w_2, w_3')$ to denote the multivalued analytic function in the duality property for the twisted intertwining operator $\mathcal{Y}$ with the preferred branch $f^e(z_1, z_2; u, w_1, w_2, w_3')$. Note that $f(z_1, z_2; u, w_1, w_2, w_3')$ in [H8] is of the particular form in the definition of twisted intertwining operator there. But in this proof, it is a maximally-extended multivalued analytic function on M^2 with regular singular points at $z_1, z_2 = 0$ and $z_1 - z_2 = 0$ and in general might not have the special form in [H8].

Define

$$g_{\pm}(z_1, z_2; u, w_2, w_1, w_3') = f(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2 L'(1)} w_3') \tag{3.4}$$

and choose the preferred branch $g^e_{\pm}(z_1, z_2; u, w_2, w_1, w_3')$ of $g_{\pm}(z_1, z_2; u, w_2, w_1, w_3')$ as follows: On the subregion $|z_1| > |z_2| > 0$, $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ and $\arg z_2 < \pi$ (for Ω_+) or $\arg z_2 \geq \pi$ (for Ω_-) of M_0^2, let

$$g^e_{\pm}(z_1, z_2; u, w_2, w_1, w_3') = f^e(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2 L'(1)} w_3'). \tag{3.5}$$

For general $(z_1, z_2) \in M_0^2$, we define $g^e_{\pm}(z_1, z_2; u, w_2, w_1, w_3')$ to be the unique analytic extension on M_0^2.

When $|z_1| > |z_2| > 0$, $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ and $\arg z_2 < \pi$ (for Ω_+) or $\arg z_2 \geq \pi$ (for Ω_-), from (3.2) and (3.3) and the $L(-1)$-derivative property for $Y^{g_3}_{W_3}$, we have

$$\begin{aligned}
&\langle w_3', Y^{g_3}_{W_3}(u, z_1)\Omega_{\pm}(\mathcal{Y})(w_2, z_2)w_1\rangle \\
&\quad = \langle w_3', Y^{g_3}_{W_3}(u, x_1)\Omega_{\pm}(\mathcal{Y})(w_2, x_2)w_1\rangle\Big|_{x_1^n = e^{n\log z_1}, \log x_1 = \log z_1, x_2^n = e^{n \log z_2}, \log x_2 = \log z_2} \\
&\quad = \langle w_3', Y^{g_3}_{W_3}(u, x_1)e^{-yL(-1)}\mathcal{Y}(w_1, y)w_2\rangle\Big|_{x_1^n = e^{n\log z_1}, \log x_1 = \log z_1, y^n = e^{n \log(-z_2)}, \log y = \log(-z_2)} \\
&\quad = \langle e^{z_2 L'(1)}w_3', (Y^{g_3}_{W_3})(u, x_1 + y)\mathcal{Y}(w_1, y)w_2\rangle\Big|_{x_1^n = e^{n\log z_1}, \log x_1 = \log z_1, y^n = e^{n \log(-z_2)}, \log y = \log(-z_2)}.
\end{aligned} \tag{3.6}$$

We first prove that the right-hand side of (3.6) is absolutely convergent on the region $|z_1| > |z_2| > 0$ and is convergent to $f^e(z_1, z_2; u, w_1, w_2, w_3')$ on the region $|z_1| > |z_2| > 0$, $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$. The proof is in fact the same as the proof that the right-hand side of (9.170) in [HLZ5] is absolutely convergent in the region $|z_2| > |z_0| > 0$. Here we give a slightly different proof.

We can always take $u \in V$, $w_1 \in W_1$, $w_2 \in W_2$ and $e^{z_2 L'(1)} w_3' \in W_3'$ to be homogeneous. Let $\Delta = -\mathrm{wt}\, e^{z_2 L'(1)} w_3' + \mathrm{wt}\, u + \mathrm{wt}\, w_1 + \mathrm{wt}\, w_2$. Let

$$D = \{n \in \mathbb{C} \mid \text{there exist } i, j \in \mathbb{N}, \text{such that } \langle e^{z_2 L'(1)} w_3', \left(Y^{g_3}_{W_3}\right)_{\Delta - n - 2, j}(u)\mathcal{Y}_{n,i}(w_1)\, w_2\rangle \neq 0\}$$

and $M, N \in \mathbb{N}$ such that $\left(Y^{g_3}_{W_3}\right)_{m,j}(u) = 0$ for $m \in \mathbb{C}$, $j > M$ and $\mathcal{Y}_{n,i}(w_1) = 0$ for $i > N$. Then by the lower truncation property of $\mathcal{Y}$, the fact that u is a finite sum of generalized eigenvectors of g_3 and the equivariance property of the g_3-twisted module W_3, we know that there exist a finite subset A of $\mathbb{C}/\mathbb{Z}$ and $R_\mu \in \mu$ for each $\mu \in A$ such that

$$D \subset \bigcup_{\mu \in A} (R_\mu - \mathbb{N}). \tag{3.7}$$

Let

$$a_{n,j,i} = \left\langle e^{z_2 L'(1)} w_3', \left(Y^{g_3}_{W_3}\right)_{\Delta - n - 2, j}(u)\, (\mathcal{Y})_{n,i}(w_1)\, w_2 \right\rangle \in \mathbb{C}$$

for $n \in D$, $0 \leq j \leq M$ and $0 \leq n \leq N$. Then, by the convergence of (2.3), the $L(-1)$-derivative properties for $Y_{W_3}^{g_3}$ and $\mathcal{Y}$, and Proposition 7.9 in [HLZ4], we know that the triple series

$$\sum_{n\in D}\sum_{j=0}^{M}\sum_{i=0}^{N} a_{n,j,i}e^{(-\Delta+n+1)\log z_1}(\log z_1)^j e^{(-n-1)\log z_2}(\log z_2)^i \tag{3.8}$$

is absolutely convergent on the region given by $|z_1| > |z_2| > 0$ and is convergent on the region $|z_1| > |z_2| > 0$, $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ to

$$f^e(z_1, z_2; u, w_1, w_2, e^{z_2L'(1)}w_3') = \langle e^{z_2L'(1)}w_3', Y_{W_3}^{g_3}(u, z_1)\mathcal{Y}(w_1, z_2)w_2\rangle. \tag{3.9}$$

For $n \in D$, $j = 0, \dots, M$, $k \in \mathbb{Z}_{\geq 0}$, $s = 0, \dots, j$, let $b_{n,j,k,s} \in \mathbb{C}$ be the numbers defined in (A.6). Then

$$\sum_{m\in D-\mathbb{N}}\sum_{s=0}^{M}\sum_{i=0}^{N}\left(\sum_{j=s}^{M}\sum_{\substack{n-k=m\\ n\in D, k\in\mathbb{N}}} a_{n,j,i}b_{n,j,k,s}\right)e^{(-\Delta+m+1)\log z_1}(\log z_1)^s e^{(-m-1)\log(-z_2)}(\log(-z_2))^i$$

is equal to the right-hand side of (3.6) and, by Lemma A.1, is absolutely convergent on the region $|z_1| > |z_2| > 0$ and is convergent to $f^e(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2L'(1)}w_3')$ on the region $|z_1| > |z_2| > 0$, $|\arg z_1 - \arg(z_1 - z_2)| < \frac{\pi}{2}$.

Now it is easy to see that the left-hand side of (3.6) is absolutely convergent on the region $|z_1| > |z_2| > 0$ and its sum is equal to $g^e_{\pm}(z_1, z_2; u, w_1, w_2, w_3')$ on the region $|z_1| > |z_2| > 0$, $|\arg z_1 - \arg(z_1 - z_2)| < \frac{\pi}{2}$. In fact, we know that the left-hand side of (3.6) as a series is equal to the right-hand side of (3.6) on the region $|z_1| > |z_2| > 0$, $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ and $\arg z_2 < \pi$ (for Ω_+) or $\arg z_2 \geq \pi$ (for Ω_-). But we have just proved that the right-hand side of (3.6) is absolutely convergent on the larger region $|z_1| > |z_2| > 0$. The left-hand side of (3.6) is also a series of the same form as (3.8). In particular, its absolute convergence on the smaller region $|z_1| > |z_2| > 0$, $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ and $\arg z_2 < \pi$ (for Ω_+) or $\arg z_2 \geq \pi$ (for Ω_-) implies its absolute convergence on the larger region $|z_1| > |z_2| > 0$.

On the region $|z_1| > |z_2| > 0$, $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ and $\arg z_2 < \pi$ (for Ω_+) or $\arg z_2 \geq \pi$ (for Ω_-), by (3.6) and the discussion above, the left-hand side of (3.6) is convergent to $f^e(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2L'(1)}w_3')$, which in turn is by definition equal to $g^e_{\pm}(z_1, z_2; u, w_1, w_2, w_3')$ on the same region. We know that the left-hand side of (3.6) on the region given by $|z_1| > |z_2| > 0$ and $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ with cuts along the positive real lines on the z_1- and z_2-planes is convergent to the analytic extension of the sum of the left-hand side of (3.6) on the smaller region $|z_1| > |z_2| > 0$, $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ and $\arg z_2 < \pi$ (for Ω_+) or $\arg z_2 \geq \pi$ (for Ω_-). Also, by definition, $g^e_{\pm}(z_1, z_2; u, w_1, w_2, w_3')$ on M_0^2 is obtained by analytically extending $g^e_{\pm}(z_1, z_2; u, w_1, w_2, w_3')$ on the smaller region $|z_1| > |z_2| > 0$, $|\arg(z_1-z_2)-\arg z_1| < \frac{\pi}{2}$ and $\arg z_2 < \pi$ (for Ω_+) or $\arg z_2 \geq \pi$ (for Ω_-). Thus the left-hand side of (3.6) on the region given by $|z_1| > |z_2| > 0$ and $|\arg(z_1-z_2)-\arg z_1| < \frac{\pi}{2}$ is absolutely convergent to $g^e_{\pm}(z_1, z_2; u, w_1, w_2, w_3')$ on M_0^2.

Generalizing the proof of the convergence of (3.6) above, we can prove that

$$\langle w_3', Y_{W_3}^{g_3}(u_1, z_1) \cdots Y_{W_3}^{g_3}(u_{k-1}, z_{k-1}) \Omega_{\pm}(\mathcal{Y})(w_2, z_k) w_1 \rangle \tag{3.10}$$

is absolutely convergent on the region $|z_1| > \cdots > |z_k| > 0$ and can be maximally extended to a multivalued analytic function on the region M^k for $k \in \mathbb{N}+3$, $u_1, \dots, u_{k-1} \in V$, $w_1 \in W_1$, $w_2 \in W_2$ and $w_3' \in W_3'$. In fact, generalizing (3.6), we see that (3.10) is equal to

$$\begin{aligned}\langle e^{z_2 L'(1)} w_3', (Y_{W_3}^{g_3})(u_1, x_1 + y) \cdots (Y_{W_3}^{g_3})(u_{k-1}, x_{k-1} + y) \cdot& \\ \cdot \mathcal{Y}(w_1, y) w_2 \rangle \Big|_{x_i^n = e^{n \log z_i}, \log x_i = \log z_i, i=1,\dots,k-1, y^n = e^{n \log(-z_k)}, \log y = \log(-z_k)}&. \end{aligned}\tag{3.11}$$

From Definition 2.1, we see that the convergence and analytic extension of (3.11) on the region $|z_1| > \cdots > |z_k| > 0$ is equivalent to the convergence and analytic extension of

$$\begin{aligned}\prod_{1 \le i < j \le k-1} (z_i - z_j)^{M_{ij}} \langle e^{z_2 L'(1)} w_3', (Y_{W_3}^{g_3})(u_1, x_1 + y) \cdots (Y_{W_3}^{g_3})(u_{k-1}, x_{k-1} + y) \cdot& \\ \cdot \mathcal{Y}(w_1, y) w_2 \rangle \Big|_{x_i^n = e^{n \log z_i}, \log x_i = \log z_i, i=1,\dots,k-1, y^n = e^{n \log(-z_k)}, \log y = \log(-z_k)}& \end{aligned}\tag{3.12}$$

on the region $|z_i| > |z_k| > 0$ for $i = 1, \dots, k-1$, $z_i \neq z_j$ for $i \neq j$, where $M_{ij} \in \mathbb{Z}_+$ for $i \neq j$ satisfy $x^{M_{ij}} Y_V(u_i, x) u_j \in V[[x]]$. Note that Lemma A.1 can be generalized to the case of more than two variables for a series of the form (3.12). Using the convergence of products with more than one twisted vertex operators for the twisted intertwining operator $\mathcal{Y}$ and this generalization of Lemma A.1, we see that (3.12) is absolutely convergent on the region $|z_i| > |z_k| > 0$ for $i = 1, \dots, k-1$, $z_i \neq z_j$ for $i \neq j$ and its sum has analytic extension on the region M^k. Thus (3.10) is absolutely convergent on the region $|z_1| > \cdots > |z_k| > 0$ and its sum has maximal analytic extension on the region M^k.

When $|z_2| > |z_1| > 0$ and $\arg z_2 \geq \pi$,

$$\begin{aligned}&\langle w_3', \Omega_-(\mathcal{Y})(w_2, z_2) Y_{W_1}^{g_1}(u, z_1) w_1 \rangle \\ &\quad = \langle w_3', e^{z_2 L(-1)} \mathcal{Y}(Y_{W_1}^{g_1}(u, z_1) w_1, -z_2) w_2 \rangle \\ &\quad = \langle e^{z_2 L'(1)} w_3', \mathcal{Y}(Y_{W_1}^{g_1}(u, (z_1 - z_2) - (-z_2)) w_1, -z_2) w_2 \rangle \end{aligned}\tag{3.13}$$

converges absolutely and if in addition, $|\arg(z_1 - z_2) - \arg(-z_2)| < \frac{\pi}{2}$, its sum is equal to

$$f^e(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2 L'(1)} w_3'). \tag{3.14}$$

Note that by definition, (3.14) is a single-valued analytic function on the set $\widetilde{M_0^2}$ given by cutting M^2 along the postive real line in the z_1- and $(z_1 - z_2)$-planes and along the negative real line in the z_2-plane, with these positive real lines in the z_1- and $(z_1 - z_2)$-planes attached to the upper half z_1- and $(z_1 - z_2)$-planes and the negative real line in the z_2-plane attached to the lower half z_2-plane. Then the subset

$$\left\{ (z_1, z_2) \in M_0^2 \;\Big|\; |z_2| > |z_1| > 0, \arg z_2 \geq \pi, |\arg(z_1 - z_2) - \arg(-z_2)| < \frac{\pi}{2} \right\} \tag{3.15}$$

of $\widetilde{M}_0^2$ is also a subset of M_0^2. But on the subset of M_0^2 given by $|z_1| > |z_2| > 0$, $\arg z_2 \geq \pi$, by definition, (3.14) is equal to $g_-^e(z_1, z_2; u, w_2, w_1, w_3')$. Since $g_-^e(z_1, z_2; u, w_2, w_1, w_3')$ on M_0^2 is obtained by analytic extension, we see that (3.14) is equal to $g_-^e(z_1, z_2; u, w_2, w_1, w_3')$ also on the subset (3.15). Thus the left-hand side of (3.13) is absolutely convergent to $g_-^e(z_1, z_2; u, w_2, w_1, w_3')$ when $|z_2| > |z_1| > 0$, $|\arg(z_1 - z_2) - \arg(-z_2)| < \frac{\pi}{2}$ and $\arg z_2 \geq \pi$. Since $g_-^e(z_1, z_2; u, w_2, w_1, w_3')$ and the sum of (3.13) are both analytic extensions of their restrictions on the subset given by $|z_2| > |z_1| > 0$, $|\arg(z_1 - z_2) - \arg(-z_2)| < \frac{\pi}{2}$ and $\arg z_2 \geq \pi$, we see that the left-hand side of (3.13) is absolutely convergent to $g_-^e(z_1, z_2; u, w_2, w_1, w_3')$ when $|z_2| > |z_1| > 0$ and $|\arg(z_1 - z_2) - \arg(-z_2)| < \frac{\pi}{2}$. But when $\arg z_2 \geq \pi$, $\arg(-z_2) = \arg z_2 - \pi$. Hence in this case, the inequality $|\arg(z_1 - z_2) - \arg(-z_2)| < \frac{\pi}{2}$ becomes $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$. Also both the left-hand side of (3.13) and $g_-^e(z_1, z_2; u, w_2, w_1, w_3')$ are single valued analytic functions in z_1 and z_2 with cuts at $z_1 \in \mathbb{R}_+$ and $z_2 \in \mathbb{R}_+$. Thus when $|z_2| > |z_1| > 0$ and $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$, the left-hand side of (3.13) is absolutely convergent to $g_-^e(z_1, z_2; u, w_2, w_1, w_3')$.

Next we discuss the iterate of $\Omega_-(\mathcal{Y})$ and the twisted vertex operator map $\phi_{g_1}(Y_{W_2}^{g_2})$. When $|z_2| > |z_1 - z_2| > 0$ and $\arg z_2 \geq \pi$,

$$\begin{aligned}
&\langle w_3', \Omega_-(\mathcal{Y})(\phi_{g_1}(Y_{W_2}^{g_2})(u, z_1 - z_2)w_2, z_2)w_1\rangle \\
&\quad = \langle w_3', \Omega_-(\mathcal{Y})((Y_{W_2}^{g_2})(g_1^{-1}u, z_1 - z_2)w_2, z_2)w_1\rangle \\
&\quad = \langle w_3', e^{z_2L(-1)}\mathcal{Y}(w_1, -z_2)(Y_{W_2}^{g_2})(g_1^{-1}u, z_1 - z_2)w_2\rangle \\
&\quad = \langle e^{z_2L'(1)}w_3', \mathcal{Y}(w_1, -z_2)(Y_{W_2}^{g_2})(g_1^{-1}u, z_1 - z_2)w_2\rangle, \qquad (3.16)
\end{aligned}$$

converges absolutely and if in addition, $-\frac{3\pi}{2} < \arg z_1 - \arg(-z_2) < -\frac{\pi}{2}$, its sum is equal to $f^e(z_1 - z_2, -z_2; g_1^{-1}u, w_1, w_2, e^{z_2L'(1)}w_3')$. Note that the proofs of Lemmas 4.5 and 4.6 in [H8] do not use the explicit form of the multivalued analytic functions in the duality property of the twisted intertwining operators introduced in [H8]. Then the same proof of Lemma 4.5 in [H8] shows that the sum of the right-hand side of (3.16) is equal to $f^{b_{12}^{-1}}(z_1 - z_2, -z_2; g_1^{-1}u, w_1, w_2, e^{z_2L'(1)}w_3')$ when $|z_2| > |z_1 - z_2| > 0$, $\arg z_2 \geq \pi$ and $\frac{\pi}{2} < \arg z_1 - \arg(-z_2) < \frac{3\pi}{2}$. The same proof of (4.4) in Lemma 4.6 in [H8] shows that

$$f^{b_{12}^{-1}}(z_1 - z_2, -z_2; g_1^{-1}u, w_1, w_2, e^{z_2L'(1)}w_3') = f^e(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2L'(1)}w_3').$$

Since when $\arg z_2 \geq \pi$, $\arg(-z_2) = \arg z_2 - \pi$ and $\frac{\pi}{2} < \arg z_1 - \arg(-z_2) < \frac{3\pi}{2}$ becomes $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$, we see that the sum of the right-hand side of (3.16) is equal to (3.14) when $|z_2| > |z_1 - z_2| > 0$, $\arg z_2 \geq \pi$ and $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$.

We now use the same argument as above to finish the proof in this case. The subset

$$\left\{(z_1, z_2) \in M_0^2 \;\middle|\; |z_2| > |z_1 - z_2| > 0, \arg z_2 \geq \pi, |\arg(z_1) - \arg z_2| < \frac{\pi}{2}\right\} \qquad (3.17)$$

of $\widetilde{M}_0^2$ is also a subset of M_0^2. By definition, on the subset of M_0^2 given by $|z_1| > |z_2| > 0$, $\arg z_2 \geq \pi$, (3.14) is equal to $g_-^e(z_1, z_2; u, w_2, w_1, w_3')$. Since $g_-^e(z_1, z_2; u, w_2, w_1, w_3')$ on M_0^2 is obtained by analytic extension, we see that (3.14) is equal to $g_-^e(z_1, z_2; u, w_2, w_1, w_3')$

also on the subset (3.17). Thus the left-hand side of (3.16) is absolutely convergent to $g^e_-(z_1, z_2; u, w_2, w_1, w_3')$ when $|z_2| > |z_1 - z_2| > 0$, $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$ and $\arg z_2 \geq \pi$. Since $g^e_-(z_1, z_2; u, w_2, w_1, w_3')$ and the sum of (3.16) are both analytic extensions of their restrictions on the subset given by $|z_2| > |z_1 - z_2| > 0$, $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$ and $\arg z_2 \geq \pi$, we see that the left-hand side of (3.16) is absolutely convergent to $g^e_-(z_1, z_2; u, w_2, w_1, w_3')$ when $|z_2| > |z_1 - z_2| > 0$ and $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$.

We still need to prove the two other cases for $\Omega_+(\mathcal{Y})$. When $|z_2| > |z_1| > 0$ and $\arg z_2 < \pi$,

$$\begin{aligned}
&\langle w_3', \Omega_+(\mathcal{Y})(w_2, z_2)\phi_{g_2^{-1}}(Y^{g_1}_{W_1})(u, z_1)w_1\rangle\\
&\quad = \langle w_3', e^{z_2L(-1)}\mathcal{Y}(\phi_{g_2^{-1}}(Y^{g_1}_{W_1})(u, z_1)w_1, -z_2)w_2\rangle\\
&\quad = \langle e^{z_2L'(1)}w_3', \mathcal{Y}((Y^{g_1}_{W_1})(g_2u, z_1)w_1, -z_2)w_2\rangle
\end{aligned} \tag{3.18}$$

converges absolutely and if in addition, $|\arg(z_1 - z_2) - \arg(-z_2)| < \frac{\pi}{2}$, its sum is equal to

$$f^e(z_1 - z_2, -z_2; g_2u, w_1, w_2, e^{z_2L'(1)}w_3'). \tag{3.19}$$

The subset

$$\left\{(z_1, z_2) \in M_0^2 \;\middle|\; |z_2| > |z_1| > 0, \arg z_2 < \pi, |\arg(z_1 - z_2) - \arg(-z_2)| < \frac{\pi}{2}\right\} \tag{3.20}$$

of $\widetilde{M}_0^2$ is also a subset of M_0^2. By definition, on the subset of M_0^2 given by $|z_1| > |z_2| > 0$, $\arg z_2 < \pi$, (3.19) is equal to $g^e_+(z_1, z_2; g_2u, w_2, w_1, w_3')$. Since $g^e_+(z_1, z_2; g_2u, w_2, w_1, w_3')$ on M_0^2 is obtained by analytic extension, (3.19) is equal to $g^e_+(z_1, z_2; g_2u, w_2, w_1, w_3')$ also on the subset (3.20). Thus the left-hand side of (3.18) is absolutely convergent to $g^e_+(z_1, z_2; g_2u, w_2, w_1, w_3')$ when $|z_2| > |z_1| > 0$, $|\arg(z_1-z_2)-\arg(-z_2)| < \frac{\pi}{2}$ and $\arg z_2 < \pi$. Since $g^e_+(z_1, z_2; g_2u, w_2, w_1, w_3')$ and the sum of (3.18) are both analytic extensions of their restrictions on the subset given by $|z_2| > |z_1| > 0$, $|\arg(z_1 - z_2) - \arg(-z_2)| < \frac{\pi}{2}$ and $\arg z_2 < \pi$, the left-hand side of (3.18) is absolutely convergent to $g^e_+(z_1, z_2; g_2u, w_2, w_1, w_3')$ when $|z_2| > |z_1| > 0$, $|\arg(z_1 - z_2) - \arg(-z_2)| < \frac{\pi}{2}$.

The same proof as that of (4.5) in Lemma 4.6 in [H8] gives

$$g^e_+(z_1, z_2; g_2u, w_1, w_2, w_3') = g^{b_{13}^{-1}}_+(z_1, z_2; u, w_1, w_2, w_3'), \tag{3.21}$$

since $\arg z_1 < \pi$, $\arg(-z_2) = \arg z_2 + \pi$ and the inequality $|\arg(z_1 - z_2) - \arg(-z_2)| < \frac{\pi}{2}$ becomes $\frac{\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < \frac{3\pi}{2}$. Also, the same proof as that of Lemma 4.5 in [H8] shows that when when $|z_2| > |z_1| > 0$ and $\frac{\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < \frac{3\pi}{2}$, the sum of left-hand side of (3.18) is equal to $g^e_+(z_1, z_2; u, w_2, w_1, w_3')$.

Finally, we discuss the iterate of $\Omega_+(\mathcal{Y})$ and the twisted vertex operator map $Y^{g_2}_{W_2}$. When $|z_2| > |z_1 - z_2| > 0$ and $\arg z_2 < \pi$,

$$\begin{aligned}
&\langle w_3', \Omega_+(\mathcal{Y})((Y^{g_2}_{W_2})(u, z_1 - z_2)w_2, z_2)w_1\rangle\\
&\quad = \langle w_3', e^{z_2L(-1)}\mathcal{Y}(w_1, -z_2)(Y^{g_2}_{W_2})(u, z_1 - z_2)w_2\rangle\\
&\quad = \langle e^{z_2L'(1)}w_3', \mathcal{Y}(w_1, -z_2)(Y^{g_2}_{W_2})(u, z_1 - z_2)w_2\rangle
\end{aligned} \tag{3.22}$$

converges absolutely and if in addition, $-\frac{3\pi}{2} < \arg z_1 - \arg(-z_2) < -\frac{\pi}{2}$, its sum is equal to (3.14). The subset

$$\left\{(z_1, z_2) \in M_0^2 \;\middle|\; |z_2| > |z_1 - z_2| > 0, \arg z_2 < \pi, -\frac{3\pi}{2} < \arg z_1 - \arg(-z_2) < -\frac{\pi}{2}\right\} \quad (3.23)$$

of $\widetilde{M}_0^2$ is also a subset of M_0^2. As we have discussed above, on the subset of M_0^2 given by $|z_1| > |z_2| > 0$, $\arg z_2 < \pi$, (3.14) is equal to $g_+^e(z_1, z_2; u, w_2, w_1, w_3')$. Since $g_+^e(z_1, z_2; u, w_2, w_1, w_3')$ on M_0^2 is obtained by analytic extension, (3.14) is equal to $g_+^e(z_1, z_2; u, w_2, w_1, w_3')$ also on the subset (3.23). Thus the left-hand side of (3.22) is absolutely convergent to $g_+^e(z_1, z_2; u, w_2, w_1, w_3')$ when $|z_2| > |z_1 - z_2| > 0$, $-\frac{3\pi}{2} < \arg z_1 - \arg(-z_2) < -\frac{\pi}{2}$ and $\arg z_2 < \pi$. Since $g_+^e(z_1, z_2; u, w_2, w_1, w_3')$ and the sum of (3.18) are both analytic extensions of their restrictions on the subset given by $|z_2| > |z_1 - z_2| > 0$, $-\frac{3\pi}{2} < \arg z_1 - \arg(-z_2) < -\frac{\pi}{2}$ and $\arg z_2 < \pi$, the left-hand side of (3.22) is absolutely convergent to $g_+^e(z_1, z_2; u, w_2, w_1, w_3')$ when $|z_2| > |z_1 - z_2| > 0$ and $-\frac{3\pi}{2} < \arg z_1 - \arg(-z_2) < -\frac{\pi}{2}$. ∎

Let $\mathcal{V}_{W_1W_2}^{W_3}$ be the space of twisted intertwining operators of type $\binom{W_3}{W_1W_2}$. Then we have:

Corollary 3.2 *The maps* $\Omega_+ : \mathcal{V}_{W_1W_2}^{W_3} \to \mathcal{V}_{W_2\phi_{g_2^{-1}}(W_1)}^{W_3}$ *and* $\Omega_- : \mathcal{V}_{W_1W_2}^{W_3} \to \mathcal{V}_{\phi_{g_1}(W_2)W_1}^{W_3}$ *are linear isomorphisms. In particular,* $\mathcal{V}_{W_1W_2}^{W_3}$, $\mathcal{V}_{\phi_{g_1}(W_2)W_1}^{W_3}$ *and* $\mathcal{V}_{W_2\phi_{g_2^{-1}}(W_1)}^{W_3}$ *are linearly isomorphic.*

Proof. It is clear that Ω_+ and Ω_- are inverse of each other. ∎

The linear isomorphisms Ω_+ and Ω_- are called the *skew-symmetry isomorphisms*.

Let g_1, g_2 be automorphisms of V, W_1, W_2 and W_3 g_1-, g_2- and g_1g_2-twisted V-modules and $\mathcal{Y}$ a twisted intertwining operator of type $\binom{W_3}{W_1W_2}$. We define linear maps

$$\begin{aligned} A_\pm(\mathcal{Y}) : W_1 \otimes W_3' &\to W_2'\{x\}[\log x] \\ w_1 \otimes w_3' &\mapsto A_\pm(\mathcal{Y})(w_1, x)w_3' \end{aligned}$$

by

$$\langle A_\pm(\mathcal{Y})(w_1, x)w_3', w_2\rangle = \langle w_3', \mathcal{Y}(e^{xL(1)}e^{\pm\pi \mathbf{i}L(0)}(x^{-L(0)})^2 w_1, x^{-1})w_2\rangle \quad (3.24)$$

for $w_1 \in W_1$ and $w_2 \in W_2$ and $w_3' \in W_3'$. Similarly to Ω_p for $p \in \mathbb{Z}$, we can also define A_p for $p \in \mathbb{Z}$ by replacing $\pm$ in (3.24) by $+p$. But we will also not discuss A_p in this paper.

Let (W, Y_W^g) be a g-twisted V-module. When $W_1 = V$, $W_2 = W_3 = W$ and $\mathcal{Y} = Y_W^g$, by definition, $A_+(Y_W^g) = A_-(Y_W^g) = (Y_W^g)'$ (see Section 4).

Let $L_{W_1}^s(0)$ be the semisimple part of $L_{W_1}(0)$. From the definition (3.24), for $p \in \mathbb{Z}$, $w_1 \in W_1$, $w_2 \in W_2$, $w_3' \in W_3'$ and $z \in \mathbb{C}^\times$, we have

$$\begin{aligned} &\langle A_\pm(\mathcal{Y})^p(w_1, z)w_3', w_2\rangle \\ &\quad = \langle A_\pm(\mathcal{Y})^p(w_1, x)w_3', w_2\rangle\Big|_{x^n = e^{nl_p(z)},\, \log x = l_p(z)} \end{aligned}$$

$$
\begin{aligned}
&= \langle w_3', \mathcal{Y}(e^{xL_{W_1}(1)}e^{\pm\pi\mathbf{i}L_{W_1}(0)}(x^{-L_{W_1}(0)})^2 w_1, x^{-1})w_2\rangle\Big|_{x^n=e^{nl_p(z)},\ \log x=l_p(z)}\\
&= \langle w_3', \mathcal{Y}(e^{xL_{W_1}(1)}e^{\pm\pi\mathbf{i}L_{W_1}(0)}(x^{-L^s_{W_1}(0)})^2\cdot\\
&\qquad\qquad \cdot (e^{-(L_{W_1}(0)-L^s_{W_1}(0))\log x})^2 w_1, x^{-1})w_2\rangle\Big|_{x^n=e^{nl_p(z)},\ \log x=l_p(z)}\\
&= \langle w_3', \mathcal{Y}(e^{zL_{W_1}(1)}e^{\pm\pi\mathbf{i}L_{W_1}(0)}(e^{-l_p(z)L^s_{W_1}(0)})^2\cdot\\
&\qquad\qquad \cdot (e^{-(L_{W_1}(0)-L^s_{W_1}(0))l_p(z)})^2 w_1, y)w_2\rangle\Big|_{y^n=e^{-nl_p(z)},\ \log y=-l_p(z)}\\
&= \langle w_3', \mathcal{Y}(e^{zL_{W_1}(1)}e^{\pm\pi\mathbf{i}L_{W_1}(0)}e^{-2l_p(z)L_{W_1}(0)} w_1, y)w_2\rangle\Big|_{y^n=e^{-nl_p(z)},\ \log y=-l_p(z)}.
\end{aligned}
\tag{3.25}
$$

When $\arg z = 0$, $\arg z^{-1} = \arg z = 0$ and $-l_p(z) = l_{-p}(z^{-1})$. When $\arg z \neq 0$, $\arg z^{-1} = -\arg z + 2\pi$ and $-l_p(z) = l_{-p-1}(z^{-1})$. Hence when $\arg z = 0$, the right-hand side of (3.25) is equal to

$$
\begin{aligned}
&\langle w_3', \mathcal{Y}(e^{zL_{W_1}(1)}e^{\pm\pi\mathbf{i}L_{W_1}(0)}e^{2l_{-p}(z^{-1})L_{W_1}(0)} w_1, y)w_2\rangle\Big|_{y^n=e^{nl_{-p}(z^{-1})},\ \log y=l_{-p}(z^{-1})}\\
&\quad = \langle w_3', \mathcal{Y}^{-p}(e^{zL_{W_1}(1)}e^{\pm\pi\mathbf{i}L_{W_1}(0)}e^{2l_{-p}(z^{-1})L_{W_1}(0)} w_1, z^{-1})w_2\rangle
\end{aligned}
\tag{3.26}
$$

and when $\arg z \neq 0$, it is equal to

$$
\begin{aligned}
&\langle w_3', \mathcal{Y}(e^{zL_{W_1}(1)}e^{\pm\pi\mathbf{i}L_{W_1}(0)}e^{2l_{-p-1}(z^{-1})L_{W_1}(0)} w_1, y)w_2\rangle\Big|_{y^n=e^{nl_{-p-1}(z^{-1})},\ \log y=l_{-p-1}(z^{-1})}\\
&\quad = \langle w_3', \mathcal{Y}^{-p-1}(e^{zL_{W_1}(1)}e^{\pm\pi\mathbf{i}L_{W_1}(0)}e^{2l_{-p-1}(z^{-1})L_{W_1}(0)} w_1, z^{-1})w_2\rangle.
\end{aligned}
\tag{3.27}
$$

From (3.25)–(3.27), for $w_1 \in W_1$, $w_2 \in W_2$, $w_3' \in W_3'$ and $z \in \mathbb{C}^\times$, we have

$$
\langle A_\pm(\mathcal{Y})^p(w_1, z)w_3', w_2\rangle = \langle w_3', \mathcal{Y}^{-p}(e^{zL_{W_1}(1)}e^{\pm\pi\mathbf{i}L_{W_1}(0)}e^{2l_{-p}(z^{-1})L_{W_1}(0)} w_1, z^{-1})w_2\rangle \tag{3.28}
$$

when $\arg z = 0$ and

$$
\langle A_\pm(\mathcal{Y})^p(w_1, z)w_3', w_2\rangle = \langle w_3', \mathcal{Y}^{-p-1}(e^{zL_{W_1}(1)}e^{\pm\pi\mathbf{i}L_{W_1}(0)}e^{2l_{-p-1}(z^{-1})L_{W_1}(0)} w_1, z^{-1})w_2\rangle \tag{3.29}
$$

when $\arg z \neq 0$.

Theorem 3.3 *The linear maps $A_+(\mathcal{Y})$ and $A_-(\mathcal{Y})$ are twisted intertwining operators of types $\binom{\phi_{g_1}(W_2')}{W_1 W_3'}$ and $\binom{W_2'}{W_1 \phi_{g_1^{-1}}(W_3')}$, respectively.*

Proof. The proof of this result is also essentially the same as the proof of Theorem 6.1 in [H8]. But the proof here is much more complicated because the correlation functions involved are not of the explicitly form as in [H8]. As in [H8], we need only prove the duality property.

We first give the multivalued analytic functions with preferred branches in the duality property. We shall denote these multivalued analytic functions for $A_+(\mathcal{Y})$ and $A_-(\mathcal{Y})$ by $h_+(z_1, z_2; u, w_1, w_2, w_3')$ and $h_-(z_1, z_2; u, w_1, w_2, w_3')$, respectively. Let $f(z_1, z_2; u, w_1, w_2, w_3')$ be the multivalued analytic function with the preferred branch $f^e(z_1, z_2; u, w_1, w_2, w_3')$ in the duality property for the twisted intertwining operator $\mathcal{Y}$. Define

$$h_\pm(z_1, z_2; u, w_2, w_1, w_3') = f(z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)}(-z_1^2)^{-L_V(0)}u, e^{z_2 L_{W_1}(1)}(-z_2^2)^{-L_{W_1}(0)}w_1, w_2, w_3') \tag{3.30}$$

and choose the preferred branch $h^e_\pm(z_1, z_2; u, w_2, w_1, w_3')$ of $h_\pm(z_1, z_2; u, w_2, w_1, w_3')$ as follows: On the subregion $|z_1| > |z_2| > 0$, $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ and $\arg(z_1 - z_2) < \pi$ (for A_+) or $\arg(z_1 - z_2) \geq \pi$ (for A_-) of M_0^2, let

$$\begin{aligned}
&h^e_\pm(z_1, z_2; u, w_2, w_1, w_3')\\
&\quad = f^{b_{13}^{-1}b_{23}^{-1}}(z_1^{-1}, z_2^{-1}; e^{z^{-1} L_V(1)}(-z_1^2)^{-L_V(0)}u, e^{z_2 L_{W_1}(1)}e^{\pm\pi \mathbf{i} L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, w_2, w_3').
\end{aligned} \tag{3.31}$$

For general $(z_1, z_2) \in M_0^2$, we define $h^e_\pm(z_1, z_2; u, w_2, w_1, w_3')$ to be the unique analytic extension on M_0^2.

Let $u \in V$, $w_1 \in W_1$, $w_2 \in W_2$ and $w_3' \in W_3'$. We consider $z_1, z_2 \in \mathbb{C}$ satisfying $|z_2^{-1}| > |z_1^{-1}| > 0$ (or equivalently $|z_1| > |z_2| > 0$) and $\arg z_1, \arg z_2 \neq 0$. Since $|z_2^{-1}| > |z_1^{-1}| > 0$ and $\arg z_1, \arg z_2 \neq 0$, from (3.29), $(Y^{g_2}_{W_2})' = A_+(Y^{g_2}_{W_2})$ and the duality property for $\mathcal{Y}$, we know that

$$\langle w_3', \mathcal{Y}(e^{z_2 L_{W_1}(1)}e^{\pi \mathbf{i} L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, z_2^{-1})Y^{g_2}_{W_2}(e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}g_1^{-1}u, z_1^{-1})w_2\rangle \tag{3.32}$$

is absolutely convergent and if in addition, $-\frac{3\pi}{2} < \arg(z_1^{-1} - z_2^{-1}) - \arg z_2^{-1} < -\frac{\pi}{2}$, its sum is equal to

$$\begin{aligned}
&f^e(z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}g_1^{-1}u, e^{z_2 L_{W_1}(1)}e^{\pi \mathbf{i} L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, w_2, w_3')\\
&\quad = f^e(z_1^{-1}, z_2^{-1}; g_1^{-1}e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}u, e^{z_2 L_{W_1}(1)}e^{\pi \mathbf{i} L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, w_2, w_3').
\end{aligned} \tag{3.33}$$

We know that

$$\begin{aligned}
&\langle \phi_{g_1}((Y^{g_2}_{W_2})')(u, z_1)A_+(\mathcal{Y})(w_1, z_2)w_3', w_2\rangle\\
&\quad = \langle((Y^{g_2}_{W_2})')(g_1^{-1}u, z_1)A_+(\mathcal{Y})(w_1, z_2)w_3', w_2\rangle\\
&\quad = \langle w_3', \mathcal{Y}^{-1}(e^{z_2 L_{W_1}(1)}e^{\pi \mathbf{i} L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, z_2^{-1})\cdot\\
&\qquad\qquad \cdot (Y^{g_2}_{W_2})^{-1}(e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}g_1^{-1}u, z_1^{-1})w_2\rangle
\end{aligned} \tag{3.34}$$

can be obtained using the multivalued analytic function (3.30) on the region $|z_2^{-1}| > |z_1^{-1}| > 0$ starting from the value given by (3.32) by letting z_1^{-1} go around 0 clockwise once (corresponding to b_{13}^{-1}), then letting z_2^{-1} go around 0 clockwise once (corresponding to $b_{23}^{-1}b_{12}^{-1}$). Then (3.34) also converges absolutely on the region $|z_2^{-1}| > |z_1^{-1}| > 0$ and if in addition, $-\frac{3\pi}{2} < \arg(z_1^{-1} - z_2^{-1}) - \arg z_2^{-1} < -\frac{\pi}{2}$, its sum is equal to

$$f^{b_{13}^{-1}b_{23}^{-1}b_{12}^{-1}}(z_1^{-1}, z_2^{-1}; g_1^{-1}e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}u, e^{z_2 L_{W_1}(1)}e^{\pi \mathbf{i} L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, w_2, w_3')$$

$$= f^{b_{12}^{-1}b_{13}^{-1}b_{23}^{-1}}(z_1^{-1}, z_2^{-1}; g_1^{-1}e^{z_1L_V(1)}(-z_1^{-2})^{L_V(0)}u, e^{z_2L_{W_1}(1)}e^{\pi \mathbf{i}L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, w_2, w_3'). \tag{3.35}$$

Using the equivariance property for W_1 and the convergence of (2.5) to $f^e(z_1, z_2; u, w_1, w_2, w_3')$ on the region $|z_2| > |z_1 - z_2| > 0$, $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$, we have

$$f^e(z_1, z_2; g_1^{-1}u, w_1, w_2, w_3') = f^{b_{12}}(z_1, z_2; u, w_1, w_2, w_3').$$

Applying $b \in \mathrm{PB}_3$ to both sides of this equality, we obtain

$$f^b(z_1, z_2; g_1^{-1}u, w_1, w_2, w_3') = f^{b_{12}b}(z_1, z_2; u, w_1, w_2, w_3') \tag{3.36}$$

for $b \in \mathrm{PB}_3$. By (3.36) with $b = b_{12}^{-1}b_{13}^{-1}b_{23}^{-1}$, we see that (3.35) is equal to

$$f^{b_{13}^{-1}b_{23}^{-1}}(z_1^{-1}, z_2^{-1}; e^{z_1L_V(1)}(-z_1^{-2})^{L_V(0)}u, e^{z_2L_{W_1}(1)}e^{\pi \mathbf{i}L_{W_1}(0)}e^{2\log(z_2^{-1})L_{W_1}(0)}w_1, w_2, w_3'), \tag{3.37}$$

which by definition, is equal to $h_+^e(z_1, z_2; u, w_2, w_1, w_3')$ when $|z_1| > |z_2| > 0$, $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ and $\arg(z_1 - z_2) < \pi$.

On the other hand, we have

$$\arg(z_1^{-1} - z_2^{-1}) = \arg\left(\frac{z_1 - z_2}{-z_1z_2}\right) = \arg(z_1 - z_2) - \arg z_1 - \arg z_2 + (2q+1)\pi \tag{3.38}$$

for some $q \in \mathbb{Z}$. But for any $z \in \mathbb{C}$, we have $0 \le \arg z < 2\pi$. In particular, we have

$$0 \le \arg(z_1 - z_2) - \arg z_1 - \arg z_2 + (2q+1)\pi < 2\pi.$$

Since we also have $0 \le \arg z_2 < 2\pi$, we obtain

$$-\pi - 2q\pi \le \arg(z_1 - z_2) - \arg z_1 < \pi - 2q\pi. \tag{3.39}$$

Therefore when $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$, we must have $q = 0$ and thus

$$\arg(z_1^{-1} - z_2^{-1}) = \arg(z_1 - z_2) - \arg z_1 - \arg z_2 + \pi.$$

Also when $\arg z_1, \arg z_2 \neq 0$, we have $\arg z_1^{-1} = -\arg z_1 + 2\pi$, $\arg z_2^{-1} = -\arg z_2 + 2\pi$. Therefore when $\arg z_1, \arg z_2 \neq 0$ and $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$, we have

$$-\frac{3\pi}{2} < \arg(z_1^{-1} - z_2^{-1}) - \arg z_2^{-1} = \arg(z_1 - z_2) - \arg z_1 - \pi < -\frac{\pi}{2}.$$

Thus when $|z_1| > |z_2| > 0$, $\arg z_1, \arg z_2 \neq 0$, $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ and $\arg(z_1 - z_2) < \pi$, the series (3.34) is absolutely convergent to $h_+^e(z_1, z_2; u, w_2, w_1, w_3')$. Since both the sum of the left-hand side of (3.34) and $h_+^e(z_1, z_2; u, w_2, w_1, w_3')$ are analytic extensions of their restrictions on the subset given by $|z_1| > |z_2| > 0$, $\arg z_1, \arg z_2 \neq 0$, $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ and $\arg(z_1 - z_2) < \pi$, the sum of the left-hand side of (3.34) is equal to $h_+^e(z_1, z_2; u, w_2, w_1, w_3')$ when $|z_1| > |z_2| > 0$ and $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$.

Generalizing the convergence and analytic extension of (3.34) above, we can prove that

$$\langle \phi_{g_1}((Y_{W_2}^{g_2})')(u_1, z_1)\cdots\phi_{g_1}((Y_{W_2}^{g_2})')(u_{k-1}, z_{k-1})A_+(\mathcal{Y})(w_1, z_k)w_3', w_2\rangle \tag{3.40}$$

is absolutely convergent on the region $|z_1| > \cdots > |z_k| > 0$ and its sum can be maximally extended to a multivalued analytic function on the region M^k for $k \in \mathbb{Z}_+ + 3$, $u_1, \dots, u_{k-1} \in V$, $w_1 \in W_1$, $w_2 \in W_2$ and $w_3' \in W_3'$. In fact, the same calculations as in (3.34) shows that (3.40) is equal to

$$\begin{aligned}\langle w_3', \mathcal{Y}^{-1}(e^{z_kL_{W_1}(1)}e^{\pi\mathbf{i}L_{W_1}(0)}e^{-2\log(z_k)L_{W_1}(0)}w_1, z_k^{-1})\cdot\\ \cdot (Y_{W_2}^{g_2})^{-1}(e^{z_{k-1}L_V(1)}(-z_{k-1}^{-2})^{L_V(0)}g_1^{-1}u_{k-1}, z_{k-1}^{-1})\cdot\\ \cdots (Y_{W_2}^{g_2})^{-1}(e^{z_1L_V(1)}(-z_1^{-2})^{L_V(0)}g_1^{-1}u_1, z_1^{-1})w_2\rangle\end{aligned} \tag{3.41}$$

From the duality properties in Definitions 2.1 and 2.5, we see that

$$\begin{aligned}\prod_{1\le i<j\le k-1}(z_i^{-1}-z_j^{-1})^{M_{ij}}\langle w_3', (Y_{W_2}^{g_2})^{-1}(e^{z_{k-1}L_V(1)}(-z_{k-1}^{-2})^{L_V(0)}g_1^{-1}u_{k-1}, z_{k-1}^{-1})\cdot\\ \cdots (Y_{W_2}^{g_2})^{-1}(e^{z_1L_V(1)}(-z_1^{-2})^{L_V(0)}g_1^{-1}u_1, z_1^{-1})\cdot\\ \cdot\mathcal{Y}^{-1}(e^{z_kL_{W_1}(1)}e^{\pi\mathbf{i}L_{W_1}(0)}e^{-2\log(z_k)L_{W_1}(0)}w_1, z_k^{-1})w_2\rangle\end{aligned} \tag{3.42}$$

is absolutely convergent on the region $|z_i^{-1}| > |z_k^{-1}| > 0$ for $i = 1, \dots, k-1$, $z_i^{-1} \ne z_j^{-1}$ for $i \ne j$, and its sum can be analytically extended to a maximal multivalued analytic function on

$$\{(z_1, \dots, z_k) \in \mathbb{C}^k \mid z_i \ne z_k, i = 1, \dots, k-1\},$$

where $M_{ij} \in \mathbb{Z}_+$ for $i \ne j$ satisfy $x^{M_{ij}}Y_V(u_i, x)u_j \in V[[x]]$. Using the duality property in Definition 2.5 repeatedly, we see that

$$\begin{aligned}\prod_{1\le i<j\le k-1}(z_i^{-1}-z_j^{-1})^{M_{ij}}\langle w_3', \mathcal{Y}^{-1}(e^{z_kL_{W_1}(1)}e^{\pi\mathbf{i}L_{W_1}(0)}e^{-2\log(z_k)L_{W_1}(0)}w_1, z_k^{-1})\cdot\\ \cdot (Y_{W_2}^{g_2})^{-1}(e^{z_{k-1}L_V(1)}(-z_{k-1}^{-2})^{L_V(0)}g_1^{-1}u_{k-1}, z_{k-1}^{-1})\cdot\\ \cdots (Y_{W_2}^{g_2})^{-1}(e^{z_1L_V(1)}(-z_1^{-2})^{L_V(0)}g_1^{-1}u_1, z_1^{-1})w_2\rangle\end{aligned} \tag{3.43}$$

is absolutely convergent on the region $|z_i^{-1}| < |z_k^{-1}| > 0$ for $i = 1, \dots, k-1$, $z_i^{-1} \ne z_j^{-1}$ for $i \ne j$, and its sum can be analytically extended to a maximal multivalued analytic function on

$$\{(z_1, \dots, z_k) \in \mathbb{C}^k \mid z_i \ne z_k, i = 1, \dots, k-1\}.$$

Thus (3.41) and consequently (3.40) is absolutely convergent on the region $|z_1| > \cdots > |z_k| > 0$ and its sum has a maximal analytic extension on the region M^k.

Next we consider the product of $A_+(\mathcal{Y})$ and the twisted vertex operator $(Y_{W_3}^{g_2})'$. Let u, w_1, w_2 and w_3' be the same as above. When $|z_1^{-1}| > |z_2^{-1}| > 0$ (or equivalently $|z_2| > |z_1| > 0$),

$$\langle w_3', Y_{W_2}^{g_2}(e^{z_1L_V(1)}(-z_1^{-2})^{L_V(0)}u, z_1^{-1})\mathcal{Y}(e^{z_2L_{W_1}(1)}e^{\pi\mathbf{i}L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, z_2^{-1})w_2\rangle \tag{3.44}$$

converges absolutely and if in addition, $|\arg(z_1^{-1}-z_2^{-1})-\arg z_1^{-1}|<\frac{\pi}{2}$, its sum is equal to

$$f^e(z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}u, e^{z_2 L_{W_1}(1)}e^{\pi \mathbf{i} L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, w_2, w_3'). \tag{3.45}$$

We know that when $\arg z_1, \arg z_2 \neq 0$,

$$\begin{aligned}&\langle A_+(\mathcal{Y})(w_1, z_2)(Y_{W_2}^{g_2})'(u, z_1)w_3', w_2\rangle\\ &\quad = \langle w_3', (Y_{W_2}^{g_2})^{-1}(e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}u, z_1^{-1})\cdot\\ &\qquad\qquad \cdot \mathcal{Y}^{-1}(e^{z_2 L_{W_1}(1)}e^{\pi \mathbf{i} L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, z_2^{-1})w_2\rangle\end{aligned} \tag{3.46}$$

can be obtained using the multivalued analytic function (3.30) on the region $|z_1^{-1}|>|z_2^{-1}|>0$ starting from the value given by (3.44) by letting z_2^{-1} go around 0 clockwise once (corresponding to b_{23}^{-1}), then letting z_1^{-1} go around 0 clockwise once (corresponding to $b_{12}^{-1}b_{13}^{-1}$). Then (3.46) also converges absolutely and if in addition, $|\arg(z_1^{-1}-z_2^{-1})-\arg z_1^{-1}|<\frac{\pi}{2}$, its sum is equal to

$$\begin{aligned}&f^{b_{23}^{-1}b_{12}^{-1}b_{13}^{-1}}(z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}u, e^{z_2 L_{W_1}(1)}e^{\pi \mathbf{i} L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, w_2, w_3')\\ &\quad = f^{b_{13}^{-1}b_{23}^{-1}b_{12}^{-1}}(z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}u, e^{z_2 L_{W_1}(1)}e^{\pi \mathbf{i} L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, w_2, w_3').\end{aligned} \tag{3.47}$$

When $-\frac{3\pi}{2}<\arg(z_1-z_2)-\arg z_2<-\frac{\pi}{2}$, we have

$$-\frac{\pi}{2}<\arg(z_1-z_2)-\arg z_2+\pi<\frac{\pi}{2}.$$

In the case $\arg z_1\neq 0$, by (3.38) and $\arg z_1^{-1}=-\arg z_1+2\pi$, we obtain

$$-\frac{\pi}{2}+2(q-1)\pi<\arg(z_1^{-1}-z_2^{-1})-\arg z_1^{-1}<\frac{\pi}{2}+2(q-1)\pi.$$

When $0\leq \arg(z_1^{-1}-z_2^{-1}), \arg z_1^{-1}<\pi$, we have

$$-\pi<\arg(z_1^{-1}-z_2^{-1})-\arg z_1^{-1}<\pi.$$

So in this case, $q=1$. Since in this case,

$$-\frac{\pi}{2}<\arg(z_1^{-1}-z_2^{-1})-\arg z_1^{-1}<\frac{\pi}{2},$$

we see that the sum of (3.46) is equal to (3.47). Using analytic extension, we see that when $|z_2|>|z_1|>0$ and $-\frac{3\pi}{2}<\arg(z_1-z_2)-\arg z_2<-\frac{\pi}{2}$, the sum of (3.46) is equal to (3.47).

On the region $|z_2|>|z_1|>0$, (3.47) is in fact equal to $h_+^e(z_1, z_2; u, w_2, w_1, w_3')$. This can be seen as follows: On the intersection of M_0^2 and the region $|z_2|>|z_1|>0$, $h_+^e(z_1, z_2; u, w_2, w_1, w_3')$ is obtained by analytically extending (3.30) defined on the intersection of the region $|z_1|>|z_2|>0$, $0\leq \arg(z_1^{-1}-z_2^{-1})<\pi$ and M_0^2. We need to find what is this analytic extension. Let $\xi, \zeta\in -\mathbb{R}_+$ satisfying $\xi<\zeta<0$. Then we have $|\xi|>|\zeta|>0$ and, by definition,

$$h_+^e(\xi, \zeta; u, w_2, w_1, w_3')$$

$$= f^{b_{13}^{-1}b_{23}^{-1}}(\xi^{-1}, \zeta^{-1}; e^{\xi^{-1}L_V(1)}(-\xi^2)^{-L_V(0)}u, e^{\zeta L_{W_1}(1)}e^{\pm\pi \mathbf{i} L_{W_1}(0)}e^{-2\log(\zeta)L_{W_1}(0)}w_1, w_2, w_3').$$

Let $\gamma = (\gamma_1, \gamma_2)$ be the path from (ξ, ζ) to (ζ, ξ) given by the upper half circle γ_1 centered at $\frac{\xi+\zeta}{2}$ with radius $\frac{-\xi+\zeta}{2}$ from ξ to ζ and the lower half circle γ_2 centered at $\frac{\xi+\zeta}{2}$ with radius $\frac{-\xi+\zeta}{2}$ from ζ to ξ. It is clear that γ is a continuous path in M_0^2. So $h_+^e(\zeta, \xi; u, w_2, w_1, w_3')$ is obtained by analytically extending the value $h_+^e(\xi, \zeta; u, w_2, w_1, w_3')$ along γ. On the other hand, γ gives a path $\gamma' = (\gamma_2^{-1}, \gamma_1^{-1})$ from (ζ^{-1}, ξ^{-1}) to (ξ^{-1}, ζ^{-1}). But γ' is not a continuous path in M_0^2 because when (z_1, z_2) goes from (ζ^{-1}, ξ^{-1}) to (ξ^{-1}, ζ^{-1}) along the path γ', $z_1 - z_2$ crosses the positive real line clockwise. Crossing the positive real line clockwise corresponds to changing the branch by an action of b_{12}^{-1}. Thus on the intersection of the region $|z_2| > |z_1| > 0$ and M_0^2, we must have

$$\begin{aligned} &h_+^e(z_1, z_2; u, w_2, w_1, w_3') \\ &\quad = f^{b_{13}^{-1}b_{23}^{-1}b_{12}^{-1}}(z_1^{-1}, z_2^{-1}; e^{z_1^{-1}L_V(1)}(-z_1^2)^{-L_V(0)}u, e^{z_2 L_{W_1}(1)}e^{\pm\pi \mathbf{i} L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, w_2, w_3'). \end{aligned} \tag{3.48}$$

Since when $|z_2| > |z_2| > 0$ and $|\arg(z_1^{-1} - z_2^{-1}) - \arg z_1^{-1}| < \frac{\pi}{2}$, the sum of (3.46) is equal to (3.47), that is, the right-hand side of (3.48), the sum of (3.46) is indeed equal to $h_+^e(z_1, z_2; u, w_2, w_1, w_3')$.

We can prove the absolute convergence of

$$\langle ((Y_{W_2}^{g_2})')(u, z_1)A_-(\mathcal{Y})^{p_2}(w_1, z_2)w_3', w_2\rangle$$

and

$$\langle A_-(\mathcal{Y})^{p_2}(w_1, z_2)\phi_{g_1^{-1}}((Y_{W_3}^{g_2})')^{p_1}(u, z_1)w_3', w_2\rangle$$

in the corresponding regions to $h_-^e(z_1, z_2; u, w_2, w_1, w_3')$ similarly by generalizing the proofs in [H8] using the same method above for $A_-(\mathcal{Y})$. Here we omit the details.

Finally we study the iterate of $A_\pm(\mathcal{Y})$ and the twisted vertex operator $Y_{W_1}^{g_1}$. When $\arg z_2 \neq 0$, from (3.29), we have

$$\begin{aligned} &\langle A_\pm(\mathcal{Y})(Y_{W_1}^{g_1}(u, z_1 - z_2)w_1, z_2)w_3', w_2\rangle \\ &\quad = \langle w_3', \mathcal{Y}^{-1}(e^{z_2 L_{W_1}(1)}e^{\pm\pi \mathbf{i} L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}Y_{W_1}^{g_1}(u, z_1 - z_2)w_1, z_2^{-1})w_2\rangle. \end{aligned} \tag{3.49}$$

As in [H8], we have in the region $|z_2| > |z_1 - z_2| > 0$

$$\begin{aligned} &e^{z_2 L_{W_1}(1)}e^{\pm\pi \mathbf{i} L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}Y_{W_1}^{g_1}(u, z_1 - z_2)w_1 \\ &\quad = Y_{W_1}^{g_1}\left(e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}u, \frac{xx_0}{(x_2 + x_0)x_2}\right)\Bigg|_{\substack{x_0^n = e^{nl_{p_{12}}(z_1 - z_2)},\ \log x_0 = \log(z_1 - z_2),\ x_2^n = e^{n\log(z_2)} \\ \log x_2 = l_{p_2}(z_2),\ x^n = e^{\pm n\pi \mathbf{i}},\ \log x = \pm\pi\mathbf{i}}} \cdot \\ &\qquad\qquad \cdot e^{z_2 L_{W_1}(1)}e^{\pm\pi \mathbf{i} L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1. \end{aligned} \tag{3.50}$$

As in the proof of the first part of Theorem 3.1, we see that

$$\mathcal{Y}^{-1}\left(Y_{W_1}^{g_1}\left(e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}u, \frac{xx_0}{(x_2 + x_0)x_2}\right)\Bigg|_{\substack{x_0^n = e^{nl_{p_{12}}(z_1 - z_2)},\ \log x_0 = \log(z_1 - z_2),\ x_2^n = e^{n\log(z_2)} \\ \log x_2 = l_{p_2}(z_2),\ x^n = e^{\pm n\pi \mathbf{i}},\ \log x = \pm\pi\mathbf{i}}} \cdot\right.$$

$$\cdot\, e^{z_2L_{W_1}(1)}e^{\pm\pi \mathbf{i}L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, z_2^{-1}\Big)$$

is an iterated series obtained by expanding one variable inside the series obtained from the iterate of $\mathcal{Y}^{-1}$ and $Y_{W_1}^{g_1}$. Using the same method as in the proof of the first part of Theorem 3.1, we can prove that the corresponding multiple series is absolutely convergent. In particular, we can calculate the sum of the series using any of the iterated sums associated with the multisum. Then the same calculations as those in the proof of Theorem 6.1 in [H8] shows that when $|z_2| > |z_1 - z_2| > 0$ and $|\arg z_1 - \arg z_2| < \frac{1}{2}$, the right-hand side of (3.49) is equal to

$$\langle w_3', \mathcal{Y}^{-1}((Y_{W_1}^{g_1})^{m+\frac{1\pm 1}{2}}(e^{z_1L_V(1)}(-z_1^{-2})^{L_V(0)}u, z_1^{-1} - z_2^{-1})\cdot \\ \cdot\, e^{z_2L_{W_1}(1)}e^{\pm\pi \mathbf{i}L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, z_2^{-1})w_2\rangle, \tag{3.51}$$

where $m \in \mathbb{Z}$ is given by

$$\log(z_1 - z_2) - \log z_1 - \log z_2 + \pi i = l_{m+\frac{1\pm 1}{2}}(z_1^{-1} - z_2^{-1}).$$

We know that

$$\langle w_3', \mathcal{Y}(Y_{W_1}^{g_1}(e^{z_1L_V(1)}(-z_1^{-2})^{L_V(0)}u, z_1^{-1} - z_2^{-1})e^{z_2L_{W_1}(1)}e^{\pm\pi \mathbf{i}L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, z_2^{-1})w_2\rangle \tag{3.52}$$

is absolutely convergent on the region $|z_2^{-1}| > |z_1^{-1} - z_2^{-1}| > 0$ and, if in addition $|\arg z_1^{-1} - \arg z_2^{-1}| < \frac{\pi}{2}$, the sum is equal to

$$f^e(z_1^{-1}, z_2^{-1}; e^{z_1L_V(1)}(-z_1^{-2})^{L_V(0)}u, e^{z_2L_{W_1}(1)}e^{\pm\pi \mathbf{i}L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, w_2, w_3').$$

Since (3.51) can be obtained from (3.52) using the multivalued analytic function (3.30) on the region $|z_2^{-1}| > |z_1^{-1} - z_2^{-1}| > 0$ starting from the value given by (3.52) by letting z_2^{-1} go around 0 clockwise once while $z_1^{-1} - z_2^{-1}$ is fixed (corresponding to $b_{13}^{-1}b_{23}^{-1}$ since to keep $z_1^{-1} - z_2^{-1}$ fixed, z_1^{-1} must also go around 0 clockwise once) and then letting z_1^{-1} go around z_2^{-1} counterclockwise $m + \frac{1\pm 1}{2}$ times (corresponding to $b_{12}^{m+\frac{1\pm 1}{2}}$), we see that (3.51) is absolutely convergent in the region $|z_2^{-1}| > |z_1^{-1} - z_2^{-1}| > 0$ and if $|\arg z_1^{-1} - \arg z_2^{-1}| < \frac{\pi}{2}$, the sum is equal to

$$f^{b_{13}^{-1}b_{23}^{-1}b_{12}^{m+\frac{1\pm 1}{2}}}(z_1^{-1}, z_2^{-1}; e^{z_1L_V(1)}(-z_1^{-2})^{L_V(0)}u, e^{z_2L_{W_1}(1)}e^{\pm\pi \mathbf{i}L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, w_2, w_3'). \tag{3.53}$$

We now consider the set given by $|z_1^{-1}| > |z_2^{-1}| > |z_1^{-1} - z_2^{-1}| > 0$, $|\arg z_1^{-1} - \arg z_2^{-1}| < \frac{\pi}{2}$, $|\arg(z_1^{-1} - z_2^{-1}) - \arg z_2^{-1}| < \frac{\pi}{2}$, $\arg z_1, \arg z_2 \neq 0$ and $m + \frac{1\pm 1}{2} = 0$. Since $|z_1^{-1}| > |z_2^{-1}| > |z_1^{-1} - z_2^{-1}| > 0$, $|\arg z_1^{-1} - \arg z_2^{-1}| < \frac{\pi}{2}$ and $|\arg(z_1^{-1} - z_2^{-1}) - \arg z_2^{-1}| < \frac{\pi}{2}$, we know that (3.52) is equal to

$$\langle w_3', Y_{W_1}^{g_3}(e^{z_1L_V(1)}(-z_1^{-2})^{L_V(0)}u, z_1^{-1})\mathcal{Y}(e^{z_2L_{W_1}(1)}e^{\pm\pi \mathbf{i}L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, z_2^{-1})w_2\rangle$$

$$= f^e(z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}u, e^{z_2 L_{W_1}(1)}e^{\pm\pi \mathbf{i} L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, w_2, w_3'). \quad (3.54)$$

Then on this set, (3.51) is absolutely convergent to

$$f^{b_{13}^{-1}b_{23}^{-1}}(z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}u, e^{z_2 L_{W_1}(1)}e^{\pm\pi \mathbf{i} L_{W_1}(0)}e^{-2\log(z_2)L_{W_1}(0)}w_1, w_2, w_3')$$
$$= h^e_{\pm}(z_1, z_2; u, w_1, w_3', w_2).$$

By analytic extension, we see that the sum of (3.51) is equal to $h^e_{\pm}(z_1, z_2; u, w_1, w_3', w_2)$ on the subregion of M_0^2 given by $|z_2^{-1}| > |z_1^{-1} - z_2^{-1}| > 0$, $|\arg z_1^{-1} - \arg z_2^{-1}| < \frac{\pi}{2}$. By analytic extension again, we see that the sum of (3.49) is equal to $h^e_{\pm}(z_1, z_2; u, w_1, w_3', w_2)$ on the subregion of M_0^2 given by $|z_2| > |z_1 - z_2| > 0$, $|\arg z_1 - \arg z_2| < \frac{1}{2}$. ∎

4 Tensor product bifunctors and some natural isomorphisms

In this section we introduce the notion of twisted $P(z)$-intertwining map and give a definition and a construction of $P(z)$-tensor product of a g_1-twisted module and a g_2-twisted V-module for g_1, g_2 in a group G of automorphisms of V in a category $\mathcal{C}$ of twisted V-modules under suitable assumptions. Using the skew-symmetry isomorphism Ω_+ given in the preceding section, we construct G-crossed commutativity isomorphisms. We also construct parallel transport isomorphisms. Using G-crossed commutativity isomorphisms and parallel transport isomorphisms, we construct G-crossed braiding isomorphisms. The material in this section is essentially the same as the corrresponding material in [HL2], [H6] and [HLZ2] except that V-modules and intertwining maps are replaced by twisted V-modules in $\mathcal{C}$ and twisted intertwining maps.

Let G be a group of automorphisms of V and $\mathcal{C}$ a category of g-twisted V-modules for $g \in G$. The category $\mathcal{C}$ can be the category of grading-restricted g-twisted V-modules for $g \in G$. But since many of the constructions in the present paper works for any category satisfying suitable conditions, we shall work with a general category $\mathcal{C}$.

Definition 4.1 Let $g_1, g_2 \in G$, W_1, W_2, W_3 g_1-, g_2-, g_1g_2-twisted V-modules, respectively, in the category $\mathcal{C}$ and $z \in \mathbb{C}^\times$. A *twisted $P(z)$-intertwining map of type* $\binom{W_3}{W_1W_2}$ is a linear map $I : W_1 \otimes W_2 \to \overline{W}_3$ given by $I(w_1 \otimes w_2) = \mathcal{Y}(w_1, z)w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$, where $\mathcal{Y}$ is a twisted intertwining operator of type $\binom{W_3}{W_1W_2}$.

Using the notion of twisted $P(z)$-intertwining map, we now define the notion of tensor product of two twisted modules in $\mathcal{C}$.

Definition 4.2 Let W_1 and W_2 be g_1- and g_2-twisted V-modules, respectively, in $\mathcal{C}$. A *$P(z)$-product of W_1 and W_2* is a pair (W_3, I) consisting of a g_1g_2-twisted V-module W_3 and a twisted $P(z)$-intertwining map I of type $\binom{W_3}{W_1W_2}$. A *$P(z)$-tensor product of W_1 and W_2*

is a $P(z)$-product $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$ satisfying the following universal property: For any $P(z)$-product (W_3, I) of W_1 and W_2, there exists a unique module map $f : W_1 \boxtimes_{P(z)} W_2 \to W_3$ such that we have the commutative diagram

$$\begin{tikzcd}
W_1 \otimes W_2 \arrow[r, "I"] \arrow[d, "\boxtimes_{P(z)}"'] & \overline{W}_3 \\
\overline{W_1 \boxtimes_{P(z)} W_2} \arrow[ur, "\bar{f}"'] &
\end{tikzcd}$$

where $\bar{f}$ is the natural extension of f to $\overline{W_1 \boxtimes_{P(z)} W_2}$.

We now give a construction of $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$ under a suitable assumption using the same method as in [HL2] and [HLZ2].

Given a $P(z)$-product (W_3, I) of W_1 and W_2, for $w_3' \in W_3'$, we have an element $\lambda_{I,w_3'} \in (W_1 \otimes W_2)^*$ defined by

$$\lambda_{I,w_3'}(w_1 \otimes w_2) = \langle w_3', I(w_1 \otimes w_2)\rangle$$

for $w_1 \in W_1$ and $w_2 \in W_2$. Let $W_1 \boxbackslash_{P(z)} W_2$ be the subspace of $(W_1 \otimes W_2)^*$ spanned by $\lambda_{I,w_3'}$ for all $P(z)$-products (W_3, I) and $w_3' \in W_3'$. We define a vertex operator map

$$Y^{g_1g_2}_{W_1 \boxbackslash_{P(z)} W_2} : V \otimes (W_1 \boxbackslash_{P(z)} W_2) \to (W_1 \boxbackslash_{P(z)} W_2)\{x\}[\log x]$$

by

$$Y^{(g_1g_2)^{-1}}_{W_1 \boxbackslash_{P(z)} W_2}(v, x)\lambda_{I,w_3'} = \lambda_{I, Y^{(g_1g_2)^{-1}}_{W_3'}(v,x)w_3'}$$

for $v \in V$ and $\lambda_{I,w_3'} \in W_1 \boxbackslash_{P(z)} W_2$.

Proposition 4.3 *The pair $(W_1 \boxbackslash_{P(z)} W_2, Y^{g_1g_2}_{W_1 \boxbackslash_{P(z)} W_2})$ is a generalized $(g_1g_2)^{-1}$-twisted V-module.*

Proof. Note that every element of $W_1 \boxbackslash_{P(z)} W_2$ is a linear combination of elements of the form $\lambda_{I,w_3'}$ for a g_1g_2-twisted V-module W_3 in $\mathcal{C}$, a $P(z)$-intertwining map I of type of $\binom{W_3}{W_1 W_2}$ and an element $w_3' \in W_3'$. For fixed W_3 and I, the space spanned by all $\lambda_{I,w_3'}$ for $w_3' \in W_3'$ is the image of W_3' under the linear map from W_3' to $W_1 \boxbackslash_{P(z)} W_2$ given by $w_3' \mapsto \lambda_{I,w_3'}$. This linear map preserve the gradings, commutes with the actions of $(g_1g_2)^{-1}$ and twisted vertex operators. So the space of spanned by all $\lambda_{I,w_3'}$ for $w_3' \in W_3'$ is a generalized $(g_1g_2)^{-1}$-twisted V-module. Thus $W_1 \boxbackslash_{P(z)} W_2$ as a sum of generalized $(g_1g_2)^{-1}$-twisted V-modules is also a generalized $(g_1g_2)^{-1}$-twisted V-module. ∎

Assumption 4.4 *We assume that the following conditions for $\mathcal{C}$ hold:*

1. *For objects W_1 and W_2 in $\mathcal{C}$, $W_1 \boxbackslash_{P(z)} W_2$ is also in $\mathcal{C}$.*
2. *The contragredient of an object in $\mathcal{C}$ is also in $\mathcal{C}$.*

3. the double contragredient of an object in $\mathcal{C}$ is equivalent to the object

From the Conditions 1 and 2 in Assumption 4.4, we see that $(W_1 \boxbackslash_{P(z)} W_2)'$ is in $\mathcal{C}$. We take $W_1 \boxtimes_{P(z)} W_2$ to be $(W_1 \boxbackslash_{P(z)} W_2)'$. We still need to give a twisted $P(z)$-intertwining map $\boxtimes_{P(z)}$ of type $\binom{W_1 \boxtimes_{P(z)} W_2}{W_1 W_2}$ or equivalently, an intertwining operator of the same type.

Let W be a $(g_1 g_2)^{-1}$-twisted V-module in $\mathcal{C}$ and $f : W \to W_1 \boxbackslash_{P(z)} W_2$ a V-module map. Since the double contragredient of an object in $\mathcal{C}$ is equivalent to the object itself by Condition 3 in Assumption 4.4, every element of $(W')'$ can be viewed as an element of W. For $w_1 \in W_1$, $w_2 \in W_2$ and $w \in W$, we define

$$\langle w, \mathcal{Y}_f(w_1, z) w_2 \rangle = (f(w))(w_1 \otimes w_2). \tag{4.1}$$

Then we define

$$\mathcal{Y}_f(w_1, x) w_2 = x^{L_{W'}(0)} e^{-(\log z) L_{W_3}(0)} \mathcal{Y}_f(x^{-L_{W_1}(0)} e^{(\log z) L_{W_1}(0)} w_1, z) x^{-L_{W_2}(0)} e^{(\log z) L_{W_2}(0)} w_2 \tag{4.2}$$

for $w_1 \in W_1$, $w_2 \in W_2$. We now have a linear map

$$\mathcal{Y}_f : W_1 \otimes W_2 \to W'\{x\}[\log x].$$

Proposition 4.5 *The linear map $\mathcal{Y}_f : W_1 \otimes W_2 \to W'\{x\}[\log x]$ given by* (4.1) *and* (4.2) *above is a $g_1 g_2$-twisted intertwining operator of type $\binom{W'}{W_1 W_2}$. In particular, in the case that $W = W_1 \boxbackslash_{P(z)} W_2$ and $f = 1_{W_1 \boxbackslash_{P(z)} W_2} : W \to W_1 \boxbackslash_{P(z)} W_2$ is the identity map, we obtain a $g_1 g_2$-twisted intertwining operator $\mathcal{Y}_{1_{W_1 \boxbackslash_{P(z)} W_2}}$ of type $\binom{W_1 \boxtimes_{P(z)} W_2}{W_1 W_2}$.*

Proof. We first verify the $L(-1)$-derivative property.

$$\begin{aligned}
&\frac{d}{dx} \mathcal{Y}_f(w_1, x) w_2 \\
&\quad = \frac{d}{dx} x^{L_{W'}(0)} e^{-(\log z) L_{W'}(0)} \mathcal{Y}_f(x^{-L_{W_1'}(0)} e^{(\log z) L_{W_1'}(0)} w_1, z) x^{-L_{W_2}(0)} e^{(\log z) L_{W_2}(0)} w_2 \\
&\quad = x^{L_{W'}(0)-1} e^{-(\log z) L_{W'}(0)} L_{W'}(0) \mathcal{Y}_f(x^{-L_{W_1}(0)} e^{(\log z) L_{W_1}(0)} w_1, z) x^{-L_{W_2}(0)} e^{(\log z) L_{W_2}(0)} w_2 \\
&\qquad - x^{L_{W'}(0)} e^{-(\log z) L_{W'}(0)} \mathcal{Y}_f(L_{W_1}(0) x^{-L_{W_1}(0)-1} e^{(\log z) L_{W_1}(0)} w_1, z) x^{-L_{W_2}(0)} e^{(\log z) L_{W_2}(0)} w_2 \\
&\qquad - x^{L_{W'}(0)} e^{-(\log z) L_{W'}(0)} \mathcal{Y}_f(x^{-L_{W_1}(0)} e^{(\log z) L_{W_1}(0)} w_1, z) L_{W_2}(0) x^{-L_{W_2}(0)-1} e^{(\log z) L_{W_2}(0)} w_2 \\
&\quad = x^{-1} x^{L_{W'}(0)} e^{-(\log z) L_{W'}(0)} \cdot \\
&\qquad\quad \cdot \big(L_{W'}(0) \mathcal{Y}_f(x^{-L_{W_1}(0)} e^{(\log z) L_{W_1}(0)} w_1, z) \\
&\qquad\qquad - \mathcal{Y}_f(L_{W_1}(0) x^{-L_{W_1}(0)} e^{(\log z) L_{W_1}(0)} w_1, z) - \mathcal{Y}_f(x^{-L_{W_1}(0)} e^{(\log z) L_{W_1}(0)} w_1, z) L_{W_2}(0) \big) \cdot \\
&\qquad\quad \cdot x^{-L_{W_2}(0)} e^{(\log z) L_{W_2}(0)} w_2.
\end{aligned} \tag{4.3}$$

Since f is a V-module map, $f(W)$ is a submodule of $W_1 \boxbackslash_{P(z)} W_2$. By the definition of $W_1 \boxbackslash_{P(z)} W_2$, it is spanned by elements of the form $\lambda_{I, w_3'}$ for a $g_1 g_2$-twisted V-module W_3, a $P(z)$-intertwining map I of type $\binom{W_3}{W_1 W_2}$ and $w_3' \in W_3'$. In particular, for $w \in W$,

$$f(w) = \sum_{i=3}^{n} \lambda_{I^i, w_i'},$$

where for $i = 3, \ldots, n$, w_i' is an element of the contragredient module W_i' of a g_1g_2-twisted V-module W_i, I^i a $P(z)$-intertwining map of type $\binom{W_i}{W_1W_2}$.

Let $\mathcal{Y}^i$ be the intertwining operator of type $\binom{W_i}{W_1W_2}$ such that $I^i = \mathcal{Y}^i(\cdot, z)\cdot$. Then we have

$$\begin{aligned}
\langle w, \mathcal{Y}_f(w_1, z)w_2\rangle &= (f(w))(w_1 \otimes w_2)\\
&= \sum_{i=3}^{n} \lambda_{I^i, w_i'}(w_1 \otimes w_2)\\
&= \sum_{i=3}^{n} \langle w_i', I^i(w_1 \otimes w_2)\rangle\\
&= \sum_{i=3}^{n} \langle w_i', \mathcal{Y}^i(w_1, z)w_2)\rangle
\end{aligned}$$

for $w_1 \in W_1$ and $w_2 \in W_2$. Also,

$$\begin{aligned}
(f(L_W(0)w))(w_1 \otimes w_2) &= ((L_{W_1 \boxtimes_{P(z)} W_2}(0)f(w))(w_1 \otimes w_2)\\
&= \sum_{i=3}^{n} (L_{W_1 \boxtimes_{P(z)} W_2}(0)\lambda_{I^i, w_i'})(w_1 \otimes w_2)\\
&= \sum_{i=3}^{n} \mathrm{Res}_x x(Y_{W'}^{(g_1g_2)^{-1}}(\omega, x)\lambda_{I^i, w_i'})(w_1 \otimes w_2)\\
&= \sum_{i=3}^{n} \mathrm{Res}_x x(\lambda_{I^i, Y_{W_i'}^{(g_1g_2)^{-1}}(\omega, x)w_i'}(w_1 \otimes w_2)\\
&= \sum_{i=3}^{n} \lambda_{I^i, L_{W_i'}(0)w_i'}(w_1 \otimes w_2).
\end{aligned}$$

for $w_1 \in W_1$ and $w_2 \in W_2$. So we have

$$f(L_W(0)w) = \sum_{i=3}^{n} \lambda_{I^i, L_{W_i'}(0)w_i'}.$$

Thus for $w_1 \in W_1$, $w_2 \in W_2$,

$$\begin{aligned}
&\langle w, (L_{W'}(0)\mathcal{Y}_f(w_1, z) - \mathcal{Y}_f(L_{W_1}(0)w_1, z) - \mathcal{Y}_f(w_1, z)L_{W_2}(0))\, w_2\rangle\\
&\quad = \langle L_W(0)w, \mathcal{Y}_f(w_1, z)w_2\rangle - \langle w, \mathcal{Y}_f(L_{W_1}(0)w_1, z)w_2\rangle - \langle w, \mathcal{Y}_f(w_1, z)L_{W_2}(0)w_2\rangle\\
&\quad = (f(L_W(0)w))(w_1 \otimes w_2) - (f(w))(L_{W_1}(0)w_1 \otimes w_2) - (f(w))(w_1 \otimes L_{W_2}(0)w_2)\\
&\quad = \sum_{i=3}^{n} \lambda_{I^i, L_{W_i'}(0)w_i'}(w_1 \otimes w_2) - \sum_{i=3}^{n} \lambda_{I, w_i'}(L_{W_1}(0)w_1 \otimes w_2) - \sum_{i=3}^{n} \lambda_{I, w_i'}(w_1 \otimes L_{W_2}(0)w_2)\\
&\quad = \sum_{i=3}^{n} \langle L_{W_i'}(0)w_i', \mathcal{Y}^i(w_1, z)w_2\rangle - \sum_{i=3}^{n} \langle w_i', \mathcal{Y}^i(L_{W_1}(0)w_1, z)w_2\rangle - \sum_{i=3}^{n} \langle w_i', \mathcal{Y}^i(w_1, z)L_{W_2}(0)w_2\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=3}^{n} \langle w_i', (L_{W_i}(0)\mathcal{Y}^i(w_1, z) - \mathcal{Y}^i(L_{W_1}(0)w_1, z) - \mathcal{Y}^i(w_1, z)L_{W_2}(0))w_2\rangle \\
&= \sum_{i=3}^{n} z\langle w_i', \mathcal{Y}^i(L_{W_1}(-1)w_1, z)w_2\rangle \\
&= z\sum_{i=3}^{n} \lambda_{I^i, w_i'}((L_{W_1}(-1)w_1 \otimes w_2) \\
&= z(f(w))(L_{W_1}(-1)w_1 \otimes w_2) \\
&= z\langle w, \mathcal{Y}_f(L_{W_1}(-1)w_1, z)w_2\rangle,
\end{aligned}
$$

where we have used the $L(0)$-commutator formula for the twisted intertwining operators $\mathcal{Y}^i$. Since $w \in W$ and $w_2 \in W_2$ are arbitrary, we obtain

$$
L_W(0)\mathcal{Y}_f(w_1, z) - \mathcal{Y}_f(L_{W_1}(0)w_1, z) - \mathcal{Y}_f(w_1, z)L_{W_2}(0) = z\mathcal{Y}_f(L_{W_1}(-1)w_1, z) \tag{4.4}
$$

for $w_1 \in W_1$.

Using (4.4), we see that the right-hand side of (4.3) is equal to

$$
\begin{aligned}
x^{-1}&x^{L_W(0)}e^{-(\log z)L_W(0)}z\mathcal{Y}_f(L_{W_1}(-1)x^{-L_{W_1}(0)}e^{(\log z)L_{W_1}(0)}w_1, z)x^{-L_{W_2}(0)}e^{(\log z)L_{W_2}(0)}w_2 \\
&= x^{L_W(0)}e^{-(\log z)L_W(0)}\mathcal{Y}_f(x^{-L_{W_1}(0)}e^{(\log z)L_{W_1}(0)}L_{W_1}(-1)w_1, z)x^{-L_{W_2}(0)}e^{(\log z)L_{W_2}(0)}w_2 \\
&= \mathcal{Y}_f(L_{W_1}(-1)w_1, x)w_2,
\end{aligned}
$$

proving the $L(-1)$-derivative property.

For $v \in V$, $w_1 \in W_1$ and $w_2 \in W_2$, we have

$$
\begin{aligned}
(f(Y_W^{(g_1g_2)^{-1}}(v, x)w))(w_1 \otimes w_2) &= (Y^{(g_1g_2)^{-1}}_{W_1 \boxtimes_{P(z)} W_2}(v, x)f(w))(w_1 \otimes w_2) \\
&= \sum_{i=3}^{n} (Y^{(g_1g_2)^{-1}}_{W_1 \boxtimes_{P(z)} W_2}(v, x)\lambda_{I^i, w_i'})(w_1 \otimes w_2) \\
&= \sum_{i=3}^{n} \lambda_{I^i, Y_{W_i}^{(g_1g_2)^{-1}}(v,x)w_i'}(w_1 \otimes w_2).
\end{aligned}
$$

Then we obtain

$$
f(Y_W^{(g_1g_2)^{-1}}(v, x)w) = \sum_{i=3}^{n} \lambda_{I^i, Y_{W_i}^{(g_1g_2)^{-1}}(v,x)w_i'}
$$

For a g_1g_2-twisted V-module W_3 in $\mathcal{C}$, a $P(z)$-intertwining map I of type $\binom{W_3}{W_1W_2}$, $u \in V$, $w_1 \in W_1$, $w_2 \in W_2$ and $w_3' \in W_3'$, we have

$$
\begin{aligned}
\langle w, &Y_{W'}^{g_1g_2}(u, z_1)\mathcal{Y}_f(w_1, z_2)w_2\rangle \\
&= \langle w, Y_{W'}^{g_1g_2}(u, z_1)e^{(\log z_2)L_{W'}(0)}e^{-(\log z)L_{W'}(0)} \cdot \\
&\qquad \cdot \mathcal{Y}_f(e^{-(\log z_2)L_{W_1}(0)}e^{(\log z)L_{W_1}(0)}w_1, z)e^{-(\log z_2)L_{W_2}(0)}e^{(\log z)L_{W_2}(0)}w_2\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \langle e^{(\log z_2)L_W(0)}e^{-(\log z)L_W(0)}Y_W^{(g_1g_2)^{-1}}(e^{z_1L_V(1)}(-z_1^{-2})^{L_V(0)}u, z_1^{-1})w,\\
&\qquad\qquad \mathcal{Y}_f(e^{-(\log z_2)L_{W_1}(0)}e^{(\log z)L_{W_1}(0)}w_1, z)e^{-(\log z_2)L_{W_2}(0)}e^{(\log z)L_{W_2}(0)}w_2\rangle\\
&= (f(e^{(\log z_2)L_W(0)}e^{-(\log z)L_W(0)}Y_W^{(g_1g_2)^{-1}}(e^{z_1L_V(1)}(-z_1^{-2})^{L_V(0)}u, z_1^{-1})w))\\
&\qquad\qquad (e^{-(\log z_2)L_{W_1}(0)}e^{(\log z)L_{W_1}(0)}w_1\otimes e^{-(\log z_2)L_{W_2}(0)}e^{(\log z)L_{W_2}(0)}w_2)\\
&= \sum_{i=3}^{n}\lambda_{I^i, e^{(\log z_2)L_{W_i'}(0)}e^{-(\log z)L_{W_i'}(0)}Y_{W_i'}^{(g_1g_2)^{-1}}(e^{z_1L_V(1)}(-z_1^{-2})^{L_V(0)}u, z_1^{-1})w_i'}\\
&\qquad\qquad (e^{-(\log z_2)L_{W_1}(0)}e^{(\log z)L_{W_1}(0)}w_1\otimes e^{-(\log z_2)L_{W_2}(0)}e^{(\log z)L_{W_2}(0)}w_2)\\
&= \sum_{i=3}^{n}\langle e^{(\log z_2)L_{W_i'}(0)}e^{-(\log z)L_{W_i'}(0)}Y_{W_i'}^{(g_1g_2)^{-1}}(e^{z_1L_V(1)}(-z_1^{-2})^{L_V(0)}u, z_1^{-1})w_i',\\
&\qquad\qquad \mathcal{Y}^i(e^{-(\log z_2)L_{W_1}(0)}e^{(\log z)L_{W_1}(0)}w_1, z)e^{-(\log z_2)L_{W_2}(0)}e^{(\log z)L_{W_2}(0)}w_2\rangle\\
&= \sum_{i=3}^{n}\langle w_i', Y_{W_i}^{g_1g_2}(u, z_1)e^{(\log z_2)L_{W_i}(0)}e^{-(\log z)L_{W_i}(0)}\cdot\\
&\qquad\qquad \cdot\mathcal{Y}^i(e^{-(\log z_2)L_{W_1}(0)}e^{(\log z)L_{W_1}(0)}w_1, z)e^{-(\log z_2)L_{W_2}(0)}e^{(\log z)L_{W_2}(0)}w_2\rangle\\
&= \sum_{i=3}^{n}\langle w_i', Y_{W_i}^{g_1g_2}(u, z_1)\mathcal{Y}^i(w_1, z_2)w_2\rangle. \qquad (4.5)
\end{aligned}
$$

Similarly, we have

$$
\langle w, \mathcal{Y}_f(w_1, z_2)Y_{W_2}^{g_2}(u, z_1)w_2\rangle = \sum_{i=3}^{n}\langle w_i', \mathcal{Y}^i(w_1, z_2)Y_{W_2}^{g_2}(u, z_1)w_2\rangle \qquad (4.6)
$$

and

$$
\langle w, \mathcal{Y}_f(Y_{W_2}^{g_1}(u, z_1-z_2)w_1, z_2)w_2\rangle = \sum_{i=3}^{n}\langle w_i', \mathcal{Y}^i(Y_{W_1}^{g_1}(u, z_1-z_2)w_1, z_2)w_2\rangle. \qquad (4.7)
$$

Since $\mathcal{Y}^i$ for $i = 1, \dots, n$ are twisted intertwining operators, the duality property for $\mathcal{Y}$ follows from (4.5), (4.6), (4.7) and the duality properties for $\mathcal{Y}^i$.

The convergence for products of more than two operators follows from the formula

$$
\begin{aligned}
&\langle w, Y_{W'}^{g_1g_2}(u_1, z_1)\cdots Y_{W'}^{g_1g_2}(u_{k-1}, z_{k-1})\mathcal{Y}_f(w_1, z_k)w_2\rangle\\
&\quad = \sum_{i=3}^{n}\langle w_i', Y_{W_i}^{g_1g_2}(u_1, z_1)\cdots Y_{W_i}^{g_1g_2}(u_{k-1}, z_{k-1})\mathcal{Y}^i(w_1, z_k)w_2\rangle,
\end{aligned}
$$

whose proof is the same as that of (4.5). ∎

Let $\boxtimes_{P(z)} = \mathcal{Y}_{1_{W_1\boxtimes_{P(z)}W_2}}(\cdot, z)\cdot$. Then $\boxtimes_{P(z)}$ is a $P(z)$-intertwining map of type $\binom{W_1\boxtimes_{P(z)}W_2}{W_1W_2}$. Let

$$
w_1\boxtimes_{P(z)} w_2 = \boxtimes_{P(z)}(w_1\otimes w_2) = \mathcal{Y}(w_1, z)w_2 \in \overline{W_1\boxtimes_{P(z)} W_2}
$$

for $w_1 \in W_1$ and $w_2 \in W_2$. We call $w_1 \boxtimes_{P(z)} w_2$ the *tensor product* of w_1 and w_2. By (4.1), we have

$$\lambda(w_1 \otimes w_2) = \langle \lambda, w_1 \boxtimes_{P(z)} w_2 \rangle \tag{4.8}$$

for $\lambda \in W_1 \boxdot_{P(z)} W_2$, $w_1 \in W_1$ and $w_2 \in W_2$.

Theorem 4.6 *The pair $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$ is a $P(z)$-tensor product of W_1 and W_2.*

Proof. Let (W_3, I) be a $P(z)$-product of W_1 and W_2. Then we have a module map $g : W_3' \to W_1 \boxdot_{P(z)} W_2$ given by $g(w_3') = \lambda_{I,w_3'}$ for $w_3' \in W_3'$. By definition, we have $(g(w_3'))(w_1 \otimes w_2) = \lambda_{I,w_3'}(w_1 \otimes w_2) = \langle w_3', I(w_1 \otimes w_2)\rangle$ for $w_1 \in W_1$ and $w_2 \in W_2$. The adjoint of this module map is a module map $f : W_1 \boxtimes_{P(z)} W_2 \to W_3$. By definitions and (4.8),

$$\begin{aligned}
\langle w_3', (\bar{f} \circ \boxtimes_{P(z)})(w_1 \otimes w_2)\rangle &= \langle w_3', \bar{f}(w_1 \boxtimes_{P(z)} w_2)\rangle \\
&= \langle g(w_3'), w_1 \boxtimes_{P(z)} w_2\rangle \\
&= (g(w_3'))(w_1 \otimes w_2) \\
&= \langle w_3', I(w_1 \otimes w_2)\rangle.
\end{aligned}$$

So we obtain $\bar{f} \circ \boxtimes_{P(z)} = I$. ∎

We have assigned each object (W_1, W_2) in the category $\mathcal{C} \times \mathcal{C}$ an object $W_1 \boxtimes_{P(z)} W_2$ in $\mathcal{C}$. To obtain a functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$, we still need to assign a morphism (f_1, f_2) in $\mathcal{C} \times \mathcal{C}$ a morphism $f_1 \boxtimes_{P(z)} f_2$ in $\mathcal{C}$.

Let $W_1, \widetilde{W_1}$ be g_1-twisted V-modules in $\mathcal{C}$ and $W_2, \widetilde{W_2}$ g_1-twisted V-modules in $\mathcal{C}$. Let $f_1 : W_1 \to \widetilde{W_1}$ and $f_2 : W_2 \to \widetilde{W_2}$ be module maps. Let $\widetilde{\mathcal{Y}}$ be the intertwining operator of type $\binom{\widetilde{W_1} \boxtimes_{P(z)} \widetilde{W_2}}{\widetilde{W_1}\widetilde{W_2}}$ such that $\tilde{w}_1 \boxtimes_{P(z)} \tilde{w}_2 = \widetilde{\mathcal{Y}}(\tilde{w}_1, z)\tilde{w}_2$. Since f_1 and f_2 are module maps, $\mathcal{Y} = \widetilde{\mathcal{Y}} \circ (f_1 \otimes f_2)$ is an intertwining operator of type $\binom{\widetilde{W_1} \boxtimes_{P(z)} \widetilde{W_2}}{W_1 W_2}$. Then $I = \mathcal{Y}(\cdot, z)\cdot$ is a $P(z)$-intertwining operator of the same type. Hence we have a $P(z)$-product $(\widetilde{W_1} \boxtimes_{P(z)} \widetilde{W_2}, I)$ of W_1 and W_2. By the universal property of the tensor product $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$, there exist a unique module map $f : W_1 \boxtimes_{P(z)} W_2 \to \widetilde{W_1} \boxtimes_{P(z)} \widetilde{W_2}$ such that $I = \bar{f} \circ \boxtimes_{P(z)}$. We define this module map f to be the $P(z)$-tensor product of f_1 and f_2 and denote it by $f_1 \boxtimes_{P(z)} f_2$.

Theorem 4.7 *The assignments given by $(W_1, W_2) \mapsto W_1 \boxtimes_{P(z)} W_2$ and $(f_1, f_2) \mapsto f_1 \boxtimes_{P(z)} f_2$ above is a functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$.*

Proof. It is easy to verify $1_{w_1} \boxtimes_{P(z)} 1_{W_2} = 1_{W_1 \boxtimes_{P(z)} W_2}$ and $(f_1 \boxtimes_{P(z)} f_2) \circ (g_1 \boxtimes_{P(z)} g_2) = (f_1 g_1) \boxtimes_{P(z)} (f_2 g_2)$ by using the construction of the tensor products of module maps. We omit the details of the proofs. ∎

We call this functor the *$P(z)$-tensor product bifunctor*.

We now give a result on Condition 1 in Assumption 4.4.

Theorem 4.8 *Let $\mathcal{C}$ be the category of grading-restricted g-twisted V-modules for $g \in G$. Assume that the following conditions are satisfied:*

1. *For $g \in G$, there are only finitely many irreducible grading-restricted g-twisted V-modules.*

2. *Every grading-restricted twisted V-module is completely reducible.*

3. *For $g_1, g_2 \in G$ and g_1-, g_2-, g_1g_2-twisted V-modules W_1, W_2, W_3 in $\mathcal{C}$, the fusion rule $N_{W_1W_2}^{W_3} = \dim \mathcal{V}_{W_1W_2}^{W_3}$ is finite.*

Then $W_1 \boxslash_{P(z)} W_2$ is in $\mathcal{C}$ for objects W_1 and W_2 in $\mathcal{C}$.

Proof. Let W_1 and W_2 be g_1- and g_2-twisted V-modules in $\mathcal{C}$. Then $W_1 \boxslash_{P(z)} W_2$ is a geberalized $(g_1g_2)^{-1}$-twisted V-module. From the construction of $W_1 \boxslash_{P(z)} W_2$, it is a sum of grading-restricted $(g_1g_2)^{-1}$-twisted V-module. By Condition 2, $W_1 \boxslash_{P(z)} W_2$ must be a direct sum of irreducible grading-restricted $(g_1g_2)^{-1}$-twisted V-modules. But by Condition 1, there are only finitely many irreducible grading-restricted $(g_1g_2)^{-1}$-twisted V-modules. If $W_1 \boxslash_{P(z)} W_2$ is an infinite direct sum of irreducible grading-restricted $(g_1g_2)^{-1}$-twisted V-modules, at least one irreducible grading-restricted $(g_1g_2)^{-1}$-twisted V-module W_3 has infinitely many copies in this decomposition of $W_1 \boxslash_{P(z)} W_2$. But then we have infinitely many linearly independent injective V-module maps from W_3 to the $W_1 \boxslash_{P(z)} W_2$. But by Proposition 4.5, these infinite injective V-module maps give linearly independent twisted intertwining operator of type $\binom{W_3'}{W_1W_2}$. Thus the fusion rule $N_{W_1W_2}^{W_3}$ is ∞. By Condition 3, this is a contradiction. So $W_1 \boxslash_{P(z)} W_2$ must be a finite direct sum of irreducible grading-restricted $(g_1g_2)^{-1}$-twisted V-modules. In particular, it is grading restricted. ∎

Corollary 4.9 *The category of grading-restricted g-twisted V-modules for $g \in G$ satisfies Assumption 4.4.*

Proof. Theorem 4.8 shows that Condition 1 holds. Conditions 2 and 3 are clearly holds for grading-restricted twisted V-modules. ∎

For the same W_1 and W_2, let $\mathcal{Y}$ be the twisted intertwining operator of type $\binom{\phi_{g_1}(W_2)\boxtimes_{P(-z)}W_1}{\phi_{g_1}(W_2)W_1}$ such that $w_2 \boxtimes_{P(-z)} w_1 = \mathcal{Y}(w_2, -z)w_1$ for $w_1 \in W_1$, $w_2 \in \phi_{g_1}(W_2) = W_2$. Then by Theorem 3.1, $\Omega_+(\mathcal{Y})$ is a twisted intertwining operator of type

$$\binom{\phi_{g_1}(W_2) \boxtimes_{P(-z)} W_1}{W_1\phi_{g_1}^{-1}(\phi_{g_1}(W_2))} = \binom{\phi_{g_1}(W_2) \boxtimes_{P(-z)} W_1}{W_1W_2}.$$

In particular, the pair $(\phi_{g_1}(W_2) \boxtimes_{P(-z)} W_1, \Omega_+(\mathcal{Y})(\cdot, z)\cdot)$ is a $P(z)$-product of W_1 and W_2. By the universal property of the tensor product $W_1 \boxtimes_{P(z)} W_2$, there exists a unique g_1g_2-twisted V-module map

$$\mathcal{R}_{P(z)} : W_1 \boxtimes_{P(z)} W_2 \to \phi_{g_1}(W_2) \boxtimes_{P(-z)} W_1$$

such that

$$\Omega(\mathcal{Y})(\cdot, z)\cdot = \overline{\mathcal{R}}_{P(z)} \circ \boxtimes_{P(z)},$$

where $\overline{\mathcal{R}}_{P(z)}$ is the natural extension of $\mathcal{R}_{P(z)}$. The g_1g_2-twisted V-module map $\mathcal{R}_{P(z)}$ has an inverse

$$\mathcal{R}_{P(z)}^{-1} : \phi_{g_1}(W_2) \boxtimes_{P(-z)} W_1 \to W_1 \boxtimes_{P(z)} W_2$$

constructed in the same as above except that we use Ω_- instead of Ω_+. Then we obtain a natural isomorphism $\mathcal{R}_{P(z)}$ called the *G-crossed commutativity isomorphism*.

As in [H6] and [HLZ2], we also have parallel transport isomorphisms. Let $z_1, z_2 \in \mathbb{C}^\times$ and γ a path in $\mathbb{C}^\times$ from z_1 to z_2. We denote the homotopy class of γ by $[\gamma]$. For the same W_1 and W_2, let $\mathcal{Y}$ be the twisted intertwining operator of type $\binom{W_1 \boxtimes_{P(z_2)} W_2}{W_1 W_2}$ such that $w_1 \boxtimes_{P(z_2)} w_2 = \mathcal{Y}(w_1, z)w_2$ for $w_1 \in W_1$, $w_2 \in W_2$. Then $(W_1 \boxtimes_{P(z_2)} W_2, \mathcal{Y}(\cdot, z_1)\cdot)$ is a $P(z_1)$-product of W_1 and W_2. By the universal property of the $P(z_1)$-tensor product $W_1 \boxtimes_{P(z_1)} W_2$, there exists a unique g_1g_2-twisted V-module map

$$\mathcal{T}_{[\gamma]} : W_1 \boxtimes_{P(z_1)} W_2 \to W_1 \boxtimes_{P(z_2)} W_2$$

such that $\overline{\mathcal{T}_{[\gamma]}} \circ \boxtimes_{P(z_1)} = \boxtimes_{P(z_2)}$. The g_1g_2-twisted twisted V-module map $\mathcal{T}_{[\gamma]}$ is invertible since the same construction also gives a g_1g_2-twisted V-module map

$$\mathcal{T}_{[\gamma^{-1}]} : W_1 \boxtimes_{P(z_2)} W_2 \to W_1 \boxtimes_{P(z_1)} W_2$$

which is clearly the inverse of $\mathcal{T}_{[\gamma]}$. Thus the natural transformation $\mathcal{T}_{[\gamma]}$ is a natural isomorphism called the *parallel transport isomorphism from z_1 to z_2 along* $[\gamma]$.

Let γ be a path from -1 to 1 in the closed upper half plane with 0 deleted. For the same W_1 and W_2, we define the *G-crossed braiding isomorphsim* $\mathcal{R} : W_1 \boxtimes_{P(1)} W_2 \to \phi_{g_1}(W_2) \boxtimes_{P(1)} W_1$ by

$$\mathcal{R} = \mathcal{T}_{[\gamma]} \circ \mathcal{R}_{P(1)}.$$

5 Compatibility condition and grading-restriction condition

In this section, we introduce $P(z)$-compatibility condition and $P(z)$-local grading-restriction condition and using these conditions to give another construction of $W_1 \boxdot_{P(z)} W_2$ for two twisted V-modules W_1 and W_2. In the untwisted case (the case that $\mathcal{C}$ is the category of (untwisted or 1_V-twisted) V-modules), these conditions and this second construction given in [HL2], [HL3] and [HLZ3] play a crucial role in the proof of the associativity of intertwining operators and the construction of associativity isomorphisms in [H1] and [HLZ5]. It is expected that they will play the same crucial role in the proof of the associativity of twisted intertwining operators and the construction of associativity isomorphisms for the $P(z)$ tensor product bifunctors on the category $\mathcal{C}$ of twisted V-modules.

In the untwisted case, The $P(z)$-compatibility condition is formulated using a formula corresponding to the Jacobi identity for intertwining operators. Even though we can obtain a Jacobi identity for the rational coefficients of the expansions in a suitable basis of products and iterates of twisted intertwining operators with twisted vertex operators as in [H2] , we do not have a Jacobi identity for the products and iterates of twisted intertwining operators with twisted vertex operators. Thus we have to use the analytic method to formulate the $P(z)$-compatibility condition and prove the main results. In particular, the formulation and proofs involving the $P(z)$-compatibility condition are completely different from those in [HL2], [HL3] and [HLZ3].

For a fixed $z \in \mathbb{C}_\times$, we need to study multivalued analytic functions on the region

$$M^n(0,z) = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n \;\middle|\; \begin{array}{l} z_i \neq 0,\; z_i \neq z, \quad i = 1, \dots, n, \\ z_i \neq z_j, \text{ for } i,j = 1,\dots,n, \text{ and } i \neq j, \text{ if } n > 1 \end{array} \right\}.$$

for $n \in \mathbb{Z}_+$ and its subregions

$$\Omega_{m,k,l}(z) = \left\{ \begin{array}{r} (z_1, \dots, z_{m+k+l}) \\ \in \mathbb{C}^{m+k+l} \end{array} \;\middle|\; \begin{array}{l} |z| < |z_m| < \dots < |z_1|, \text{ if } m > 0, \\ 0 < |z_{m+k} - z| < \dots < |z_{m+1} - z| < |z|, \text{ if } k > 0, \\ 0 < |z_{m+k+l}| < \dots < |z_{m+k+1}| < |z|, \text{ if } l > 0, \\ |z_{m+1} - z| + |z_{m+k+1}| < |z|, \quad \text{if } k > 0, l > 0, \\ |z_{m+1} - z| + |z_1| < |z|, \quad \text{if } m > 0, k > 0 \end{array} \right\},$$

$$\Omega^{(1)}_{m,k,l}(z) = \left\{ \begin{array}{r} (z_1, \dots, z_{m+k+l}) \\ \in \Omega_{m,k,l}(z) \end{array} \;\middle|\; \begin{array}{l} |\arg(z_j - z) - \arg(z_j)| < \dfrac{\pi}{2}, \\ \qquad j = 1, \dots, m, \text{if } m > 0 \\ |\arg(z_j) - \arg(z)| < \dfrac{\pi}{2}, \\ \qquad j = m+1, \dots, m+k, \text{if } k > 0 \\ -\dfrac{3\pi}{2} < \arg(z_j - z) - \arg(z) < -\dfrac{\pi}{2}, \\ \qquad j = m+k+1, \dots, m+k+l, \quad \text{if } l > 0 \end{array} \right\},$$

$$\Omega^{(2)}_{m,k,l}(z) = \left\{ \begin{array}{r} (z_1, \dots, z_{m+k+l}) \\ \in \Omega_{m,k,l}(z) \end{array} \;\middle|\; \begin{array}{l} |\arg(z_j - z) - \arg(z_j)| < \dfrac{\pi}{2}, \\ \qquad j = 1, \dots, m, \text{if } m > 0 \\ |\arg(z_j) - \arg(z)| < \dfrac{\pi}{2}, \\ \qquad j = m+1, \dots, m+k, \text{if } k > 0 \\ \dfrac{\pi}{2} < \arg(z_j - z) - \arg(z) < \dfrac{3\pi}{2}, \\ \qquad j = m+k+1, \dots, m+k+l, \quad \text{if } l > 0 \end{array} \right\}.$$

for $m, k, l \in \mathbb{N}$. Also, we define $M^n_0(0,z)$ to be the simply-connected region given by cutting $M^n(0,z)$ along the positive real lines in the z_i-planes, $z_i - z_j$-planes and $z_i - z$-planes, that is, the sets

$$\{(z_1, \dots, z_n) \in M^n(0,z) \mid z_i \in \mathbb{R}_+\}, \quad i = 1, \dots, n,$$
$$\{(z_1, \dots, z_n) \in M^n(0,z) \mid z_i - z_j \in \mathbb{R}_+\}, \quad i,j = 1, \dots, n, i \neq j,$$

$$\{(z_1, \dots, z_n) \in M^n(0, z) \mid z_i - z \in \mathbb{R}_+\}, \quad i = 1, \dots, n,$$

with these sets attached to the upper half z_i-planes, $z_i - z_j$-planes and $z_i - z$-planes.

To formulate the $P(z)$-compatibility condition, we need a generalization of the notion of isolated singularity in the theory of one complex variable to several complex variables. Let $b = (b_1, \dots, b_n) \in (\mathbb{C} \cup \{\infty\})^n$ and $r = (r_1, \dots, r_n) \in \mathbb{R}_+^n$. Let $I \subset \{1, \dots, n\}$. We use the following notation for polydisks and polycircular domains:

$$\Delta^I(b, r) = \left\{ (z_1, \dots, z_n) \in (\mathbb{C} \cup \{0\})^n \;\middle|\; \begin{array}{l} |z_i| > r_i \text{ or } z_i = \infty \text{ if } b_i = \infty, \\ \qquad |z_i - b_i| < r_i \text{ if } b_i \in \mathbb{C}, \text{ for } i \in I, \\ |z_j| > r_j \text{ if } b_j = \infty, \\ \qquad 0 < |z_j - b_j| < r_j \text{ if } b_j \in \mathbb{C}, \text{ for } j \notin I. \end{array} \right\},$$

$$\Delta(b, r) = \Delta^{\{1, \dots, n\}}(b, r),$$
$$\Delta^\times(b, r) = \Delta^\varnothing(b, r).$$

We shall use $b_{z_i, z}$ to denote the homotopy class of loop in $M^l(0, z)$ with z_i going around z counterclockwise once with z_j for $j \neq i$ fixed.

Let g_1 and g_2 be automorphisms of V, W_1, W_2, W_3 g_1-, g_2-, g_1g_2-twisted V-modules, respectively, in the category $\mathcal{C}$ and $z \in \mathbb{C}^\times$. Let I be a twisted $P(z)$-intertwining map of type $\binom{W_3}{W_1 W_2}$ and $w_3' \in W_3'$. Then we have an element $\lambda_{I, w_3'} \in W_1 \boxtimes_{P(z)} W_2$.

Proposition 5.1 *The element $\lambda_{I, w_3'}$ has the following property: For $l \in \mathbb{N}$, $u_1, \dots, u_l \in V$, $w_1 \in W_1$, and $w_2 \in W_2$, there exists a multivalued analytic function*

$$f_l(z_1, \dots, z_l; u_1, \dots, u_l, w_1, w_2; \lambda_{I, w_3'}) \tag{5.1}$$

on $M^l(0, z)$ with a preferred branch

$$f_l^e(z_1, \dots, z_l; u_1, \dots, u_l, w_1, w_2; \lambda_{I, w_3'}) \tag{5.2}$$

on $M_0^l(0, z)$, satisfying the following:

1. (a) *For $i, j = 1, \dots, l$, $i \neq j$, $z_i - z_j = 0$ are poles of* (5.1). *In particular, there exist $M_{ij} \in \mathbb{Z}_+$ such that*

$$\left(\prod_{1 \leq i < j \leq l} (z_i - z_j)^{M_{ij}} \right) f_l(z_1, \dots, z_l; u_1, \dots, u_l, w_1, w_2; \lambda_{I, w_3'}) \tag{5.3}$$

can be analytically extended to a multivalued analytic function on

$$\{(z_1, \dots, z_l) \in \mathbb{C}^l \mid z_i \neq 0, \ z_i \neq z, \ i = 1, \dots, l\}$$

(b) *All component-isolated singularities of* (5.3) *are regular singularities.*

2. For $u_1, \ldots, u_l \in V$, $w_1 \in W_1$, and $w_2 \in W_2$,

(a) The series

$$\begin{aligned}
&\lambda_{I,w_3'}\Big(Y^{g_1}(u_1, z_1 - z)w_1 \otimes w_2\Big)\\
&\quad = \lambda_{I,w_3'}\Big(Y^{g_1}(u_1, x)w_1 \otimes w_2\Big)\Big|_{x^n = e^{nl_0(z_1 - z)},\ \log x = l_0(z_1 - z)}
\end{aligned} \tag{5.4}$$

is absolutely convergent on the region $\Omega_{0,1,0}(z)$. Moreover, it is convergent to $f_1^e(z_1; u_1, w_1, w_2; \lambda_{I,w_3'})$ on the region $\Omega^{(1)}_{0,1,0}(z) = \Omega^{(2)}_{0,1,0}(z)$.

(b) For $l \in \mathbb{N}$, the multiple series

$$\begin{aligned}
&\lambda_{I,w_3'}\Big(w_1 \otimes Y^{g_2}(u_1, z_1) \cdots Y^{g_2}(u_l, z_l)w_2\Big)\\
&\quad = \lambda_{I,w_3'}\Big(w_1 \otimes Y^{g_2}(u_1, x_1) \cdots Y^{g_2}(u_l, x_l)w_2\Big)\Big|_{x_i^n = e^{nl_0(z_1)},\ \log(x_i) = l_0(z_1),\ i = 1, \ldots, l}
\end{aligned} \tag{5.5}$$

in powers and logarithms of $z_1, \ldots, z_l$ is absolutely convergent on the region $\Omega_{0,0,l}(z)$. Moreover, it is absolutely convergent to (5.2) on the region $\Omega^{(1)}_{0,0,l}(z)$ and to

$$f_l^{b_{z_1,z}^{-1} \cdots b_{z_l,z}^{-1}}(z_1, \ldots, z_l; u_1, \ldots, u_l, w_1, w_2; \lambda)$$

on the region $\Omega^{(2)}_{0,0,l}(z)$.

Proof. This result can be easily verified by using the definitions of $\lambda_{I,w_3'}$ and $P(z)$-intertwining maps and the properties of twisted intertwining operators. We omit the details. ∎

Let g_1 and g_2 be automorphisms of V, W_1, W_2 g_1-, g_2-twisted V-modules, respectively, in the category $\mathcal{C}$ and $z \in \mathbb{C}^\times$. Motivated by Proposition 5.1, we formulate the following condition for $\lambda \in (W_1 \otimes W_2)^*$:

$P(z)$-**Compatibility condition** A element $\lambda \in (W_1 \otimes W_2)^*$ is said to be $P(z)$-*compatible* if for $l \in \mathbb{N}$, $u_1, \ldots, u_l \in V$, $w_1 \in W_1$, and $w_2 \in W_2$, there exists a multivalued analytic function

$$f_l(z_1, \ldots, z_l; u_1, \ldots, u_l, w_1, w_2; \lambda) \tag{5.6}$$

on $M^l(0, z)$ with a preferred branch

$$f_l^e(z_1, \ldots, z_l; u_1, \ldots, u_l, w_1, w_2; \lambda) \tag{5.7}$$

on $M_0^l(0, z)$, satisfying the following:

1. (a) For $i, j = 1, \ldots, l$, $i \neq j$, $z_1 - z_j = 0$ are poles of (5.6). In particular, there exists $M_{ij} \in \mathbb{Z}_+$ such that

$$\prod_{1 \leq i < j \leq n} (z_i - z_j)^{M_{ij}} f_l(z_1, \ldots, z_n; u_1, \ldots, u_n, w_1, w_2; \lambda) \tag{5.8}$$

can be analytically extended to a multivalued analytic function on

$$\{(z_1, \dots, z_l) \in \mathbb{C}^l \mid z_i \neq 0,\ z_i \neq z,\ i = 1, \dots, l\}.$$

(b) All component-isolated singularities of (5.8) are regular singularities.

2. For $u_1, \dots, u_l \in V$, $w_1 \in W_1$, and $w_2 \in W_2$,

(a) The series

$$\lambda\Big(Y^{g_1}(u_1, z_1 - z)w_1 \otimes w_2\Big) = \lambda\Big(Y^{g_1}(u_1, x)w_1 \otimes w_2\Big)\Big|_{x^n = e^{nl_0(z_1 - z)},\ \log x = l_0(z_1 - z)} \tag{5.9}$$

is absolutely convergent on the region $\Omega_{0,1,0}(z)$. Moreover, it is convergent to $f_1^e(z_1; u_1, w_1, w_2; \lambda)$ on the region $\Omega^{(1)}_{0,1,0}(z) = \Omega^{(2)}_{0,1,0}(z)$.

(b) For $l \in \mathbb{N}$, the multiple series

$$\begin{aligned} &\lambda\Big(w_1 \otimes Y^{g_2}(u_1, z_1) \cdots Y^{g_2}(u_l, z_l)w_2\Big) \\ &\quad = \lambda\Big(w_1 \otimes Y^{g_2}(u_1, x_1) \cdots Y^{g_2}(u_l, x_l)w_2\Big)\Big|_{x_i^n = e^{nl_0(z_1)},\ \log(x_i) = l_0(z_1),\ i = 1, \dots, l} \end{aligned} \tag{5.10}$$

in powers and logarithms of $z_1, \dots, z_l$ is absolutely convergent on the region $\Omega_{0,0,l}(z)$. Moreover, it is absolutely convergent to (5.7) on the region $\Omega^{(1)}_{0,0,l}(z)$ and to

$$f_l^{b^{-1}_{z_1,z} \cdots b^{-1}_{z_l,z}}(z_1, \dots, z_l; u_1, \dots, u_l, w_1, w_2; \lambda)$$

on the region $\Omega^{(2)}_{0,0,l}(z)$.

We denote the subspace of $P(z)$-compatible functionals in $(W_1 \otimes W_2)^*$ as

$$\mathrm{COMP}_{P(z)}((W_1 \otimes W_2)^*)$$

or COMP for short.

Remark 5.2 Note that the following are component-isolated singularities (and therefore regular singular points) of (5.6) and (5.8):

- $(z_1, \dots, z_l) - \beta = \delta$, for any $\beta \in \{0, z\}^l$, and $\delta \in \{0, \infty\}^l$ are isolated singularities (and therefore regular singular points) of (5.6) and (5.8).
- $(z_1 - z_2, z_2) = (0, \infty)$ and $(z_1 - z_2, z_2 - z) = (0, \infty)$ are regular singular points of $f_2(z_1, z_2; u_1, u_2, w_1, w_2; \lambda)$.

Remark 5.3 In 2(b) of Proposition 5.1 and 2.(b) of the $P(z)$-compatibility condition, the reason that we involve two different regions $\Omega^{(1)}_{0,0,l}(z)$ and $\Omega^{(2)}_{0,0,l}(z)$ is because either of these two regions could be empty. Notice that here z is a *fixed* nonzero complex number. Actually, when $\arg z \in [0, \pi/2]$, the region $\Omega^{(1)}_{0,0,l}(z)$ is empty. When $\arg z \in [3\pi/2, 2\pi)$, the region $\Omega^{(2)}_{0,0,l}(z)$ is empty. When $\Omega^{(1)}_{0,0,l}(z)$ and $\Omega^{(2)}_{0,0,l}(z)$ are both nonempty, the absolute convergence of (5.10) on these two regions are equivalent.

Remark 5.4 Because of the definition of the domain of (5.7), i.e. $M^l_0(0, z)$, and the fact that its singularities at $z_i = z_j$ for $i \neq j$ are poles, branches of (5.6) can be indexed by elements in the fundamental group of the space

$$\{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq 0,\ z_i \neq z,\ i = 1, \ldots, n\} = \prod_{i=1}^{n} \{z_i \in \mathbb{C} |\ z_i \neq 0,\ z_i \neq z\}.$$

A set of generators of this fundamental group can be chosen to be $b_{z_i,0}$, $b_{z_i,z}$, $i = 1, \ldots, n$. For each i, the elements $b_{z_i,0}$ and $b_{z_i,z}$ corresponds to b_{13} and b_{12} defined in section 2, and they freely generate $\pi_1(\{z_i \in \mathbb{C}|\ z_i \neq 0,\ z_i \neq z\})$. Notice that

$$\pi_1\left(\prod_{i=1}^{n}\{z_i \in \mathbb{C}|\ z_i \neq 0,\ z_i \neq z\}\right) = \prod_{i=1}^{n} \pi_1\big(\{z_i \in \mathbb{C}|\ z_i \neq 0,\ z_i \neq z\}\big) = \prod_{i=1}^{n} \langle b_{i,0}, b_{i,z}\rangle. \quad (5.11)$$

For $\lambda \in \mathrm{COMP}$, we want to define $Y^{(g_1g_2)^{-1}}_{P(z)}(u, x)\lambda \in (W_1 \otimes W_2)^*\{x\}[\log x]$. We first define $Y^{(g_1g_2)^{-1}}_{P(z)}(u, x)\lambda \in (W_1 \otimes W_2)^*\{x\}[\log x]$ for λ in a larger subspace of $(W_1 \otimes W_2)^*$ than COMP. Let $\mathrm{COM}_{P(z)}((W_1 \otimes W_2)^*)$ or simply COM be the subspace of $(W_1 \otimes W_2)^*$ consisting of λ satisfying 1.(a), (b) and 2.(b) in the $P(z)$-compatibility condition. By definition, $\mathrm{COMP} \subset \mathrm{COM}$. To define $Y^{(g_1g_2)^{-1}}_{P(z)}(u, x)\lambda \in (W_1 \otimes W_2)^*\{x\}[\log x]$ for $u \in V$ and $\lambda \in \mathrm{COM}$ is equivalent to define

$$Y^{(g_1g_2)^{-1}}_{P(z)}(e^{xL(1)}(-x^2)^{-L(0)}u, x^{-1})\lambda = \left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^o (u, x)\lambda \in (W_1 \otimes W_2)^*\{x\}[\log x].$$

Since $z_1 = \infty$ is a regular singular point of $f_1(z_1; u_1, w_1, w_2; \lambda)$, we know that there exist unique $a_{i,n,j}(u_1, w_1, w_2; \lambda) \in \mathbb{C}$ and $r_i \in \mathbb{C}$, for $i, j = 0, \ldots, K$ and $n \in \mathbb{N}$, such that on $\Omega^{(1)}_{1,0,0}(z) = \Omega^{(2)}_{1,0,0}(z)$ (i.e. the region given by $|z_1| > |z|$, $|\arg(z_1 - z) - \arg(z_1)| < \frac{\pi}{2}$),

$$f^e_1(z_1; u, w_1, w_2; \lambda) = \sum_{i,j=0}^{K} \sum_{n \in \mathbb{N}} a_{i,n,j}(u, w_1, w_2; \lambda) z_1^{r_i - n} (\log z_1)^j.$$

For $i, j = 0, \ldots, K$, $n \in \mathbb{N}$ and $u \in V$, we define $\left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^o_{-r_i+n-1,j}(u)\lambda \in (W_1 \otimes W_2)^*$ by

$$\left(\left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^o_{-r_i+n-1,k}(u)\lambda\right)(w_1 \otimes w_2) = a_{i,n,j}(u, w_1, w_2; \lambda)$$

for $w_1 \in W_1$ and $w_2 \in W_2$. Then we define $Y_{P(z)}^{(g_1g_2)^{-1}}(e^{xL(1)}(-x^2)^{L(0)}u, x^{-1})\lambda$ to be

$$\sum_{i,j=0}^{N}\sum_{n\in\mathbb{N}}\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^{o}_{-r_i+n-1,j}(u)\lambda x^{r_i-n}(\log x)^j \in (W_1\otimes W_2)^*\{x\}[\log x],$$

that is,

$$\left(Y_{P(z)}^{(g_1g_2)^{-1}}(e^{xL(1)}(-x^2)^{-L(0)}u, x^{-1})\lambda\right)(w_1\otimes w_2) = \sum_{i,j=0}^{N}\sum_{n\in\mathbb{N}} a_{i,n,j}(u, w_1, w_2; \lambda)x^{r_i-n}(\log x)^j. \tag{5.12}$$

By definition,

$$\left(\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^{o}(u, z_1)\lambda\right)(w_1\otimes w_2)$$

is absolutely convergent on the region $|z_1| > |z|$ and its sum on $\Omega^{(1)}_{1,0,0}(z) = \Omega^{(2)}_{1,0,0}(z)$ is equal to $f_1^e(z_1; u, w_1, w_2; \lambda)$. For simplicity, let $\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^{o}_{m,k}(u) = 0$ for $m \in \mathbb{C}$, $m \neq -r_i + n - 1$ for $i = 0, \ldots, N$ and $n \in \mathbb{N}$ and $k = 0, \ldots N$. Then we have

$$\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^{o}(u, x) = \sum_{m\in\mathbb{C}}\sum_{k=0}^{N}\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^{o}_{m,k}(u)x^{-m-1}(\log x)^k.$$

We have the following result:

Proposition 5.5 *The space* COM *is invariant under the action of the components of the twisted vertex operators* $Y_{P(z)}^{(g_1g_2)^{-1}}(u, x)$ *for* $u \in V$.

Proof. We need to show that for $n_1 \in \mathbb{C}$, $k = 1, \ldots, K$, $u_1 \in V$ and $\lambda \in \mathrm{COM}$,

$$\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^{o}_{n_1,k_1}(u_1)\lambda \in \mathrm{COM}. \tag{5.13}$$

For $u_2, \ldots, u_l \in V$, we have

$$\begin{aligned}Y^{g_2}(u_2, x_2)\cdots Y^{g_2}(u_l, x_l) = \sum_{n_2,\ldots,n_l\in\mathbb{C}}\sum_{k_2,\ldots,k_l=0}^{K} & Y^{g_2}_{n_2,k_2}(u_2)\cdots Y^{g_2}_{n_l,k_l}(u_l)\cdot\\ & x_2^{-n_2-1}\cdots x_{l+1}^{-n_l-1}(\log x_2)^{k_2}\cdots(\log x_l)^{k_l}.\end{aligned}$$

Since $\lambda \in \mathrm{COM}$, for $u_2, \ldots, u_l \in V$, $n_2, \ldots, n_l \in \mathbb{C}$, $k_2, \ldots, k_l = 0, \ldots, K$, $w_1 \in W_1$ and $w_2 \in W_2$,

$$\lambda(w_1 \otimes Y^{g_2}(u_1, z_1)Y^{g_2}_{n_2,k_2}(u_2)\cdots Y^{g_2}_{n_l,k_l}(u_l)w_2)$$

is absolutely convergent on the region $\Omega_{0,0,1}(z)$ and is absolutely convergent on the region $\Omega^{(1)}_{0,0,1}(z)$ to the preferred single-valuued branches

$$f_1^e(z_1; u_1, w_1, Y^{g_2}_{n_2,k_2}(u_2)\cdots Y^{g_2}_{n_l,k_l}(u_l)w_2) \tag{5.14}$$

defined on $M_0^1(0, z)$ of the multivalued analytic function

$$f_1(z_1; u_1, w_1, Y^{g_2}_{n_2,k_2}(u_2) \cdots Y^{g_2}_{n_l,k_l}(u_l)w_2) \tag{5.15}$$

defined on $M^1(0, z)$. By definition, $(Y_{P(z)}^{(g_1g_2)^{-1}})^o_{-r_i+n-1,j}(u)\lambda$ is obtained by expanding (5.15) on the region $\Omega^{(1)}_{1,0,0}(z) = \Omega^{(2)}_{1,0,0}(z)$ as a series in suitable powers of z_1 and $\log z_1$ and then taking the corresponding coefficients.

We now consider the series

$$f_1^e(z_1; u_1, w_1, Y^{g_2}(u_2, z_2) \cdots Y^{g_2}(u_l, z_l)w_2) \tag{5.16}$$

in suitable powers of $z_2, \dots, z_l$ and $\log z_1, \dots, \log z_l$. The series (5.16) on the region $\Omega^{(1)}_{0,0,1}(z)$ can be further expanded as the series

$$\lambda(w_1 \otimes Y^{g_2}(u_1, z_1)Y^{g_2}(u_2, z_2) \cdots Y^{g_2}(u_l, z_l)w_2). \tag{5.17}$$

But (5.17) is absolutely convergent on the region $\Omega_{0,0,l+1}(z)$ and is absolutely convergent either to

$$f_l^e(z_1, z_2, \dots, z_l; u_1, \dots, u_l, w_1, w_2; \lambda) \tag{5.18}$$

on the region $\Omega^{(1)}_{0,0,l}(z)$ or to

$$f_l^{b^{-1}_{z_1,z} \cdots b^{-1}_{b_l,z}}(z_1, z_2, \dots, z_l; u_1, \dots, u_l, w_1, w_2; \lambda) \tag{5.19}$$

on the region $\Omega^{(2)}_{0,0,l+1}(z)$. Thus for z_1 satisfying $|z| > |z_1| > 0$, the series (5.16) as the sum of (5.17) viewed as a series in suitable powers of z_1 and $\log z_1$ must be absolutely convergent on the region $|z_1| > |z_2| > \cdots > |z_l| > 0$ and if in addition, $-\frac{3\pi}{2} < \arg(z_i - z) < -\frac{\pi}{2}$ or $\frac{\pi}{2} < \arg(z_i - z) < \frac{3\pi}{2}$ for $i = 1, \dots, l+1$, its sum must also be equal to (5.18) or (5.19), respectively. But (5.18) can also be expanded on the region $|z_1| > |z| > |z_2| > \cdots > |z_l| > 0$ as a series in suitable powers of $z_1, \dots, z_l$ and $\log z_1, \dots, \log z_l$. This fact can be seen as follows: By 1.(a) in the $P(z)$-compatibility condition, we know that

$$\prod_{1 \le i < j \le n} (z_i - z_j)^{M_{ij}} f_l(z_1, z_2, \dots, z_n; u_1, \dots, u_n, w_1, w_2; \lambda)$$

can be analytically extended to an analytic function on the region

$$\{(z_1, \dots, z_l) \in \mathbb{C}^l \mid z_i \neq 0,\ z_i \neq z,\ i = 1, \dots, l\}$$

Moreover, this analytic function has a regular singularity at $(\infty, 0, \dots, 0)$. In particular, this function can be analytically expanded on the region $|z| < |z_1,\ 0 < |z_i| < |z|$ for $i = 2, \dots, l$ as a series in suitable powers of $z_1, \dots, z_l$ and $\log z_1, \dots, \log z_l$. Thus (5.18) can be expanded on the region $|z_1| > |z| > |z_2| > \cdots > |z_l| > 0$ as a series in suitable powers of $z_1, \dots, z_l$ and $\log z_1, \dots, \log z_l$. By definition, this expansion is equal to

$$\left(\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o(u_1, z_1)\lambda\right)(w_1 \otimes Y^{g_2}(u_2, z_2) \cdots Y^{g_2}(u_l, z_l)w_2) \tag{5.20}$$

The coefficients of the powers of z_1 and $\log z_1$ in (5.20) are exactly

$$\left(\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o_{-r_i+n-1,j}(u_1)\lambda\right)(w_1\otimes Y^{g_2}(u_2,z_2)\cdots Y^{g_2}(u_l,z_l)w_2)$$

for $i,j=0,\dots,N$, $n\in\mathbb{N}$. Since these are expansions of single-valued branches of multivalued analytic functions on $\Omega_{0,0,l}$, they are absolutely convergent to these multivalued analytic functions on the same region and are absolutely convergent to their corresponding single-valued branches on $\Omega^{(1)}_{0,0,l}$ or $\Omega^{(2)}_{0,0,l}$. Moreover, since $f_l(z_1,\dots,z_l;u_1,\dots,u_l,w_1,w_2;\lambda)$ satisfies 1.(a) and 1.(b) in $P(z)$-compatibility condition, the coefficients of its expansion on the region $|z_1|>|z|>|z_2|>\cdots>|z_l|>0$ also satisfy these conditions. This finishes the proof of (5.13). ∎

Proposition 5.6 *For $u_1,\dots,u_{m+l}\in V$, $w_1\in W_1$, $w_2\in W_2$ and $\lambda\in \mathrm{COM}$, the series*

$$\left(\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o(u_m,z_m)\cdots\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o(u_1,z_1)\lambda\right)(w_1\otimes Y^{g_2}(u_{m+1},z_{m+1})\cdots Y^{g_2}(u_{m+l},z_{m+l})w_2) \tag{5.21}$$

is absolutely convergent on the region $|z_1|>\cdots>|z_m|>|z|>|z_{m+1}|>\cdots>|z_{l+m}|>0$ and its sum is equal to

$$f_l^e(z_1,\dots,z_{m+l};u_1,\dots,u_{m+l};w_1,w_2;\lambda) \tag{5.22}$$

on $\Omega^{(1)}_{m,0,l}(z)$ or to

$$f_l^{b^{-1}_{z_{m+1},z}\cdots b^{-1}_{z_{m+l},z}}(z_1,\dots,z_{m+l};u_1,\dots,u_{m+l};w_1,w_2;\lambda) \tag{5.23}$$

on $\Omega^{(2)}_{m,0,l}(z)$. Moreover, we have the following commutativity for $Y_{P(z)}^{(g_1g_2)^{-1}}$: For $u_1,\dots,u_m\in V$, $w_1\in W_1$, $w_2\in W_2$ and $\lambda\in\mathrm{COM}$, the series

$$\left(Y_{P(z)}^{(g_1g_2)^{-1}}(u_1,z_1)\cdots Y_{P(z)}^{(g_1g_2)^{-1}}(u_m,z_m)\lambda\right)(w_1\otimes w_2) \tag{5.24}$$

is absolutely convergent on the region $|z^{-1}|>|z_1|>\cdots>|z_m|>0$ and for $\sigma\in S_m$, the sums of (5.24) and

$$\left(Y_{P(z)}^{(g_1g_2)^{-1}}(u_{\sigma(1)},z_{\sigma(1)})\cdots Y_{P(z)}^{(g_1g_2)^{-1}}(u_{\sigma(m)},z_{\sigma(l)})\lambda\right)(w_1\otimes w_2) \tag{5.25}$$

are analytic extensions of each other. We also have the weak commutativity for $Y_{P(z)}^{(g_1g_2)^{-1}}$: For $u,v\in V$, there exists $M\in\mathbb{Z}_+$ such that

$$(x_1-x_2)^MY_{P(z)}^{(g_1g_2)^{-1}}(u,x_1)Y_{P(z)}^{(g_1g_2)^{-1}}(v,x_2)=(x_1-x_2)^MY_{P(z)}^{(g_1g_2)^{-1}}(v,x_2)Y_{P(z)}^{(g_1g_2)^{-1}}(u,x_1). \tag{5.26}$$

Proof. To prove the convergence of (5.21), we use induction on m. In the case $m = 0$, the convergence of (5.21) is given by Condition 2.(b). Assume that the convergence of (5.21) in the case that m is $m-1$ holds. Then

$$\begin{aligned}
&\left(\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o(u_{m-1}, z_{m-1})\cdots\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o(u_1, z_1)\lambda\right)\\
&\qquad\qquad(w_1\otimes Y^{g_2}(u_m, z_m)\cdots Y^{g_2}(u_{m+l}, z_{m+l})w_2)\\
&\quad=\left(\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o(u_{m-1}, x_{m-1})\cdots\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o(u_1, x_1)\lambda\right)\\
&\qquad\qquad(w_1\otimes Y^{g_2}(u_m, x_m)\cdots Y^{g_2}(u_{m+l}, x_{m+l})w_2)\Big|_{x_i^n=e^{n\log z_i},\log x_i=\log z_i, i=1,\dots,m+l}
\end{aligned}\tag{5.27}$$

is absolutely convergent on the region $|z_1| > \cdots > |z_{m-1}| > |z| > |z_m| > \cdots > |z_{m+l}| > 0$ and its sum is equal to (5.22) on $\Omega^{(1)}_{m-1,0,l+1}(z)$ or to

$$f_l^{b_{z_m,z}^{-1}\cdots b_{z_{m+l},z}^{-1}}(z_1,\dots,z_{m+l}; u_1,\dots,u_{m+l}; w_1, w_2;\lambda)\tag{5.28}$$

on $\Omega^{(2)}_{m-1,0,l+1}(z)$.

Write

$$\begin{aligned}
&\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o(u_{m-1}, x_{m-1})\cdots\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o(u_1, x_1)\\
&\quad=\sum_{n_{m-1},\dots,n_1\in\mathbb{C}}\sum_{k_{m-1},\dots,k_1=0}^{K}\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o_{n_{m-1},k_{m-1}}(u_{m-1})\cdots\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o_{n_1,k_1}(u_1)\cdot\\
&\qquad\qquad\cdot x_{m-1}^{-n_{m-1}-1}\cdots x_1^{-n_1-1}(\log x_{m-1})^{k_{m-1}}\cdots(\log x_1)^{k_1}
\end{aligned}$$

and

$$\begin{aligned}
&Y^{g_2}(u_m+1, x_m+1)\cdots Y^{g_2}(u_{m+l}, x_{m+l})\\
&\quad=\sum_{n_{m+1},\dots,n_{m+l}\in\mathbb{C}}\sum_{k_{m+1},\dots,k_{m+l}=0}^{K}Y^{g_2}_{n_{m+1},k_{m+1}}(u_{m+1})\cdots Y^{g_2}_{n_{m+l},k_{m+l}}(u_{m+l})\cdot\\
&\qquad\qquad\cdot x_2^{-n_{m+1}-1}\cdots x_{m+l}^{-n_{m+l}-1}(\log x_{m+1})^{k_{m+1}}\cdots(\log x_{m+l})^{k_{m+l}}.
\end{aligned}$$

By Proposition (5.5),

$$\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o_{n_{m-1},k_{m-1}}(u_{m-1})\cdots\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o_{n_1,k_1}(u_1)\lambda\in\mathrm{COM}.$$

Then

$$\begin{aligned}
&\left(\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o_{n_{m-1},k_{m-1}}(u_{m-1})\cdots\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o_{n_1,k_1}(u_1)\lambda\right)\\
&\qquad\qquad(w_1\otimes Y^{g_2}(u_m, z_m)Y^{g_2}_{n_{m+1},k_{m+1}}(u_m)\cdots Y^{g_2}_{n_{m+l},k_{m+l}}(u_{m+l})w_2)
\end{aligned}$$

is absolutely convergent on the region $|z| > |z_m| > 0$ and its sum is equal to

$$f^e\Big(z_m; u_m, w_1, Y^{g_2}_{n_{m+1},k_{m+1}}(u_m)\cdots Y^{g_2}_{n_{m+l},k_{m+l}}(u_{m+l})w_2;\\ \Big(Y^{(g_1g_2)^{-1}}_{P(z)}\Big)^o_{n_{m-1},k_{m-1}}(u_{m-1})\cdots\Big(Y^{(g_1g_2)^{-1}}_{P(z)}\Big)^o_{n_1,k_1}(u_1)\lambda\Big) \tag{5.29}$$

on the region $|z| > |z_m| > 0$, $-\frac{3\pi}{2} < \arg(z_1 - z) - \arg z < -\frac{\pi}{2}$ and to

$$f^{b^{-1}_{z_m,z}}\Big(z_m; u_m, w_1, Y^{g_2}_{n_{m+1},k_{m+1}}(u_m)\cdots Y^{g_2}_{n_{m+l},k_{m+l}}(u_{m+l})w_2;\\ \Big(Y^{(g_1g_2)^{-1}}_{P(z)}\Big)^o_{n_{m-1},k_{m-1}}(u_{m-1})\cdots\Big(Y^{(g_1g_2)^{-1}}_{P(z)}\Big)^o_{n_1,k_1}(u_1)\lambda\Big)$$

on the region $|z| > |z_m| > 0$, $\frac{\pi}{2} < \arg(z_1 - z) - \arg z < \frac{3\pi}{2}$. By definition,

$$\bigg(\Big(Y^{(g_1g_2)^{-1}}_{P(z)}\Big)^o(u_m, z_m)\Big(Y^{(g_1g_2)^{-1}}_{P(z)}\Big)^o_{n_{m-1},k_{m-1}}(u_{m-1})\cdots\Big(Y^{(g_1g_2)^{-1}}_{P(z)}\Big)^o_{n_1,k_1}(u_1)\lambda\bigg)\\ (w_1 \otimes Y^{g_2}_{n_{m+1},k_{m+1}}(u_m)\cdots Y^{g_2}_{n_{m+l},k_{m+l}}(u_{m+l})w_2) \tag{5.30}$$

is absolutely convergent on the region $|z_m| > |z|$ and its sum is equal to (5.29) on the region $|z_m| > |z|$, $|\arg(z_m - z) - \arg z_m| < \frac{\pi}{2}$.

But we know that

$$\sum_{n_1,\dots,n_{m-1},n_{m+1},\dots,n_{m+l}\in\mathbb{C}}\ \sum_{k_1,\dots,k_{m-1},k_{m+1},\dots,k_{m+l}=0}^{K}\\ f^e\Big(z_m; u_m, w_1, Y^{g_2}_{n_{m+1},k_{m+1}}(u_m)\cdots Y^{g_2}_{n_{m+l},k_{m+l}}(u_{m+l})w_2;\\ \Big(Y^{(g_1g_2)^{-1}}_{P(z)}\Big)^o_{n_{m-1},k_{m-1}}(u_{m-1})\cdots\Big(Y^{(g_1g_2)^{-1}}_{P(z)}\Big)^o_{n_1,k_1}(u_1)\lambda\Big)\cdot\\ \cdot z_1^{-n_1-1}\cdots z_{m-1}^{-n_{m-1}-1}z_{m+1}^{-n_1-1}\cdots z_{m+l}^{-n_{m+l}-1}\cdot\\ \cdot(\log z_1)^{k_1}\cdots(\log z_{m-1})^{k_{m-1}}(\log z_{m+1})^{k_{m+1}}\cdots(\log z_{m+l})^{k_{m+l}}\\ = \sum_{n_1,\dots,n_{m-1},n_{m+1},\dots,n_{m+l}\in\mathbb{C}}\ \sum_{k_1,\dots,k_{m-1},k_{m+1},\dots,k_{m+l}=0}^{K}\\ \bigg(\Big(Y^{(g_1g_2)^{-1}}_{P(z)}\Big)^o_{n_{m-1},k_{m-1}}(u_{m-1})\cdots\Big(Y^{(g_1g_2)^{-1}}_{P(z)}\Big)^o_{n_1,k_1}(u_1)\lambda\bigg)\\ (w_1 \otimes Y^{g_2}(u_m, z_m)Y^{g_2}_{n_{m+1},k_{m+1}}(u_m)\cdots Y^{g_2}_{n_{m+l},k_{m+l}}(u_{m+l})w_2)\cdot\\ \cdot z_1^{-n_1-1}\cdots z_{m-1}^{-n_{m-1}-1}z_{m+1}^{-n_1-1}\cdots z_{m+l}^{-n_{m+l}-1}\cdot\\ \cdot(\log z_1)^{k_1}\cdots(\log z_{m-1})^{k_{m-1}}(\log z_{m+1})^{k_{m+1}}\cdots(\log z_{m+l})^{k_{m+l}} \tag{5.31}$$

as an iterated series of the multi-series (5.27) is absolutely convergent on the region $|z_1| > \cdots > |z_{m-1}| > |z| > |z_m| > \cdots > |z_{m+l}| > 0$ and its sum is equal to (5.22) on $\Omega^{(1)}_{m-1,0,l+1}(z)$.

In other words, the expansion of (5.22) on $\Omega^{(1)}_{m-1,0,l+1}(z)$ can also be written as the iterated series (5.31). By the discussion above, the coefficients of the left-hand side of (5.31) is equal to (5.30) on the region $|z_m| > |z|$, $|\arg(z_m - z) - \arg z_m| < \frac{\pi}{2}$. So the expansion of (5.22) on the region $\Omega^{(1)}_{m,0,l}(z)$ can be written as the iterated series

$$\begin{aligned}\sum_{n_1,\dots,n_{m-1},n_{m+1},\dots,n_{m+l}\in\mathbb{C}}\ \sum_{k_1,\dots,k_{m-1},k_{m+1},\dots,k_{m+l}=0}^{K}&\\ \Bigg(\left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^{o}(u_m, z_m)\left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^{o}_{n_{m-1},k_{m-1}}&(u_{m-1})\cdots\left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^{o}_{n_1,k_1}(u_1)\lambda\Bigg)\\ &(w_1\otimes Y^{g_2}_{n_{m+1},k_{m+1}}(u_m)\cdots Y^{g_2}_{n_{m+l},k_{m+l}}(u_{m+l})w_2)\cdot\\ &\cdot z_1^{-n_1-1}\cdots z_{m-1}^{-n_{m-1}-1}z_{m+1}^{-n_1-1}\cdots z_{m+l}^{-n_{m+l}-1}\cdot\\ &\cdot(\log z_1)^{k_1}\cdots(\log z_{m-1})^{k_{m-1}}(\log z_{m+1})^{k_{m+1}}\cdots(\log z_{m+l})^{k_{m+l}}.\end{aligned}\tag{5.32}$$

Since the expansion of of (5.22) on the region $\Omega^{(1)}_{m,0,l}(z)$ must be absolutely convergent as a multisum, we see that the multi-series (5.21) corresponding to (5.32) must be absolutely convergent on the region $\Omega^{(1)}_{m,0,l}(z)$ to (5.22). Similarly, we can show that (5.21) is absolutely convergent on the region $\Omega^{(2)}_{m,0,l}(z)$ to (5.23).

This convergence result implies in particular the absolute convergence of (5.24) on the region $|z^{-1}| > |z_1| > \cdots > |z_l| > 0$. Using the commutativity for the twisted vertex operators Y^{g_2}, we see that for $\sigma \in S_m$, the sums of (5.24) and (5.25) are analytic extensions of each other.

For $u, v \in V$, since λ satisfies the condition 1.(a) in the $P(z)$-compatibility condition, there exists $M \in \mathbb{Z}_+$ such that $z_1 - z_2 = 0$ is not a singularity of

$$(z_1 - z_2)^M f^e_2(z_1, z_2; u, u_1, w_1, w_2; \lambda)$$

and we have

$$(z_1 - z_2)^M f^e_2(z_1, z_2; u, v, w_1, w_2; \lambda) = (z_1 - z_2)^M f^e_2(z_2, z_1; v, u, w_1, w_2; \lambda). \tag{5.33}$$

Since the expansion of $f^e_2(z_1, z_2; u, v, w_1, w_2; \lambda)$ on the region $\Omega^{(1)}_{2,0,0}(z)$ is

$$\left(\left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^{o}(v, z_2)\left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^{o}(u, z_1)\lambda\right)(w_1\otimes w_2),$$

we see that

$$\begin{aligned}&(z_1 - z_2)^M\left(\left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^{o}(v, z_2)\left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^{o}(u, z_1)\lambda\right)(w_1\otimes w_2)\\ &\quad=\sum_{i=0}^{M}\sum_{n\in\mathbb{C}}\sum_{k=0}^{K_1}\sum_{n_1\in\mathbb{C}}\sum_{k_1=0}^{K_1}\binom{M}{i}\left(\left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^{o}_{n_1,k_1}(v)\left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^{o}_{n,k}(u)\lambda\right)(w_1\otimes w_2)\cdot\\ &\qquad\qquad\cdot e^{(n+M-i)\log z_1}(\log z_1)^{k}e^{(n_1+i)\log z_2}(\log z_2)^{k_1}\end{aligned}\tag{5.34}$$

must be convergent absolutely to the left-hand side of (5.33) on $\Omega^{(1)}_{2,0,0}(z)$ (the region given by $|z_1| > |z_2| > |z|$, $|\arg(z_1-z)-\arg z_1| < \frac{\pi}{2}$, $|\arg(z_2-z)-\arg z_2| < \frac{\pi}{2}$). On the other hand, $(z_1-z_2)^M f^e_2(z_1, z_2; u, v, w_1, w_2; \lambda)$ is analytic at $z_1 - z_2 = 0$. So (5.34) is in fact absolutely convergent to the left-hand side of (5.33) on the region $|z_1|, |z_2| > |z|$, $|\arg(z_1-z)-\arg z_1| < \frac{\pi}{2}$, $|\arg(z_2-z)-\arg z_2| < \frac{\pi}{2}$. Thus

$$(z_1-z_2)^M \left(\left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^o (u, z_1) \left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^o (v, z_2)\lambda\right)(w_1 \otimes w_2) \tag{5.35}$$

is absolutely convergent to the right-hand side of (5.33) also on the region $|z_1|, |z_2| > |z|$, $|\arg(z_1-z)-\arg z_1| < \frac{\pi}{2}$, $|\arg(z_2-z)-\arg z_2| < \frac{\pi}{2}$. From (5.33), (5.34) and (5.35), we obtain

$$\begin{aligned}&(z_1-z_2)^M \left(\left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^o (v, z_2) \left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^o (u, z_1)\lambda\right)(w_1 \otimes w_2)\\ &\quad= (z_1-z_2)^M \left(\left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^o (u, z_1) \left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^o (v, z_2)\lambda\right)(w_1 \otimes w_2)\end{aligned}$$

on the region $|z_1|, |z_2| > |z|$, $|\arg(z_1-z)-\arg z_1| < \frac{\pi}{2}$, $|\arg(z_2-z)-\arg z_2| < \frac{\pi}{2}$ for $\lambda \in \mathrm{COM}$, $u_1, v \in V$, $w_1 \in W_1$ and $w_2 \in W_2$, which, by the definition of $\left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^o (u, x)$ for $u \in V$ above, is equivalent to (5.26). ∎

Since COMP ⊂ COM, by Proposition 5.5, $Y^{(g_1g_2)^{-1}}_{P(z)}(u, x)\lambda$ for $u \in V$ and $\lambda \in \mathrm{COMP}$ is in $\mathrm{COM}\{x\}[\log x]$. We now prove the following stronger result:

Proposition 5.7 *The space* COMP *is invariant under the action of the components of the twiste vertex operators* $Y^{(g_1g_2)^{-1}}_{P(z)}(u, x)$ *for* $u \in V$.

Proof. Let $\lambda \in \mathrm{COMP}$. Then λ satisfies 2.(a) in the $P(z)$-compatibility condition. We need only show that for $v \in V$, $n \in \mathbb{C}$, $k = 0, \dots, K$, $\left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^o_{n,k}(v)\lambda$ also satisfies 2.(a) in the $P(z)$-compatibility condition.

By the definition of $\left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^o (u, z_1)$ for $u \in V$ and the $P(z)$-compatibility condition for λ,

$$\left(\left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^o (u, z_1)\lambda\right)(w_1 \otimes Y^{g_2}_{n,k}(v)w_2)$$

and

$$\lambda(Y^{g_1}(u, z_1-z)w_1 \otimes Y^{g_2}_{n,k}(v)w_2)$$

for $v \in V$, $n \in \mathbb{C}$, $k = 0, \dots, K$, $w_1 \in W_1$ and $w_2 \in W_2$ are absolutely convergent to $f^e_1(z_1; u, w_1, Y^{g_2}_{n,k}(v)w_2; \lambda)$ on the region $|z_1| > |z|$, $|\arg(z_1-z)-\arg z_1| < \frac{\pi}{2}$ and the region $|z| > |z_1-z| > 0$, $|\arg z_1 - \arg z| < \frac{\pi}{2}$, respectively. By Proposition (5.6),

$$\left(\left(Y^{(g_1g_2)^{-1}}_{P(z)}\right)^o (u, z_1)\lambda\right)(w_1 \otimes Y^{g_2}(v, z_2)w_2)$$

$$= \sum_{n\in\mathbb{C}}\sum_{k=0}^{K}\left(\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^{o}(u,z_1)\lambda\right)(w_1\otimes Y_{n,k}^{g_2}(v)w_2)e^{n\log z_2}(\log z_2)^k \tag{5.36}$$

is absolutely convergent on the region $|z_1|>|z|>|z_2|>0$ and its sum is equal to

$$f_2^{e}(z_1,z_2;u,v;w_1,w_2;\lambda) \tag{5.37}$$

on $\Omega_{1,0,1}^{(1)}(z)$ and to

$$f_2^{b_{z_2,z}^{-1}}(z_1,z_2;u,v;w_1,w_2;\lambda) \tag{5.38}$$

on $\Omega_{1,0,1}^{(2)}(z)$. Thus

$$\begin{aligned}\sum_{n\in\mathbb{C}}\sum_{k=0}^{K}&\lambda(Y^{g_1}(u,z_1-z)w_1\otimes Y_{n,k}^{g_2}(v)w_2)e^{n\log z_2}(\log z_2)^k\\&=\lambda(Y^{g_1}(u,z_1-z)w_1\otimes Y^{g_2}(v,z_2)w_2)\end{aligned} \tag{5.39}$$

is absolutely convergent on the region $|z_1|>|z|>|z_1-z|+|z_2|>0$ and its sum is equal to (5.37) on the region $|z_1|>|z|>|z_1-z|+|z_2|>0$, $|\arg(z_1-z)-\arg z_1|<\frac{\pi}{2}$, $|\arg z_1-\arg z|<\frac{\pi}{2}$, $-\frac{3\pi}{2}<\arg(z_2-z)-\arg z<-\frac{\pi}{2}$ and is equal to (5.38) on the region $|z_1|>|z|>|z_1-z|+|z_2|>0$, $|\arg(z_1-z)-\arg z_1|<\frac{\pi}{2}$, $|\arg z_1-\arg z|<\frac{\pi}{2}$, $\frac{\pi}{2}<\arg(z_2-z)-\arg z<\frac{3\pi}{2}$. On the other hand, since $(z_1-z,z_2)=(0,0)$ is a regular singular point of (5.37) and (5.38), we can expand them on the regions $|z|>|z_2|,|z_1-z|>0$ to obtain a series of the same form as (5.39). Thus we see that (5.39) must be absolutely convergent on the region $|z|>|z_2|,|z_1-z|>0$ and its sum is equal to (5.37) and (5.38) on the regions $|z|>|z_1-z|+|z_2|>0$, $|\arg z_1-\arg z|<\frac{\pi}{2}$, $-\frac{3\pi}{2}<|\arg(z_2-z)-\arg z<-\frac{\pi}{2}$ and on the region $|z|>|z_1-z|+|z_2|>0$, $|\arg z_1-\arg z|<\frac{\pi}{2}$, $\frac{\pi}{2}<\arg(z_2-z)-\arg z<\frac{3\pi}{2}$, respectively.

The right-hand side of (5.39) is equal to

$$\sum_{n\in\mathbb{C}}\sum_{k=0}^{K}\lambda(Y_{n,k}^{g_1}(u)w_1\otimes Y^{g_2}(v,z_2)w_2)e^{n\log(z_1-z)}(\log(z_1-z))^k. \tag{5.40}$$

We know that the series $\lambda(Y_{n,k}^{g_1}(u)w_1\otimes Y^{g_2}(v,z_2)w_2)$ is absolutely convergent on the region $|z|>|z_2|>0$ and its sum is equal to $f_1^{e}(z_2;v,Y_{n,k}^{g_1}(u)w_1,w_2;\lambda)$ on the region $|z|>|z_2|>0$, $-\frac{3\pi}{2}<\arg(z_2-z)-\arg z_2<-\frac{\pi}{2}$ and to $f_1^{b_{z_2,z}}(z_2;v,Y_{n,k}^{g_1}(u)w_1,w_2;\lambda)$ on the region $|z|>|z_2|>0$, $\frac{\pi}{2}<\arg(z_2-z)-\arg z_2<\frac{3\pi}{2}$. We also know that the series

$$\left(\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^{o}(v,z_2)\lambda\right)(Y_{n,k}^{g_1}(u)w_1\otimes w_2)$$

is absolutely convergent to $f_1^{e}(z_2;v,Y_{n,k}^{g_1}(u)w_1,w_2;\lambda)$ on the region $|z|>|z_2|>0$, $|\arg(z_2-z)-\arg z_2|<\frac{\pi}{2}$. From this discussion, (5.41) and the convergence of (5.39), we see that

$$\sum_{n\in\mathbb{C}}\sum_{k=0}^{K}\left(\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^{o}(v,z_2)\lambda\right)(Y_{n,k}^{g_1}(u)w_1\otimes w_2)e^{n\log(z_1-z)}(\log(z_1-z))^k$$

$$= \left(\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o (v, z_2)\lambda\right) (Y^{g_1}(u, z_1 - z)w_1 \otimes w_2) \tag{5.41}$$

is absolutely convergent on the region $|z_2| > |z| > |z_1 - z| > 0$ and its sum is equal to (5.37) on the region $|z_2| > |z| > |z_1 - z| > 0$, $|\arg(z_2 - z) - \arg z_2| < \frac{\pi}{2}$, $|\arg z_1 - \arg z| < \frac{\pi}{2}$.

On the other hand, by Proposition 5.6,

$$\left(\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o (u, z_1) \left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o (v, z_2)\lambda\right) (w_1 \otimes w_2) \tag{5.42}$$

is absolutely convergent on the region $|z_2| > |z_1| > |z|$ and its sum is equal to (5.37) on the region $|z_2| > |z_1| > |z|$, $|\arg(z_1 - z) - \arg z_1| < \frac{\pi}{2}$, $|\arg(z_2 - z) - \arg z_2| < \frac{\pi}{2}$. Taking the coefficients of $\left(Y_{P(z)}^{(g_1g_2)^{-1}}(v, z_2)\right)^o$ in both (5.41) and (5.42), we see that

$$\left(\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o_{n,k} (v)\lambda\right) (Y^{g_1}(u, z_1 - z)w_1 \otimes w_2)$$

and

$$\left(\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o (u, z_1) \left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o_{n,k} (v)\lambda\right) (w_1 \otimes w_2)$$

are absolutely convergent to the coefficients of $e^{n \log z_2}(\log z_2)^k$ in the expansion of (5.37) near the singularity $z_2 = \infty$ on the region $|z| > |z_1 - z| > 0$, $|\arg z_1 - \arg z| < \frac{\pi}{2}$ and on the region $|z_1| > |z|$, $|\arg(z_1 - z) - \arg z_1| < \frac{\pi}{2}$, respectively. This is equivalent to 2.(a) in the $P(z)$-compatibility condition for $\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o_{n,k} (v)\lambda$. ∎

For a $P(z)$-intertwining map I of type $\binom{W_3}{W_1W_2}$ and an element w_3' of W_3', the element $\lambda_{I,w_3'} \in \mathrm{COM}$ also have the following property:

Proposition 5.8 *Consider the subspace $W_{\lambda_{I,w_3'}}$ of* COM *obtained by applying the coefficients of the vertex operators $Y_{P(z)}^{(g_1g_2)^{-1}}(u, x)$ for all $u \in V$ to $\lambda_{I,w_3'}$. Then $W_{\lambda_{I,w_3'}}$ equipped with $Y_{P(z)}^{(g_1g_2)^{-1}}$ is a generalized $(g_1g_2)^{-1}$-twisted V-module in the category $\mathcal{C}$.*

Proof. The proof of this result is a straightforward verification. We omit the details. ∎

Motivated by Proposition 5.8, we also introduce the following condition for $\lambda \in \mathrm{COM}$:

$P(z)$**-local-grading-restriction condition**

(a) The *$P(z)$-grading condition*: λ is a (finite) sum of generalized eigenvectors for the operator $L(0)'_{P(z)}$.

(b) Let W_λ be the smallest subspace of $(W_1 \otimes W_2)^*$ containing λ and stable under the action of all the coefficients of the vertex operators $Y_{P(z)}^{(g_1g_2)^{-1}}(u, x)$ for all $u \in V$. We know that W_λ can be decomposed into generalized eigenspaces of $L(0)'_{P(z)}$, which we write as $W_\lambda = \coprod_{n\in\mathbb{C}}(W_\lambda)_{[n]}$. Then

$$\dim(W_\lambda)_{[n]} < \infty, \tag{5.43}$$

$$(W_\lambda)_{[n]} = 0, \quad \text{for } \Re(n) \text{ sufficiently negative.} \tag{5.44}$$

We denote the subspace of $P(z)$-local grading restricted functionals in $\mathrm{COM} \subset (W_1 \otimes W_2)^*$ as $\mathrm{LGR}_{P(z)}((W_1 \otimes W_2)^*)$, or LGR for short. Clearly, the space LGR is closed under the action of $Y_{P(z)}^{(g_1g_2)^{-1}}(u, x)$, $u \in V$.

Theorem 5.9 *For λ satisfying the $P(z)$-compatibility condition and the $P(z)$-local-grading-restriction condition, the graded space W_λ equipped with $Y_{P(z)}^{(g_1g_2)^{-1}}$ is a grading-restricted $(g_1g_2)^{-1}$-twisted generalized module. An element $\lambda \in (W_1 \otimes W_2)^*$ is in $W_1 \boxtimes_{P(z)} W_2$ if and only if λ satisfies the $P(z)$-compatibility condition and the $P(z)$-local-grading-restriction condition. In other words,*

$$W_1 \boxtimes_{P(z)} W_2 = \mathrm{COMP} \cap \mathrm{LGR}.$$

Proof. The identity property follows immediately from the definition of the twisted vertex operator map $Y_{P(z)}^{(g_1g_2)^{-1}}$. The $L(0)$-grading condition follows from the definition of the twisted vertex operator map $Y_{P(z)}^{(g_1g_2)^{-1}}$ and the $P(z)$-grading condition in the $P(z)$-local-grading-restriction condition. The $(g_1g_2)^{-1}$-grading condition also follows from the the definition of $Y_{P(z)}^{(g_1g_2)^{-1}}$. The $L(-1)$-derivative property follows from the the definition of $Y_{P(z)}^{(g_1g_2)^{-1}}$ and the $L(-1)$-derivative property of the twisted vertex operator Y^{g_2}. We omit the details of the proofs of these properties.

We prove the equivariance property for $Y_{P(z)}^{(g_1g_2)^{-1}}$ now. It is equivalent to

$$\left(\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o\right)^{p+1}(g_1g_2u, z_1)\tilde{\lambda} = \left(\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o\right)^p(u, z_1)\tilde{\lambda} \tag{5.45}$$

for $u \in V$, $\tilde{\lambda} \in W_\lambda$. By the definition of $\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o$, we know that for $w_1 \in W_1$, $w_2 \in W_2$,

$$\left(\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o(u, z)\tilde{\lambda}\right)(w_1 \otimes w_2)$$

is absolutely convergent on the region $|z_1| > |z|$, $|\arg(z_1 - z) - \arg(z_1)| < \frac{\pi}{2}$ to $f_1^e(z_1; u, w_1, w_2; \tilde{\lambda})$. Let b_3 be the homotopy class containing a loop given by a circle centered at 0 with radius larger than $|z|$ in the counterclockwise direction on the complex z_1 plane. Then (5.45) is equivalent to

$$f_1^{b_3}(z_1; g_1g_2u, w_1, w_2; \tilde{\lambda}) = f_1^e(z_1; u, w_1, w_2; \tilde{\lambda}) \tag{5.46}$$

for $u \in V$, $w_1 \in W_1$, $w_2 \in W_2$ and $\tilde{\lambda} \in W_\lambda$.

By the equivariance property for Y^{g_2} and the convergence of $\tilde{\lambda}(w_1 \otimes Y^{g_2}(v, z_1) \otimes w_2)$ to $f_1^e(z_1; v, w_1, w_2; \tilde{\lambda})$ for $v \in V$ on the region $|z| > |z_1| > 0$, $-\frac{3\pi}{2} < \arg(z_1 - z) - \arg z_1 < -\frac{\pi}{2}$, we obtain

$$f_1^{b_{z_1,0}}(z_1; g_2v, w_1, w_2; \tilde{\lambda}) = f_1^e(z_1; v, w_1, w_2; \tilde{\lambda}).$$

Using the equivariance property for Y^{g_2} and the convergence of $\tilde{\lambda}(w_1 \otimes Y^{g_2}(v, z_1) \otimes w_2)$, we obtain similarly

$$f_1^{b_{z_1,z}}(z_1; g_1v, w_1, w_2; \tilde{\lambda}) = f_1^e(z_1; v, w_1, w_2; \tilde{\lambda}).$$

Applying any homotopy class b of loops in the z_1 complex plane with z and 0 deleted to both sides of this equality, we obtain

$$f_1^{b_{z_1,z}b}(z_1; g_1 v, w_1, w_2; \tilde{\lambda}) = f_1^{b}(z_1; v, w_1, w_2; \tilde{\lambda})$$

for $v \in V$. Then we have

$$\begin{aligned} f_1^{b_{z_1,z}b_{z_1,0}}(z_1; g_1 g_2 u, w_1, w_2; \tilde{\lambda}) &= f_1^{b_{z_1,0}}(z_1; g_2 u, w_1, w_2; \tilde{\lambda}) \\ &= f_1^{e}(z_1; u, w_1, w_2; \tilde{\lambda}). \end{aligned}$$

But it is easy to see that $b_{z_1,z}b_{z_1,0} = b_3$. Thus we have proved (5.46).

For $u, v \in V$, $w \in W_\lambda$, $w' \in W'_\lambda$, by Proposition 5.5, we have

$$\begin{aligned} &(x_1 - x_2)^M \langle w', Y_{P(z)}^{(g_1g_2)^{-1}}(u, x_1) Y_{P(z)}^{(g_1g_2)^{-1}}(v, x_2) w\rangle \\ &\quad = (x_1 - x_2)^M \langle w', Y_{P(z)}^{(g_1g_2)^{-1}}(v, x_2) Y_{P(z)}^{(g_1g_2)^{-1}}(u, x_1) w\rangle, \end{aligned} \qquad (5.47)$$

where $M \in \mathbb{Z}_+$ depending on only on u and v. Since W_λ is lower bounded, by (5.47), the left-hand side of (5.47) has only finitely many terms in complex powers of x_1, x_2 and integer powers of $\log x_1, \log x_2$. Then

$$\langle w', Y_{P(z)}^{(g_1g_2)^{-1}}(u, x_1) Y_{P(z)}^{(g_1g_2)^{-1}}(v, x_2) w\rangle$$

is equal to this finite sum multiplied by $(x_1 - x_2)^{-M}$, which is is expanded in nonnegative powers of x_2. Thus we have a multivalued function of the form

$$f(z_1, z_2) = \sum_{i,j,k,l=0}^{N} a_{ijkl} z_1^{m_i} z_2^{n_j} (\log z_1)^k (\log z_2)^l (z_1 - z_2)^{-M},$$

where $m_i, n_i \in \mathbb{C}$ for $i = 0, \ldots, N$, with the preferred branch

$$f^e(z_1, z_2) = \sum_{i,j,k,l=0}^{N} a_{ijkl} e^{m_i \log z_1} e^{n_j \log z_2} (\log z_1)^k (\log z_2)^l (z_1 - z_2)^{-M}$$

such that

$$\langle w', Y_{P(z)}^{(g_1g_2)^{-1}}(u, z_1) Y_{P(z)}^{(g_1g_2)^{-1}}(v, z_2) w\rangle$$

is absolutely convergent on the region $|z_1| > |z_2| > 0$ to $f^e(z_1, z_2)$. From (5.47), we also obtain the commutativity, that is,

$$\langle w', Y_{P(z)}^{(g_1g_2)^{-1}}(v, x_2) Y_{P(z)}^{(g_1g_2)^{-1}}(u, x_1) w\rangle$$

is absolutely convergent on the region $|z_2| > |z_1| > 0$ to $f^e(z_1, z_2)$.

We now prove the associativity for $Y_{P(z)}^{(g_1g_2)^{-1}}$. Since the associativity for $Y_{P(z)}^{(g_1g_2)^{-1}}$ is equivalent to the associativity for $(Y_{P(z)}^{(g_1g_2)^{-1}})^o$, we prove this associativity. For $u, v \in V$, $w_1 \in W_1$, $w_2 \in W_2$ and $\tilde{\lambda} \in W_\lambda$, by Proposition 5.6,

$$\left(\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o(v, z_2)\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o(u, z_1)\tilde{\lambda}\right)(w_1 \otimes w_2) \tag{5.48}$$

is absolutely convergent on the region $|z_1| > |z_2| > |z|$ and its sum is equal to

$$f_2^e(z_1, z_2; u, v, w_1, w_2; \tilde{\lambda}) \tag{5.49}$$

on the region $|z_1| > |z_2| > |z|$, $|\arg(z_1 - z) - \arg z_1| < \frac{\pi}{2}$, $|\arg(z_2 - z) - \arg z_2| < \frac{\pi}{2}$. By the definition of $\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o$,

$$\left(\left(Y_{P(z)}^{(g_1g_2)^{-1}}\right)^o((Y_V)_n(u)v, z_2)\tilde{\lambda}\right)(w_1 \otimes w_2)$$

is absolutely convergent on the region $|z_2| > |z|$ and its sum is equal to

$$f_1^e(z_2; (Y_V)_n(u)v, w_1, w_2; \tilde{\lambda}) \tag{5.50}$$

on the region $|z_2| > |z|$, $|\arg(z_2 - z) - \arg z_2| < \frac{\pi}{2}$. Also

$$\tilde{\lambda}(w_1 \otimes Y^{g_2}((Y_V)_n(u)v, z_2)w_2)$$

is absolutely convergent on the region $|z| > |z_2| > 0$ and its sum is equal to (5.50) and to

$$f_1^{b_{z_2,z}^{-1}}(z_2; (Y_V)_n(u)v, w_1, w_2; \tilde{\lambda})$$

on the region $|z| > |z_2| > 0$, $-\frac{3\pi}{2} < \arg(z_2 - z) - \arg z < -\frac{\pi}{2}$ and $|z| > |z_2| > 0$, $\frac{\pi}{2} < \arg(z_2 - z) - \arg z < \frac{3\pi}{2}$, respectively.

By the $P(z)$-compatibility condition for $\tilde{\lambda}$,

$$\tilde{\lambda}(w_1 \otimes Y^{g_2}(u, z_1)Y^{g_2}(v, z_2)w_2)$$

is absolutely convergent to (5.49) and

$$f_2^{b_{z_1,z}^{-1}b_{z_2,z}^{-1}}(z_1, z_2; u, v, w_1, w_2; \tilde{\lambda}) \tag{5.51}$$

on the region $|z| > |z_1| > |z_2| > 0$, $-\frac{3\pi}{2} < \arg(z_1 - z) - \arg z, \arg(z_2 - z) - \arg z < -\frac{\pi}{2}$ and on the region $|z| > |z_1| > |z_2| > 0$, $\frac{\pi}{2} < \arg(z_1 - z) - \arg z, \arg(z_2 - z) - \arg z < \frac{3\pi}{2}$, respectively. By the associativity of the twisted vertex operator map Y^{g_2},

$$\sum_{n\in\mathbb{Z}} \tilde{\lambda}(w_1 \otimes Y^{g_2}((Y_V)^n(u)v, z_2)w_2)(z_1 - z_2)^{-n-1}$$
$$= \tilde{\lambda}(w_1 \otimes Y^{g_2}(Y_V(u, z_1 - z_2)v, z_2)w_2)$$

$$= \tilde{\lambda}(w_1 \otimes Y^{g_2}(u, z_1) Y^{g_2}(v, z_2) w_2) \tag{5.52}$$

on the region $|z| > |z_1| > |z_2| > |z_1 - z_2| > 0$, $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$. Thus the left-hand side of (5.52) is absolutely convergent on the region $|z| > |z_1| > |z_2| > |z_1 - z_2| > 0$ and its sum is equal to (5.49) and (5.51) on the region $|z| > |z_1| > |z_2| > |z_1 - z_2| > 0$, $-\frac{3\pi}{2} < \arg(z_1 - z) - \arg z, \arg(z_2 - z) - \arg z < -\frac{\pi}{2}$, $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$ and on the region $|z| > |z_1| > |z_2| > |z_1 - z_2| > 0$, $\frac{\pi}{2} < \arg(z_1 - z) - \arg z, \arg(z_2 - z) - \arg z < \frac{3\pi}{2}$, $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$, respectively. Then by the definition of $\left(Y_{P(z)}^{(g_1 g_2)^{-1}}\right)^o$,

$$\begin{aligned}
&\left(\left(Y_{P(z)}^{(g_1 g_2)^{-1}}\right)^o (Y_V(u, z_1 - z_2) v, z_2) \tilde{\lambda}\right)(w_1 \otimes w_2) \\
&\quad = \sum_{n \in \mathbb{Z}} \left(\left(Y_{P(z)}^{(g_1 g_2)^{-1}}\right)^o ((Y_V)^n(u) v, z_2) \tilde{\lambda}\right)(w_1 \otimes w_2)(z_1 - z_2)^{-n-1} \\
&\quad = \sum_{n \in \mathbb{Z}} f_1^e(z_2; (Y_V)_n(u) v, w_1, w_2; \tilde{\lambda})(z_1 - z_2)^{-n-1}
\end{aligned} \tag{5.53}$$

is in fact the expansion of (5.49) as a Laurent series in $z_1 - z_2$ near $z_1 - z_2 = 0$ and then expand the coefficients as a series in powers of z_2 and $\log z_2$ near $z_2 = \infty$. Thus we have shown that (5.48) and the left-hand side of (5.53) are convergent on the region $|z_1| > |z_2| > |z|$ and $|z_2| > |z_1 - z_2|, |z|$, respectively and their sums are equal to (5.49) on the region $|z_1| > |z_2| > |z|$, $|\arg(z_1 - z) - \arg z_1|, |\arg(z_2 - z) - \arg z_2| < \frac{\pi}{2}$. and $|z_2| > |z_1 - z_2|, |z|$, $|\arg(z_2 - z) - \arg z_2| < \frac{\pi}{2}$, $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$.

Since W_λ is lower-bounded and the singularities of (5.49) are all regular, (5.48) is a series with only finitely many terms in negative powers of z_2. Since (5.48) is absolutely convergent on the region $|z_1| > |z_2| > |z|$, it must also absolutely convergent on the region $|z_1| > |z_2| > 0$. Similarly, we see that the left-hand side of (5.53) is also absolutely convergent on the region $|z_2| > |z_1 - z_2| > 0$. Thus we have proved the associativity for $(Y_{P(z)}^{(g_1 g_2)^{-1}})^o$, which is equivalent to the associativity for $Y_{P(z)}^{(g_1 g_2)^{-1}}$. This finishes the proof that W_λ equipped with $Y_{P(z)}^{(g_1 g_2)^{-1}}$ is a grading-restricted generalized $(g_1 g_2)^{-1}$-twisted V-module.

By Propositions 5.1 and 5.8, an element of $W_1 \boxbackslash_{P(z)} W_2$ satisfies the $P(z)$-compatibility condition and the $P(z)$-local-grading-restriction condition. We still need to prove an element λ of $(W_1 \otimes W_2)^*$ satisfying the $P(z)$-compatibility condition and the $P(z)$-local-grading-restriction condition is in $W_1 \boxbackslash_{P(z)} W_2$.

Since W_λ is grading restricted, W_λ^* is linearly isomorphic to $\overline{W_\lambda'}$. We shall identify W_λ^* with $\overline{W_\lambda'}$. We define a map linear map $I : W_1 \otimes W_2 \to W_\lambda^*$ by

$$\langle \mu, I(w_1 \otimes w_2) \rangle = \mu(w_1 \otimes w_2)$$

for $\mu \in W_\lambda$, $w_1 \in W_1$ and $w_2 \in W_2$. We define a linear map

$$\begin{aligned}
\mathcal{Y}_I : W_1 \otimes W_2 &\to (W_\lambda')\{x\}[\log x] \\
w_1 \otimes w_2 &\mapsto \mathcal{Y}_I(w_1, x) w_2
\end{aligned}$$

by
$$\mathcal{Y}_I(w_1, x)w_2 = x^{L(0)}e^{-(\log z)L(0)}I\left(x^{-L(0)}e^{(\log z)L(0)}w_1 \otimes x^{-L(0)}e^{(\log z)L(0)}w_2\right)$$
for $w_1 \in W_1$ and $w_2 \in W_2$.

Using the definition of $Y_{P(z)}^{(g_1g_2)^{-1}}$, it is easy to see that $Y_{P(z)}^{(g_1g_2)^{-1}}$ is an intertwining operator of type $\binom{W'_\lambda}{W_1 W_2}$ when W'_λ, W_1 and W_2 are viewed as modules for the fixed point subalgebra of V under g_1 and g_2. In particular, $Y_{P(z)}^{(g_1g_2)^{-1}}$ satisfies the $L(-1)$-derivative property.

For $u_1, \dots, u_{k-1} \in V$, $w_1 \in W_1$, $w_2 \in W_2$ and $w \in W_\lambda$, we have
$$\begin{aligned}
&\langle w, Y_{P(z)}^{(g_1g_2)^{-1}}(u_1, z_1)\cdots Y_{P(z)}^{(g_1g_2)^{-1}}(u_{k-1}, z_{k-1})\mathcal{Y}_I(w_1, z_k)w_2\rangle\\
&\quad= \Big\langle w, Y_{P(z)}^{(g_1g_2)^{-1}}(u_1, z_1)\cdots Y_{P(z)}^{(g_1g_2)^{-1}}(u_{k-1}, z_{k-1})\cdot\\
&\qquad\qquad \cdot e^{(\log z_k-\log z)L(0)}I\left(e^{-(\log z_k-\log z)L(0)}w_1 \otimes e^{-(\log z_k-\log z)L(0)}w_2\right)\Big\rangle\\
&\quad= \Big\langle e^{(\log z_k-\log z)L(0)}w, Y_{P(z)}^{(g_1g_2)^{-1}}(z_k^{-L(0)}z^{L(0)}u_1, e^{-(\log z_k-\log z-\log z_1)})\cdot\\
&\qquad\qquad \cdots Y_{P(z)}^{(g_1g_2)^{-1}}(z_k^{-L(0)}z^{L(0)}u_{k-1}, e^{-(\log z_k-\log z-\log z_{k-1})})\cdot\\
&\qquad\qquad \cdot I\left(e^{-(\log z_k-\log z)L(0)}w_1 \otimes e^{-(\log z_k-\log z)L(0)}w_2\right)\Big\rangle\\
&\quad= \Big\langle (Y_{P(z)}^{(g_1g_2)^{-1}})^o(z_k^{-L(0)}z^{L(0)}u_{k-1}, e^{-(\log z_k-\log z-\log z_{k-1})})\cdot\\
&\qquad \cdots (Y_{P(z)}^{(g_1g_2)^{-1}})^o(z_k^{-L(0)}z^{L(0)}u_1, e^{-(\log z_k-\log z-\log z_1)})e^{(\log z_k-\log z)L(0)}w,\\
&\qquad\qquad I\left(e^{-(\log z_k-\log z)L(0)}w_1 \otimes e^{-(\log z_k-\log z)L(0)}w_2\right)\Big\rangle\\
&\quad= \Big((Y_{P(z)}^{(g_1g_2)^{-1}})^o(z_k^{-L(0)}z^{L(0)}u_{k-1}, e^{-(\log z_k-\log z-\log z_{k-1})})\cdot\\
&\qquad \cdots (Y_{P(z)}^{(g_1g_2)^{-1}})^o(z_k^{-L(0)}z^{L(0)}u_1, e^{-(\log z_k-\log z-\log z_1)})e^{(\log z_k-\log z)L(0)}w\Big)\\
&\qquad\qquad \left(e^{-(\log z_k-\log z)L(0)}w_1 \otimes e^{-(\log z_k-\log z)L(0)}w_2\right).
\end{aligned} \tag{5.54}$$

By (5.12), the right-hand side of (5.54) is equal to the series obtained by expanding the function
$$f_l^e(\xi_1, \dots, \xi_l; z_k^{-L(0)}z^{L(0)}u_1, \dots, z_k^{-L(0)}z^{L(0)}u_{k-1}, z_k^{-L(0)}e^{(\log z)L(0)}w_1, z_k^{-L(0)}e^{(\log z)L(0)}w_2; w)$$
on the region $|\xi_1| > \cdots > |\xi_{k-1}| > |z|$ as series in powers of $\xi_1, \dots, \xi_{k-1}$ and nonnegative integer powers of $\log\xi_1, \dots, \log\xi_1$ and then substituting $e^{-n(\log z_k-\log z-\log z_1)}, \dots, e^{-n(\log z_k-\log z-\log z_{k-1})}$ ($n \in \mathbb{C}$) for $e^{n\log\xi_1}, \dots, e^{n\log\xi_{k-1}}$ and $\log z_k - \log z - \log z_1, \dots, \log z_k - \log z - \log z_{k-1}$ for $\log\xi_1, \dots, \log\xi_{k-1}$, respectively. Then the right-hand side of (5.54) is absolutely convergent on the region $|zz_1z_k^{-1}| > \cdots > |zz_{k-1}z_k^{-1}| > |z|$ or equivalently the region $|z_1| > \cdots > |z_{k-1}| > |z_k| > 0$ and can be analytically extended to a multivalued analytic function on $M^{k-1}(0, z)$ with a preferred branch. By (5.54),
$$\langle w, Y_{P(z)}^{(g_1g_2)^{-1}}(u_1, z_1)\cdots Y_{P(z)}^{(g_1g_2)^{-1}}(u_{k-1}, z_{k-1})\mathcal{Y}_I(w_1, z_k)w_2\rangle$$

is absolutely convergent on the region $|z_1| > \cdots > |z_{k-1}| > |z_k| > 0$ and can be analytically extended to a multivalued analytic function on $M^k(0, z)$ with a preferred branch.

We now prove the duality property for $\mathcal{Y}_I$. By (5.54), for $v \in V$, $w_1 \in W_1$ and $w_2 \in W_2$,

$$\begin{aligned}
&\langle w, Y_{P(z)}^{(g_1g_2)^{-1}}(v, z_1)\mathcal{Y}_I(w_1, z_2)w_2\rangle \\
&\quad = \Big((Y_{P(z)}^{(g_1g_2)^{-1}})^o(z_2^{-L(0)}z^{L(0)}v, e^{-(\log z_2 - \log z - \log z_1)})e^{(\log z_2 - \log z)L(0)}w\Big) \\
&\qquad\qquad\qquad\qquad \big(e^{-(\log z_2 - \log z)L(0)}w_1 \otimes e^{-(\log z_2 - \log z)L(0)}w_2\big),
\end{aligned}$$

which is absolutely convergent on the region $|z_2^{-1}zz_1| > |z|$ or equivalently $|z_1| > |z_2| > 0$ and its sum is equal to

$$f_1^e(e^{-(\log z_2 - \log z - \log z_1)}; z_2^{-L(0)}z^{L(0)}v, e^{-(\log z_2 - \log z)L(0)}w_1, e^{-(\log z_2 - \log z)L(0)}w_2; e^{(\log z_2 - \log z)L(0)}w) \tag{5.55}$$

on the region $|z_2^{-1}zz_1| > |z|$, $|\arg(z_2^{-1}zz_1 - z) - \arg z_2^{-1}zz_1| < \frac{\pi}{2}$ or equivalently, $|z_1| > |z_2| > 0$, $|\arg(z_1 - z_2) - \arg z_1|$. By definition,

$$\begin{aligned}
&\langle w, \mathcal{Y}_I(w_1, z_2)Y^{g_2}(v, z_1)w_2\rangle \\
&\quad = \langle e^{(\log z_2 - \log z)L(0)}w, \mathcal{Y}_I(e^{-(\log z_2 - \log z)L(0)}w_1, z)\cdot \\
&\qquad\qquad\qquad \cdot Y^{g_2}(z_2^{-L(0)}z^{L(0)}v, e^{-(\log z_2 - \log z - \log z_1)})e^{-(\log z_2 - \log z)L(0)}w_2\rangle \\
&\quad = (e^{(\log z_2 - \log z)L(0)}w)(e^{-(\log z_2 - \log z)L(0)}w_1 \\
&\qquad\qquad\qquad \otimes Y^{g_2}(z_2^{-L(0)}z^{L(0)}v, e^{-(\log z_2 - \log z - \log z_1)})e^{-(\log z_2 - \log z)L(0)}w_2)
\end{aligned}$$

is absolutely convergent on the region $|z| > |z_2^{-1}zz_1| > 0$ or equivalently $|z_2| > |z_1| > 0$ and its sum is equal to to (5.55) on the region $|z_2| > |z_1| > 0$, $-\frac{3\pi}{2} < \arg(z_2^{-1}zz_1 - z) - \arg z < -\frac{\pi}{2}$ or equivalently, $|z_2| > |z_1| > 0$, $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$. By 2.(a) in the compatibility condition, we see that

$$\begin{aligned}
&\langle w, \mathcal{Y}_I(Y^{g_1}(v, z_1 - z_2)w_1, z_2)w_2\rangle \\
&\quad = \langle e^{(\log z_2 - \log z)L(0)}w, \\
&\qquad\qquad \mathcal{Y}_I(Y^{g_1}(z_2^{-L(0)}z^{L(0)}v, z_2^{-1}zz_1 - z)e^{-(\log z_2 - \log z)L(0)}w_1, z)e^{-(\log z_2 - \log z)L(0)}w_2\rangle \\
&\quad = (e^{(\log z_2 - \log z)L(0)}w)(Y^{g_1}(z_2^{-L(0)}z^{L(0)}v, z_2^{-1}zz_1 - z)e^{-(\log z_2 - \log z)L(0)}w_1 \otimes e^{-(\log z_2 - \log z)L(0)}w_2),
\end{aligned}$$

which is absolutely convergent on the region $|z| > |z_2^{-1}zz_1 - z| > 0$ or equivalently $|z_2| > |z_1 - z_2| > 0$ and its sum is equal to (5.55) on the region $|z| > |z_2^{-1}zz_1 - z| > 0$, $|\arg z_2^{-1}zz_1 - \arg z| < \frac{\pi}{2}$ or equivalently $|z_2| > |z_1 - z_2| > 0$, $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$. Thus the duality property for $\mathcal{Y}_I$ is proved and $\mathcal{Y}_I$ is a twisted intertwining operator. Then I is a twisted $P(z)$-intertwining map.

Now we have

$$\lambda(w_1 \otimes w_2) = \langle \lambda, I(w_1 \otimes w_2)\rangle = \lambda_{I,\lambda}(w_1 \otimes w_2)$$

for $w_1 \in W_1$ and $w_2 \in W_2$. In particular, $\lambda = \lambda_{I,\lambda} \in W_1 \boxtimes_{P(z)} W_2$. ∎

A A convergence lemma

Let A be a finite subset of $\mathbb{C}/\mathbb{Z}$, $R_\mu \in \mu$ for $\mu \in A$, D a subset of $\cup_{\mu\in A}(R_\mu - \mathbb{N})$, $\Delta \in \mathbb{C}$ and $a_{n,j,i} \in \mathbb{C}$ for $n \in D$, $j = 0, \dots, M$ and $i = 1, \dots, N$. Consider the triple series

$$\sum_{n\in D}\sum_{j=0}^{M}\sum_{i=0}^{N} a_{n,j,i}e^{(-\Delta+n+1)\log z_0}(\log z_0)^j e^{(-n-1)\log z_2}(\log z_2)^i \tag{A.1}$$

for $z_0, z_2 \in \mathbb{C}^\times$.

For any z_1, z_2 satisfying $|z_1| > |z_2| > 0$, $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$, we have

$$e^{\alpha \log(z_1-z_2)} = \sum_{k\in\mathbb{N}} \binom{\alpha}{k}(-1)^k e^{(\alpha-k)\log z_1} z_2^k\ , \tag{A.2}$$

$$\log(z_1 - z_2) = \log z_1 + \sum_{k\in\mathbb{Z}_+} \frac{(-1)^k}{k} z_1^{-k} z_2^k\ , \tag{A.3}$$

$$e^{\alpha l_{q_2}(-z_2)} = e^{\alpha\pi\mathbf{i}} e^{\alpha\log z_2}, \tag{A.4}$$

$$l_{q_2}(-z_2) = \log z_2 + \pi\mathbf{i}, \tag{A.5}$$

where $\alpha \in \mathbb{C}$ and $q_2 = 0, 1$ if $\arg z_2 < \pi$, $\arg z_2 \geq \pi$, respectively. Note that in our notations, $\log z = l_0(z)$ for $z \in \mathbb{C}^\times$.

For $n \in D$, $j = 0, \dots, M$, $k \in \mathbb{Z}_{\geq 0}$, $s = 0, \dots, j$, define $b_{n,j,k,s} \in \mathbb{C}$ as the coefficients of the following formal power series expansion

$$(x + y)^{-\Delta+n+1}\log(x + y)^j = \sum_{k\in\mathbb{N}}\sum_{s=0}^{j} b_{n,j,k,s}x^{-\Delta+n+1-k}y^k \log(x)^s, \tag{A.6}$$

where x and y are formal variables. From (A.2), (A.3) and (A.6), when $|z_1| > |z_2| > 0$ and $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$, we have the following expansion

$$e^{(-\Delta+n+1)\log(z_1-z_2)}(\log(z_1 - z_2))^j = \sum_{k\in\mathbb{N}}\sum_{s=0}^{j}(-1)^k b_{n,j,k,s}e^{(-\Delta+n+1-k)\log z_1} z_2^k(\log z_1)^s. \tag{A.7}$$

Now consider

$$\sum_{n\in D}\sum_{j=0}^{M}\sum_{i=0}^{N} a_{n,j,i}e^{(-\Delta+n+1)\log(z_1-z_2)}(\log(z_1 - z_2))^j e^{(-n-1)\log(-z_2)}(\log(-z_2))^i. \tag{A.8}$$

Using (A.7), we can further expand each term in the right-hand side of (A.8) so that the right-hand side of (A.8) becomes the iterated sum

$$\sum_{n\in D}\sum_{j=0}^{M}\sum_{i=0}^{N} a_{n,j,i}\left(\sum_{k\in\mathbb{N}}\sum_{s=0}^{j}(-1)^k b_{n,j,k,s}e^{(-\Delta+n+1-k)\log z_1} z_2^k(\log z_1)^s\right).$$

$$\cdot e^{(-n-1)\log(-z_2)}(\log(-z_2))^i$$

$$= \sum_{n\in D}\sum_{j=0}^{M}\sum_{i=0}^{N}\left(\sum_{k\in\mathbb{N}}\sum_{s=0}^{j} a_{n,j,i}(-1)^k b_{n,j,k,s} e^{(-\Delta+n+1-k)\log z_1} z_2^k \cdot \right.$$

$$\left. \cdot (\log z_1)^s e^{(-n-1)\log(-z_2)}(\log(-z_2))^i \right), \tag{A.9}$$

where the inner sum is absolutely convergent in the region $|z_1| > |z_2| > 0$.

We are interested in the convergence of the multisum

$$\sum_{n\in D}\sum_{j=0}^{M}\sum_{i=0}^{N}\sum_{k\in\mathbb{N}}\sum_{s=0}^{j} a_{n,j,i}(-1)^k b_{n,j,k,s} e^{(-\Delta+n+1-k)\log z_1} z_2^k (\log z_1)^s e^{(-n-1)\log(-z_2)}(\log(-z_2))^i \tag{A.10}$$

and the corresponding series

$$\sum_{m\in D-\mathbb{N}}\sum_{s=0}^{M}\sum_{i=0}^{N}\left(\sum_{j=s}^{M}\sum_{\substack{n-k=m\\ n\in D, k\in\mathbb{N}}} a_{n,j,i} b_{n,j,k,s}\right) e^{(-\Delta+m+1)\log z_1}(\log z_1)^s e^{(-m-1)\log(-z_2)}(\log(-z_2))^i \tag{A.11}$$

in powers of z_1 and z_2 and nonnegative integral powers of $\log z_1$ and $\log z_2$.

Lemma A.1 *Assume that the triple series* (A.1) *and the series obtained from* (A.1) *by taking derivatives of each term in* (A.1) *with respect to z_1 and z_2 are absolutely convergent on the region given by $|z_1| > |z_2| > 0$. Then the multisum* (A.10) *is absolutely convergent on the region $|z_1| > 2|z_2| > 0$. Assume in addition that* (A.1) *is convergent on the region $|z_1| > |z_2| > 0$, $|\arg(z_1-z_2)-\arg z_1| < \frac{\pi}{2}$ to a single-valued analytic branch $f^e(z_1, z_2)$ on M_0^2 of a maximally extended multivalued analytic function on M^2 such that $f^e(z_1-z_2, -z_2)$ has no singular point in the region $|z_1| > |z_2| > 0$. Then the iterated sum* (A.11) *are absolutely convergent on the region $|z_1| > |z_2| > 0$, $|\arg z_1 - \arg(z_1-z_2)| < \frac{\pi}{2}$ to $f^e(z_1-z_2, -z_2)$.*

Proof. Let $\tilde{n} = R_\mu - n$. Then the sum $\sum_{n\in D}$ in (A.1) can be written as the same as $\sum_{\mu\in A}\sum_{\tilde{n}\in\mathbb{N}}$. So the series (A.1) can be written as

$$\sum_{\mu\in A}\sum_{\tilde{n}\in\mathbb{N}}\sum_{j=0}^{M}\sum_{i=0}^{N} a_{R_\mu-\tilde{n},j,i} e^{(-\Delta+R_\mu-\tilde{n}+1)\log z_1}(\log z_1)^j e^{(-R_\mu+\tilde{n}-1)\log z_2}(\log z_2)^i. \tag{A.12}$$

For any $r > 1$, consider

$$\sum_{\mu\in A}\sum_{\tilde{n}\in\mathbb{N}}\sum_{j=0}^{M}\sum_{i=0}^{N} a_{R_\mu-\tilde{n},j,i} e^{(-\Delta+R_\mu-\tilde{n}+1)\log z_1}(\log z_1)^j e^{(-R_\mu+\tilde{n}-1)\log z_2}(\log z_2)^i r^{\tilde{n}}. \tag{A.13}$$

Then the absolute convergence of (A.12) on the region $|z_1| > |z_2| > 0$ is equivalent to the absolute convergence of (A.13) on the region $|z_1| > r|z_2| > 0$. But the absolute convergence of (A.13) on the region $|z_1| > r|z_2| > 0$ is in turn equivalent to the absolute convergence of

$$\sum_{\tilde{n}\in\mathbb{N}} a_{R_\mu-\tilde{n},j,i}\left(\frac{z_2}{z_1}\right)^{\tilde{n}} r^{\tilde{n}} \tag{A.14}$$

on the same region $|z_1| > r|z_2| > 0$. But as a power series in $\frac{z_2}{z_1}$, (A.14) has a removable singularity at $\frac{z_2}{z_1} = 0$. Then we see that (A.14) is absolutely convergent on the region $|\frac{z_2}{z_1}| < \frac{1}{r}$ and is hence also uniformly convergent on the closed region $|\frac{z_2}{z_1}| \leq r_2$ for any positive $r_2 < \frac{1}{r}$. Thus for such r_2, (A.13) is a (finite) linear combination of the uniformly convergent series (A.14) on the region $|\frac{z_2}{z_1}| \leq r_2$ with analytic functions of z_1 and z_2 on M_0^2 as coefficients.

Substituting $z_1 - z_2$ and $-z_2$ for z_1 and z_2, respectively, in (A.13), we see that

$$\sum_{\mu\in A}\sum_{\tilde{n}\in\mathbb{N}}\sum_{j=0}^{M}\sum_{i=0}^{N} a_{n,j,i}e^{(-\Delta+n+1)\log(z_1-z_2)}(\log(z_1-z_2))^j e^{(-n-1)\log(-z_2)}(\log(-z_2))^i r^{\tilde{n}}. \tag{A.15}$$

is absolutely convergent on the region $|z_1-z_2| > r|z_2| > 0$ and is a (finite) linear combination of uniformly convergent series on the region $|\frac{z_2}{z_1-z_2}| \leq r_2$ with analytic coefficients on M_0^2 for any positive $r_2 < \frac{1}{r}$. Now we expand each term in (A.15) using (A.7) to obtain the iterated sum

$$\begin{aligned}
&\sum_{\mu\in A}\sum_{\tilde{n}\in\mathbb{N}}\sum_{j=0}^{M}\sum_{i=0}^{N} a_{R_\mu-\tilde{n},j,i}\left(\sum_{k\in\mathbb{N}}\sum_{s=0}^{j}(-1)^k b_{R_\mu-\tilde{n},j,k,s}e^{(-\Delta+R_\mu-\tilde{n}+1-k)\log z_1}z_2^k(\log z_1)^s\right)\cdot\\
&\qquad\qquad\cdot e^{(-R_\mu+\tilde{n}-1)\log(-z_2)}(\log(-z_2))^i r^{\tilde{n}}\\
&=\sum_{\mu\in A}\sum_{\tilde{n}\in\mathbb{N}}\sum_{j=0}^{M}\sum_{i=0}^{N}\Bigg(\sum_{k\in\mathbb{N}}\sum_{s=0}^{j} a_{R_\mu-\tilde{n},j,i}(-1)^k b_{R_\mu-\tilde{n},j,k,s}e^{(-\Delta+R_\mu-\tilde{n}+1-k)\log z_1}z_2^k\cdot\\
&\qquad\qquad\cdot(\log z_1)^s e^{(-R_\mu+\tilde{n}-1)\log(-z_2)}(\log(-z_2))^i\Bigg) r^{\tilde{n}},
\end{aligned} \tag{A.16}$$

where the inner sum is absolutely convergent on the region $|z_1| > |z_2| > 0$. Then as a subseries of (A.16) divided by $e^{(-\Delta+R_\mu+1)\log z_1}e^{(-R_\mu-1)\log(-z_2)}(\log(-z_2))^i$, the series

$$\begin{aligned}
&\sum_{\tilde{n}\in\mathbb{N}}\left(\sum_{k\in\mathbb{N}}\sum_{s=0}^{j}(-1)^{\tilde{n}+k}a_{R_\mu-\tilde{n},j,i}b_{R_\mu-\tilde{n},j,k,s}\left(\frac{z_2}{z_1}\right)^{\tilde{n}+k}(\log z_1)^s\right) r^{\tilde{n}}\\
&\quad=\sum_{\tilde{n}\in\mathbb{N}}\left(\sum_{\tilde{k}\in\mathbb{N}}\sum_{s=0}^{j}(-1)^{\tilde{k}}a_{R_\mu-\tilde{n},j,i}b_{R_\mu-\tilde{n},j,\tilde{k}-\tilde{n},s}\left(\frac{z_2}{z_1}\right)^{\tilde{k}}(\log z_1)^s\right) r^{\tilde{n}}
\end{aligned} \tag{A.17}$$

is also absolutely convergent in the region $|z_1-z_2|>r|z_2|$ and is uniformly convergent in the closed region $|\frac{z_2}{z_1-z_2}|\leq r_2$ any positive $r_2<\frac{1}{r}$, where the inner sum is absolutely convergent in the region $|z_1|>|z_2|$. In the region $|z_1|>(1+r)|z_2|>0$, we have $|z_1-z_2|\geq |z_1|-|z_2|> r|z_2|>0$. Then (A.17) is absolutely convergent in the region $|z_1|>(1+r)|z_2|$ or $|\frac{z_2}{z_1}|<\frac{1}{1+r}$ with the inner sum absolutely convergent in the region $|z_1|>|z_2|$ or $|\frac{z_2}{z_1}|<1$. Note that for fixed $\frac{z_2}{z_1}$, z_1 can be any complex number and thus $\log z_1$ can also be any nonzero complex number. This means that $\zeta_1=\frac{z_2}{z_1}$ and $\zeta_2=\log z_1$ can be viewed as independent variables and

$$\sum_{\tilde{n}\in\mathbb{N}}\left(\sum_{\tilde{k}\in\mathbb{N}}\sum_{s=0}^{j}(-1)^{\tilde{k}}a_{R_\mu-\tilde{n},j,i}b_{R_\mu-\tilde{n},j,\tilde{k}-\tilde{n},s}\zeta_1^{\tilde{k}}\zeta_2^{s}\right)\zeta_3^{\tilde{n}}, \tag{A.18}$$

where $b_{R_\mu-\tilde{n},j,\tilde{k}-\tilde{n},s}$ is defined to be 0 when $\tilde{k}<\tilde{n}$, is absolutely convergent on the region given by $|\zeta_1|<\frac{1}{1+r}$, $\zeta_2\in\mathbb{C}$ and $|\zeta_3|<r$.

On the other hand, on the region $|\frac{z_2}{z_1}|\leq\frac{r_2}{1+r_2}$ for positive $r_2<\frac{1}{r}$,

$$\left|\frac{z_2}{z_1-z_2}\right|\leq\frac{|z_2|}{|z_1|-|z_2|}\leq\frac{|z_2|}{\frac{1+r_2}{r_2}|z_2|-|z_2|}=r_2.$$

Then for $\frac{z_2}{z_1}\in\mathbb{C}$ satisfying $|\frac{z_2}{z_1}|\leq\frac{r_2}{1+r_2}$ for positive $r_2<\frac{1}{r}$ and $\log z_1\in\mathbb{C}$, (A.17) is uniformly convergent with the inner sum also uniformly convergent on the same closed region. Thus (A.18) is uniformly convergent on the region $|\zeta_1|\leq\frac{r_2}{1+r_2}$, $\zeta_2\in\mathbb{C}$ and $|\zeta_3|\leq r$ with the inner sum also uniformly convergent on the same closed region. Starting with the absolute convergence of the series obtained from (A.1) by taking derivatives of each term in (A.1) with respect to z_1 and z_2 and using the completely same proof of the uniform convergence of (A.18), we can show that the series obtained from (A.18) by taking derivatives of each term in (A.18) with respect to ζ_1, ζ_2 and ζ_3 is also uniformly convergent on the region $|\zeta_1|\leq\frac{r_2}{1+r_2}$, $\zeta_2\in\mathbb{C}$ and $|\zeta_3|\leq r$. In particular, the derivatives of the sum of (A.18) with respect to ζ_1, ζ_2 and ζ_3 exist and is equal to the sum of the series obtained by taking the corresponding derivatives of each term in (A.18) on the region $|\zeta_1|<\frac{r_2}{1+r_2}$, $\zeta_2\in\mathbb{C}$ and $|\zeta_3|<r$. Then the sum of (A.18) is an analytic function of ζ_1, ζ_2 and ζ_3 on the same open region.

For $r>1$, let ζ_1,ζ_2,ζ_3 be complex numbers satisfying $|\zeta_1|<\frac{1}{(1+r)r}$, $\zeta_2\in\mathbb{C}$ and $|\zeta_3|<r$. Then we have $\frac{|\zeta_1|}{1-|\zeta_1|}<(1+r)|\zeta_1|<\frac{1}{r}$. We choose r_2 such that $\frac{|\zeta_1|}{1-|\zeta_1|}<r_2<(1+r)|\zeta_1|$. From $\frac{|\zeta_1|}{1-|\zeta_1|}<r_2$, we obtain $|\zeta_1|<\frac{r_2}{1+r_2}$. We also have $0<r_2<(1+r)|\zeta_1|<\frac{1}{r}$. Now ζ_1,ζ_3 satisfy $|\zeta_1|\leq\frac{r_2}{1+r_2}$ and $|\zeta_3|<r$. This means that the sum of (A.18) is analytic and the derivatives can be calculated term by term at ζ_1,ζ_2,ζ_3. So the sum of (A.18) is analytic and the derivatives can be calculated term by term on the polydisc $|\zeta_1|<\frac{1}{(1+r)r}$, $\zeta_2\in\mathbb{C}$ and $|\zeta_3|<r$ for any $r>1$. Since analytic functions on polydiscs can be expanded as power series, the sum of (A.18) can be expanded as a power series in ζ_1, ζ_2 and ζ_3 and the coefficients of the power series expansion can be obtained using its derivatives with respect to ζ_1, ζ_2 and ζ_3 evaluated at $\zeta_1=\zeta_2=\zeta_3=0$. By taking the derivatives term by term, we see that the coefficients of the power series expansion of this analytic function are equal to the coefficients

$(-1)^{\tilde{k}} a_{R_\mu-\tilde{n},j,i} b_{R_\mu-\tilde{n},j,\tilde{k}-\tilde{n},s}$ of the iterated series (A.18). Since the power series expansion of an analytic function is an absolutely convergent multisum, we see that the triple series

$$\sum_{\tilde{n}\in\mathbb{N}} \sum_{\tilde{k}\in\mathbb{N}} \sum_{s=0}^{j} (-1)^{\tilde{k}} a_{R_\mu-\tilde{n},j,i} b_{R_\mu-\tilde{n},j,\tilde{k}-\tilde{n},s} \zeta_1^{\tilde{k}} \zeta_2^s \zeta_3^{\tilde{n}} \tag{A.19}$$

is absolutely convergent on the region $|\zeta_1| < \frac{1}{(1+r)r}$, $\zeta_2 \in \mathbb{C}$ and $|\zeta_3| < r$.

In particular, on the region $|z_1| > (1+r)r|z_2|$, taking $\zeta_1 = \frac{z_2}{z_1}$, $\zeta_2 = \log z_1$ and $\zeta_3 = 1$ in (A.19), we see that the triple series

$$\sum_{\tilde{n}\in\mathbb{N}} \sum_{\tilde{k}\in\mathbb{N}} \sum_{s=0}^{j} (-1)^{\tilde{k}} a_{R_\mu-\tilde{n},j,i} b_{R_\mu-\tilde{n},j,\tilde{k}-\tilde{n},s} \left(\frac{z_2}{z_1}\right)^{\tilde{k}} (\log z_1)^s \tag{A.20}$$

is absolutely convergent. Since r is an arbitrary real number satisfying $r > 1$, (A.20) is in fact absolutely convergent on the region $|z_1| > 2|z_2|$. Multiplying

$$e^{(-\Delta+R_\mu+1)\log z_1} e^{(-R_\mu-1)\log(-z_2)} (\log(-z_2))^i$$

to (A.20) and summing over A, $j = 0, \dots, M$ and $i = 0, \dots, N$, we see that the multiseires

$$\begin{aligned}
&\sum_{\mu\in A} \sum_{\tilde{n}\in\mathbb{N}} \sum_{j=0}^{M} \sum_{i=0}^{N} \sum_{\tilde{k}\in\mathbb{N}} \sum_{s=0}^{j} (-1)^{\tilde{k}} a_{R_\mu-\tilde{n},j,i} b_{R_\mu-\tilde{n},j,\tilde{k}-\tilde{n},s} e^{(-\Delta+R_\mu+1)\log z_1} \cdot \\
&\qquad\qquad \cdot e^{(-R_\mu-1)\log(-z_2)} (\log(-z_2))^i \left(\frac{z_2}{z_1}\right)^{\tilde{k}} (\log z_1)^s \\
&= \sum_{n\in D} \sum_{j=0}^{M} \sum_{i=0}^{N} \sum_{k\in\mathbb{N}} \sum_{s=0}^{j} a_{n,j,i} (-1)^k b_{n,j,k,s} e^{(-\Delta+n+1-k)\log z_1} z_2^k (\log z_1)^s e^{(-n-1)\log(-z_2)} (\log(-z_2))^i
\end{aligned} \tag{A.21}$$

is absolutely convergent on the region $|z_1| > 2|z_2| > 0$, proving the first part of the lemma.

In the case that the addional assumption holds, from the proof above and the additional assumptuion, the multisum (A.21), which is equal to the iterated series in the right-hand side of (A.9), is absolutely convergent on the region $|z_1| > 2|z_2| > 0$, $|\arg z_1 - \arg(z_1 - z_2)| < \frac{\pi}{2}$ to $f^e(z_1 - z_2, -z_2)$. In particular, (A.11) as an iterated sum of (A.21) is also absolutely convergent on the region $|z_1| > 2|z_2| > 0$, $|\arg z_1 - \arg(z_1 - z_2)| < \frac{\pi}{2}$ to $f^e(z_1 - z_2, -z_2)$. Since there is no singular point of $f^e(z_1 - z_2, -z_2)$ in the region $|z_1| > |z_2| > 0$, (A.11) must also be absolutely convergent when $|z_1| > |z_2| > 0$ and is thus absolutely convergent on the region $|z_1| > |z_2| > 0$, $|\arg z_1 - \arg(z_1 - z_2)| < \frac{\pi}{2}$ to $f^e(z_1 - z_2, -z_2)$. ∎

Remark A.2 There is a subtlety about the convergence regions for (A.21) and (A.11). Note that in general the multisum (A.21) might not be absolutely convergent on the larger region $|z_1| > |z_2| > 0$. This is because (A.21) is not a series in powers of z_1 and z_2 and nonnegative integral powers of $\log z_1$ and $\log z_2$. Even if we restore the variable ζ_3 in the proof of the lemma above to obtain a series in powers of z_1, z_2, ζ_3 and nonnegative integral powers of $\log z_1$ and $\log z_2$, since we do not have the assumption that the sum of this series can be analytically extended to a region containing $|z_1| > |z_2| > 0$, this series might not be absolutely convergent on any region containing $|z_1| > |z_2| > 0$.

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Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019
E-mail address: jd1422@scarletmail.rutgers.edu, yzhuang@math.rutgers.edu