Logarithmic tensor category theory, VII: Convergence and extension properties and applications to expansion for intertwining maps

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Abstract

This is the seventh part in a series of papers in which we introduce and develop a natural, general tensor category theory for suitable module categories for a vertex (operator) algebra. In this paper (Part VII), we give sufficient conditions for the existence of the associativity isomorphisms.

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In this paper, Part VII of a series of eight papers on logarithmic tensor category theory, we give sufficient conditions for the existence of the associativity isomorphisms. The sections, equations, theorems and so on are numbered globally in the series of papers rather than within each paper, so that for example equation (a.b) is the b-th labeled equation in Section a, which is contained in the paper indicated as follows: In Part I [HLZ1], which contains Sections 1 and 2, we give a detailed overview of our theory, state our main results and introduce the basic objects that we shall study in this work. We include a brief discussion of some of the recent applications of this theory, and also a discussion of some recent literature. In Part II [HLZ2], which contains Section 3, we develop logarithmic formal calculus and study logarithmic intertwining operators. In Part III [HLZ3], which contains Section 4, we introduce and study intertwining maps and tensor product bifunctors. In Part IV [HLZ4], which contains Sections 5 and 6, we give constructions of the P(z)- and Q(z)-tensor product bifunctors using what we call "compatibility conditions" and certain other conditions. In Part V [HLZ5], which contains Sections 7 and 8, we study products and iterates of intertwining maps and of logarithmic intertwining operators and we begin the development of our analytic approach. In Part VI [HLZ6], which contains Sections 9 and 10, we construct the appropriate natural associativity isomorphisms between triple tensor product functors. The present paper, Part VII, contains Section 11. In Part VIII [HLZ7], which contains Section 12, we construct braided tensor category structure.

Acknowledgments The authors gratefully acknowledge partial support from NSF grants DMS-0070800 and DMS-0401302. Y.-Z. H. is also grateful for partial support from NSF grant PHY-0901237 and for the hospitality of Institut des Hautes Études Scientifiques in the fall of 2007.

11 The convergence and extension properties and differential equations

In the construction of the associativity isomorphisms we have needed, and assumed, the convergence and expansion conditions for intertwining maps in \mathcal{C} . In this section we will follow [H1] to give certain sufficient conditions for a category \mathcal{C} to have these properties. In Section 11.1, by exhibiting explicitly the shapes of products and iterates of logarithmic intertwining operators as analytic functions, we give what we call, as in [H1], the "convergence and extension properties" and show that they imply the convergence and expansion conditions for intertwining maps in \mathcal{C} . In our general setting, these properties in general involve both the logarithm function and the abelian group gradings. These properties are formulated *globally* using analytic functions defined on the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, while the expansion condition is formulated *locally* near a point (z_1, z_2) in the intersection of these regions, that is, such that $|z_1| > |z_2| > |z_1 - z_2| > 0$. In Section 11.2, we show that the proofs in [H2], deriving and using differential equations, can be adapted to generalize the results of [H2] to result in the logarithmic generality. In particular, we will see that two purely algebraic conditions, the " C_1 -cofiniteness condition" and the "quasi-finitedimensionality condition," for all objects of \mathcal{C} imply the convergence and extension properties and thus also imply the convergence and expansion conditions for intertwining maps in \mathcal{C} .

11.1 The convergence and extension properties

Given objects W_1 , W_2 , W_3 , W_4 , M_1 and M_2 of the category \mathcal{C} , let \mathcal{Y}_1 , \mathcal{Y}_2 , \mathcal{Y}^1 and \mathcal{Y}^2 be logarithmic intertwining operators of types $\binom{W_4}{W_1M_1}$, $\binom{M_1}{W_2W_3}$, $\binom{W_4}{M_2W_3}$ and $\binom{M_2}{W_1W_2}$, respectively. We now consider certain natural conditions on the product of \mathcal{Y}_1 and \mathcal{Y}_2 and on the iterate of \mathcal{Y}^1 and \mathcal{Y}^2 . These conditions require that the product of \mathcal{Y}_1 and \mathcal{Y}_2 (respectively, the iterate of \mathcal{Y}^1 and \mathcal{Y}^2) be absolutely convergent in the region $|z_1| > |z_2| > 0$ (respectively, $|z_2| > |z_1 - z_2| > 0$) and that it can be analytically extended to a function of the same shape as (finite) sums of iterates (respectively, products) of logarithmic intertwining operators in the region $|z_2| > |z_1 - z_2| > 0$ (respectively, $|z_1| > |z_2| > 0$) with finitely-generated "lower bounded doubly-graded generalized V-modules" as intermediate generalized V-modules (see Definition 11.3 below). In fact, when the intermediate generalized V-module M_2 is a finitelygenerated lower bounded doubly-graded generalized V-module, the sum defining the iterate (11.7) in the region $|z_2| > |z_1 - z_2| > 0$ is a branch of a multivalued analytic function of the form (11.4), and analogously, when the intermediate generalized V-module M_1 is a finitelygenerated lower bounded doubly-graded generalized V-module, the sum defining the product (11.3) in the region $|z_1| > |z_2| > 0$ is a branch of a multivalued analytic function of the form (11.8). In the following conditions, we recall from the beginning of Section 7, in particular, (7.1), (7.2), (7.5) and (7.8), the meaning of the absolute convergence of (11.3) and (11.7) below; as in (7.13) and (7.14), we are taking p = 0 in the notation of (4.12).

Convergence and extension property for products For any $\beta \in \tilde{A}$, there exists an integer N_{β} depending only on \mathcal{Y}_1 , \mathcal{Y}_2 and β , and for any weight-homogeneous elements $w_{(1)} \in (W_1)^{(\beta_1)}$ and $w_{(2)} \in (W_2)^{(\beta_2)}$ $(\beta_1, \beta_2 \in \tilde{A})$ and any $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$ such that

$$\beta_1 + \beta_2 = -\beta, \tag{11.1}$$

there exist $M \in \mathbb{N}$, $r_k, s_k \in \mathbb{R}$, $i_k, j_k \in \mathbb{N}$, $k = 1, \ldots, M$; $K \in \mathbb{Z}_+$ independent of $w_{(1)}$ and $w_{(2)}$ such that each $i_k < K$; and analytic functions $f_k(z)$ on |z| < 1, $k = 1, \ldots, M$, satisfying

wt
$$w_{(1)}$$
 + wt $w_{(2)}$ + $s_k > N_\beta$, $k = 1, \dots, M$, (11.2)

such that

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_1 = z_1, x_2 = z_2}$$
 (11.3)

is absolutely convergent when $|z_1| > |z_2| > 0$ and can be analytically extended to the multivalued analytic function

$$\sum_{k=1}^{M} z_2^{r_k} (z_1 - z_2)^{s_k} (\log z_2)^{i_k} (\log (z_1 - z_2))^{j_k} f_k \left(\frac{z_1 - z_2}{z_2}\right)$$
(11.4)

(here $\log(z_1 - z_2)$ and $\log z_2$, and in particular, the powers of the variables, mean the multivalued functions, not the particular branch we have been using) in the region $|z_2| > |z_1 - z_2| > 0$.

- Convergence and extension property without logarithms for products When $i_k = j_k = 0$ for k = 1, ..., M, we call the property above the convergence and extension property without logarithms for products.
- Convergence and extension property for iterates For any $\beta \in A$, there exists an integer \tilde{N}_{β} depending only on \mathcal{Y}^1 , \mathcal{Y}^2 and β , and for any $w_{(1)} \in W_1$, $w_{(2)} \in (W_2)^{(\beta_2)}$, $w_{(3)} \in (W_3)^{(\beta_3)}$ $(\beta_2, \beta_3 \in \tilde{A})$ and $w'_{(4)} \in W'_4$, with $w_{(2)}$ and $w_{(3)}$ weight-homogeneous, such that

$$\beta_2 + \beta_3 = -\beta, \tag{11.5}$$

there exist $\tilde{M} \in \mathbb{N}$, $\tilde{r}_k, \tilde{s}_k \in \mathbb{R}$, $\tilde{i}_k, \tilde{j}_k \in \mathbb{N}$, $k = 1, \ldots, \tilde{M}$; $\tilde{K} \in \mathbb{Z}_+$ independent of $w_{(2)}$ and $w_{(3)}$ such that each $\tilde{i}_k < \tilde{K}$; and analytic functions $\tilde{f}_k(z)$ on $|z| < 1, k = 1, \ldots, \tilde{M}$, satisfying

wt
$$w_{(2)}$$
 + wt $w_{(3)}$ + $\tilde{s}_k > \tilde{N}_\beta$, $k = 1, \dots, \tilde{M}$, (11.6)

such that

$$\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle_{W_4}\Big|_{x_0=z_1-z_2, x_2=z_2}$$
 (11.7)

is absolutely convergent when $|z_2| > |z_1 - z_2| > 0$ and can be analytically extended to the multivalued analytic function

$$\sum_{k=1}^{M} z_1^{\tilde{r}_k} z_2^{\tilde{s}_k} (\log z_1)^{\tilde{i}_k} (\log z_2)^{\tilde{j}_k} \tilde{f}_k \left(\frac{z_2}{z_1}\right)$$
(11.8)

(here log z_1 and log z_2 , and in particular, the powers of the variables, mean the multivalued functions, not the particular branches we have been using) in the region $|z_1| > |z_2| > 0$.

Convergence and extension property without logarithms for iterates When $i_k = j_k = 0$ for k = 1, ..., M, we call the property above the convergence and extension property without logarithms for iterates.

If the convergence and extension property (with or without logarithms) for products holds for any objects W_1 , W_2 , W_3 , W_4 and M_1 of C and any logarithmic intertwining operators \mathcal{Y}_1 and \mathcal{Y}_2 of the types as above, we say that the (corresponding) convergence and extension property for products holds in C. We similarly define the meaning of the phrase the (corresponding) convergence and extension property for iterates holds in C.

Remark 11.1 When we verify these properties using differential equations below, we will actually be verifying stronger absolute convergence assertions than as in (11.3) and (11.7) (that is, as in (7.5) and (7.8)): When all four vectors are weight-homogeneous, (11.3) and (11.7) will be proved to be absolutely convergent in the substitution-sense described in the statement of Proposition 7.20, and this implies the absolute convergence of (11.3) and (11.7) as above (again, as in (7.5) and (7.8)), because the absolute convergence of the triple series (7.46) implies the absolute convergence of the corresponding iterated series.

Lemma 11.2 If the convergence and extension property for products holds, then, with all four vectors assumed weight-homogeneous, we can always find M, r_k , s_k , i_k , j_k and $f_k(z)$, $k = 1, \ldots, M$, such that

$$r_k + s_k = \Delta, \quad k = 1, \dots, M, \tag{11.9}$$

where

$$\Delta = -\mathrm{wt} \ w_{(1)} - \mathrm{wt} \ w_{(2)} - \mathrm{wt} \ w_{(3)} + \mathrm{wt} \ w'_{(4)}. \tag{11.10}$$

Analogously, if the convergence and extension property for iterates holds, then, with all four vectors assumed weight-homogeneous, we can always find \tilde{M} , \tilde{r}_k , \tilde{s}_k , \tilde{i}_k , \tilde{j}_k and $\tilde{f}_k(z)$, $k = 1, \ldots, \tilde{M}$, such that

$$\tilde{r}_k + \tilde{s}_k = \Delta, \quad k = 1, \dots, \tilde{M}. \tag{11.11}$$

Proof Here we prove the assertion for products; the proof for iterates is entirely analogous. By (3.61), the expression (11.3) but without the evaluation at z_1 and z_2 is

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle = \langle y^{L'(0)} w'_{(4)}, \mathcal{Y}_1(y^{-L(0)} w_{(1)}, x_1 y^{-1}) \mathcal{Y}_2(y^{-L(0)} w_{(2)}, x_2 y^{-1}) y^{-L(0)} w_{(3)} \rangle,$$

and we know that this formal series in x_1 , $\log x_1$, x_2 , $\log x_2$, y and $\log y$ is constant with respect to the formal variables y and $\log y$. This formal series equals

$$\begin{split} \langle y^{\operatorname{wt} w'_{(4)}} e^{\log y(L'(0) - \operatorname{wt} w'_{(4)})} w'_{(4)}, \mathcal{Y}_1(y^{-\operatorname{wt} w_{(1)}} e^{-\log y(L(0) - \operatorname{wt} w_{(1)})} w_{(1)}, x_1 y^{-1}) \cdot \\ \cdot \mathcal{Y}_2(y^{-\operatorname{wt} w_{(2)}} e^{-\log y(L(0) - \operatorname{wt} w_{(2)})} w_{(2)}, x_2 y^{-1}) y^{-\operatorname{wt} w_{(3)}} e^{-\log y(L(0) - \operatorname{wt} w_{(3)})} w_{(3)} \rangle \\ = y^{\Delta} \langle e^{\log y(L'(0) - \operatorname{wt} w'_{(4)})} w'_{(4)}, \mathcal{Y}_1(e^{-\log y(L(0) - \operatorname{wt} w_{(1)})} w_{(1)}, x_1 y^{-1}) \cdot \\ \cdot \mathcal{Y}_2(e^{-\log y(L(0) - \operatorname{wt} w_{(2)})} w_{(2)}, x_2 y^{-1}) e^{-\log y(L(0) - \operatorname{wt} w_{(3)})} w_{(3)} \rangle, \end{split}$$

and the coefficient of each monomial $x_1^m x_2^n$ $(m, n \in \mathbb{R})$ in x_1 and x_2 is a real power of y times a polynomial in $\log x_1$, $\log x_2$ and $\log y$, and is in fact constant with respect to y and $\log y$; in particular, $m + n = \Delta$ in each nonzero term. Thus this expression equals the result of specializing y to z_2 , with $\log y$ specialized to $\log z_2$ (using our usual branch of log), namely,

$$e^{\Delta \log z_2} \langle e^{(\log z_2)(L'(0) - \operatorname{wt} w'_{(4)})} w'_{(4)}, \mathcal{Y}_1(e^{-(\log z_2)(L(0) - \operatorname{wt} w_{(1)})} w_{(1)}, e^{-\log z_2} x_1) \cdot \\ \cdot \mathcal{Y}_2(e^{-(\log z_2)(L(0) - \operatorname{wt} w_{(2)})} w_{(2)}, e^{-\log z_2} x_2) e^{-(\log z_2)(L(0) - \operatorname{wt} w_{(3)})} w_{(3)} \rangle,$$

using the notation (3.76) for each of \mathcal{Y}_1 and \mathcal{Y}_2 . Thus, using the notation (7.14) with p = 0, we have that

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_1 = z_1, x_2 = z_1}$$

an absolutely convergent sum dictated by the monomials in x_1 and x_2 , equals

$$e^{\Delta \log z_2} \langle e^{(\log z_2)(L'(0) - \operatorname{wt} w'_{(4)})} w'_{(4)}, \mathcal{Y}_1(e^{-(\log z_2)(L(0) - \operatorname{wt} w_{(1)})} w_{(1)}, \xi_1) \cdot \\ \cdot \mathcal{Y}_2(e^{-(\log z_2)(L(0) - \operatorname{wt} w_{(2)})} w_{(2)}, \xi_2) e^{-(\log z_2)(L(0) - \operatorname{wt} w_{(3)})} w_{(3)} \rangle \Big|_{\xi_1 = e^{\log z_1 - \log z_2}, \, \xi_2 = e^0}$$

Thus (11.3) as an analytic function defined on $|z_1| > |z_2| > 0$ can be analytically extended to the analytic function

$$e^{\Delta \log z_2} \langle e^{(\log z_2)(L'(0) - \operatorname{wt} w'_{(4)})} w'_{(4)}, \mathcal{Y}_1(e^{(\log z_2)(-L(0) + \operatorname{wt} w_{(1)})} w_{(1)}, 1 + (z_1 - z_2)/z_2) \cdot \mathcal{Y}_2(e^{(\log z_2)(-L(0) + \operatorname{wt} w_{(2)})} w_{(2)}, 1) e^{(\log z_2)(-L(0) + \operatorname{wt} w_{(3)})} w_{(3)} \rangle$$
(11.12)

on the same region, where as usual, we are using the particular branch of log and the corresponding branch of each power function.

On the other hand, by the convergence and extension property for products, (11.3) can be analytically extended to the multivalued function

$$\sum_{k=1}^{M} z_{2}^{r_{k}} (z_{1} - z_{2})^{s_{k}} (\log z_{2})^{i_{k}} (\log (z_{1} - z_{2}))^{j_{k}} f_{k} \left(\frac{z_{1} - z_{2}}{z_{2}}\right)$$
$$= \sum_{k=1}^{M} z_{2}^{r_{k} + s_{k}} \left(\frac{z_{1} - z_{2}}{z_{2}}\right)^{s_{k}} (\log z_{2})^{i_{k}} (\log (z_{1} - z_{2}))^{j_{k}} f_{k} \left(\frac{z_{1} - z_{2}}{z_{2}}\right)$$
(11.13)

in the region $|z_2| > |z_1 - z_2| > 0$. Thus the right-hand side of (11.12) can be analytically extended to the right-hand side of (11.13) in the region $|z_2| > |z_1 - z_2| > 0$. Fix $w \in \mathbb{C}$ such that |1 + w| > 1 > |w| > 0. Then for any $z_2 \in \mathbb{C}^{\times}$, if we let $z_1 = z_2 + z_2 w$, we have

$$|z_1| = |z_2||1 + w| > |z_2| > |z_2||w| = |z_1 - z_2| > 0,$$

and thus

$$e^{\Delta \log z_{2}} \langle e^{(\log z_{2})(L'(0) - \operatorname{wt} w'_{(4)})} w'_{(4)}, \mathcal{Y}_{1}(e^{(\log z_{2})(-L(0) + \operatorname{wt} w_{(1)})} w_{(1)}, 1 + w) \rangle \\ \cdot \mathcal{Y}_{2}(e^{(\log z_{2})(-L(0) + \operatorname{wt} w_{(2)})} w_{(2)}, 1) e^{(\log z_{2})(-L(0) + \operatorname{wt} w_{(3)})} w_{(3)} \rangle$$

as an analytic function of z_2 in the region $z_2 \in \mathbb{C}^{\times}$ can be analytically extended to the multivalued analytic function

$$\sum_{k=1}^{M} z_2^{r_k + s_k} w^{s_k} (\log z_2)^{i_k} (\log(z_2 w))^{j_k} f_k(w)$$

in the same region. In particular, we can change f_k if necessary so that

$$e^{\Delta \log z_{2}} \langle e^{(\log z_{2})(L'(0) - \operatorname{wt} w'_{(4)})} w'_{(4)}, \mathcal{Y}_{1}(e^{(\log z_{2})(-L(0) + \operatorname{wt} w_{(1)})} w_{(1)}, 1 + w) \cdot \mathcal{Y}_{2}(e^{(\log z_{2})(-L(0) + \operatorname{wt} w_{(2)})} w_{(2)}, 1) e^{(\log z_{2})(-L(0) + \operatorname{wt} w_{(3)})} w_{(3)} \rangle$$

$$= \sum_{k=1}^{M} e^{(r_{k} + s_{k}) \log z_{2}} w^{s_{k}} (\log z_{2})^{i_{k}} (\log(z_{2}w))^{j_{k}} f_{k}(w) \qquad (11.14)$$

in the region $z_2 \in \mathbb{C}^{\times}$. By Proposition 7.8 (applied to a finite sum), we obtain from (11.14) that for $l \neq \Delta$,

$$\sum_{r_k+s_k=l} w^{s_k} (\log z_2)^{i_k} (\log (z_2 w))^{j_k} f_k(w) = 0$$

in the region $z_2 \in \mathbb{C}^{\times}$. Thus we see that (11.3) can be analytically extended to the multivalued function (11.4) where $r_k + s_k = \Delta$. \Box

Recall the notion of lower bounded (strongly A-graded) generalized V-module in Definition 2.25. We also need the following more general notion, for which we recall the notions of doubly-graded generalized V-module and doubly-graded V-module in Definition 9.14 (for which the grading restrictions (2.85) and (2.86) are not assumed):

Definition 11.3 If a doubly-graded generalized V-module $W = \coprod_{\beta \in \tilde{A}} \coprod_{n \in \mathbb{R}} W_{[n]}^{(\beta)}$ satisfies the condition that for $\beta \in \tilde{A}$, $W_{[n]}^{(\beta)} = 0$ for n sufficiently negative, we say that W is a *lower bounded doubly-graded generalized* V-module. We define the notion of *lower bounded doublygraded* V-module $W = \coprod_{\beta \in \tilde{A}} \coprod_{n \in \mathbb{R}} W_{(n)}^{(\beta)}$ analogously. Such a structure is a lower bounded generalized V-module (respectively, lower bounded V-module) if and only if each space $W_{[n]}^{(\beta)}$ (respectively, $W_{(n)}^{(\beta)}$) is finite dimensional. We use such lower boundedness to give conditions for insuring that the convergence and the expansion condition for intertwining maps in C hold:

Theorem 11.4 Suppose that the following two conditions are satisfied:

- Every finitely-generated lower bounded doubly-graded generalized V-module is an object of C (or every finitely-generated lower bounded doubly-graded V-module is an object of C, when C is in M_{sq}).
- The convergence and extension property for either products or iterates holds in C (or the convergence and extension property without logarithms for either products or iterates holds in C, when C is in M_{sg}).

Then the convergence and expansion conditions for intertwining maps in C both hold (recall Definitions 7.4 and 9.28).

Proof By the convergence and extension property for either products or iterates, the convergence condition for intertwining maps in C holds (recall Definition 7.4).

We shall now use the convergence and extension property for products to prove the first of the two equivalent conditions in Theorem 9.27; the convergence and extension property for iterates can be used analogously to prove the second condition in Theorem 9.27. We shall work in the general (logarithmic) case; the argument for the case C in \mathcal{M}_{sg} is analogous (and shorter). That is, we shall prove: For any objects W_1, W_2, W_3, W_4 and M_1 of C, any nonzero complex numbers z_1 and z_2 satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$, any $P(z_1)$ -intertwining map I_1 of type $\binom{W_4}{W_1M_1}$ and $P(z_2)$ -intertwining map I_2 of type $\binom{M_1}{W_2W_3}$, and any $w'_{(4)} \in W'_4$, $(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}) \in (W_1 \otimes W_2 \otimes W_3)^*$ satisfies the $P^{(2)}(z_1 - z_2)$ -local grading restriction condition. Moreover, for any $w_{(3)} \in W_3$ and $n \in \mathbb{R}$, the smallest doubly graded subspace of $W^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}),w_{(3)}}$ containing the term $\lambda_n^{(2)}$ of the (unique) series $\sum_{n \in \mathbb{R}} \lambda_n^{(2)}$ weakly absolutely convergent to $\mu^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}),w_{(3)}}$ as indicated in the $P^{(2)}(z_0)$ -grading condition and stable under the action of V and of $\mathfrak{sl}(2)$ is a generalized V-submodule of some object of C included in $(W_1 \otimes W_2)^*$.

We may and do take the elements $w_{(1)} \in W_1, w_{(2)} \in W_2, w_{(3)} \in W_3, w'_{(4)} \in W'_4$ to be weight-homogeneous, since it is enough to prove the required properties for such elements. We also recall the notation Δ from Lemma 11.2. Let $\mathcal{Y}_1 = \mathcal{Y}_{I_1,0}$ and $\mathcal{Y}_2 = \mathcal{Y}_{I_2,0}$. By Condition 2 (for products) and Lemma 11.2, for $\beta \in \tilde{A}$, there exists $N_\beta \in \mathbb{Z}$ and for any $w_{(1)} \in (W_1)_{[n_1]}^{(\beta_1)}, w_{(2)} \in (W_2)_{[n_2]}^{(\beta_2)}, w_{(3)} \in (W_3)_{[n_3]}, w'_{(4)} \in (W'_4)_{[n_4]}$ such that (11.1) holds, there exist $M \in \mathbb{N}, r_k, s_k \in \mathbb{R}, i_k, j_k \in \mathbb{N}, k = 1, \ldots, M$; $K \in \mathbb{Z}_+$ independent of $w_{(1)}$ and $w_{(2)}$ such that each $i_k < K$; and analytic functions $f_k(z)$ on $|z| < 1, k = 1, \ldots, M$, such that (11.9) holds and (11.3) is absolutely convergent when $|z_1| > |z_2| > 0$ and can be analytically extended to the multivalued analytic function (11.4) in the region $|z_2| > |z_1 - z_2| > 0$. Then we can always find $f_k(z)$ for $k = 1, \ldots, M$ such that (11.3) is equal to (11.4) when we choose the values of $\log z_2$ and $\log(z_1 - z_2)$ to be $\log |z_2| + i \arg z_2$ and $\log |z_1 - z_2| + i \arg(z_1 - z_2)$ where $0 \leq \arg z_2, \arg(z_1 - z_2) < 2\pi$ and the values of $z_2^{r_k}$ and $(z_1 - z_2)^{s_k}$ to be $e^{r_k \log z_2}$ and $e^{s_k \log(z_1-z_2)}$ (recalling that in the convergence and extension properties for products and iterates, $\log z_2$ and $\log(z_1-z_2)$ mean the multivalued logarithm functions).

Expanding $f_k(z)$, k = 1, ..., M, we see that in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$, (11.3) is equal to the absolutely convergent sum

$$\sum_{k=1}^{M} \sum_{m \in \mathbb{N}} C_{m;k}(w'_{(4)}, w_{(1)}, w_{(2)}, w_{(3)}) \cdot e^{(r_k - m)\log z_2} e^{(s_k + m)\log(z_1 - z_2)} (\log z_2)^{i_k} (\log(z_1 - z_2))^{j_k}, \qquad (11.15)$$

where, as in this work except in a few identified places, $\log z_2$ and $\log(z_1 - z_2)$ are the values of the logarithm function at z_2 and $z_1 - z_2$ such that $0 \leq \arg z_2, \arg(z_1 - z_2) < 2\pi$.

The expression (11.15) can be written as

$$\sum_{i,j\in\mathbb{N}}\sum_{n\in\mathbb{R}}a_{n;i,j}(w'_{(4)},w_{(1)},w_{(2)},w_{(3)})\cdot e^{(\Delta+n+1)\log(z_1-z_2)}e^{(-n-1)\log z_2}(\log z_2)^i(\log(z_1-z_2))^j,$$
(11.16)

where

$$a_{n;i,j}(w'_{(4)}, w_{(1)}, w_{(2)}, w_{(3)}) = 0$$
(11.17)

whenever

$$n \neq -r_k - 1 = -\Delta + s_k - 1 \quad \text{or} \quad i \neq i_k \quad \text{or} \quad j \neq j_k \tag{11.18}$$

for each $k = 1, \dots, M$. Also, if $w_{(1)} \in (W_1)^{(\beta_1)}, w_{(2)} \in (W_2)^{(\beta_2)}, w_{(3)} \in (W_3)^{(\beta_3)}$ and $w'_{(4)} \in (W'_4)^{(\beta_4)}$ and

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 \neq 0, \tag{11.19}$$

then (11.17) holds. From (11.18), (11.19), (11.2), (11.9) we see that for $n \in \mathbb{R}$, if

$$n + 1 + \operatorname{wt} w'_{(4)} - \operatorname{wt} w_{(3)} \le N_{\beta_3 + \beta_4},$$
 (11.20)

then (11.17) holds.

Since $|z_1| > |z_2| > |z_1 - z_2| > 0$, we know that

$$\sum_{i,j\in\mathbb{N}}\sum_{n\in\mathbb{R}}a_{n;i,j}(w'_{(4)},w_{(1)},w_{(2)},w_{(3)})\cdot e^{(\Delta+n+1)\log(z_1-z_2)}e^{(-n-1)\log z_2}(\log z_2)^i(\log(z_1-z_2))^j$$
(11.21)

converges absolutely to (11.3). For $i, j \in \mathbb{N}$, $w'_{(4)} \in W'_4$, $w_{(3)} \in W_3$, let $\beta_{n;i,j}(w'_{(4)}, w_{(3)}) \in (W_1 \otimes W_2)^*$ be defined by

$$(\beta_{n;i,j}(w'_{(4)}, w_{(3)}))(w_{(1)} \otimes w_{(2)}) = a_{n;i,j}(e^{L(0)_s \log(z_1 - z_2)}w'_{(4)}, e^{-L(0)_s \log(z_1 - z_2)}w_{(1)}, e^{-L(0)_s \log(z_1 - z_2)}w_{(2)}, e^{-L(0)_s \log(z_1 - z_2)}w_{(3)})$$
(11.22)

for all $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. By definition, the series

$$\sum_{i,j\in\mathbb{N}}\sum_{n\in\mathbb{R}}(\beta_{n;i,j}(w'_{(4)},w_{(3)}))(w_{(1)}\otimes w_{(2)})e^{(n+1)\log(z_1-z_2)}e^{(-n-1)\log z_2}(\log z_2)^i(\log(z_1-z_2))^j(\log(z_1-z_2)))^$$

is absolutely convergent to $\mu_{I_1 \circ (1_{W_2} \otimes I_2))'(w'_{(4)}),w_{(3)}}(w_{(1)} \otimes w_{(2)})$ for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. Also since (11.17) holds when (11.19) holds, we have

$$\beta_{n;i,j}(w'_{(4)}, w_{(3)}) \in ((W_1 \otimes W_2)^*)^{(\beta_3 + \beta_4)}$$
(11.23)

for $w_{(3)} \in (W_3)^{(\beta_3)}$ and $w'_{(4)} \in (W'_4)^{(\beta_4)}$.

To show that $(I_1 \circ (1_{W_2} \otimes I_2))'(w'_{(4)})$ satisfies the $P^{(2)}(z_1 - z_2)$ -local grading restriction condition, we need to calculate

$$(v_1)_{m_1}\cdots(v_r)_{m_r}\beta_{n;i,j}(w'_{(4)},w_{(3)})$$

and its weight for any $r \in \mathbb{N}, v_1, \ldots, v_r \in V, m_1, \ldots, m_r \in \mathbb{Z}, n \in \mathbb{R}$, where $(v_1)_{m_1}, \ldots, (v_r)_{m_r}$, $m_1, \dots, m_r \in \mathbb{Z}$, are the components of $Y'_{P(z_1-z_2)}(v_1, x), \dots, Y'_{P(z_1-z_2)}(v_r, x)$, respectively, on $(W_1 \otimes W_2)^*$. As in [H1], for convenience, we instead calculate

$$(v_1^o)_{m_1}\cdots(v_r^o)_{m_r}\beta_{n;i,j}(w'_{(4)},w_{(3)}),$$

where $(v_1^o)_{m_1}, \ldots, (v_r^o)_{m_r}, m_1, \cdots, m_r \in \mathbb{Z}$, are the components of the opposite vertex operators $Y_{P(z_1-z_2)}^{\prime o}(v_1, x), \ldots, Y_{P(z_1-z_2)}^{\prime o}(v_r, x)$, respectively. By the definition (5.87) of $Y_{P(z_1-z_2)}^{\prime}(v, x)$ and

$$Y_{P(z_1-z_2)}^{\prime o}(v,x) = Y_{P(z_1-z_2)}^{\prime}(e^{xL(1)}(-x^{-2})^{L(0)}v,x^{-1}),$$

we have

$$\begin{aligned} (Y_{P(z_{1}-z_{2})}^{\prime o}(v,x)\beta_{n;i,j}(w_{(4)}^{\prime},w_{(3)}))(w_{(1)}\otimes w_{(2)}) \\ &= (\beta_{n;i,j}(w_{(4)}^{\prime},w_{(3)}))(w_{(1)}\otimes Y_{2}(v,x)w_{(2)}) \\ &+ \operatorname{Res}_{x_{0}}(z_{1}-z_{2})^{-1}\delta\left(\frac{x-x_{0}}{z_{1}-z_{2}}\right)(\beta_{n;i,j}(w_{(4)}^{\prime},w_{(3)}))(Y_{1}(v,x_{0})w_{(1)}\otimes w_{(2)}) \\ &= a_{n;i,j}(e^{L(0)_{s}\log(z_{1}-z_{2})}w_{(4)}^{\prime},e^{-L(0)_{s}\log(z_{1}-z_{2})}w_{(1)},e^{-L(0)_{s}\log(z_{1}-z_{2})}Y_{2}(v,x)w_{(2)},e^{-L(0)_{s}\log(z_{1}-z_{2})}w_{(3)}) \\ &+ \operatorname{Res}_{x_{0}}(z_{1}-z_{2})^{-1}\delta\left(\frac{x-x_{0}}{z_{1}-z_{2}}\right) \cdot \\ &\cdot a_{n;i,j}(e^{L(0)_{s}\log(z_{1}-z_{2})}w_{(4)}^{\prime},e^{-L(0)_{s}\log(z_{1}-z_{2})}Y_{1}(v,x_{0})w_{(1)},e^{-L(0)_{s}\log(z_{1}-z_{2})}w_{(2)},e^{-L(0)_{s}\log(z_{1}-z_{2})}w_{(3)}). \end{aligned}$$

On the other hand, since (11.21) is absolutely convergent to (11.3) when $|z_1| > |z_2| >$ $|z_1 - z_2| > 0$, we have

$$\sum_{i,j\in\mathbb{N}}\sum_{n\in\mathbb{R}}e^{(n+1)\log(z_1-z_2)}e^{(-n-1)\log z_2}(\log z_2)^i(\log(z_1-z_2))^j$$

$$\begin{aligned} \cdot a_{n;i,j} \left(e^{L(0)_{s} \log(z_{1}-z_{2})} w_{(4)}^{\prime}, e^{-L(0)_{s} \log(z_{1}-z_{2})} w_{(1)}, e^{-L(0)_{s} \log(z_{1}-z_{2})} Y_{2}(v, x) w_{(2)}, e^{-L(0)_{s} \log(z_{1}-z_{2})} w_{(3)} \right) \\ + \operatorname{Res}_{x_{0}} (z_{1}-z_{2})^{-1} \delta \left(\frac{x-x_{0}}{z_{1}-z_{2}} \right) \sum_{i,j\in\mathbb{N}} \sum_{n\in\mathbb{R}} e^{(n+1)\log(z_{1}-z_{2})} e^{(-n-1)\log z_{2}} (\log z_{2})^{i} (\log(z_{1}-z_{2}))^{j} \cdot \\ \cdot a_{n;i,j} (e^{L(0)_{s} \log(z_{1}-z_{2})} w_{(4)}^{\prime}, e^{-L(0)_{s} \log(z_{1}-z_{2})} Y_{1}(v, x_{0}) w_{(1)}, e^{-L(0)_{s} \log(z_{1}-z_{2})} w_{(2)}, e^{-L(0)_{s} \log(z_{1}-z_{2})} w_{(3)}) \\ = \langle w_{(4)}^{\prime}, \mathcal{Y}_{1}(w_{(1)}, x_{1}) \mathcal{Y}_{2}(Y_{2}(v, x) w_{(2)}, x_{2}) w_{(3)} \rangle_{W_{4}} \Big|_{x_{1}=z_{1}, x_{2}=z_{2}} \\ + \operatorname{Res}_{x_{0}} (x_{1}-x_{2})^{-1} \delta \left(\frac{x-x_{0}}{x_{1}-x_{2}} \right) \cdot \\ \cdot \langle w_{(4)}^{\prime}, \mathcal{Y}_{1}(w_{(1)}, x_{1}) \mathcal{Y}_{2}(Y_{2}(v, x) w_{(2)}, x_{2}) w_{(3)} \rangle_{W_{4}} \Big|_{x_{1}=z_{1}, x_{2}=z_{2}} \\ = \langle w_{(4)}^{\prime}, \mathcal{Y}_{1}(w_{(1)}, x_{1}) \mathcal{Y}_{2}(Y_{2}(v, x) w_{(2)}, x_{2}) w_{(3)} \rangle_{W_{4}} \Big|_{x_{1}=z_{1}, x_{2}=z_{2}} \\ + \operatorname{Res}_{x_{0}} x_{1}^{-1} \delta \left(\frac{(x+x_{2})-x_{0}}{x_{1}} \right) \cdot \\ \cdot \langle w_{(4)}^{\prime}, \mathcal{Y}_{1}(Y_{1}(v, x_{0}) w_{(1)}, x_{1}) \mathcal{Y}_{2}(w_{(2)}, x_{2}) w_{(3)} \rangle_{W_{4}} \Big|_{x_{1}=z_{1}, x_{2}=z_{2}}. \end{aligned}$$

$$(11.25)$$

Using the Jacobi identity for the logarithmic intertwining operators \mathcal{Y}_1 and \mathcal{Y}_2 and the properties of the formal δ -function, the right-hand side of (11.25) is equal to

$$\operatorname{Res}_{y} x^{-1} \delta\left(\frac{y-x_{2}}{x}\right) \langle w_{(4)}', \mathcal{Y}_{1}(w_{(1)}, x_{1})Y_{5}(v, y)\mathcal{Y}_{2}(w_{(2)}, x_{2})w_{(3)}\rangle_{W_{4}}\Big|_{x_{1}=z_{1}, x_{2}=z_{2}} \\ -\operatorname{Res}_{y} x^{-1} \delta\left(\frac{x_{2}-y}{-x}\right) \langle w_{(4)}', \mathcal{Y}_{1}(w_{(1)}, x_{1})\mathcal{Y}_{2}(w_{(2)}, x_{2})Y_{3}(v, y)w_{(3)}\rangle_{W_{4}}\Big|_{x_{1}=z_{1}, x_{2}=z_{2}} \\ + \langle w_{(4)}', Y_{4}(v, x+x_{2})\mathcal{Y}_{1}(w_{(1)}, x_{1})\mathcal{Y}_{2}(w_{(2)}, x_{2})w_{(3)}\rangle_{W_{4}}\Big|_{x_{1}=z_{1}, x_{2}=z_{2}} \\ - \langle w_{(4)}', \mathcal{Y}_{1}(w_{(1)}, x_{1})Y_{5}(v, x+x_{2})\mathcal{Y}_{2}(w_{(2)}, x_{2})w_{(3)}\rangle_{W_{4}}\Big|_{x_{1}=z_{1}, x_{2}=z_{2}} \\ = \langle w_{(4)}', Y_{4}(v, x+x_{2})\mathcal{Y}_{1}(w_{(1)}, x_{1})\mathcal{Y}_{2}(w_{(2)}, x_{2})w_{(3)}\rangle_{W_{4}}\Big|_{x_{1}=z_{1}, x_{2}=z_{2}} \\ -\operatorname{Res}_{y} x^{-1} \delta\left(\frac{x_{2}-y}{-x}\right) \langle w_{(4)}', \mathcal{Y}_{1}(w_{(1)}, x_{1})\mathcal{Y}_{2}(w_{(2)}, x_{2})w_{(3)}\rangle_{W_{4}}\Big|_{x_{1}=z_{1}, x_{2}=z_{2}} \\ -\operatorname{Res}_{y} x^{-1} \delta\left(\frac{y-x_{2}}{x}\right) \langle w_{(4)}', \mathcal{Y}_{1}(w_{(1)}, x_{1})\mathcal{Y}_{2}(w_{(2)}, x_{2})w_{(3)}\rangle_{W_{4}}\Big|_{x_{1}=z_{1}, x_{2}=z_{2}} \\ -\operatorname{Res}_{y} x^{-1} \delta\left(\frac{x_{2}-y}{-x}\right) \langle w_{(4)}', \mathcal{Y}_{1}(w_{(1)}, x_{1})\mathcal{Y}_{2}(w_{(2)}, x_{2})w_{(3)}\rangle_{W_{4}}\Big|_{x_{1}=z_{1}, x_{2}=z_{2}} \\ -\operatorname{Res}_{y} x^{-1} \delta\left(\frac{x_{2}-y}{-x}\right) \langle w_{(4)}', \mathcal{Y}_{1}(w_{(1)}, x_{1})\mathcal{Y}_{2}(w_{(2)}, x_{2})w_{(3)}\rangle_{W_{4}}\Big|_{x_{1}=z_{1}, x_{2}=z_{2}} \\ -\operatorname{Res}_{y} x^{-1} \delta\left(\frac{x_{2}-y}{-x}\right) \langle w_{(4)}', \mathcal{Y}_{1}(w_{(1)}, x_{1})\mathcal{Y}_{2}(w_{(2)}, x_{2})w_{(3)}\rangle_{W_{4}}\Big|_{x_{1}=z_{1}, x_{2}=z_{2}} \\ -\operatorname{Res}_{y} x^{-1} \delta\left(\frac{x_{2}-y}{-x}\right) \langle w_{(4)}', \mathcal{Y}_{1}(w_{(1)}, x_{1})\mathcal{Y}_{2}(w_{(2)}, x_{2})W_{(3)}\rangle_{W_{4}}\Big|_{x_{1}=z_{1}, x_{2}=z_{2}} \\ -\operatorname{Res}_{y} x^{-1} \delta\left(\frac{x_{2}-y}{-x}\right) \langle w_{(4)}', \mathcal{Y}_{1}(w_{(1)}, x_{1})\mathcal{Y}_{2}(w_{(2)}, x_{2})Y_{3}(v, y)w_{(3)}\rangle_{W_{4}}\Big|_{x_{1}=z_{1}, x_{2}=z_{2}} \\ -\operatorname{Res}_{y} x^{-1} \delta\left(\frac{x_{2}-y}{-x}\right) \langle w_{(4)}', \mathcal{Y}_{1}(w_{(1)}, x_{1})\mathcal{Y}_{2}(w_{(2)}, x_{2})Y_{3}(v, y)w_{(3)}\rangle_{W_{4}}\Big|_{x_{1}=z_{1}, x_{2}=z_{2}} \\ -\operatorname{Res}_{y} x^{-1} \delta\left(\frac{x_{2}-y}{-x}\right) \langle w_{(4)}', \mathcal{Y}_{1}(w_{(1)}, x_{2})\mathcal{Y}_{2}(w_{(2)}, x_{2})Y_{3}(v, y)w_{(3)}\rangle_{W_{4}}\Big|_{x_{1}=z_{1}, x_{2}=z_{$$

From (11.25) and (11.26), we obtain

$$\sum_{i,j\in\mathbb{N}}\sum_{n\in\mathbb{R}}e^{(n+1)\log(z_1-z_2)}e^{(-n-1)\log z_2}(\log z_2)^i(\log(z_1-z_2))^j$$

$$\begin{aligned} \cdot a_{n;i,j} \left(e^{L(0)_s \log(z_1 - z_2)} w'_{(4)}, e^{-L(0)_s \log(z_1 - z_2)} w_{(1)}, e^{-L(0)_s \log(z_1 - z_2)} Y_2(v, x) w_{(2)}, e^{-L(0)_s \log(z_1 - z_2)} w_{(3)} \right) \\ + \operatorname{Res}_{x_0} \left(z_1 - z_2 \right)^{-1} \delta \left(\frac{x - x_0}{z_1 - z_2} \right) \sum_{i,j \in \mathbb{N}} \sum_{n \in \mathbb{R}} e^{(n+1) \log(z_1 - z_2)} e^{(-n-1) \log z_2} (\log z_2)^i (\log(z_1 - z_2))^j \cdot \\ \cdot a_{n;i,j} \left(e^{L(0)_s \log(z_1 - z_2)} w'_{(4)}, e^{-L(0)_s \log(z_1 - z_2)} Y_1(v, x_0) w_{(1)}, e^{-L(0)_s \log(z_1 - z_2)} w_{(2)}, e^{-L(0)_s \log(z_1 - z_2)} w_{(3)} \right) \\ = \operatorname{Res}_y x^{-1} \delta \left(\frac{y - x_2}{x} \right) \langle w'_{(4)}, Y(v, y) \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_1 = z_1, x_2 = z_2} \\ - \operatorname{Res}_y x^{-1} \delta \left(\frac{x_2 - y}{x} \right) \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) Y(v, y) w_{(3)} \rangle_{W_4} \Big|_{x_1 = z_1, x_2 = z_2} \\ = \sum_{m \in \mathbb{Z}} \sum_{l \in \mathbb{N}} (-1)^l \binom{m}{l} x^{-m-1} x_2^{m-l} \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) v_{(3)} \rangle_{W_4} \Big|_{x_1 = z_1, x_2 = z_2} \\ - \sum_{m \in \mathbb{Z}} \sum_{l \in \mathbb{N}} (-1)^{l+m} \binom{m}{l} x^{-m-1} x_2^{m-l} \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) v_{(3)} \rangle_{W_4} \Big|_{x_1 = z_1, x_2 = z_2} \\ = \sum_{m \in \mathbb{Z}} \sum_{l \in \mathbb{N}} (-1)^l \binom{m}{l} x^{-m-1} \sum_{i,j \in \mathbb{N}} \sum_{n \in \mathbb{R}} e^{(n+1) \log(z_1 - z_2)} e^{(l-n-1) \log z_2} (\log z_2)^i (\log(z_1 - z_2))^j \cdot \\ \cdot a_{n;i,j} (e^{L(0)_s \log(z_1 - z_2)} v_{m-l}^o w'_{(4)}, e^{-L(0)_s \log(z_1 - z_2)} w_{(1)}, e^{-L(0)_s \log(z_1 - z_2)} w_{(2)}, e^{-L(0)_s \log(z_1 - z_2)} w_{(3)}) \\ - \sum_{m \in \mathbb{Z}} \sum_{l \in \mathbb{N}} (-1)^{l+m} \binom{m}{l} x^{-m-1} \sum_{i,j \in \mathbb{N}} \sum_{n \in \mathbb{R}} e^{(n+1) \log(z_1 - z_2)} e^{(m-l-n-1) \log z_2} (\log z_2)^i (\log(z_1 - z_2))^j \cdot \\ \cdot a_{n;i,j} (e^{L(0)_s \log(z_1 - z_2)} w'_{(4)}, e^{-L(0)_s \log(z_1 - z_2)} w_{(1)}, e^{-L(0)_s \log(z_1 - z_2)} w_{(2)}, e^{-L(0)_s \log(z_1 - z_2)} v_{(3)}). \end{aligned}$$

$$(11.27)$$

The intermediate steps in the equality (11.27) hold only when $|z_1| > |z_2| > |z_1 - z_2| > 0$. But since both sides of (11.27) can be analytically extended to the region $|z_2| > |z_1 - z_2| > 0$, (11.27) must hold when $|z_2| > |z_1 - z_2| > 0$. By Proposition 7.8, the coefficients of both sides of (11.27) in powers of $e^{\log z_2}$, $\log z_2$ and $\log(z_1 - z_2)$ are equal, that is,

$$a_{n;i,j}(e^{L(0)_{s}\log(z_{1}-z_{2})}w'_{(4)}, e^{-L(0)_{s}\log(z_{1}-z_{2})}w_{(1)}, e^{-L(0)_{s}\log(z_{1}-z_{2})}Y_{2}(v, x)w_{(2)}, e^{-L(0)_{s}\log(z_{1}-z_{2})}w_{(3)}) + \operatorname{Res}_{y}(z_{1}-z_{2})^{-1}\delta\left(\frac{x-y}{z_{1}-z_{2}}\right) \cdot \\ \cdot a_{n;i,j}(e^{L(0)_{s}\log(z_{1}-z_{2})}w'_{(4)}, e^{-L(0)_{s}\log(z_{1}-z_{2})}Y_{1}(v, y)w_{(1)}, e^{-L(0)_{s}\log(z_{1}-z_{2})}w_{(2)}, e^{-L(0)_{s}\log(z_{1}-z_{2})}w_{(3)}) \\ = \sum_{m\in\mathbb{Z}}\sum_{l\in\mathbb{N}}(-1)^{l}\binom{m}{l}x^{-m-1}(z_{1}-z_{2})^{l} \cdot \\ \cdot a_{n+l;i,j}(e^{L(0)_{s}\log(z_{1}-z_{2})}v'_{m-l}w'_{(4)}, e^{-L(0)_{s}\log(z_{1}-z_{2})}w_{(1)}, e^{-L(0)_{s}\log(z_{1}-z_{2})}w_{(2)}, e^{-L(0)_{s}\log(z_{1}-z_{2})}w_{(3)}) \\ - \sum_{m\in\mathbb{Z}}\sum_{l\in\mathbb{N}}(-1)^{l+m}\binom{m}{l}x^{-m-1}(z_{1}-z_{2})^{m-l} \cdot \\ \cdot a_{n+m-l;i,j}(e^{L(0)_{s}\log(z_{1}-z_{2})}w'_{(4)}, e^{-L(0)_{s}\log(z_{1}-z_{2})}w_{(1)}, e^{-L(0)_{s}\log(z_{1}-z_{2})}w_{(2)}, e^{-L(0)_{s}\log(z_{1}-z_{2})}v_{l}w_{(3)}).$$

$$(11.28)$$

By (11.24) and (11.28), we obtain

$$(v_{m}^{o}\beta_{n;i,j}(w_{(4)}',w_{(3)}))(w_{(1)}\otimes w_{(2)})$$

$$= \sum_{l\in\mathbb{N}} (-1)^{l} \binom{m}{l} (\beta_{n+l;i,j}(v_{m-l}^{o}w_{(4)}',w_{(3)}))(w_{(1)},w_{(2)})(z_{1}-z_{2})^{l}$$

$$- \sum_{l\in\mathbb{N}} (-1)^{l+m} \binom{m}{l} (\beta_{n+m-l;i,j}(w_{(4)}',v_{l}w_{(3)}))(w_{(1)},w_{(2)})(z_{1}-z_{2})^{m-l}.$$
(11.29)

By induction, we obtain

$$((v_{1}^{o})_{m_{1}}\cdots(v_{r}^{o})_{m_{r}}\beta_{n;i,j}(w_{(4)}',w_{(3)})(w_{(1)}\otimes w_{(2)}) =$$

$$= \sum_{s=0}^{t} \sum_{\substack{j_{1} > \cdots > j_{s} \\ j_{s+1} > \cdots > j_{r} \\ \{j_{1},\ldots,j_{r}\} = \{1,\ldots,r\}} \sum_{\substack{l_{1},\ldots,l_{r} \ge 0}} (-1)^{l_{1}+\cdots+l_{r}+(m_{j_{s+1}}+1)+\cdots+(m_{j_{r}}+1)} \binom{m_{j_{1}}}{l_{1}}\cdots\binom{m_{j_{r}}}{l_{r}} \cdot \binom{m_{j_{r}}}{l_{r}} \cdot \binom{m_{$$

for $m_1, \ldots, m_r \in \mathbb{Z}$, and any $v_1, \ldots, v_r \in V$. In particular, when V is a conformal vertex algebra,

$$\begin{aligned} (L'_{P(z_{1}-z_{2})}(0)\beta_{n;i,j}(w'_{(4)},w_{(3)}))(w_{(1)}\otimes w_{(2)}) \\ &= (\beta_{n;i,j}(L'(0)w'_{(4)},w_{(3)}))(w_{(1)}\otimes w_{(2)}) \\ &- (\beta_{n+1;i,j}(L'(1)w'_{(4)},w_{(3)}))(w_{(1)}\otimes w_{(2)}) \cdot (z_{1}-z_{2}) \\ &- (\beta_{n;i,j}(w'_{(4)},L(0)w_{(3)}))(w_{(1)}\otimes w_{(2)}) \\ &+ (\beta_{n+1;i,j}(w'_{(4)},L(-1)w_{(3)}))(w_{(1)}\otimes w_{(2)}) \cdot (z_{1}-z_{2}) \\ &= (\operatorname{wt} w'_{(4)} - \operatorname{wt} w_{(3)})(\beta_{n;i,j}(w'_{(4)},w_{(3)}))(w_{(1)}\otimes w_{(2)}) \\ &+ (\beta_{n;i,j}((L'(0) - \operatorname{wt} w'_{(4)})w'_{(4)},w_{(3)}))(w_{(1)}\otimes w_{(2)}) \\ &- (\beta_{n;i,j}(w'_{(4)},(L(0) - \operatorname{wt} w_{(3)})w_{(3)}))(w_{(1)}\otimes w_{(2)}) \\ &- (\beta_{n+1;i,j}(L'(1)w'_{(4)},w_{(3)}))(w_{(1)}\otimes w_{(2)}) \cdot (z_{1}-z_{2}) \\ &+ (\beta_{n+1;i,j}(w'_{(4)},L(-1)w_{(3)}))(w_{(1)}\otimes w_{(2)}) \cdot (z_{1}-z_{2}); \end{aligned}$$

$$(11.31)$$

by a similar argument, (11.31) holds for a Möbius vertex algebra as well. From (11.30) and (11.23), we see that when $v_1 \in V_{(m_1)}^{(\alpha_1)}, \ldots, v_r \in V_{(m_r)}^{(\alpha_r)}, w_{(3)} \in (W_3)_{[n_3]}^{(\beta_3)}$ and $w'_{(4)} \in (W'_4)_{[n_4]}^{(\beta_4)}$,

$$(v_1^o)_{m_1}\cdots(v_r^o)_{m_r}\beta_{n;i,j}(w'_{(4)},w_{(3)})\in((W_1\otimes W_2)^*)^{(\alpha_1+\cdots+\alpha_r+\beta_3+\beta_4)}.$$
(11.32)

Let
$$z_{0} = z_{1} - z_{2}$$
. When $|z_{1}| > |z_{2}| > |z_{0}| > 0$,

$$-\sum_{i,j \in \mathbb{N}} \sum_{n \in \mathbb{R}} a_{n;i,j} (L'(1)w'_{(4)}, w_{(1)}, w_{(2)}, w_{(3)}) \cdot e^{((\Delta-1)+n+1)\log(z_{1}-z_{2})}e^{(-n-1)\log z_{2}} (\log z_{2})^{i} (\log(z_{1}-z_{2}))^{j} + \sum_{i,j \in \mathbb{N}} \sum_{n \in \mathbb{R}} a_{n;i,j} (w'_{(4)}, w_{(1)}, w_{(2)}, L(-1)w_{(3)}) \cdot e^{((\Delta-1)+n+1)\log(z_{1}-z_{2})}e^{(-n-1)\log z_{2}} (\log z_{2})^{i} (\log(z_{1}-z_{2}))^{j} = -\langle L'(1)w'_{(4)}, \mathcal{Y}_{1}(w_{(1)}, x_{1})\mathcal{Y}_{2}(w_{(2)}, x_{2})w_{(3)}\rangle_{W_{4}}\Big|_{x_{1}=z_{2}+z_{0}, x_{2}=z_{2}} + \langle w'_{(4)}, \mathcal{Y}_{1}(w_{(1)}, x_{1})\mathcal{Y}_{2}(w_{(2)}, x_{2})L(-1)w_{(3)}\rangle_{W_{4}}\Big|_{x_{1}=z_{2}+z_{0}, x_{2}=z_{2}}.$$
(11.33)

By the commutator formula for L(-1) and intertwining operators and the L(-1)-derivative property for intertwining operators, the right-hand side of (11.33) is equal to

$$-\left\langle w_{(4)}^{\prime}, \left(\frac{d}{dx_{1}}(\mathcal{Y}_{1}(w_{(1)},x_{1}))\right) \mathcal{Y}_{2}(w_{(2)},x_{2})w_{(3)}\right) \right\rangle_{W_{4}} \Big|_{x_{1}=z_{2}+z_{0}, x_{2}=z_{2}} \\ -\left\langle w_{(4)}^{\prime}, \mathcal{Y}_{1}(w_{(1)},x_{1})\left(\frac{d}{dx_{2}}(\mathcal{Y}_{2}(w_{(2)},x_{2})\right)w_{(3)}\right) \right\rangle_{W_{4}} \Big|_{x_{1}=z_{2}+z_{0}, x_{2}=z_{2}} \\ = -\frac{\partial}{\partial z_{2}} \left(\left\langle w_{(4)}^{\prime}, (\mathcal{Y}_{1}(w_{(1)},x_{1})\mathcal{Y}_{2}(w_{(2)},x_{2})w_{(3)}) \right\rangle_{W_{4}} \Big|_{x_{1}=z_{2}+z_{0}, x_{2}=z_{2}} \right) \\ = -\frac{\partial}{\partial z_{2}} \left(\sum_{i,j\in\mathbb{N}}\sum_{n\in\mathbb{R}}a_{n;i,j}(w_{(4)}^{\prime},w_{(1)},w_{(2)},w_{(3)}) \cdot \\ \cdot e^{(\Delta+n+1)\log z_{0}}e^{(-n-1)\log z_{2}}(\log z_{2})^{i}(\log z_{0})^{j} \right) \\ = -\sum_{i,j\in\mathbb{N}}\sum_{n\in\mathbb{R}}(-n-1)a_{n;i,j}(w_{(4)}^{\prime},w_{(1)},w_{(2)},w_{(3)}) \cdot \\ \cdot e^{(\Delta+n+1)\log z_{0}}e^{(-n-2)\log z_{2}}(\log z_{2})^{i}(\log z_{0})^{j} \\ -\sum_{i,j\in\mathbb{N}}\sum_{n\in\mathbb{R}}a_{n;i,j}(w_{(4)}^{\prime},w_{(1)},w_{(2)},w_{(3)}) \cdot \\ \cdot e^{(\Delta+n+1)\log z_{0}}e^{(-n-1)\log z_{2}}\left(\frac{\partial}{\partial z_{2}}(\log z_{2})^{i}\right)(\log z_{0})^{j}, \quad (11.34)$$

where $\frac{\partial}{\partial z_2}$ is partial differentiation with respect to z_2 acting on functions of z_0 and z_2 rather than on functions of z_1 and z_2 .

From (11.33) and (11.34), we obtain

$$-\sum_{i,j\in\mathbb{N}}\sum_{n\in\mathbb{R}}\beta_{n;i,j}(L'(1)w'_{(4)},w_{(3)})(w_{(1)}\otimes w_{(2)})\cdot e^{(n+1)\log(z_1-z_2)}e^{(-n-1)\log z_2}(\log z_2)^i(\log(z_1-z_2))^j$$

$$+ \sum_{i,j\in\mathbb{N}}\sum_{n\in\mathbb{R}}\beta_{n;i,j}(w'_{(4)}, L(-1)w_{(3)})(w_{(1)}\otimes w_{(2)}) \cdot \\ \cdot e^{(n+1)\log(z_1-z_2)}e^{(-n-1)\log z_2}(\log z_2)^i(\log(z_1-z_2))^j \\ = -\sum_{i,j\in\mathbb{N}}\sum_{n\in\mathbb{R}}(-n-1)\beta_{n;i,j}(w'_{(4)}, w_{(3)})(w_{(1)}\otimes w_{(2)}) \cdot \\ e^{(n+1)\log(z_1-z_2)}e^{(-n-2)\log z_2}(\log z_2)^i(\log(z_1-z_2))^j \\ -\sum_{i,j\in\mathbb{N}}\sum_{n\in\mathbb{R}}\beta_{n;i,j}(w'_{(4)}, w_{(3)})(w_{(1)}\otimes w_{(2)}) \cdot \\ e^{(n+1)\log(z_1-z_2)}e^{(-n-1)\log z_2}\left(\frac{\partial}{\partial z_2}(\log z_2)^i\right)(\log(z_1-z_2))^j$$
(11.35)

when $|z_1| > |z_2| > |z_1 - z_2| > 0$. Since both sides of (11.35) can be analytically extended to the region $|z_2| > |z_1 - z_2| > 0$, it also holds when $|z_2| > |z_1 - z_2| > 0$. Thus the expansion coefficients of the two sides of (11.35) as series in z_2 and $z_1 - z_2$ are equal. So for $i, j \in \mathbb{N}$,

$$(z_{1}-z_{2})\Big(-\beta_{n+1;i,j}(L'(1)w'_{(4)},w_{(3)})(w_{(1)}\otimes w_{(2)})+\beta_{n+1;i,j}(w'_{(4)},L(-1)w_{(3)})(w_{(1)}\otimes w_{(2)})\Big)$$

= -(-n-1)\beta_{n;i,j}(w'_{(4)},w_{(3)})(w_{(1)}\otimes w_{(2)})-(i+1)\beta_{n;i+1,j}(w'_{(4)},w_{(3)})(w_{(1)}\otimes w_{(2)}).
(11.36)

From (11.31) and (11.36), we obtain

$$((L'_{P(z_1-z_2)}(0) - \operatorname{wt} w'_{(4)} - n - 1 + \operatorname{wt} w_{(3)})\beta_{n;i,j}(w'_{(4)}, w_{(3)}))(w_{(1)} \otimes w_{(2)}) = (\beta_{n;i,j}((L'(0) - \operatorname{wt} w'_{(4)})w'_{(4)}, w_{(3)}))(w_{(1)} \otimes w_{(2)}) -(\beta_{n;i,j}(w'_{(4)}, (L(0) - \operatorname{wt} w_{(3)})w_{(3)}))(w_{(1)} \otimes w_{(2)}) -(i+1)\beta_{n;i+1,j}(w'_{(4)}, (L(0) - \operatorname{wt} w_{(3)})w_{(3)}))(w_{(1)} \otimes w_{(2)}).$$
(11.37)

From (11.37), we see that there exists $N \in \mathbb{Z}_+$ independent of $w_{(1)} \in W_{(1)}$ and $w_{(2)} \in W_{(2)}$ such that for $n \in \mathbb{R}$ and $i, j \in \mathbb{N}$,

$$(L'_{P(z_1-z_2)}(0) - \operatorname{wt} w'_{(4)} + \operatorname{wt} w_{(3)} - n - 1)^N \beta_{n;i,j}(w'_{(4)}, w_{(3)}) = 0.$$
(11.38)

Thus we obtain the following conclusion: For $v_1 \in V_{(m_1)}^{(\alpha_1)}, \ldots, v_r \in V_{(m_r)}^{(\alpha_r)}, w_{(3)} \in (W_3)_{[n_3]}^{(\beta_3)}$ and $w'_{(4)} \in (W'_4)_{[n_4]}^{(\beta_4)}$, the element

$$(v_1^o)_{m_1}\cdots(v_r^o)_{m_r}\beta_{n;i,j}(w'_{(4)},w_{(3)})$$

is homogeneous of (generalized) weight

 $-(\text{wt } v_1 - m_1 - 1) - \dots - (\text{wt } v_r - m_r - 1) + \text{wt } w'_{(4)} + n + 1 - \text{wt } w_{(3)}$ (11.39)

and of \tilde{A} -degree

$$\alpha_1 + \dots + \alpha_r + \beta_3 + \beta_4,$$

and we have

$$(v_1^o)_{m_1} \cdots (v_r^o)_{m_r} \beta_{n;i,j}(w'_{(4)}, w_{(3)}) = 0$$

when wt $(v_1^o)_{m_1} \cdots (v_r^o)_{m_r} \beta_{n;i,j}(w'_{(4)}, w_{(3)}) \le N_{\alpha_1 + \dots + \alpha_r + \beta_3 + \beta_4}.$ (11.40)

In particular, the (generalized) weight of $\beta_{n;i,j}(w'_{(4)}, w_{(3)})$ is wt $w'_{(4)} + n + 1 - \text{wt } w_{(3)}$. For $n \in \mathbb{R}$, let $\lambda_n^{(2)}(w'_{(4)}, w_{(3)}) \in (W_1 \otimes W_2)^*$ be defined by

for homogeneous $w'_{(4)} \in W'_4$ and $w_{(3)} \in W_3$. Then

$$\sum_{n \in \mathbb{R}} (\lambda_n^{(2)}(w'_{(4)}, w_{(3)}))(w_{(1)} \otimes w_{(2)})$$

is absolutely convergent to $\mu_{(I_1 \circ (1_{W_2} \otimes I_2))'(w'_{(4)}),w_{(3)}}^{(2)}(w_{(1)} \otimes w_{(2)})$ for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. Since (11.38) holds for $n \in \mathbb{R}$ and $i, j \in \mathbb{N}$,

$$(L'_{P(z_1-z_2)}(0)-n)^N \lambda_n^{(2)}(w'_{(4)},w_{(3)}) = 0$$

for $n \in \mathbb{R}$.

Moreover, by (11.41) and (11.38),

$$\sum_{n \in \mathbb{R}} (e^{z'L'_{P(z_1-z_2)}(0)} \lambda_n^{(2)}(w'_{(4)}, w_{(3)}))(w_{(1)} \otimes w_{(2)})$$

$$= \sum_{n \in \mathbb{R}} \sum_{i,j \in \mathbb{N}} (e^{z'L'_{P(z_1-z_2)}(0)} \beta_{n-\text{wt}} w'_{(4)} - 1 + \text{wt} w_{(3)}; i, j}(w'_{(4)}, w_{(3)}))(w_{(1)} \otimes w_{(2)}) \cdot \cdot e^{(n-\text{wt}} w'_{(4)} + \text{wt} w_{(3)})\log(z_1 - z_2)} e^{(-n+\text{wt}} w'_{(4)} - \text{wt} w_{(3)})\log z_2} (\log z_2)^i (\log(z_1 - z_2))^j$$

$$= \sum_{n \in \mathbb{R}} \sum_{i,j \in \mathbb{N}} \sum_{l=0}^{N-1} \frac{1}{l!} ((L'_{P(z_1-z_2)}(0) - n)^l \beta_{n-\text{wt}} w'_{(4)} - 1 + \text{wt} w_{(3)}; i, j}(w'_{(4)}, w_{(3)}))(w_{(1)} \otimes w_{(2)}) \cdot \cdot e^{nz'}(z')^l e^{(n-\text{wt}} w'_{(4)} + \text{wt} w_{(3)})\log(z_1 - z_2)} e^{(-n+\text{wt}} w'_{(4)} - \text{wt} w_{(3)})\log z_2} (\log z_2)^i (\log(z_1 - z_2))^j.$$
(11.42)

By (11.37), for $n \in \mathbb{R}$, $i, j \in \mathbb{N}$ and $l = 0, \dots, N - 1$,

$$\left(\left(L'_{P(z_1-z_2)}(0)-n\right)^l\beta_{n-\operatorname{wt} w'_{(4)}-1+\operatorname{wt} w_{(3)};i,j}(w'_{(4)},w_{(3)})\right)(w_{(1)}\otimes w_{(2)})\right)$$

is a linear combination of

$$(\beta_{n-\text{wt }w'_{(4)}-1+\text{wt }w_{(3)};i',j}((L'(0)-\text{wt }w'_{(4)})^pw'_{(4)},(L(0)-\text{wt }w_{(3)})^qw_{(3)}))(w_{(1)}\otimes w_{(2)})$$

for $i', p, q \in \mathbb{N}$ such that $p, q \leq N - 1$. For $p, q \in \mathbb{N}$,

$$\sum_{i,j\in\mathbb{N}}\sum_{n\in\mathbb{R}} (\beta_{n;i,j}((L'(0) - \operatorname{wt} w'_{(4)})^{p}w'_{(4)}, (L(0) - \operatorname{wt} w_{(3)})^{q}w_{(3)}))(w_{(1)}\otimes w_{(2)}) \cdot e^{(n+1)\log(z_{1}-z_{2})}e^{(-n-1)((\log z_{2})-z')}((\log z_{2})-z')^{i}(\log(z_{1}-z_{2}))^{j}$$
$$= \sum_{i,j\in\mathbb{N}}\sum_{n\in\mathbb{R}}a_{n;i,j}((L'(0) - \operatorname{wt} w'_{(4)})^{p}w'_{(4)}, w_{(1)}, w_{(2)}, (L(0) - \operatorname{wt} w_{(3)})^{q}w_{(3)}) \cdot e^{(\Delta+n+1)\log(z_{1}-z_{2})}e^{(-n-1)((\log z_{2})-z')}((\log z_{2})-z')^{i}(\log(z_{1}-z_{2}))^{j}$$

is absolutely convergent when z' is in a sufficiently small neighborhood of z' = 0 such that $|e^{z'}z_2| > |z_0| > 0$. In particular, for $i, j, l \in \mathbb{N}$ and $p, q \in \mathbb{N}$ such that $p, q \leq N - 1$,

$$\sum_{n \in \mathbb{R}} \beta_{n-\operatorname{wt} w'_{(4)}-1+\operatorname{wt} w_{(3)}; i, j} ((L'(0) - \operatorname{wt} w'_{(4)})^p w'_{(4)}, (L(0) - \operatorname{wt} w^q_{(3)} w_{(3)}))(w_{(1)} \otimes w_{(2)}) \cdot e^{nz'}(z')^l e^{(n-\operatorname{wt} w'_{(4)}+\operatorname{wt} w_{(3)})\log(z_1-z_2)} e^{(-n+\operatorname{wt} w'_{(4)}-\operatorname{wt} w_{(3)})\log z_2} (\log z_2)^i (\log(z_1-z_2))^j$$

is absolutely convergent in the same neighborhood of z' = 0. Thus the right-hand side of (11.42) is absolutely convergent. Hence the left-hand side of (11.42) is absolutely convergent, completing the proof that $(I_1 \circ (1_{W_2} \otimes F_2))'(w'_{(4)})$ satisfies Part (a) of the $P^{(2)}(z_1 - z_2)$ -local grading restriction condition.

Recall the space W_{λ} for $\lambda \in (W_1 \otimes W_2)^*$ in the P(z)-local grading restriction condition in Section 5 and the space $W^{(2)}_{\lambda,w_{(3)}}$ for $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ and $w_{(3)} \in W_3$ in the $P^{(2)}(z)$ -local grading restriction condition in Section 9. For fixed $n \in \mathbb{R}, w_{(3)} \in (W_3)_{[n_3]}^{(\beta_3)}$ and $w'_{(4)} \in (W'_4)^{(\beta_4)}_{[n_4]}$, (11.39) and (11.40) show that for $\beta \in \tilde{A}$, the homogeneous subspace $(W_{\lambda_n^{(2)}(w'_{(4)},w_{(3)})})_{[l]}^{(\beta)} \text{ of } W_{\lambda_n^{(2)}(w'_{(4)},w_{(3)})} \text{ is } 0 \text{ when } l \leq N_{\beta}. \text{ In particular, each } \lambda_n^{(2)}(w'_{(4)},w_{(3)}),$ $n \in \mathbb{R}$, satisfies the $P(z_1 - z_2)$ -lower truncation condition. By Theorem 9.17, $W_{\lambda_n^{(2)}(w'_{(4)}, w_{(3)})}$ is a doubly-graded generalized V-module (a doubly-graded V-module when \mathcal{C} is in \mathcal{M}_{sg}). Since for $\beta \in \tilde{A}$, $(W_{\lambda_n^{(2)}(w'_{(\ell)},w_{(3)})})_{[l]}^{(\beta)} = 0$ when $l \leq N_{\beta}$, it is in fact lower bounded. Thus $W_{\lambda_n^{(2)}(w'_{(4)},w_{(3)})}$, generated by $\lambda_n^{(2)}(w'_{(4)},w_{(3)})$, is a finitely-generated lower bounded doublygraded generalized V-module (or V-module), and so by assumption it is in fact an object of \mathcal{C} . Thus $W^{(2)}_{(I_1 \circ (1_{W_2} \otimes I_2))'(w'_{(4)}),w_{(3)}}$, as a sum of these objects of \mathcal{C} , lies in $W_1 \boxtimes_{P(z_1-z_2)} W_2$, which, by assumption, is an object of \mathcal{C} . Since \mathcal{C} is a subcategory of \mathcal{GM}_{sg} , $W^{(2)}_{(I_1 \circ (1_{W_2} \otimes I_2))'(w'_{(4)}),w_{(3)}}$, as a generalized V-module of $W_1 \square_{P(z_1-z_2)} W_2$, satisfies the two grading-restriction conditions, proving that the element $I_1 \circ (1_{W_2} \otimes I_2))'(w'_{(4)})$ of $(W_1 \otimes W_2 \otimes W_3)^*$ satisfies the $P^{(2)}(z_1 - z_2)$ local grading-restriction condition.

11.2 Differential equations

In this section, we assume for simplicity that A and A are trivial. We would like to emphasize that this assumption is not essential since all the results can be generalized to the case that Aand \tilde{A} are not trivial. To avoid spending too many pages to straighforwardly generalize many definitions and results to the general case (for example, the definition of C_1 -cofiniteness), and to allow the use of most of the arguments in [H2], for which both A and \tilde{A} are trivial, we choose to simply assume that A and \tilde{A} are trivial.

We use differential equations to prove the convergence and extension property, given in the preceding section, when the objects of C satisfy natural conditions. The results and the proofs here are in fact the same as those in [H2], except that in this section, we consider objects of C, not just ordinary V-modules, and we consider logarithmic intertwining operators, not just ordinary intertwining operators. So in the proofs of the results in this section, we shall indicate only how the proofs here differ from the corresponding ones in [H2] and refer the reader to [H2] for more details.

Let V be a Möbius or conformal vertex algebra with A the trivial group and let

$$V_+ = \coprod_{n>0} V_{(n)}$$

For a generalized V-module W, set

$$C_1(W) = \operatorname{span}\{u_{-1}w \mid u \in V_+, w \in W\}$$

Definition 11.5 If $W/C_1(W)$ is finite dimensional, we say that W is C_1 -cofinite or satisfies the C_1 -cofiniteness condition. If for any $N \in \mathbb{R}$, $\prod_{n < N} W_{[n]}$ is finite dimensional, we say that W is quasi-finite dimensional or satisfies the quasi-finite-dimensionality condition.

We have:

Theorem 11.6 Let W_i for i = 0, ..., n + 1 be generalized V-modules satisfying the C_1 cofiniteness condition and the quasi-finite-dimensionality condition. Then for any $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1, ..., w_{(n+1)} \in W_{n+1}$, there exist

$$a_{k,l}(z_1,\ldots,z_n) \in \mathbb{C}[z_1^{\pm 1},\ldots,z_n^{\pm 1},(z_1-z_2)^{-1},(z_1-z_3)^{-1},\ldots,(z_{n-1}-z_n)^{-1}]$$

for k = 1, ..., m and l = 1, ..., n, such that the following holds: For any generalized V-modules $\widetilde{W}_1, ..., \widetilde{W}_{n-1}$, and any logarithmic intertwining operators

$$\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_{n-1}, \mathcal{Y}_n$$

of types

$$\binom{W_0}{W_1\widetilde{W}_1}, \binom{\widetilde{W}_1}{W_2\widetilde{W}_2}, \dots, \binom{\widetilde{W}_{n-2}}{W_{n-1}\widetilde{W}_{n-1}}, \binom{\widetilde{W}_{n-1}}{W_nW_{n+1}},$$

respectively, the series

$$\langle w'_{(0)}, \mathcal{Y}_1(w_{(1)}, z_1) \cdots \mathcal{Y}_n(w_{(n)}, z_n) w_{(n+1)} \rangle$$
 (11.43)

satisfies the system of differential equations

$$\frac{\partial^m \varphi}{\partial z_l^m} + \sum_{k=1}^m \iota_{|z_1| > \dots > |z_n| > 0} (a_{k,l}(z_1, \dots, z_n)) \frac{\partial^{m-k} \varphi}{\partial z_l^{m-k}} = 0, \quad l = 1, \dots, n$$

in the region $|z_1| > \cdots > |z_n| > 0$, where

$$\iota_{|z_1| > \cdots > |z_n| > 0}(a_{k, l}(z_1, \dots, z_n))$$

for k = 1, ..., m and l = 1, ..., n are the (unique) Laurent expansions of $a_{k, l}(z_1, ..., z_n)$ in the region $|z_1| > \cdots > |z_n| > 0$. Moreover, for any set of possible singular points of the system

$$\frac{\partial^m \varphi}{\partial z_l^m} + \sum_{k=1}^m a_{k,l}(z_1, \dots, z_n) \frac{\partial^{m-k} \varphi}{\partial z_l^{m-k}} = 0, \quad l = 1, \dots, n$$
(11.44)

such that either $z_i = 0$ or $z_i = \infty$ for some i or $z_i = z_j$ for some $i \neq j$, the $a_{k,l}(z_1, \ldots, z_n)$ can be chosen for $k = 1, \ldots, m$ and $l = 1, \ldots, n$ so that these singular points are regular.

Proof Proposition 1.1, Corollary 1.2 and Corollary 1.3 in [H2] and their proofs still hold when all V-modules are replaced by strongly-graded generalized V-modules. Theorem 1.4 in [H2] also still holds, except that in its proof all V-modules are replaced by strongly-graded generalized V-modules and the spaces

$$z_1^{\Delta} \mathbb{C}(\{z_2/z_1\})[z_1^{\pm 1}, z_2^{\pm 1}],$$

$$z_2^{\Delta} \mathbb{C}(\{(z_1-z_2)/z_1\})[z_2^{\pm 1}, (z_1-z_2)^{\pm 1}],$$

$$z_2^{\Delta} \mathbb{C}(\{z_1/z_2\})[z_1^{\pm 1}, z_2^{\pm 1}]$$

are replaced by

$$z_{1}^{\Delta}\mathbb{C}(\{z_{2}/z_{1}\})[z_{1}^{\pm1}, z_{2}^{\pm1}][\log z_{1}, \log z_{2}],$$

$$z_{2}^{\Delta}\mathbb{C}(\{(z_{1}-z_{2})/z_{1}\})[z_{2}^{\pm1}, (z_{1}-z_{2})^{\pm1}][\log z_{2}, \log(z_{1}-z_{2})],$$

$$z_{2}^{\Delta}\mathbb{C}(\{z_{1}/z_{2}\})[z_{1}^{\pm1}, z_{2}^{\pm1}][\log z_{1}, \log z_{2}],$$

respectively. Theorem 1.6 and its proof also still hold in our setting here. Thus the first conclusion holds.

Proposition 2.1, Lemma 2.2, Theorem 2.3, Remark 2.4 and Theorem 2.5 in [H2] and their proofs still hold here. Thus the second conclusion holds. \Box

Remark 11.7 In [H2] and in Theorem 11.6 above, we are using the definition of the notion of system of differential equations with regular singular points (or regular singularities) given in Appendix B of the book [Kn]. This definition defines such a system in terms of the form of the expansions of the solutions at the singular points. What we need in the present section is precisely this form of the expansions of the solutions of such a system. However, we warn the reader that in some books and papers, a system of differential equations with regular singular points is instead defined in terms of the coefficients of the system. In fact, there are systems with regular singular points that do not satisfy the definition in terms of the coefficients of the system. See for example [Lut] for a study of the problem of analyzing under what conditions the singular points of a system of differential equations that do not satisfy the regularity condition in terms of the coefficients are still regular. In Section B.5 of [Kn], systems with simple singular points (called simple sigularities there and defined in terms of the coefficients) are discussed and it is proved there that simple singular points are indeed regular singular points. From Theorem B.16 in Section B.5 of [Kn], together with (2.2) and (2.3) in [H2] and their obvious extensions to the case of more than two intertwining operators, it is clear that for any set of possible of singular points of (11.44) of any of the indicated types, the $a_{k,l}(z_1,\ldots,z_n)$ can be chosen for $k=1,\ldots,m$ and $l=1,\ldots,n$ so that these singular points are simple singular points and thus are regular singular points.

We now have:

Theorem 11.8 Suppose that all generalized V-modules in C satisfy the C_1 -cofiniteness condition and the quasi-finite-dimensionality condition. Then:

- 1. The convergence and extension properties for products and iterates hold in C. If C is in \mathcal{M}_{sg} and if every object of C is a direct sum of irreducible objects of C and there are only finitely many irreducible objects of C (up to equivalence), then the convergence and extension properties without logarithms for products and iterates hold in C.
- 2. For any $n \in \mathbb{Z}_+$, any objects W_1, \ldots, W_{n+1} and $\widetilde{W}_1, \ldots, \widetilde{W}_{n-1}$ of \mathcal{C} , any logarithmic intertwining operators

$$\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_{n-1}, \mathcal{Y}_n$$

of types

$$\binom{W_0}{W_1\widetilde{W}_1}, \binom{\widetilde{W}_1}{W_2\widetilde{W}_2}, \dots, \binom{\widetilde{W}_{n-2}}{W_{n-1}\widetilde{W}_{n-1}}, \binom{\widetilde{W}_{n-1}}{W_nW_{n+1}}$$

respectively, and any $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1, \ldots, w_{(n+1)} \in W_{n+1}$, the series

$$\langle w'_{(0)}, \mathcal{Y}_1(w_{(1)}, z_1) \cdots \mathcal{Y}_n(w_{(n)}, z_n) w_{(n+1)} \rangle$$
 (11.45)

is absolutely convergent in the region $|z_1| > \cdots > |z_n| > 0$ and its sum can be analytically extended to a multivalued analytic function on the region given by $z_i \neq 0$, $i = 1, \ldots, n, z_i \neq z_j, i \neq j$, such that for any set of possible singular points with either $z_i = 0, z_i = \infty$ or $z_i = z_j$ for $i \neq j$, this multivalued analytic function can be expanded near the singularity as a series having the same form as the expansion near the singular points of a solution of a system of differential equations with regular singular points (as defined in Appendix B of [Kn]; recall Remark 11.7).

Proof The first statement in the first part, except for the existence of the upper bound K for i_k , and the statement in the second part of the theorem follow directly from Theorem 11.6 and the theory of differential equations with regular singular points. To prove the existence of K, note that in [H2], T/J is a finitely generated R-module and the map ϕ_{y_1,y_2} with domain T induces a map $\bar{\phi}_{y_1,y_2}$. Since T/J such that in particular the image of ϕ_{y_1,y_2} is equal to the image of $\bar{\phi}_{y_1,y_2}$. Since T/J is a finitely-generated $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, (z_1 - z_2)^{-1}]$ -module, its image under $\bar{\phi}_{y_1,y_2}$ is also a finitely-generated $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, (z_1 - z_2)^{-1}]$ -module. Any set of generators of this image has been proved to have analytic extensions in the region $|z_2| > |z_1 - z_2| > 0$ of the form (11.4), and for finitely many such generators, we have an upper bound K for the powers of $\log z_2$ in their analytic extensions in this region. Thus the powers of $\log z_2$ in the analytic extension of any element of the image of the map ϕ_{y_1,y_2} in this region, as a linear combination of the analytic extensions of these generators with coefficients in $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, (z_1 - z_2)^{-1}]$, are also less than K.

The second statement in the first part was in fact proved in [H2]. \Box

Remark 11.9 Note that the first statement in the first part of Theorem 11.8 follows immediately from Theorem 11.6. To prove the second statement in the first part of Theorem 11.8, we start with the product of two intertwining operators without logarithms in the region $|z_1| > |z_2| > 0$, and we have to prove that its analytic extension in the region $|z_2| > |z_1 - z_2| > 0$ has no terms involving logarithms. This is the main hard part of the proof of Theorem 3.5 in [H2].

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