Logarithmic tensor category theory, V: Convergence condition for intertwining maps and the corresponding compatibility condition

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Abstract

This is the fifth part in a series of papers in which we introduce and develop a natural, general tensor category theory for suitable module categories for a vertex (operator) algebra. In this paper (Part V), we study products and iterates of intertwining maps and of logarithmic intertwining operators and we begin the development of our analytic approach.

Contents

In this paper, Part V of a series of eight papers on logarithmic tensor category theory, we study products and iterates of intertwining maps and of logarithmic intertwining operators and we begin the development of our analytic approach. The sections, equations, theorems and so on are numbered globally in the series of papers rather than within each paper, so that for example equation (a.b) is the b-th labeled equation in Section a, which is contained in the paper indicated as follows: In Part I [HLZ1], which contains Sections 1 and 2, we give a detailed overview of our theory, state our main results and introduce the basic objects that we shall study in this work. We include a brief discussion of some of the recent applications of this theory, and also a discussion of some recent literature. In Part II [HLZ2], which contains Section 3, we develop logarithmic formal calculus and study logarithmic intertwining operators. In Part III [HLZ3], which contains Section 4, we introduce and study intertwining maps and tensor product bifunctors. In Part IV [HLZ4], which contains Sections 5 and 6, we give constructions of the $P(z)$- and $Q(z)$-tensor product bifunctors using what we call "compatibility conditions" and certain other conditions. The present paper, Part V, contains Sections 7 and 8. In Part VI [HLZ5], which contains Sections 9 and 10, we construct the appropriate natural associativity isomorphisms between triple tensor product functors. In Part VII [HLZ6], which contains Section 11, we give sufficient conditions for the existence of the associativity isomorphisms. In Part VIII [HLZ7], which contains Section 12, we construct braided tensor category structure.

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7 The convergence condition for intertwining maps and convergence and analyticity for logarithmic intertwining operators

Now that we have constructed tensor product modules and functors, our next goal is to construct natural associativity isomorphisms for our category $\mathcal{C}$ (recall Assumption 5.30). More precisely, under suitable conditions, we shall construct a natural isomorphism between two functors from $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$, one given by

$$(W_1, W_2, W_3) \mapsto (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3,$$

and the other by

$$(W_1, W_2, W_3) \mapsto W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3),$$

where W_1, W_2 and W_3 are objects of $\mathcal{C}$ and z_1 and z_2 are suitable complex numbers. This will give us natural module isomorphisms

$$\alpha_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)} : (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \to W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$$

and their inverses, which we will call the "associativity isomorphisms." We have seen that geometric data plays a crucial role in the tensor product itself, and we will see that it continues to be a crucial ingredient in the construction of the associativity isomorphisms.

We will mainly follow, and considerably generalize, the ideas developed in [H2], and it will be natural for us to work only in the case where all tensor products involved are of type $P(z)$, for various nonzero complex numbers z (recall Remark 2.4).

As we have stated in Sections 4 and 5, in the remainder of this work, in particular in this section, Assumptions 4.1 and 5.30 hold. We shall also introduce a new one, Assumption 7.11, in this section.

In this section we study one of the prerequisites for the existence of the associativity isomorphisms. As we have discussed in Section 1.4, in order to construct the associativity isomorphisms between tensor products of three objects, we must have that the intertwining maps involved are "composable," which means that certain convergence conditions have to be satisfied. We formulate two such conditions—one for products of suitable intertwining maps and the other for iterates—and we prove their equivalence (Proposition 7.3); we call the resulting single condition the "convergence condition for intertwining maps" (Definition 7.4).

Then we develop crucial analytic principles, including Proposition 7.8 on what we call "unique expansion sets" (Definition 7.5), and Proposition 7.9 and Corollary 7.10 on ensuring absolute convergence of double sums involving powers of both z and $\log z$. These principles enable us to uniquely determine the coefficients of the monomials in suitable variables and their logarithms obtained from products and iterates of logarithmic intertwining operators. We establish the fundamental analyticity properties of suitably-evaluated products and iterates of logarithmic intertwining operators we derive consequences of this analyticity.

More precisely, we first need to consider the composition of a $P(z_1)$-intertwining map and a $P(z_2)$-intertwining map for suitable nonzero complex numbers z_1 and z_2. Geometrically, these compositions correspond to sewing operations (see [H1] and [H3]) of Riemann surfaces with punctures and local coordinates. Compositions (that is, products and iterates) of maps of this type have been defined in [H2] for intertwining maps among ordinary modules. The same definitions carry over to the greater generality of this work:

Recall from Definition 4.2 the space

$$\mathcal{M}[P(z)]^{W_3}_{W_1W_2}$$

of $P(z)$-intertwining maps of type $\binom{W_3}{W_1W_2}$ for $z \in \mathbb{C}^\times$ and W_1, W_2, W_3 objects of $\mathcal{C}$. Let W_1, W_2, W_3, W_4 and M_1 be objects of $\mathcal{C}$. Let $z_1, z_2 \in \mathbb{C}^\times$, $I_1 \in \mathcal{M}[P(z_1)]^{W_4}_{W_1M_1}$ and $I_2 \in \mathcal{M}[P(z_2)]^{M_1}_{W_2W_3}$. If for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$, the series

$$\sum_{n\in\mathbb{C}} \langle w'_{(4)}, I_1(w_{(1)} \otimes \pi_n(I_2(w_{(2)} \otimes w_{(3)})))\rangle_{W_4} \tag{7.1}$$

(recall the notation π_n from (2.44) and Definition 2.18 and note that $\pi_n(I_2(w_{(2)}\otimes w_{(3)})) \in M_1$) is absolutely convergent, then the sums of these series give a linear map

$$W_1 \otimes W_2 \otimes W_3 \to (W'_4)^*.$$

Recalling the arguments in Lemmas 4.41 and 5.17, we see that the image of this map is actually in $\overline{W_4}$, so that we obtain a linear map

$$W_1 \otimes W_2 \otimes W_3 \to \overline{W_4}.$$

Analogously, let W_1, W_2, W_3, W_4 and M_2 be objects of $\mathcal{C}$. let $z_2, z_0 \in \mathbb{C}^\times$, $I^1 \in \mathcal{M}[P(z_2)]^{W_4}_{M_2W_3}$ and $I^2 \in \mathcal{M}[P(z_0)]^{M_2}_{W_1W_2}$. If for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$, the series

$$\sum_{n\in\mathbb{C}} \langle w'_{(4)}, I^1(\pi_n(I^2(w_{(1)} \otimes w_{(2)})) \otimes w_{(3)})\rangle_{W_4} \tag{7.2}$$

is absolutely convergent, then the sums of these series also give a linear map

$$W_1 \otimes W_2 \otimes W_3 \to \overline{W_4}.$$

Definition 7.1 Let W_1, W_2, W_3, W_4 and M_1 be objects of $\mathcal{C}$. Let $z_1, z_2 \in \mathbb{C}^\times$, $I_1 \in \mathcal{M}[P(z_1)]_{W_1 M_1}^{W_4}$ and $I_2 \in \mathcal{M}[P(z_2)]_{W_2 W_3}^{M_1}$. We say that *the product of* I_1 *and* I_2 *exists* if for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$, the series (7.1) is absolutely convergent. In this case, we denote the sum (7.1) by

$$\langle w'_{(4)}, I_1(w_{(1)} \otimes I_2(w_{(2)} \otimes w_{(3)}))\rangle. \tag{7.3}$$

We call the map

$$W_1 \otimes W_2 \otimes W_3 \to \overline{W}_4,$$

defined by (7.3) the *product* of I_1 and I_2 and denote it by

$$I_1 \circ (1_{W_1} \otimes I_2).$$

In particular, we have

$$\langle w'_{(4)}, I_1(w_{(1)} \otimes I_2(w_{(2)} \otimes w_{(3)}))\rangle = \langle w'_{(4)}, (I_1 \circ (1_{W_1} \otimes I_2))(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})\rangle.$$

Analogously, let W_1, W_2, W_3, W_4 and M_2 be objects of $\mathcal{C}$, and let $z_2, z_0 \in \mathbb{C}^\times$, $I^1 \in \mathcal{M}[P(z_2)]_{M_2 W_3}^{W_4}$ and $I^2 \in \mathcal{M}[P(z_0)]_{W_1 W_2}^{M_2}$. We say that *the iterate of* I^1 *and* I^2 *exists* if for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$, the series (7.2) is absolutely convergent. In this case, we denote the sum (7.2) by

$$\langle w'_{(4)}, I^1(I^2(w_{(1)} \otimes w_{(2)}) \otimes w_{(3)})\rangle \tag{7.4}$$

and we call the map

$$W_1 \otimes W_2 \otimes W_3 \to \overline{W}_4$$

defined by (7.4) the *iterate* of I^1 and I^2 and denote it by

$$I^1 \circ (I^2 \otimes 1_{W_3}).$$

In particular, we have

$$\langle w'_{(4)}, I^1(I^2(w_{(1)} \otimes w_{(2)}) \otimes w_{(3)})\rangle = \langle w'_{(4)}, (I^1 \circ (I^2 \otimes 1_{W_3}))(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})\rangle.$$

Remark 7.2 Note that from the grading compatibility condition (4.2) for $P(z)$-intertwining maps, the product and the iterate defined above, when they exist, also satisfy the following *grading compatibility conditions*: With the notation as in Definition 7.1, suppose that $w_{(1)} \in W_1^{(\beta)}$, $w_{(2)} \in W_2^{(\gamma)}$ and $w_{(3)} \in W_3^{(\delta)}$, where $\beta, \gamma, \delta \in \tilde{A}$. Then

$$(I_1 \circ (1_{W_1} \otimes I_2))(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \in \overline{W_4^{(\beta+\gamma+\delta)}}$$

if the product of I_1 and I_2 exists, and

$$(I^1 \circ (I^2 \otimes 1_{W_3}))(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \in \overline{W_4^{(\beta+\gamma+\delta)}}$$

if the iterate of I^1 and I^2 exists.

Proposition 7.3 *The following two conditions are equivalent:*

1. *Let W_1, W_2, W_3, W_4 and M_1 be arbitrary objects of $\mathcal{C}$ and let z_1 and z_2 be arbitrary nonzero complex numbers satisfying*
$$|z_1| > |z_2| > 0.$$
Then for any $I_1 \in \mathcal{M}[P(z_1)]_{W_1 M_1}^{W_4}$ and $I_2 \in \mathcal{M}[P(z_2)]_{W_2 W_3}^{M_1}$, the product of I_1 and I_2 exists.

2. *Let W_1, W_2, W_3, W_4 and M_2 be arbitrary objects of $\mathcal{C}$ and let z_0 and z_2 be arbitrary nonzero complex numbers satisfying*
$$|z_2| > |z_0| > 0.$$
Then for any $I^1 \in \mathcal{M}[P(z_2)]_{M_2 W_3}^{W_4}$ and $I^2 \in \mathcal{M}[P(z_0)]_{W_1 W_2}^{M_2}$, the iterate of I^1 and I^2 exists.

Proof We shall use the isomorphism Ω_0 given by (3.77) and its inverse Ω_{-1} (recall Proposition 3.44) to prove this result. Suppose that Condition 1 holds. Let z_0 and z_2 be any nonzero complex numbers. For any intertwining maps I^1 and I^2 as in the statement of Condition 2, let $\mathcal{Y}^1 = \mathcal{Y}_{I^1,0}$ and $\mathcal{Y}^2 = \mathcal{Y}_{I^2,0}$ be the logarithmic intertwining operators corresponding to I^1 and I^2, respectively, according to Proposition 4.8. We need to prove that when $|z_2| > |z_0| > 0$, the series (7.2), which can now be written as
$$\sum_{n\in\mathbb{C}} \left(\langle w'_{(4)}, \mathcal{Y}^1(\pi_n(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}), x_2)w_{(3)}\rangle_{W_4}\Big|_{x_0=z_0,\, x_2=z_2} \right) \tag{7.5}$$
(recall the "substitution" notation from (4.12), where we choose $p = 0$ for both substitutions), is absolutely convergent for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$.

Using the linear isomorphism $\Omega_{-1} : \mathcal{V}_{W_3 M_2}^{W_4} \to \mathcal{V}_{M_2 W_3}^{W_4}$ (see (3.77)),
$$\Omega_{-1}(\mathcal{Y})(w, x)w_{(3)} = e^{xL(-1)}\mathcal{Y}(w_{(3)}, e^{-\pi i}x)w,$$
for $\mathcal{Y} \in \mathcal{V}_{W_3\, M_2}^{W_4}$, $w \in M_2$ and $w_{(3)} \in W_3$, and its inverse $\Omega_0 : \mathcal{V}_{M_2 W_3}^{W_4} \to \mathcal{V}_{W_3 M_2}^{W_4}$, we have
$$\begin{aligned}
&\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle_{W_4}\\
&\quad = \langle w'_{(4)}, \Omega_{-1}(\Omega_0(\mathcal{Y}^1))(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle_{W_4}\\
&\quad = \langle w'_{(4)}, e^{x_2L(-1)}\Omega_0(\mathcal{Y}^1)(w_{(3)}, e^{-\pi i}x_2)\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}\rangle_{W_4}\\
&\quad = \langle e^{x_2L'(1)}w'_{(4)}, \Omega_0(\mathcal{Y}^1)(w_{(3)}, e^{-\pi i}x_2)\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}\rangle_{W_4}
\end{aligned} \tag{7.6}$$
for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. Hence for $n \in \mathbb{C}$,
$$\begin{aligned}
&\langle w'_{(4)}, \mathcal{Y}^1(\pi_n(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}), x_2)w_{(3)}\rangle_{W_4}\Big|_{x_0=z_0,\, x_2=z_2}\\
&\quad = \langle e^{x_2L'(1)}w'_{(4)}, \Omega_0(\mathcal{Y}^1)(w_{(3)}, e^{-\pi i}x_2)\pi_n(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)})\rangle_{W_4}\Big|_{x_0=z_0,\, x_2=z_2}\\
&\quad = \langle e^{z_2L'(1)}w'_{(4)}, \Omega_0(\mathcal{Y}^1)(w_{(3)}, x_2)\pi_n(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)})\rangle_{W_4}\Big|_{x_0=z_0,\, x_2=-z_2},
\end{aligned}$$

where in the last expression we take $p = 0$ (respectively, $p = -1$) in (4.12) and (4.15) for the substitution $x_2 = -z_2$ when $\pi \leq \arg z_2 < 2\pi$, in which case $\log(-z_2) = \log z_2 - \pi i$ (respectively, when $0 \leq \arg z_2 < \pi$, in which case $\log(-z_2) = \log z_2 + \pi i$); cf. the corresponding considerations in Example 4.28. For brevity, let us write this last expression as

$$\langle e^{z_2 L'(1)} w'_{(4)}, \Omega_0(\mathcal{Y}^1)(w_{(3)}, x_2)\pi_n(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)})\rangle_{W_4}\Big|_{x_0=z_0,\, x_2=e^{-\pi i}z_2};$$

that is, the substitution $x_2 = e^{-\pi i}z_2$ refers to the indicated procedure, which amounts to substituting

$$e^{\log z_2 - \pi i}$$

for x_2. Thus

$$\sum_{n\in\mathbb{C}} \left(\langle w'_{(4)}, \mathcal{Y}^1(\pi_n(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}), x_2)w_{(3)}\rangle_{W_4}\Big|_{x_0=z_0,\, x_2=z_2} \right)$$
$$= \sum_{n\in\mathbb{C}} \left(\langle e^{z_2 L'(1)} w'_{(4)}, \Omega_0(\mathcal{Y}^1)(w_{(3)}, x_2)\pi_n(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)})\rangle_{W_4}\Big|_{x_0=z_0,\, x_2=e^{-\pi i}z_2} \right). \tag{7.7}$$

Since the last expression is equal to the product of a $P(-z_2)$-intertwining map and a $P(z_0)$-intertwining map evaluated at $w_{(3)}\otimes w_{(1)}\otimes w_{(2)} \in W_3\otimes W_1\otimes W_2$ and paired with $e^{z_2 L'(1)}w'_{(4)} \in W'_4$, it converges absolutely when $|-z_2| > |z_0| > 0$, or equivalently, when $|z_2| > |z_0| > 0$.

Conversely, suppose that Condition 2 holds, and let z_1 and z_2 be any nonzero complex numbers. For any intertwining maps I_1 and I_2 as in the statement of Condition 1, let $\mathcal{Y}_1 = \mathcal{Y}_{I_1,0}$ and $\mathcal{Y}_2 = \mathcal{Y}_{I_2,0}$ be the logarithmic intertwining operators corresponding to I_1 and I_2, respectively. We need to prove that when $|z_1| > |z_2| > 0$, the series (7.1), which can now be written as

$$\sum_{n\in\mathbb{C}} \left(\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\pi_n(\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)})\rangle_{W_4}\Big|_{x_1=z_1,\, x_2=z_2} \right), \tag{7.8}$$

is absolutely convergent for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$.

Using the linear isomorphism $\Omega_0 : \mathcal{V}^{W_4}_{M_1W_1} \to \mathcal{V}^{W_4}_{W_1M_1}$,

$$\Omega_0(\mathcal{Y})(w_{(1)}, x)w = e^{xL(-1)}\mathcal{Y}(w, e^{\pi i}x)w_{(1)},$$

for $\mathcal{Y} \in \mathcal{V}^{W_4}_{M_1W_1}$, $w_{(1)} \in W_1$ and $w \in M_1$, and its inverse $\Omega_{-1} : \mathcal{V}^{W_4}_{W_1M_1} \to \mathcal{V}^{W_4}_{M_1W_1}$, we have

$$\begin{aligned}
&\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle_{W_4}\\
&\quad = \langle w'_{(4)}, \Omega_0(\Omega_{-1}(\mathcal{Y}_1))(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle_{W_4}\\
&\quad = \langle w'_{(4)}, e^{x_1L(-1)}\Omega_{-1}(\mathcal{Y}_1)(\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}, e^{\pi i}x_1)w_{(1)}\rangle_{W_4}\\
&\quad = \langle e^{x_1L'(1)}w'_{(4)}, \Omega_{-1}(\mathcal{Y}_1)(\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}, e^{\pi i}x_1)w_{(1)}\rangle_{W_4}
\end{aligned} \tag{7.9}$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. Hence for $n \in \mathbb{C}$,

$$\begin{aligned}
&\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\pi_n(\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)})\rangle_{W_4}\Big|_{x_1=z_1,\ x_2=z_2}\\
&\quad = \langle e^{x_1 L'(1)} w'_{(4)}, \Omega_{-1}(\mathcal{Y}_1)(\pi_n(\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}), e^{\pi i}x_1)w_{(1)}\rangle_{W_4}\Big|_{x_1=z_1,\ x_2=z_2}\\
&\quad = \langle e^{z_1 L'(1)} w'_{(4)}, \Omega_{-1}(\mathcal{Y}_1)(\pi_n(\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}), x_1)w_{(1)}\rangle_{W_4}\Big|_{x_1=e^{\pi i}z_1,\ x_2=z_2},
\end{aligned}$$

where the substitution $x_1 = e^{\pi i}z_1$ is interpreted as above, namely, we substitute

$$e^{\log z_1 + \pi i}$$

for x_1; here $p = 0$ (respectively, $p = 1$) when $0 \leq \arg z_1 < \pi$ (respectively, when $\pi \leq \arg z_1 < 2\pi$) (cf. above). Thus

$$\begin{aligned}
&\sum_{n\in\mathbb{C}} \left(\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\pi_n(\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)})\rangle_{W_4}\Big|_{x_1=z_1,\ x_2=z_2} \right)\\
&\quad = \sum_{n\in\mathbb{C}} \left(\langle e^{z_1 L'(1)} w'_{(4)}, \Omega_{-1}(\mathcal{Y}_1)(\pi_n(\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}), x_1)w_{(1)}\rangle_{W_4}\Big|_{x_1=e^{\pi i}z_1,\ x_2=z_2} \right).
\end{aligned}\tag{7.10}$$

Since the last expression is equal to the iterate of a $P(-z_1)$-intertwining map and a $P(z_2)$-intertwining map evaluated at $w_{(2)} \otimes w_{(3)} \otimes w_{(1)} \in W_2 \otimes W_3 \otimes W_1$ and paired with $e^{z_1 L'(1)} w'_{(4)} \in W'_4$, it converges absolutely when $|-z_1| > |z_2| > 0$, or equivalently, when $|z_1| > |z_2| > 0$. □

For convenience, we shall use the notations

$$\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle_{W_4}\Big|_{x_0^n=e^{nl_p(z_0)},\ \log x_0=l_p(z_0),\ \ x_2^n=e^{nl_q(z_2)},\ \log x_2=l_q(z_2)} \tag{7.11}$$

and

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle_{W_4}\Big|_{x_1^n=e^{nl_p(z_1)},\ \log x_1=l_p(z_1),\ \ x_2^n=e^{nl_q(z_2)},\ \log x_2=l_q(z_2)} \tag{7.12}$$

to denote

$$\sum_{n\in\mathbb{C}} \left(\langle w'_{(4)}, \mathcal{Y}^1(\pi_n(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}), x_2)w_{(3)}\rangle_{W_4}\Big|_{x_0^n=e^{nl_p(z_0)},\ \log x_0=l_p(z_0),\ \ x_2^n=e^{nl_q(z_2)},\ \log x_2=l_q(z_2)} \right)$$

and

$$\sum_{n\in\mathbb{C}} \left(\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\pi_n(\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)})\rangle_{W_4}\Big|_{x_1^n=e^{nl_p(z_1)},\ \log x_1=l_p(z_1),\ \ x_2^n=e^{nl_q(z_2)},\ \log x_2=l_q(z_2)} \right),$$

respectively. We shall further use the notations

$$\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle_{W_4}\Big|_{x_0=z_0,\, x_2=z_2} \tag{7.13}$$

and

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle_{W_4}\Big|_{x_1=z_1,\, x_2=z_2}, \tag{7.14}$$

or even more simply, the notations

$$\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, z_0)w_{(2)}, z_2)w_{(3)}\rangle_{W_4} \tag{7.15}$$

and

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, z_1)\mathcal{Y}_2(w_{(2)}, z_2)w_{(3)}\rangle_{W_4} \tag{7.16}$$

to denote (7.5) and (7.8), respectively, where we are taking $p = 0$ in the notation of (4.12) for both substitutions, except for occasions when we explicitly specify different values of p, such as in the proof above. We shall also use similar notations to denote series obtained from products and iterates of more than two intertwining operators.

Definition 7.4 We call either of the two equivalent conditions in Proposition 7.3 the *convergence condition for intertwining maps in the category* $\mathcal{C}$.

We need the following concept concerning unique expansion of an analytic function in terms of powers of z and $\log z$ (recall our choice of the branch of $\log z$ in (4.9) and thus the branch of z^α, $\alpha \in \mathbb{C}$):

Definition 7.5 We call a subset $\mathcal{S}$ of $\mathbb{C} \times \mathbb{C}$ a *unique expansion set* if the absolute convergence to 0 on some nonempty open subset of $\mathbb{C}^\times$ of any series

$$\sum_{(\alpha,\beta)\in\mathcal{S}} a_{\alpha,\beta} z^\alpha (\log z)^\beta, \quad a_{\alpha,\beta} \in \mathbb{C},$$

implies that $a_{\alpha,\beta} = 0$ for all $(\alpha, \beta) \in \mathcal{S}$.

Of course, a subset of a unique expansion set is again a unique expansion set.

Remark 7.6 It is easy to show that $\mathbb{Z} \times \{0, \dots, N\}$ is a unique expansion set for any $N \in \mathbb{N}$; this is also a consequence of Proposition 7.8 below. On the other hand, it is known that $\mathbb{C} \times \{0\}$ is *not* a unique expansion set[1].

For the reader's convenience, we give the following generalization of a standard result about Laurent series:

[1] We thank A. Eremenko for informing us of this result.

Lemma 7.7 *Let D be a subset of $\mathbb{R}$ and let*

$$\sum_{\alpha \in D} a_\alpha z^\alpha \quad (a_\alpha \in \mathbb{C})$$

be absolutely convergent on a (nonempty) open subset of $\mathbb{C}^\times$. Then

$$\sum_{\alpha \in D} a_\alpha \alpha z^\alpha$$

is absolutely and uniformly convergent near any z in the open subset. In particular, the sum $\sum_{\alpha \in D} a_\alpha z^\alpha$ as a function of z is analytic in the sense that it is analytic at z when z is in the open subset of $\mathbb{C}^\times$ and $\arg z > 0$, and that it can be analytically extended to an analytic function in a neighborhood of z when z is in the intersection of the open subset and the positive real line. More generally, let E be an index set and let the multisum

$$\sum_{\alpha \in D} \sum_{\beta \in E} a_{\alpha,\beta} z^\alpha \quad (a_{\alpha,\beta} \in \mathbb{C})$$

converge absolutely on a (nonempty) open subset of $\mathbb{C}^\times$. Then the conclusions above hold for the multisums $\sum_{\alpha \in D} \sum_{\beta \in E} a_{\alpha,\beta} \alpha z^\alpha$ and $\sum_{\alpha \in D} \sum_{\beta \in E} a_{\alpha,\beta} z^\alpha$.

Proof We prove only the case for the series $\sum_{\alpha \in D} a_\alpha z^\alpha$; the general case is completely analogous. We need only prove that $\sum_{\alpha \in D} a_\alpha \alpha z^\alpha$ is absolutely and uniformly convergent near any z in the open subset. Note that since the original series is absolutely convergent on an open subset of $\mathbb{C}^\times$, $\sum_{\alpha \in D,\, \alpha \geq 0} a_\alpha z^\alpha$ and $\sum_{\alpha \in D,\, \alpha < 0} a_\alpha z^\alpha$ are also absolutely convergent on the set. For any fixed z_0 in the set, we can always find z_1 and z_2 in the set such that $|z_1| < |z_0| < |z_2|$ and both $\sum_{\alpha \in D,\, \alpha \geq 0} a_\alpha z_2^\alpha$ and $\sum_{\alpha \in D,\, \alpha < 0} a_\alpha z_1^\alpha$ are absolutely convergent. Let r_1 and r_2 be numbers such that $|z_1| < r_1 < |z_0| < r_2 < |z_2|$. Since

$$\lim_{\alpha \to \infty} \sqrt[\alpha]{\alpha} = 1,$$

we can find $M > 0$ such that

$$\sqrt[\alpha]{\alpha} < \min\left(\frac{|z_2|}{r_2}, \frac{r_1}{|z_1|}\right)$$

when $\alpha > M$. But when $r_1 < |z| < r_2$,

$$\sqrt[\alpha]{\alpha} < \min\left(\frac{|z_2|}{r_2}, \frac{r_1}{|z_1|}\right) < \min\left(\frac{|z_2|}{|z|}, \frac{|z|}{|z_1|}\right)$$

for $\alpha > M$, so for z in the open subset and satisfying $r_1 < |z| < r_2$, we have

$$\begin{aligned} \sum_{\alpha \in D,\, \alpha > M} |a_\alpha \alpha z^\alpha| &= \sum_{\alpha \in D,\, \alpha > M} |a_\alpha| |\sqrt[\alpha]{\alpha} z|^\alpha \\ &\leq \sum_{\alpha \in D,\, \alpha > M} |a_\alpha z_2^\alpha| \end{aligned} \tag{7.17}$$

and

$$\begin{aligned}\sum_{\alpha\in D,\, \alpha<-M} |a_\alpha \alpha z^\alpha| &= \sum_{\alpha\in D,\, \alpha<-M} |a_\alpha| \left|\frac{\sqrt[-\alpha]{-\alpha}}{z}\right|^{-\alpha}\\ &\leq \sum_{\alpha\in D,\, \alpha<-M} |a_\alpha z_1^\alpha|. \end{aligned}\tag{7.18}$$

On the other hand, we have

$$\begin{aligned}\sum_{\alpha\in D,\, 0\leq\alpha\leq M} |a_\alpha \alpha z^\alpha| &\leq M \sum_{\alpha\in D,\, 0\leq\alpha\leq M} |a_\alpha z^\alpha|\\ &\leq M \sum_{\alpha\in D,\, 0\leq\alpha\leq M} |a_\alpha z_2^\alpha| \end{aligned}\tag{7.19}$$

and

$$\begin{aligned}\sum_{\alpha\in D,\, 0>\alpha\geq -M} |a_\alpha \alpha z^\alpha| &\leq M \sum_{\alpha\in D,\, 0>\alpha\geq -M} |a_\alpha z^\alpha|\\ &\leq M \sum_{\alpha\in D,\, 0>\alpha\geq -M} |a_\alpha z_1^\alpha|. \end{aligned}\tag{7.20}$$

From (7.17)–(7.20), we see that $\sum_{\alpha\in D} a_\alpha \alpha z^\alpha$ is absolutely and uniformly convergent in the neighborhood of z_0 consisting of z in the open subset satisfying $r_1 < |z| < r_2$. $\square$

Proposition 7.8 *For any $N\in\mathbb{N}$, $\mathbb{R}\times\{0,\dots,N\}$ is a unique expansion set. In particular, for any subset D of $\mathbb{R}$, $D\times\{0,\dots,N\}$ is a unique expansion set.*

Proof Let $a_{n,i}\in\mathbb{C}$ for $n\in\mathbb{R}$ and $i=0,\dots,N$, and suppose that

$$\sum_{n\in\mathbb{R}}\sum_{i=0}^{N} a_{n,i} z^n (\log z)^i = \sum_{n\in\mathbb{R}}\sum_{i=0}^{N} a_{n,i} e^{n\log z} (\log z)^i$$

is absolutely convergent to 0 for z in some nonempty open subset of $\mathbb{C}^\times$. We want to prove that each $a_{n,i}=0$.

Fix $n_0\in\mathbb{R}$. We shall prove that $a_{n_0,N}=0$, and thus the result will follow by induction on N; the case $N=0$ is a special case of the proof below.

On the given open set, both

$$\sum_{n\geq n_0}\sum_{i=0}^{N} a_{n,i} e^{(n-n_0)\log z} (\log z)^i$$

and

$$-\sum_{n<n_0}\sum_{i=0}^{N} a_{n,i} e^{(n-n_0)\log z} (\log z)^i$$

are absolutely convergent and we have

$$\sum_{n\geq n_0}\sum_{i=0}^{N} a_{n,i}e^{(n-n_0)\log z}(\log z)^i = -\sum_{n<n_0}\sum_{i=0}^{N} a_{n,i}e^{(n-n_0)\log z}(\log z)^i. \tag{7.21}$$

Moreover, deleting $z = 1$ from the open set if necessary, we observe that for each $i = 0, \ldots, N$, the series

$$\sum_{n\in\mathbb{R}} a_{n,i}e^{n\log z}$$

is absolutely convergent on our open set.

Choose z_1 and z_2 in the open set satisfying $|z_1| > |z_2| > 0$; then for each $i = 0, \ldots, N$, $\sum_{n\in\mathbb{R}} a_{n,i}e^{n\log z_1}$ and $\sum_{n\in\mathbb{R}} a_{n,i}e^{n\log z_2}$ are absolutely convergent.

Since for $z' \in \mathbb{C}$ satisfying $|e^{z'}| \leq |z_1|$ and $i = 0, \ldots, N$,

$$\begin{aligned}\sum_{n\geq n_0} |a_{n,i}||e^{(n-n_0)z'}| &= \sum_{n\geq n_0} |a_{n,i}||e^{z'}|^{n-n_0}\\ &\leq \sum_{n\geq n_0} |a_{n,i}||z_1|^{n-n_0},\end{aligned}$$

which is convergent, the series $\sum_{n\geq n_0} a_{n,i}e^{(n-n_0)z'}$ is absolutely convergent, and in particular, the series

$$\sum_{n\geq n_0} a_{n,i}e^{(n-n_0)\log z} \tag{7.22}$$

is absolutely convergent for $z \in \mathbb{C}^\times$ satisfying $|z| \leq |z_1|$. Thus by Lemma 7.7, (7.22) defines an analytic function on the region $0 < |z| < |z_1|$, with $0 \leq \arg z < 2\pi$. Hence we have a single-valued analytic function

$$f_1(z') = \sum_{n\geq n_0}\sum_{i=0}^{N} a_{n,i}e^{(n-n_0)z'}(z')^i$$

on the region $|e^{z'}| < |z_1|$, or equivalently, $\Re(z') < \log|z_1|$.

Similarly, for z' in the region $|e^{z'}| \geq |z_2|$ and $i = 0, \ldots, N$,

$$\begin{aligned}\sum_{n<n_0} |a_{n,i}||e^{(n-n_0)z'}| &= \sum_{n<n_0} |a_{n,i}||e^{z'}|^{n-n_0}\\ &\leq \sum_{n<n_0} |a_{n,i}||z_2|^{n-n_0},\end{aligned}$$

so that the series $-\sum_{n<n_0} a_{n,i}e^{(n-n_0)z'}$ is absolutely convergent. Thus the series

$$-\sum_{n<n_0} a_{n,i}e^{(n-n_0)\log z}$$

is absolutely convergent for $z \in \mathbb{C}^\times$ satisfying $|z| \geq |z_2|$, defining, as above, a multivalued analytic function on the region $|z| > |z_2|$ and hence a single-valued analytic function

$$f_2(z') = -\sum_{n<n_0} \sum_{i=0}^{N} a_{n,i} e^{(n-n_0)z'} (z')^i$$

on the region $|e^{z'}| > |z_2|$, or equivalently, $\Re(z') > \log|z_2|$.

We now define a single-valued analytic function $f(z')$ of z' on the whole plane $\mathbb{C}$ as follows: For z' satisfying $|z_2| < |e^{z'}| < |z_1|$, or equivalently, $\log|z_2| < \Re(z') < \log|z_1|$, we have $f_1(z') = f_2(z')$, since f_1 and f_2, defined and analytic on this region, agree on a nonempty open subset of this region, in view of (7.21). Thus we obtain a single-valued analytic function $f(z')$ defined on the whole z'-plane by

$$f(z') = \begin{cases} f_1(z'), & \Re(z') < \log|z_1| \\ f_2(z'), & \Re(z') > \log|z_2|. \end{cases}$$

When $\Re(z') < \log|z_1|$, we have

$$\begin{aligned} |f(z')| &\leq \sum_{n\geq n_0} \sum_{i=0}^{N} |a_{n,i}||e^{(n-n_0)z'}||z'|^i \\ &\leq \sum_{n\geq n_0} \sum_{i=0}^{N} |a_{n,i}||z_1|^{n-n_0}|z'|^i \\ &= \sum_{i=0}^{N} \left(\sum_{n\geq n_0} |a_{n,i}||z_1|^{n-n_0} \right) |z'|^i \end{aligned}$$

and when $\Re(z') > \log|z_2|$,

$$\begin{aligned} |f(z')| &\leq \sum_{n<n_0} \sum_{i=0}^{N} |a_{n,i}||e^{(n-n_0)z'}||z'|^i \\ &\leq \sum_{n<n_0} \sum_{i=0}^{N} |a_{n,i}||z_2|^{n-n_0}|z'|^i \\ &= \sum_{i=0}^{N} \left(\sum_{n<n_0} |a_{n,i}||z_2|^{n-n_0} \right) |z'|^i. \end{aligned}$$

Let

$$M_i = \max\left(\sum_{n\geq n_0} |a_{n,i}||z_1|^{n-n_0}, \sum_{n<n_0} |a_{n,i}||z_2|^{n-n_0} \right)$$

for $i = 0, \ldots, N$. Then for $z' \in \mathbb{C}$ with $|z'| \geq 1$,

$$|f(z')| \leq \sum_{i=0}^{N} M_i |z'|^i \leq \left(\sum_{i=0}^{N} M_i \right) |z'|^N,$$

so that $f(z')$ is a polynomial of degree at most N and in particular, $\lim_{z'\to\infty}(z')^{-N}f(z')$ exists.

We now take the limit of $(z')^{-N}f(z')$ as $z'\to\infty$ along the positive real line. Let $M > \max(0, \log|z_2|)$. When $z' \geq M$, $f(z') = f_2(z')$, and for such z',

$$\begin{aligned}\left|\sum_{n<n_0}\sum_{i=0}^{N} a_{n,i}e^{(n-n_0)z'}(z')^{i-N}\right| &\leq \sum_{n<n_0}\sum_{i=0}^{N}|a_{n,i}|e^{(n-n_0)z'}(z')^{i-N}\\ &\leq \sum_{n<n_0}\sum_{i=0}^{N}|a_{n,i}|e^{(n-n_0)M}M^{i-N}.\end{aligned}$$

Since the right-hand side is convergent, the series

$$-\sum_{n<n_0}\sum_{i=0}^{N} a_{n,i}e^{(n-n_0)z'}(z')^{i-N}$$

is uniformly convergent for $z' \geq M$. Thus

$$\begin{aligned}&\lim_{z'\geq M,\, z'\to\infty} -\sum_{n<n_0}\sum_{i=0}^{N} a_{n,i}e^{(n-n_0)z'}(z')^{i-N}\\ &= -\sum_{n<n_0}\sum_{i=0}^{N}\lim_{z'\geq M,\, z'\to\infty} a_{n,i}e^{(n-n_0)z'}(z')^{i-N}\\ &= 0\end{aligned}$$

and so

$$\lim_{z'\to\infty}(z')^{-N}f(z') = \lim_{z'>0,\, z'\to\infty}(z')^{-N}f(z') = 0. \tag{7.23}$$

Now let $M' < \min(0, \log|z_1|)$. When $z' \leq M'$, $f(z') = f_1(z')$, and for such z',

$$\begin{aligned}\left|\sum_{n\geq n_0}\sum_{i=0}^{N} a_{n,i}e^{(n-n_0)z'}(z')^{i-N}\right| &\leq \sum_{n\geq n_0}\sum_{i=0}^{N}|a_{n,i}|e^{(n-n_0)z'}(-z')^{i-N}\\ &\leq \sum_{n\geq n_0}\sum_{i=0}^{N}|a_{n,i}|e^{(n-n_0)M'}(-M')^{i-N}.\end{aligned}$$

Since the right-hand side is convergent, the series

$$\sum_{n\geq n_0}\sum_{i=0}^{N} a_{n,i}e^{(n-n_0)z'}(z')^{i-N}$$

is uniformly convergent for $z' \leq M'$. Thus

$$\begin{aligned}
&\lim_{z' \leq M',\, z' \to -\infty} \sum_{n \geq n_0} \sum_{i=0}^{N} a_{n,i} e^{(n-n_0)z'} (z')^{i-N} \\
&\quad = \sum_{n \geq n_0} \sum_{i=0}^{N} \lim_{z' \leq M',\, z' \to -\infty} a_{n,i} e^{(n-n_0)z'} (z')^{i-N} \\
&\quad = a_{n_0,N}
\end{aligned}$$

and so we also have

$$\lim_{z' \to \infty} (z')^{-N} f(z') = \lim_{z' < 0,\, z' \to -\infty} (z')^{-N} f(z') = a_{n_0,N}. \tag{7.24}$$

From (7.23) and (7.24), we obtain $a_{n_0,N} = 0$. $\square$

We will also need the following proposition and corollary, which ensure that certain double sums converge when the corresponding iterated sums and their derivatives converge:

Proposition 7.9 *Let D be a subset of $\mathbb{R}$ and N a nonnegative integer. Then the series*

$$\sum_{\alpha \in D} a_{\alpha,\beta} z^{\alpha} \tag{7.25}$$

for $\beta = 0, \dots, N$ are all absolutely convergent on some (nonempty) open subset of $\mathbb{C}^{\times}$ if and only if the series

$$\sum_{\alpha \in D} \left(\sum_{\beta=0}^{N} a_{\alpha,\beta} (\log z)^{\beta} \right) z^{\alpha} \tag{7.26}$$

and the corresponding series of first and higher derivatives with respect to z, viewed as series whose terms are the expressions

$$\left(\sum_{\beta=0}^{N} a_{\alpha,\beta} (\log z)^{\beta} \right) z^{\alpha}$$

and their derivatives with respect to z, are absolutely convergent on the same open subset. The series of derivatives of (7.26) have the same format as (7.26), except that for the n-th derivative, the outer sum is over the set $D - n$ and the inner sum has new coefficients in $\mathbb{C}$.

Proof The last assertion is clear.

Assume the absolute convergence of (7.25) for $\beta = 0, \dots, N$. Then the double series

$$\sum_{\alpha \in D} \sum_{\beta=0}^{N} a_{\alpha,\beta} z^{\alpha} (\log z)^{\beta} \tag{7.27}$$

is absolutely convergent. We also know that the absolute convergence of (7.27) and its (higher) derivatives implies the absolute convergence of (7.26) and its derivatives. But using Lemma 7.7 we see that the (higher) derivatives of (7.25) are absolutely convergent. Since the (higher) derivatives of

$$\sum_{\alpha\in D} a_{\alpha,\beta} z^{\alpha}(\log z)^{\beta} \tag{7.28}$$

are (finite) linear combinations of the (higher) derivatives of (7.25) with coefficients containing integer powers of $\log z$ and z, the (higher) derivatives of (7.28) are also absolutely convergent. Thus the (higher) derivatives of (7.27) are also absolutely convergent, and so (7.26) and its derivatives are absolutely convergent.

Conversely, assume that (7.26) and its derivatives are absolutely convergent. We need to show that (7.25) is absolutely convergent at any z_0 in the open subset. We consider the series

$$\sum_{\alpha\in D,\ \alpha\geq 0} \left(\sum_{\beta=0}^{N} a_{\alpha,\beta} z_2^{\beta}\right) z_1^{\alpha} \tag{7.29}$$

of functions

$$\left(\sum_{\beta=0}^{N} a_{\alpha,\beta} z_2^{\beta}\right) z_1^{\alpha}$$

in two variables z_1 and z_2. Since z_0 is in the open subset, we can find a smaller open subset inside the original one such that for z in this smaller one, $|z_0| < |z|$ and $|\log z_0| < |\log z|$. We know that the series (7.29) is absolutely convergent when $z_1 = z$, $z_2 = \log z$ and z is in the original open subset. For any z_1 and z_2 satisfying $0 < |z_1| < |z|$ and $z_2 = \log z$ where z is in the smaller open subset,

$$\sum_{\alpha\in D,\ \alpha\geq 0} \left|\sum_{\beta=0}^{N} a_{\alpha,\beta} z_2^{\beta}\right| |z_1^{\alpha}| \leq \sum_{\alpha\in D,\ \alpha\geq 0} \left|\sum_{\beta=0}^{N} a_{\alpha,\beta} (\log z)^{\beta}\right| |z^{\alpha}|$$

is convergent. So in this case (7.29) is absolutely convergent. Since for any fixed $z_2 = \log z$ where z is in the smaller open subset, the numbers z_1 satisfying $0 < |z_1| < |z|$ form an open subset, we can apply Lemma 7.7 to obtain that

$$\sum_{\alpha\in D,\ \alpha\geq 0} \frac{\partial}{\partial z_1} \left(\left(\sum_{\beta=0}^{N} a_{\alpha,\beta} z_2^{\beta}\right) z_1^{\alpha}\right) \tag{7.30}$$

is also absolutely convergent for any z_1 and z_2 satisfying $0 < |z_1| < |z|$ and $z_2 = \log z$ where z is in the smaller open subset.

Also, by assumption,

$$\sum_{\alpha\in D,\ \alpha\geq 0} \left(\frac{\partial}{\partial z_1} + \frac{1}{z_1}\frac{\partial}{\partial z_2}\right) \left(\left(\sum_{\beta=0}^{N} a_{\alpha,\beta} z_2^{\beta}\right) z_1^{\alpha}\right) \tag{7.31}$$

is absolutely convergent when $z_1 = z$ and $z_2 = \log z$ when z is in the original open subset. For z in the smaller open subset and any z_1 and z_2 satisfying $0 < |z_1| < |z|$ and $z_2 = \log z$,

$$\begin{aligned}
&\sum_{\alpha\in D,\ \alpha\geq 0}\left|\left(\frac{\partial}{\partial z_1}+\frac{1}{z_1}\frac{\partial}{\partial z_2}\right)\left(\left(\sum_{\beta=0}^{N}a_{\alpha,\beta}z_2^{\beta}\right)z_1^{\alpha}\right)\right|\\
&\quad=\sum_{\alpha\in D,\ \alpha\geq 0}\left|\left(\left(\sum_{\beta=0}^{N}a_{\alpha,\beta}z_2^{\beta}\right)\alpha z_1^{\alpha-1}\right)+\left(\left(\sum_{\beta=0}^{N}a_{\alpha,\beta}\beta z_2^{\beta-1}\right)z_1^{\alpha-1}\right)\right|\\
&\quad=\sum_{\alpha\in D,\ \alpha\geq 0}\left|\left(\left(\sum_{\beta=0}^{N}a_{\alpha,\beta}z_2^{\beta}\right)\alpha+\sum_{\beta=0}^{N}a_{\alpha,\beta}\beta z_2^{\beta-1}\right)\right|\left|z_1^{\alpha-1}\right|\\
&\quad\leq |zz_1{}^{-1}|\sum_{\alpha\in D,\ \alpha\geq 0}\left|\left(\left(\sum_{\beta=0}^{N}a_{\alpha,\beta}(\log z)^{\beta}\right)\alpha+\sum_{\beta=0}^{N}a_{\alpha,\beta}\beta(\log z)^{\beta-1}\right)\right|\left|z^{\alpha-1}\right|\\
&\quad=|zz_1{}^{-1}|\sum_{\alpha\in D,\ \alpha\geq 0}\left|\left(\left(\sum_{\beta=0}^{N}a_{\alpha,\beta}(\log z)^{\beta}\right)\alpha z^{\alpha-1}+\left(\sum_{\beta=0}^{N}a_{\alpha,\beta}\beta(\log z)^{\beta-1}\right)z^{\alpha-1}\right)\right|\\
&\quad=|zz_1{}^{-1}|\sum_{\alpha\in D,\ \alpha\geq 0}\left|\frac{\partial}{\partial z}\left(\left(\sum_{\beta=0}^{N}a_{\alpha,\beta}(\log z)^{\beta}\right)z^{\alpha}\right)\right|
\end{aligned}$$

(where we keep in mind that $\alpha-1$ could be negative) is convergent, so that (7.31) is absolutely convergent for such z_1 and z_2. Thus, subtracting, we see that

$$\sum_{\alpha\in D,\ \alpha\geq 0}\frac{\partial}{\partial z_2}\left(\left(\sum_{\beta=0}^{N}a_{\alpha,\beta}z_2^{\beta}\right)z_1^{\alpha}\right)=\sum_{\alpha\in D,\ \alpha\geq 0}\left(\frac{\partial}{\partial z_2}\left(\sum_{\beta=0}^{N}a_{\alpha,\beta}z_2^{\beta}\right)\right)z_1^{\alpha} \tag{7.32}$$

is also absolutely convergent for such z_1 and z_2. By Lemma 7.7,

$$\sum_{\alpha\in D,\ \alpha\geq 0}\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}\left(\left(\sum_{\beta=0}^{N}a_{\alpha,\beta}z_2^{\beta}\right)z_1^{\alpha}\right)$$

is absolutely convergent for such z_1 and z_2.

Since $\frac{\partial}{\partial z_1}+\frac{1}{z_1}\frac{\partial}{\partial z_2}$ and $\frac{\partial}{\partial z_2}$ commute with each other, we have

$$\begin{aligned}
&\sum_{\alpha\in D,\ \alpha\geq 0}\left(\frac{\partial}{\partial z_1}+\frac{1}{z_1}\frac{\partial}{\partial z_2}\right)\frac{\partial}{\partial z_2}\left(\left(\sum_{\beta=0}^{N}a_{\alpha,\beta}z_2^{\beta}\right)z_1^{\alpha}\right)\\
&\quad=\sum_{\alpha\in D,\ \alpha\geq 0}\frac{\partial}{\partial z_2}\left(\frac{\partial}{\partial z_1}+\frac{1}{z_1}\frac{\partial}{\partial z_2}\right)\left(\left(\sum_{\beta=0}^{N}a_{\alpha,\beta}z_2^{\beta}\right)z_1^{\alpha}\right)\\
&\quad=z_1\sum_{\alpha\in D,\ \alpha\geq 0}\left(\frac{\partial}{\partial z_1}+\frac{1}{z_1}\frac{\partial}{\partial z_2}\right)^2\left(\left(\sum_{\beta=0}^{N}a_{\alpha,\beta}z_2^{\beta}\right)z_1^{\alpha}\right)
\end{aligned}$$

$$-z_1 \sum_{\alpha\in D,\, \alpha\geq 0} \frac{\partial}{\partial z_1}\left(\frac{\partial}{\partial z_1}+\frac{1}{z_1}\frac{\partial}{\partial z_2}\right)\left(\left(\sum_{\beta=0}^{N} a_{\alpha,\beta} z_2^{\beta}\right) z_1^{\alpha}\right). \tag{7.33}$$

By assumption, the first term on the right-hand side of (7.33) is absolutely convergent when $z_1 = z$ and $z_2 = \log z$ and z is in the original open subset, and then, by the same argument as above, is also absolutely convergent for z_1 and z_2 satisfying $0 < |z_1| < |z|$, $z_2 = \log z$ and z in the smaller open subset. By Lemma 7.7 and the absolute convergence of (7.31) for such z_1 and z_2, the second term on the right-hand side of (7.33) is also absolutely convergent for z_1 and z_2 satisfying $0 < |z_1| < |z|$, $z_2 = \log z$ and z in the smaller open subset. So the left-hand side of (7.33) is absolutely convergent for such z_1 and z_2. Thus

$$\begin{aligned}
&\sum_{\alpha\in D,\, \alpha\geq 0}\left(\left(\frac{\partial}{\partial z_2}\right)^2\left(\sum_{\beta=0}^{N} a_{\alpha,\beta} z_2^{\beta}\right)\right) z_1^{\alpha}\\
&\quad= \sum_{\alpha\in D,\, \alpha\geq 0}\left(\frac{\partial}{\partial z_2}\right)^2\left(\left(\sum_{\beta=0}^{N} a_{\alpha,\beta} z_2^{\beta}\right) z_1^{\alpha}\right)\\
&\quad= z_1 \sum_{\alpha\in D,\, \alpha\geq 0}\left(\frac{\partial}{\partial z_1}+\frac{1}{z_1}\frac{\partial}{\partial z_2}\right)\frac{\partial}{\partial z_2}\left(\left(\sum_{\beta=0}^{N} a_{\alpha,\beta} z_2^{\beta}\right) z_1^{\alpha}\right)\\
&\qquad -z_1 \sum_{\alpha\in D,\, \alpha\geq 0}\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}\left(\left(\sum_{\beta=0}^{N} a_{\alpha,\beta} z_2^{\beta}\right) z_1^{\alpha}\right)
\end{aligned}$$

is absolutely convergent for z in the smaller open subset and any z_1 and z_2 satisfying $0 < |z_1| < |z|$ and $z_2 = \log z$.

Repeating these arguments, we obtain that

$$\sum_{\alpha\in D,\, \alpha\geq 0}\left(\left(\frac{\partial}{\partial z_2}\right)^k\left(\sum_{\beta=0}^{N} a_{\alpha,\beta} z_2^{\beta}\right)\right) z_1^{\alpha} \tag{7.34}$$

is absolutely convergent for such z_1 and z_2 and for $k \in \mathbb{N}$. Taking $k = N$, we see that

$$\sum_{\alpha\in D,\, \alpha\geq 0} a_{\alpha,N} z_1^{\alpha}$$

is absolutely convergent for such z_1. Continuing this process with $k = N-1, \dots, 0$ we obtain that

$$\sum_{\alpha\in D,\, \alpha\geq 0} a_{\alpha,\beta} z_1^{\alpha}$$

is absolutely convergent for such z_1 and each $\beta = 0, \dots, N$. Since $0 < |z_0| < |z|$, we see that in the case $z_1 = z_0$,

$$\sum_{\alpha\in D,\, \alpha\geq 0} a_{\alpha,\beta} z_0^{\alpha}$$

is absolutely convergent for $\beta = 0, \dots, N$.

We also need to prove the absolute convergence of

$$\sum_{\alpha\in D,\, \alpha<0} a_{\alpha,\beta} z_0^{\alpha}$$

for $\beta = 0, \dots, N$. The proof is completely analogous to the proof above except that we take a smaller open subset such that for z in this smaller one, $|z_0| > |z| > 0$ and $|\log z_0| > |\log z|$ instead of $|z_0| < |z|$ and $|\log z_0| < |\log z|$. Thus

$$\sum_{\alpha\in D} a_{\alpha,\beta} z_0^{\alpha}$$

is absolutely convergent for $\beta = 0, \dots, N$. □

Corollary 7.10 *Let D be a subset of $\mathbb{R}$ and N a nonnegative integer. Then the double series (7.27) is absolutely convergent on some (nonempty) open subset of $\mathbb{C}^{\times}$ if and only if the series (7.26) and the corresponding series of first and higher derivatives with respect to z, viewed as series whose terms are the expressions*

$$\left(\sum_{\beta=0}^{N} a_{\alpha,\beta} (\log z)^{\beta}\right) z^{\alpha}$$

and their derivatives with respect to z, are absolutely convergent on the same open subset.

Proof By Proposition 7.9, we need only prove that the absolute convergence of the double series (7.27) is equivalent to the absolute convergence of each of the series (7.25).

It is clear that the absolute convergence of each of the series (7.25) implies the absolute convergence of the double series (7.27). Now assume the absolute convergence of (7.27). If $z \neq 1$, it is clear that each of the series (7.25) is absolutely convergent. If $z = 1$ is in the open subset, we can find z_1 and z_2 in the open subset such that $|z_1| < 1 < |z_2|$. Then

$$\begin{aligned}\sum_{\alpha\in D} |a_{\alpha,\beta}| &= \sum_{\alpha\in D,\, \alpha\leq 0} |a_{\alpha,\beta}| + \sum_{\alpha\in D,\, \alpha>0} |a_{\alpha,\beta}| \\ &\leq \sum_{\alpha\in D,\, \alpha\leq 0} |a_{\alpha,\beta}||z_1|^{\alpha} + \sum_{\alpha\in D,\, \alpha>0} |a_{\alpha,\beta}||z_2|^{\alpha}.\end{aligned}$$

Since the right-hand side is convergent, the left-hand side is also convergent. Thus each of the series (7.25) is absolutely convergent for all z in the open subset. □

Assumption 7.11 *Throughout the remainder of this work, we shall assume that $\mathcal{C}$ satisfies the condition that for any object of $\mathcal{C}$, all the (generalized) weights are real numbers and in addition there exists $K \in \mathbb{Z}_+$ such that*

$$(L(0) - L(0)_s)^K = 0$$

on the generalized module; when $\mathcal{C}$ is in $\mathcal{M}_{sg}$ (recall Notation 2.36), the latter assertion holds vacuously.

In practice, "virtually all the interesting examples" satisfy this assumption.

Proposition 7.12 *We have:*

1. *For any object W of $\mathcal{C}$, the set $\{(n,i) \in \mathbb{C} \times \mathbb{N} \mid (L(0)-n)^i W_{[n]} \neq 0\}$ is included in a (unique expansion) set of the form $\mathbb{R} \times \{0,\dots,N\}$; when $\mathcal{C}$ is in $\mathcal{M}_{sg}$, the set $\{(n,0) \in \mathbb{C} \times \mathbb{N} \mid W_{(n)} \neq 0\}$ is included in the (unique expansion) set $\mathbb{R} \times \{0\}$.*

2. *For any objects W_1, W_2 and W_3 of $\mathcal{C}$, any logarithmic intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$, and any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$, the powers of x and $\log x$ occurring in*
$$\langle w'_{(3)}, \mathcal{Y}(w_{(1)}, x) w_{(2)} \rangle \tag{7.35}$$
form a subset of a (unique expansion) set of the form $\mathbb{R} \times \{0,\dots,N\}$, where N depends only on W_1, W_2 and W_3 (and is independent of the three elements and independent of $\mathcal{Y}$); when $\mathcal{C}$ is in $\mathcal{M}_{sg}$, the powers of x in (7.35) form a (unique expansion) set of real numbers.

Proof This result follows immediately from Proposition 7.8 and Proposition 3.20 (see Remark 3.24, which specifies a value of N for the second assertion). □

Remark 7.13 The first assertion in Proposition 7.12 is a restatement of Assumption 7.11, in view of Propostion 7.8.

Recall again the projections π_n for $n \in \mathbb{C}$ from (2.44) and Definition 2.18 and also recall the notations (7.11) and (7.12). We now prove the analyticity of products and iterates of intertwining maps, in the following sense:

Proposition 7.14 *Assume the convergence condition for intertwining maps in $\mathcal{C}$ (recall Definition 7.4), and let W_1, W_2, W_3, W_4, M_1 and M_2 be objects of $\mathcal{C}$.*

1. *Let $\mathcal{Y}_1 \in \mathcal{V}^{W_4}_{W_1 M_1}$ and $\mathcal{Y}_2 \in \mathcal{V}^{M_1}_{W_2 W_3}$. Then for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$, $w'_{(4)} \in W'_4$ and $p, q \in \mathbb{Z}$, the sum of the absolutely convergent series*
$$\begin{aligned}&\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2) w_{(3)}\rangle\Big|_{x_1^n = e^{nl_p(z_1)},\ \log x_1 = l_p(z_1),\ x_2^n = e^{nl_q(z_2)},\ \log x_2 = l_q(z_2)}\\ &\quad = \sum_{n\in\mathbb{R}} \left(\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\pi_n(\mathcal{Y}_2(w_{(2)}, x_2) w_{(3)})\rangle\Big|_{x_1^n = e^{nl_p(z_1)},\ \log x_1 = l_p(z_1),\ x_2^n = e^{nl_q(z_2)},\ \log x_2 = l_q(z_2)} \right)\end{aligned} \tag{7.36}$$
is a single-valued analytic function on the region given by $|z_1| > |z_2| > 0$ and $0 < \arg z_1, \arg z_2 < 2\pi$, and for $k, l \in \mathbb{N}$,
$$\begin{aligned}&\frac{\partial^{k+l}}{\partial z_1^k \partial z_2^l} \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2) w_{(3)}\rangle\Big|_{x_1^n = e^{nl_p(z_1)},\ \log x_1 = l_p(z_1),\ x_2^n = e^{nl_q(z_2)},\ \log x_2 = l_q(z_2)}\\ &= \langle w'_{(4)}, \mathcal{Y}_1(L(-1)^k w_{(1)}, x_1)\mathcal{Y}_2(L(-1)^l w_{(2)}, x_2) w_{(3)}\rangle\Big|_{x_1^n = e^{nl_p(z_1)},\ \log x_1 = l_p(z_1),\ x_2^n = e^{nl_q(z_2)},\ \log x_2 = l_q(z_2)}.\end{aligned} \tag{7.37}$$

Moreover, these analytic functions in (7.36) for different $p, q \in \mathbb{Z}$ *are different branches of a multivalued analytic function defined on the region* $|z_1| > |z_2| > 0$ *with the cut* $\arg z_1 = 0$, $\arg z_2 = 0$*; similarly for each* k *and* l *for the analytic functions in (7.37).*

2. *Analogously, let* $\mathcal{Y}^1 \in \mathcal{V}_{M_2 W_3}^{W_4}$ *and* $\mathcal{Y}^2 \in \mathcal{V}_{W_1 W_2}^{M_2}$ *Then for any* $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$, $w'_{(4)} \in W'_4$ *and* $p, q \in \mathbb{Z}$*, the sum of the absolutely convergent series*

$$\begin{aligned}&\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_0^n = e^{nl_p(z_0)},\, \log x_0 = l_p(z_0),\, x_2^n = e^{nl_q(z_2)},\, \log x_2 = l_q(z_2)}\\&\quad = \sum_{n\in\mathbb{R}}\left(\langle w'_{(4)}, \mathcal{Y}^1(\pi_n(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}), x_2)w_{(3)}\rangle\Big|_{x_0^n = e^{nl_p(z_0)},\, \log x_0 = l_p(z_0),\, x_2^n = e^{nl_q(z_2)},\, \log x_2 = l_q(z_2)}\right)\end{aligned} \tag{7.38}$$

is a single-valued analytic function on the region given by $|z_2| > |z_0| > 0$ *and* $0 < \arg z_0, \arg z_2 < 2\pi$*, and for* $k, l \in \mathbb{N}$*,*

$$\begin{aligned}&\frac{\partial^{k+l}}{\partial z_0^k \partial z_2^l}\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_0^n = e^{nl_p(z_0)},\, \log x_0 = l_p(z_0),\, x_2^n = e^{nl_q(z_2)},\, \log x_2 = l_q(z_2)}\\&= \sum_{j=0}^{l}\binom{l}{j}\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(L(-1)^{k+j}w_{(1)}, x_0)\cdot\\&\qquad\qquad \cdot L(-1)^{l-j}w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_0^n = e^{nl_p(z_0)},\, \log x_0 = l_p(z_0),\, x_2^n = e^{nl_q(z_2)},\, \log x_2 = l_q(z_2)}.\end{aligned} \tag{7.39}$$

Moreover, these analytic functions in (7.38) for different $p, q \in \mathbb{Z}$*, are different branches of a multivalued analytic function defined on the region* $|z_2| > |z_0| > 0$ *with the cut* $\arg z_0 = 0$, $\arg z_2 = 0$*; similarly for each* k *and* l *for the analytic functions in (7.39).*

Proof We assume that $w_{(1)}$, $w_{(2)}$, $w_{(3)}$, $w'_{(4)}$ are homogeneous with respect to the generalized weight grading. The general case follows by linearity.

Since the series (7.36) is absolutely convergent in the region $|z_1| > |z_2| > 0$ and every term of this series is a single-valued function on the region given by $|z_1| > |z_2| > 0$ and $0 < \arg z_1, \arg z_2 < 2\pi$, its sum gives a single-valued function in the same region. By the $L(-1)$-derivative property for logarithmic intertwining operators,

$$\begin{aligned}&\sum_{n\in\mathbb{R}}\frac{\partial^{k+l}}{\partial z_1^k \partial z_2^l}\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\pi_n(\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)})\rangle\Big|_{x_1^n = e^{nl_p(z_1)},\, \log x_1 = l_p(z_1),\, x_2^n = e^{nl_q(z_2)},\, \log x_2 = l_q(z_2)}\\&\quad = \sum_{n\in\mathbb{R}}\langle w'_{(4)}, \mathcal{Y}_1(L(-1)^k w_{(1)}, x_1)\cdot\\&\qquad\qquad \cdot\pi_n(\mathcal{Y}_2(L(-1)^l w_{(2)}, x_2)w_{(3)})\rangle\Big|_{x_1^n = e^{nl_p(z_1)},\, \log x_1 = l_p(z_1),\, x_2^n = e^{nl_q(z_2)},\, \log x_2 = l_q(z_2)}\end{aligned} \tag{7.40}$$

for $k, l \in \mathbb{N}$, and by assumption, the series (7.40) are absolutely convergent in the region $|z_1| > |z_2| > 0$.

By Proposition 7.12 and the assumption that the vectors are homogeneous, for fixed z_2 (so that $l_q(z_2)$ is also fixed), (7.36) is a series of the form

$$\sum_{n\in\mathbb{R}} \left(\sum_{i=0}^{N} a_{n,i} e^{nl_p(z_1)} (l_p(z_1))^i \right) \tag{7.41}$$

with $a_{n,i} \in \mathbb{C}$. This series is absolutely convergent in the region for z_1 given by $|z_1| > |z_2| > 0$ and $0 < \arg z_1 < 2\pi$. Replacing the logarithmic intertwining operator $\mathcal{Y}_1(\cdot, x)$ in (7.36) by the logarithmic intertwining operator $\mathcal{Y}_1(\cdot, e^{-2\pi i p}x)$ (recall Remark 3.28; cf. Remark 4.11), we see that the resulting analogue of (7.36) equals

$$\sum_{n\in\mathbb{R}} \left(\sum_{i=0}^{N} a_{n,i} z_1^n (\log z_1)^i \right), \tag{7.42}$$

with the same coefficients $a_{n,i}$ as in (7.41) but using the principal branch $\log z_1$ instead of $l_p(z_1)$, and this series is also absolutely convergent in the region for z_1 given by $|z_1| > |z_2| > 0$ and $0 < \arg z_1 < 2\pi$. Since for each $k \in \mathbb{N}$ and $l = 0$, (7.40) is absolutely convergent in the region $|z_1| > |z_2| > 0$, as is the analogue of (7.40) with $\mathcal{Y}_1(\cdot, x)$ replaced by $\mathcal{Y}_1(\cdot, e^{-2\pi i p}x)$ as above, the series

$$\sum_{n\in\mathbb{R}} a_{n,i} z_1^n$$

for $i = 0, \ldots, N$ are all absolutely convergent in the region for z_1 given by $|z_1| > |z_2| > 0$ and $0 < \arg z_1 < 2\pi$, by Proposition 7.9, and in particular, the double series

$$\sum_{n\in\mathbb{R}} \sum_{i=0}^{N} a_{n,i} z_1^n (\log z_1)^i \tag{7.43}$$

is also absolutely convergent in the same region (as Corollary 7.10 states). Then by Lemma 7.7, $\sum_{n\in\mathbb{R}} a_{n,i} z_1^n$ for $i = 0, \ldots, N$ as functions of z_1 are analytic in the same region. Since (7.42) as a function of z_1 is equal to (7.43), it is also analytic in the same region. Thus for fixed z_2, the sum of the analogue of (7.36) as a function of z_1 is analytic in the region given by $|z_1| > |z_2| > 0$ and $0 < \arg z_1 < 2\pi$, and thus so is (7.36) itself; moreover, (7.36) for different values of p are different branches of the same multivalued analytic function (7.43) (with z_1 replaced by $l_p(z_1)$) in the region for z_1 given by $|z_1| > |z_2| > 0$ with the cut $\arg z_1 = 0$, and the derivatives of this function are given by the branches of the multivalued analytic function (7.37). Across the cut $\arg z_1 = 0$, the analytic function is given by (7.36) for adjacent values of p.

The same argument shows that for fixed z_1, the sum of (7.36) as a function of z_2 for different values of q are different branches of the same multivalued analytic function in the region for z_2 given by $|z_1| > |z_2| > 0$ with the cut $\arg z_2 = 0$, with derivatives given by

(7.37). Thus the sum of (7.36) as a function of z_1 and z_2 for different values of p and q are the branches of a multivalued analytic function in the region $|z_1| > |z_2| > 0$, with derivatives given by (7.37).

An analogous argument proves the second half of the proposition, for $\mathcal{Y}^1$ and $\mathcal{Y}^2$. □

Remark 7.15 As usual, we shall use the same notation to denote an absolutely convergent series and its sum. In particular, (7.36) and (7.38) denote either the series or the sums of the series. The proposition above says that these sums are in fact analytic functions in z_1 and z_2 and can be analytically extended to multivalued analytic functions on the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_0| > 0$, respectively.

Again recall (2.44) and Definition 2.18, and recall the notations (7.15) and (7.16). Using Corollary 7.10, Propositions 7.12 and 7.14, we shall prove the following result:

Proposition 7.16 *Assume the convergence condition for intertwining maps in $\mathcal{C}$. Let z_1, z_2 be two nonzero complex numbers satisfying*

$$|z_1| > |z_2| > 0,$$

and let $I_1 \in \mathcal{M}[P(z_1)]_{W_1 M_1}^{W_4}$ and $I_2 \in \mathcal{M}[P(z_2)]_{W_2 W_3}^{M_1}$. Let $w_{(1)} \in W_1$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$ be homogeneous elements with respect to the (generalized) weight gradings. Suppose that for all homogeneous $w_{(2)} \in W_2$,

$$\langle w'_{(4)}, I_1(w_{(1)} \otimes I_2(w_{(2)} \otimes w_{(3)}))\rangle = 0.$$

Then

$$\langle w'_{(4)}, I_1(w_{(1)} \otimes \pi_p I_2(w_{(2)} \otimes w_{(3)}))\rangle = 0$$

for all $p \in \mathbb{R}$ and all $w_{(2)} \in W_2$. In particular,

$$\langle w'_{(4)}, \pi_p I_1(w_{(1)} \otimes \pi_q I_2(w_{(2)} \otimes w_{(3)}))\rangle = 0$$

for all $p, q \in \mathbb{R}$ and all $w_{(2)} \in W_2$.

Proof Recall the correspondence between $P(z)$-intertwining maps and logarithmic intertwining operators of the same type (Proposition 4.8), and the notation $\mathcal{Y}_{I,p}$, $p \in \mathbb{Z}$, for the logarithmic intertwining operators corresponding to a $P(z)$-intertwining map I ((4.17) and (4.18)).

By Proposition 7.14,

$$\langle w'_{(4)}, \mathcal{Y}_{I_1,0}(w_{(1)}, z_1)\mathcal{Y}_{I_2,0}(w_{(2)}, z)w_{(3)}\rangle \tag{7.44}$$

is a single-valued analytic function of z on the region given by $|z_1| > |z| > 0$ and $0 < \arg z < 2\pi$, and its derivatives are given by (7.37) with $p = q = 0$ and $k = 0$, and with z_2 replaced by z. If $0 < \arg z_2 < 2\pi$, then by the Taylor expansion of this analytic function of z at $z = z_2$

and (7.37), we see that for all $w_{(2)} \in W_2$ and for all z in a sufficiently small neighborhood of z_2,

$$\begin{aligned}
&\langle w'_{(4)}, \mathcal{Y}_{I_1,0}(w_{(1)}, z_1)\mathcal{Y}_{I_2,0}(w_{(2)}, z)w_{(3)}\rangle \\
&\quad = \sum_{i\in\mathbb{N}} \frac{(z-z_2)^i}{i!}\langle w'_{(4)}, \mathcal{Y}_{I_1,0}(w_{(1)}, z_1)\mathcal{Y}_{I_2,0}(L(-1)^i w_{(2)}, z_2)w_{(3)}\rangle \\
&\quad = \sum_{i\in\mathbb{N}} \frac{(z-z_2)^i}{i!} 0 = 0. \qquad (7.45)
\end{aligned}$$

If $\arg z_2 = 0$, that is, if z_2 is a positive real number, then by Proposition 7.14, (7.44) can be analytically extended to a single-valued analytic function of z on a neighborhood of z_2 such that in the intersection of this neighborhood with the region $0 < \arg z < \pi$, this function is equal to (7.44). Then the same argument as above shows that in this intersection, (7.45) holds.

In either case, we see that (7.45) holds in a nonempty open subset of the region $0 < \arg z < 2\pi$. Now by Proposition 7.12, Proposition 3.20(b) and the meaning of the absolutely convergent series on the left-hand side of (7.45), for fixed $z_1 \neq 0$, this series on the left-hand side is of the form (7.26), with $D \subset \mathbb{R}$, a unique expansion set. By Proposition 7.14, the higher-derivative series of the left-hand side of (7.45) are absolutely convergent. Thus we can apply Corollary 7.10 to obtain that the double series obtained from the left-hand side of (7.45) by taking the terms to be monomials in z and $\log z$ is also absolutely convergent to 0 for z in the open subset. By the definition of unique expansion set, we see that all of the coefficients of the monomials in z and $\log z$ of this double series must be zero. Hence we get

$$\langle w'_{(4)}, \mathcal{Y}_{I_1,0}(w_{(1)}, z_1)(w_{(2)}{}^{\mathcal{Y}_{I_2,0}}_{n;\,k} w_{(3)})\rangle = 0$$

for any homogeneous $w_{(2)} \in W_2$, $n \in \mathbb{R}$ and $k \in \mathbb{N}$. Since $w_{(1)}$, $w_{(3)}$ and $w'_{(4)}$ are homogeneous, we obtain

$$\langle w'_{(4)}, I_1(w_{(1)} \otimes \pi_p I_2(w_{(2)} \otimes w_{(3)}))\rangle = 0$$

for any homogeneous $w_{(2)} \in W_2$ and $p \in \mathbb{R}$, in view of Proposition 3.20(b) and Proposition 4.8, and this remains true for any $w_{(2)} \in W_2$. The last statement is clear. □

Corollary 7.17 *Assume the convergence condition for intertwining maps in $\mathcal{C}$ (recall Definition 7.4). Let z_1, z_2 be two nonzero complex numbers satisfying*

$$|z_1| > |z_2| > 0.$$

Suppose that the $P(z_2)$-tensor product of W_2 and W_3 and the $P(z_1)$-tensor product of W_1 and $W_2 \boxtimes_{P(z_2)} W_3$ both exist (recall Definition 4.15). Then $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ is spanned (as a vector space) by all the elements of the form

$$\pi_n(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}))$$

where $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$ are homogeneous with respect to the (generalized) weight gradings and $n \in \mathbb{R}$ (recall the notation (4.31)).

Proof Let $w'_{(4)} \in (W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))'$ be homogeneous such that

$$\langle w'_{(4)}, w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})\rangle = 0$$

for all homogeneous $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. From Proposition 7.16 we see that

$$\langle w'_{(4)}, \pi_p(w_{(1)} \boxtimes_{P(z_1)} \pi_q(w_{(2)} \boxtimes_{P(z_2)} w_{(3)}))\rangle = 0$$

for all $p, q \in \mathbb{R}$ and all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. Since by Proposition 4.23, the set

$$\{\pi_p(w_{(1)} \boxtimes_{P(z_1)} \pi_q(w_{(2)} \boxtimes_{P(z_2)} w_{(3)}))|\ p, q \in \mathbb{R}, w_{(1)} \in W_1, w_{(2)} \in W_2, w_{(3)} \in W_3\}$$

spans the space $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$, we must have $w'_{(4)} = 0$, and the result follows. □

Analogously, by similar proofs we have:

Proposition 7.18 *Assume the convergence condition for intertwining maps in $\mathcal{C}$. Let z_0, z_2 be two nonzero complex numbers satisfying*

$$|z_2| > |z_0| > 0,$$

and let $I^1 \in \mathcal{M}[P(z_2)]^{W_4}_{M_2 W_3}$ and $I^2 \in \mathcal{M}[P(z_0)]^{M_2}_{W_1 W_2}$. Let $w'_{(4)} \in W'_4$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$ be homogeneous with respect to the (generalized) weight gradings. Suppose that for all homogeneous $w_{(1)} \in W_1$,

$$\langle w'_{(4)}, I^1(I^2(w_{(1)} \otimes w_{(2)}) \otimes w_{(3)})\rangle = 0.$$

Then

$$\langle w'_{(4)}, I^1(\pi_p I^2(w_{(1)} \otimes w_{(2)}) \otimes w_{(3)})\rangle = 0$$

for all $p \in \mathbb{R}$ and all $w_{(1)} \in W_1$. In particular,

$$\langle w'_{(4)}, \pi_p I^1(\pi_q I^2(w_{(1)} \otimes w_{(2)}) \otimes w_{(3)})\rangle = 0$$

for all $p, q \in \mathbb{R}$ and all $w_{(1)} \in W_1$. □

Corollary 7.19 *Assume the convergence condition for intertwining maps in $\mathcal{C}$. Let z_0, z_2 be two nonzero complex numbers satisfying*

$$|z_2| > |z_0| > 0.$$

Suppose that the $P(z_0)$-tensor product of W_1 and W_2 and the $P(z_2)$-tensor product of $W_1 \boxtimes_{P(z_0)} W_2$ and W_3 both exist. Then $(W_1 \boxtimes_{P(z_0)} W_2) \boxtimes_{P(z_2)} W_3$ is spanned by all the elements of the form

$$\pi_n((w_{(1)} \boxtimes_{P(z_0)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)})$$

where $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$ are homogeneous and $n \in \mathbb{R}$. □

In addition to the definitions (7.36) and (7.38) of the indicated products and iterates of intertwining maps, there is a different, natural candidate for interpretations of the left-hand sides of (7.36) and (7.38), involving multiple as opposed to iterated sums, and we now show that these other interpretations indeed agree with the definitions of these expressions:

Proposition 7.20 *Assume the convergence condition for intertwining maps in $\mathcal{C}$. Let z_1, z_2 be two nonzero complex numbers satisfying $|z_1| > |z_2| > 0$ and let $\mathcal{Y}_1 \in \mathcal{V}_{W_1 M_1}^{W_4}$ and $\mathcal{Y}_2 \in \mathcal{V}_{W_2 W_3}^{M_1}$. Then for any $p, q \in \mathbb{Z}$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$, the series obtained by substituting $e^{nl_p(z_1)}$, $e^{nl_q(z_2)}$, $l_p(z_1)$ and $l_q(z_2)$ for x_1^n, x_2^n, $\log x_1$ and $\log x_2$, respectively, in the formal series*

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle$$

is absolutely convergent and its sum is equal to

$$\begin{aligned}&\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_1^n=e^{nl_p(z_1)},\ \log x_1=l_p(z_1),\ x_2^n=e^{nl_q(z_2)},\ \log x_2=l_q(z_2)}\\&\quad=\sum_{n\in\mathbb{R}}\left(\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\pi_n(\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)})\rangle\Big|_{x_1^n=e^{nl_p(z_1)},\ \log x_1=l_p(z_1),\ x_2^n=e^{nl_q(z_2)},\ \log x_2=l_q(z_2)}\right).\end{aligned}$$

Analogously, let z_0, z_2 be two nonzero complex numbers satisfying $|z_2| > |z_0| > 0$ and let $\mathcal{Y}^1 \in \mathcal{V}_{M_2 W_3}^{W_4}$ and $\mathcal{Y}^2 \in \mathcal{V}_{W_1 W_2}^{M_2}$. Then for any $p, q \in \mathbb{Z}$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$, the series obtained by substituting $e^{nl_p(z_0)}$, $e^{nl_q(z_2)}$, $l_p(z_0)$ and $l_q(z_2)$ for x_0^n, x_2^n, $\log x_0$ and $\log x_2$, respectively, in the formal series

$$\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle$$

is absolutely convergent and its sum is equal to

$$\begin{aligned}&\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_0^n=e^{nl_p(z_0)},\ \log x_0=l_p(z_0),\ x_2^n=e^{nl_q(z_2)},\ \log x_2=l_q(z_2)}\\&\quad=\sum_{n\in\mathbb{R}}\left(\langle w'_{(4)}, \mathcal{Y}^1(\pi_n(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}), x_2)w_{(3)}\rangle\Big|_{x_0^n=e^{nl_p(z_0)},\ \log x_0=l_p(z_0),\ x_2^n=e^{nl_q(z_2)},\ \log x_2=l_q(z_2)}\right).\end{aligned}$$

Proof We prove only the first part, the second part being similar.

If $w_{(1)}$, $w_{(2)}$, $w_{(3)}$ and $w'_{(4)}$ are homogeneous with respect to the generalized weight gradings, then by Proposition 7.12, the first series is the triple series (recall (3.24) and Proposition 3.20(b))

$$\sum_{n\in\mathbb{R}}\sum_{j=0}^{M}\sum_{i=0}^{N}\langle w'_{(4)}, (w_{(1)})^{\mathcal{Y}_1}_{\Delta-n-2,j}(w_{(2)})^{\mathcal{Y}_2}_{n,i}w_{(3)}\rangle e^{(-\Delta+n+1)l_p(z_1)}l_p(z_1)^j e^{(-n-1)l_q(z_2)}l_q(z_2)^i \qquad (7.46)$$

and the second series is the corresponding iterated series

$$\sum_{n\in\mathbb{R}}\left(\sum_{j=0}^{M}\sum_{i=0}^{N}\langle w'_{(4)}, (w_{(1)})^{\mathcal{Y}_1}_{\Delta-n-2,j}(w_{(2)})^{\mathcal{Y}_2}_{n,i}w_{(3)}\rangle l_p(z_1)^j l_q(z_2)^i\right) e^{(-\Delta+n+1)l_p(z_1)}e^{(-n-1)l_q(z_2)},$$

with

$$\Delta = -\text{wt } w'_{(4)} + \text{wt } w_{(1)} + \text{wt } w_{(2)} + \text{wt } w_{(3)} \in \mathbb{R}.$$

By assumption, the iterated series is absolutely convergent. By Proposition 7.14, all the derivatives of the iterated series are also absolutely convergent. Replacing $\mathcal{Y}_2(w_{(2)}, x)$ by $\mathcal{Y}_2(w_{(2)}, e^{-2\pi i q}x)$, we see that the resulting iterated series

$$\sum_{n\in\mathbb{R}} \left(\sum_{j=0}^{M} \sum_{i=0}^{N} \langle w'_{(4)}, (w_{(1)})^{\mathcal{Y}_1}_{\Delta-n-2,j}(w_{(2)})^{\mathcal{Y}_2}_{n,i} w_{(3)}\rangle l_p(z_1)^j (\log z_2)^i \right) e^{(-\Delta+n+1)l_p(z_1)} z_2^{-n-1}$$

and its derivatives are still absolutely convergent. Then by Proposition 7.9, with $z = z_2$, the iterated series

$$\sum_{n\in\mathbb{R}} \left(\sum_{j=0}^{M} \langle w'_{(4)}, (w_{(1)})^{\mathcal{Y}_1}_{\Delta-n-2,j}(w_{(2)})^{\mathcal{Y}_2}_{n,i} w_{(3)}\rangle l_p(z_1)^j \right) e^{(-\Delta+n+1)l_p(z_1)} e^{(-n-1)l_q(z_2)} l_q(z_2)^i$$

for $i = 0, \dots, N$ are absolutely convergent. By the same argument but with $\mathcal{Y}_1(w_{(1)}, x)$ replaced by $\mathcal{Y}_1(w_{(1)}, e^{-2\pi i p}x)$ and with $z = z_1$ in Proposition 7.9, we see that the double series

$$\sum_{n\in\mathbb{R}} \sum_{j=0}^{M} \langle w'_{(4)}, (w_{(1)})^{\mathcal{Y}_1}_{\Delta-n-2,j}(w_{(2)})^{\mathcal{Y}_2}_{n,i} w_{(3)}\rangle l_p(z_1)^j e^{(-\Delta+n+1)l_p(z_1)} e^{(-n-1)l_q(z_2)} l_q(z_2)^i$$

for $i = 0, \dots, N$ are absolutely convergent. Thus the triple series (7.46), as a finite sum of these series, is also absolutely convergent.

In the general case, $w_{(1)}$, $w_{(2)}$, $w_{(3)}$ and $w'_{(4)}$ are finite sums of homogeneous vectors. Thus we have

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle = \sum_{i=1}^{k} \langle w^{i\prime}_{(4)}, \mathcal{Y}_1(w^i_{(1)}, x_1)\mathcal{Y}_2(w^i_{(2)}, x_2)w^i_{(3)}\rangle \tag{7.47}$$

and

$$\begin{aligned} &\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_1^n = e^{nl_p(z_1)},\ \log x_1 = l_p(z_1),\ x_2^n = e^{nl_q(z_2)},\ \log x_2 = l_q(z_2)} \\ &\quad = \sum_{i=1}^{k} \langle w^{i\prime}_{(4)}, \mathcal{Y}_1(w^i_{(1)}, x_1)\mathcal{Y}_2(w^i_{(2)}, x_2)w^i_{(3)}\rangle\Big|_{x_1^n = e^{nl_p(z_1)},\ \log x_1 = l_p(z_1),\ x_2^n = e^{nl_q(z_2)},\ \log x_2 = l_q(z_2)}, \end{aligned} \tag{7.48}$$

where $w^i_{(1)} \in W_1$, $w^i_{(2)} \in W_2$, $w^i_{(3)} \in W_3$ and $w^{i\prime}_{(4)} \in W'_4$ are homogeneous with respect to the generalized weight gradings. Substituting $e^{nl_p(z_1)}$, $e^{nl_q(z_2)}$, $l_p(z_1)$ and $l_q(z_2)$ for x_1^n, x_2^n, $\log x_1$ and $\log x_2$, respectively, in each term in the right-hand side of (7.47) gives an absolutely convergent series whose sum is equal to the corresponding term in the right-hand side of (7.48), and so making these substitutions in the left-hand side of (7.47) gives an absolutely convergent series whose sum is equal to the left-hand side of (7.48). □

Remark 7.21 Proposition 7.20 in fact justifies the notations that we have introduced in (7.11) and (7.12) (and in particular, in (7.13)–(7.16)). That is, with z_1 and z_2 satisfying the appropriate inequality, for any $p, q \in \mathbb{Z}$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$, (7.12) also means the absolutely convergent sum of the multiple series obtained by substituting $e^{nl_p(z_1)}$, $e^{nl_q(z_2)}$, $l_p(z_1)$ and $l_q(z_2)$ for x_1^n, x_2^n, $\log x_1$ and $\log x_2$, respectively, in the formal series

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle;$$

similarly for z_0 and z_2, (7.11) and the formal series

$$\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle.$$

When $p = q = 0$, (7.14) (or (7.16)) also means the absolutely convergent sum of the multiple series obtained by substituting $e^{n \log z_1}$, $e^{n \log z_2}$, $\log z_1$ and $\log z_2$ for x_1^n, x_2^n, $\log x_1$ and $\log x_2$, respectively, in the first formal series above; similarly for (7.13) (or (7.15)) and the second formal series above. Since for an absolutely convergent series, we use the same notation to denote the series and its sum, (7.11) and (7.12) (and in particular, (7.13)–(7.16)) also denote the analytic functions given by the sums of the corresponding series. Moreover, if $w_{(1)}$, $w_{(2)}$, $w_{(3)}$ and $w'_{(4)}$ are finite sums of elements of the same generalized modules, then the same notations also mean the finite sum of the series or sums obtained from the summands of $w_{(1)}$, $w_{(2)}$, $w_{(3)}$ and $w'_{(4)}$. In the rest of this work, we shall use these notations to mean any one of these things, depending on what we need.

From Proposition 7.20, we immediately obtain:

Corollary 7.22 *Assume the convergence condition for intertwining maps in $\mathcal{C}$. Let $\mathcal{Y}_1 \in \mathcal{V}_{W_1 M_1}^{W_4}$, $\mathcal{Y}_2 \in \mathcal{V}_{W_2 W_3}^{M_1}$, $\mathcal{Y}_3 \in \mathcal{V}_{W_1 \widetilde{M}_1}^{W_4}$, $\mathcal{Y}_4 \in \mathcal{V}_{W_2 W_3}^{\widetilde{M}_1}$ and let $w_{(1)}, \widetilde{w}_{(1)} \in W_1$, $w_{(2)}, \widetilde{w}_{(2)} \in W_2$, $w_{(3)}, \widetilde{w}_{(3)} \in W_3$ and $w'_{(4)}, \widetilde{w}'_{(4)} \in W'_4$. If*

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle = \langle \widetilde{w}'_{(4)}, \mathcal{Y}_3(\widetilde{w}_{(1)}, x_1)\mathcal{Y}_4(\widetilde{w}_{(2)}, x_2)\widetilde{w}_{(3)}\rangle,$$

then for any $p, q \in \mathbb{Z}$ and $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > 0$,

$$\begin{aligned} &\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_1^n = e^{nl_p(z_1)},\ \log x_1 = l_p(z_1),\ x_2^n = e^{nl_q(z_2)},\ \log x_2 = l_q(z_2)} \\ &\quad = \langle \widetilde{w}'_{(4)}, \mathcal{Y}_3(\widetilde{w}_{(1)}, x_1)\mathcal{Y}_4(\widetilde{w}_{(2)}, x_2)\widetilde{w}_{(3)}\rangle\Big|_{x_1^n = e^{nl_p(z_1)},\ \log x_1 = l_p(z_1),\ x_2^n = e^{nl_q(z_2)},\ \log x_2 = l_q(z_2)}. \end{aligned} \tag{7.49}$$

Analogously, let $\mathcal{Y}^1 \in \mathcal{V}_{M_2 W_3}^{W_4}$, $\mathcal{Y}^2 \in \mathcal{V}_{W_1 W_2}^{M_2}$, $\mathcal{Y}^3 \in \mathcal{V}_{\widetilde{M}_2 W_3}^{W_4}$ and $\mathcal{Y}^4 \in \mathcal{V}_{W_1 W_2}^{\widetilde{M}_2}$ and let $w_{(1)}, \widetilde{w}_{(1)} \in W_1$, $w_{(2)}, \widetilde{w}_{(2)} \in W_2$, $w_{(3)}, \widetilde{w}_{(3)} \in W_3$ and $w'_{(4)}, \widetilde{w}'_{(4)} \in W'_4$. If

$$\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle = \langle \widetilde{w}'_{(4)}, \mathcal{Y}^3(\mathcal{Y}^4(\widetilde{w}_{(1)}, x_0)\widetilde{w}_{(2)}, x_2)\widetilde{w}_{(3)}\rangle,$$

then for $p, q \in \mathbb{Z}$ and $z_0, z_2 \in \mathbb{C}$ satisfying $|z_0| > |z_2| > 0$,

$$\begin{aligned}
&\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_0^n=e^{nl_p(z_0)},\ \log x_0=l_p(z_0),\ x_2^n=e^{nl_q(z_2)},\ \log x_2=l_q(z_2)}\\
&\quad= \langle \widetilde{w}'_{(4)}, \mathcal{Y}^3(\mathcal{Y}^4(\widetilde{w}_{(1)}, x_0)\widetilde{w}_{(2)}, x_2)\widetilde{w}_{(3)}\rangle\Big|_{x_0^n=e^{nl_p(z_0)},\ \log x_0=l_p(z_0),\ x_2^n=e^{nl_q(z_2)},\ \log x_2=l_q(z_2)}. \qquad \square
\end{aligned} \tag{7.50}$$

Remark 7.23 One can generalize the convergence condition for two intertwining maps and the results above to products and iterates of any number of intertwining maps. The convergence conditions for three intertwining maps and the spanning properties in the case of four generalized modules will be needed in Section 12 in the proof of the commutativity of the pentagon diagram, and we will discuss these conditions and properties in Section 12.

Remark 7.24 The convergence studied in this section can easily be formulated as special cases of the following general notion: Let W be a (complex) vector space and let $\langle \cdot, \cdot \rangle$: $W^* \times W \to \mathbb{C}$ be the pairing between the dual space W^* and W. Consider the weak topology on W^* defined by this pairing, so that W^* becomes a Hausdorff locally convex topological vector space. Let $\sum_{n\in I} w_n^*$ be a formal series in W^*, where I is an index set. We say that $\sum_{n\in I} w_n^*$ is *weakly absolutely convergent* if for all $w \in W$, the formal series

$$\sum_{n\in I} \langle w_n^*, w\rangle \tag{7.51}$$

of complex numbers is absolutely convergent. Note that if $\sum_{n\in I} w_n^*$ is weakly absolutely convergent, then (7.51) as w ranges through W defines a (unique) element of W^*, and the formal series is in fact convergent to this element in the weak topology. This element is the *sum* of the series and is denoted using the same notation $\sum_{n\in I} w_n^*$. In this section, the convergence that we have been discussing amounts to the weak absolute convergence of formal series in $(W')^*$ for an object W of $\mathcal{C}$, and this kind of convergence will again be used in Section 8. In Section 9, we will use this notion for $(W_1 \otimes W_2)^*$ where W_1 and W_2 are generalized V-modules, and in Section 12, we will be using more general cases.

8 $P(z_1, z_2)$-intertwining maps and the corresponding compatibility condition

In this section we first prove some natural identities satisfied by products and iterates of logarithmic intertwining operators and of intertwining maps. These identities were first proved in [H2] for intertwining operators and intertwining maps among ordinary modules. We also prove a list of identities relating products of formal delta functions, as was done in [H2]. Using all these identities as motivation, we define "$P(z_1, z_2)$-intertwining maps" and study their basic properties, by analogy with the relevant parts of the study of $P(z)$-intertwining maps in Sections 4 and 5. The notion of $P(z_1, z_2)$-intertwining map is new; the treatment in this section is different from that in [H2], even for the case of ordinary intertwining operators.

At the end of this section, we show that products and iterates of intertwining maps or of logarithmic intertwining operators "factor through" suitable tensor product modules in a unique way.

It is possible to define "tensor products of three modules," as opposed to iterated tensor products, and $P(z_1, z_2)$-intertwining maps would play the same role for such tensor products of three modules that $P(z)$-intertwining maps play for tensor products of two modules. However, one would of course in addition need appropriate natural isomorphisms between triple tensor products and the corresponding iterated tensor products, and much more than $P(z_1, z_2)$-intertwining maps (as defined here) would be necessary for this; see Section 9 below, in particular. Since we do not need "tensor products of three modules" in this work, we will not formally introduce and study them.

We recall our continuing Assumptions 4.1, 5.30 and 7.11 concerning our category $\mathcal{C}$.

Recall the Jacobi identity (3.26) in the definition of the notion of logarithmic intertwining operator associated with generalized modules (W_1, Y_1), (W_2, Y_2) and (W_3, Y_3) for a Möbius (or conformal) vertex algebra V. Suppose that we also have generalized modules (W_4, Y_4), (M_1, Y_{M_1}) and (M_2, Y_{M_2}). Then from (3.26) we see that a product of logarithmic intertwining operators of types $\binom{W_4}{W_1 M_1}$ and $\binom{M_1}{W_2 W_3}$ satisfies an identity analogous to (3.26), as does an iterate of logarithmic intertwining operators of types $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$:

Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be logarithmic intertwining operators of types $\binom{W_4}{W_1 M_1}$ and $\binom{M_1}{W_2 W_3}$, respectively. Then for $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$, the product of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ satisfies the identity

$$\begin{aligned}
&x_1^{-1}\delta\Big(\frac{x_0 - y_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0 - y_2}{x_2}\Big)Y_4(v, x_0)\mathcal{Y}_1(w_{(1)}, y_1)\mathcal{Y}_2(w_{(2)}, y_2)w_{(3)}\\
&\quad = y_1^{-1}\delta\Big(\frac{x_0 - x_1}{y_1}\Big)x_2^{-1}\delta\Big(\frac{x_0 - y_2}{x_2}\Big)\mathcal{Y}_1(Y_1(v, x_1)w_{(1)}, y_1)\mathcal{Y}_2(w_{(2)}, y_2)w_{(3)}\\
&\qquad +x_1^{-1}\delta\Big(\frac{-y_1 + x_0}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0 - y_2}{x_2}\Big)\mathcal{Y}_1(w_{(1)}, y_1)Y_{M_1}(v, x_0)\mathcal{Y}_2(w_{(2)}, y_2)w_{(3)}\\
&\quad = y_1^{-1}\delta\Big(\frac{x_0 - x_1}{y_1}\Big)x_2^{-1}\delta\Big(\frac{x_0 - y_2}{x_2}\Big)\mathcal{Y}_1(Y_1(v, x_1)w_{(1)}, y_1)\mathcal{Y}_2(w_{(2)}, y_2)w_{(3)}
\end{aligned}$$

$$+x_1^{-1}\delta\Big(\frac{-y_1+x_0}{x_1}\Big)y_2^{-1}\delta\Big(\frac{x_0-x_2}{y_2}\Big)\mathcal{Y}_1(w_{(1)},y_1)\mathcal{Y}_2(Y_2(v,x_2)w_{(2)},y_2)w_{(3)}$$
$$+x_1^{-1}\delta\Big(\frac{-y_1+x_0}{x_1}\Big)x_2^{-1}\delta\Big(\frac{-y_2+x_0}{x_2}\Big)\mathcal{Y}_1(w_{(1)},y_1)\mathcal{Y}_2(w_{(2)},y_2)Y_3(v,x_0)w_{(3)}. \tag{8.1}$$

In addition, let $\mathcal{Y}^1$ and $\mathcal{Y}^2$ be logarithmic intertwining operators of types $\binom{W_4}{M_2W_3}$ and $\binom{M_2}{W_1W_2}$, respectively. Then for $v\in V$, $w_{(1)}\in W_1$, $w_{(2)}\in W_2$ and $w_{(3)}\in W_3$, the iterate of $\mathcal{Y}^1$ and $\mathcal{Y}^2$ satisfies the identity

$$\begin{aligned}
&x_2^{-1}\delta\Big(\frac{x_0-y_2}{x_2}\Big)x_1^{-1}\delta\Big(\frac{x_2-y_0}{x_1}\Big)Y_4(v,x_0)\mathcal{Y}^1(\mathcal{Y}^2(w_{(1)},y_0)w_{(2)},y_2)w_{(3)}\\
&\quad= y_2^{-1}\delta\Big(\frac{x_0-x_2}{y_2}\Big)x_1^{-1}\delta\Big(\frac{x_2-y_0}{x_1}\Big)\mathcal{Y}^1(Y_{M_2}(v,x_2)\mathcal{Y}^2(w_{(1)},y_0)w_{(2)},y_2)w_{(3)}\\
&\qquad+x_2^{-1}\delta\Big(\frac{-y_2+x_0}{x_2}\Big)x_1^{-1}\delta\Big(\frac{x_2-y_0}{x_1}\Big)\mathcal{Y}^1(\mathcal{Y}^2(w_{(1)},y_0)w_{(2)},y_2)Y_3(v,x_0)w_{(3)}\\
&\quad= y_2^{-1}\delta\Big(\frac{x_0-x_2}{y_2}\Big)y_0^{-1}\delta\Big(\frac{x_2-x_1}{y_0}\Big)\mathcal{Y}^1(\mathcal{Y}^2(Y_1(v,x_1)w_{(1)}),y_0)w_{(2)},y_2)w_{(3)}\\
&\qquad+y_2^{-1}\delta\Big(\frac{x_0-x_2}{y_2}\Big)x_1^{-1}\delta\Big(\frac{-y_0+x_2}{x_1}\Big)\mathcal{Y}^1(\mathcal{Y}^2(w_{(1)},y_1)Y_2(v,x_2)w_{(2)},y_2)w_{(3)}\\
&\qquad+x_2^{-1}\delta\Big(\frac{-y_2+x_0}{x_2}\Big)x_1^{-1}\delta\Big(\frac{x_2-y_0}{x_1}\Big)\mathcal{Y}^1(\mathcal{Y}^2(w_{(1)},y_0)w_{(2)},y_2)Y_3(v,x_0)w_{(3)}.
\end{aligned} \tag{8.2}$$

Under natural hypotheses motivated by Section 7, we will need to specialize the formal variables y_1, y_2 and y_0 to complex numbers z_1, z_2 and z_0, respectively, in (8.1) and (8.2), when $|z_1|>|z_2|>0$ and $|z_2|>|z_0|>0$. For this, following [H2], we will need the next lemma, on products of formal delta functions, with certain of the variables being complex variables in suitable domains. Our formulation and proof here are different from those in [H2]. In addition to justifying the specializations just indicated, this lemma will give us the natural relation between the specialized expressions (8.9) and (8.10) below.

Lemma 8.1 *Let z_1 and z_2 be complex numbers and set $z_0=z_1-z_2$. Then the left-hand sides of the following expressions converge absolutely in the indicated domains, in the sense that the coefficient of each monomial in the formal variables x_0, x_1 and x_2 is an absolutely convergent series in the two variables related by the inequalities, and the following identities hold:*

$$x_1^{-1}\delta\Big(\frac{x_0-z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0-z_2}{x_2}\Big) = x_2^{-1}\delta\Big(\frac{x_0-z_2}{x_2}\Big)x_1^{-1}\delta\Big(\frac{x_2-z_0}{x_1}\Big)\quad\text{for arbitrary } z_1,z_2; \tag{8.3}$$

$$z_1^{-1}\delta\Big(\frac{x_0-x_1}{z_1}\Big)x_2^{-1}\delta\Big(\frac{x_0-z_2}{x_2}\Big) = x_0^{-1}\delta\Big(\frac{z_1+x_1}{x_0}\Big)x_2^{-1}\delta\Big(\frac{z_0+x_1}{x_2}\Big)\quad\text{if } |z_1|>|z_2|; \tag{8.4}$$

$$
\begin{aligned}
z_2^{-1}\delta\Big(\frac{x_0-x_2}{z_2}\Big)z_0^{-1}\delta\Big(\frac{x_2-x_1}{z_0}\Big) &= x_0^{-1}\delta\Big(\frac{z_1+x_1}{x_0}\Big)x_2^{-1}\delta\Big(\frac{z_0+x_1}{x_2}\Big)\\
&\quad \text{if } |z_2|>|z_0|>0; \qquad (8.5)\\
x_1^{-1}\delta\Big(\frac{-z_1+x_0}{x_1}\Big)z_2^{-1}\delta\Big(\frac{x_0-x_2}{z_2}\Big) &= x_0^{-1}\delta\Big(\frac{z_2+x_2}{x_0}\Big)x_1^{-1}\delta\Big(\frac{-z_0+x_2}{x_1}\Big)\\
&\quad \text{if } |z_1|>|z_2|>0; \qquad (8.6)\\
x_2^{-1}\delta\Big(\frac{-z_2+x_0}{x_2}\Big)x_1^{-1}\delta\Big(\frac{x_2-z_0}{x_1}\Big) &= x_1^{-1}\delta\Big(\frac{-z_1+x_0}{x_1}\Big)x_2^{-1}\delta\Big(\frac{-z_2+x_0}{x_2}\Big)\\
&\quad \text{if } |z_2|>|z_0|. \qquad (8.7)
\end{aligned}
$$

(Note that the first identity does not require a restricted domain for z_1, z_2 and z_0, while the others need certain conditions among the complex numbers z_i in order for the expressions on the left-hand sides to be well defined, that is, absolutely convergent. None of the five expressions on the right-hand sides require restricted domains for absolute convergence.)

Proof In this proof we will use additional formal variables y_0, y_1, y_2, and repeatedly use Remark 2.3.25 in [LL] about delta function substitution.

First, we have

$$
\begin{aligned}
x_1^{-1}\delta\Big(\frac{x_0-y_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0-y_2}{x_2}\Big) &= x_1^{-1}\delta\Big(\frac{x_0-y_1}{x_1}\Big)x_0^{-1}\delta\Big(\frac{x_2+y_2}{x_0}\Big)\\
&= x_1^{-1}\delta\Big(\frac{x_2+y_2-y_1}{x_1}\Big)x_0^{-1}\delta\Big(\frac{x_2+y_2}{x_0}\Big)\\
&= x_2^{-1}\delta\Big(\frac{x_0-y_2}{x_2}\Big)x_1^{-1}\delta\Big(\frac{x_2-(y_1-y_2)}{x_1}\Big). \qquad (8.8)
\end{aligned}
$$

(Note that the notation $(x_0+y_2-y_1)^n$ is unambiguous: it is the power series expansion in nonnegative powers of y_1 and y_2.) Since it is clear that the left-hand side of this identity lies in

$$\mathbb{C}[y_1,y_2]((x_0^{-1}))[[x_1,x_1^{-1},x_2,x_2^{-1}]],$$

one can substitute any complex numbers z_1, z_2 for y_1, y_2, respectively, and get the identity (8.3).

For (8.4), we have

$$
\begin{aligned}
y_1^{-1}\delta\Big(\frac{x_0-x_1}{y_1}\Big)x_2^{-1}\delta\Big(\frac{x_0-y_2}{x_2}\Big) &= x_0^{-1}\delta\Big(\frac{y_1+x_1}{x_0}\Big)x_2^{-1}\delta\Big(\frac{x_0-y_2}{x_2}\Big)\\
&= x_0^{-1}\delta\Big(\frac{y_1+x_1}{x_0}\Big)x_2^{-1}\delta\Big(\frac{y_1+x_1-y_2}{x_2}\Big)\\
&= x_0^{-1}\delta\Big(\frac{y_1+x_1}{x_0}\Big)x_2^{-1}\delta\Big(\frac{(y_1-y_2)+x_1}{x_2}\Big),
\end{aligned}
$$

and the right-hand side and hence the left-hand side lies in

$$\mathbb{C}[y_1,y_1^{-1},(y_1-y_2),(y_1-y_2)^{-1}][[x_0,x_0^{-1},x_1,x_2,x_2^{-1}]].$$

Thus if $|z_1| > |z_2| > 0$, so that the binomial expansion of $(z_1 - z_2)^n$ converges for all n, we can substitute z_1, z_2 for y_1, y_2 and obtain (8.4). On the other hand,

$$\begin{aligned} y_2^{-1}\delta\Big(\frac{x_0 - x_2}{y_2}\Big)y_0^{-1}\delta\Big(\frac{x_2 - x_1}{y_0}\Big) &= x_0^{-1}\delta\Big(\frac{y_2 + x_2}{x_0}\Big)x_2^{-1}\delta\Big(\frac{y_0 + x_1}{x_2}\Big) \\ &= x_0^{-1}\delta\Big(\frac{y_2 + y_0 + x_1}{x_0}\Big)x_2^{-1}\delta\Big(\frac{y_0 + x_1}{x_2}\Big). \end{aligned}$$

It is clear from the right-hand side that both sides lie in

$$\mathbb{C}[y_0, y_0^{-1}, (y_2 + y_0), (y_2 + y_0)^{-1}][[x_0, x_0^{-1}, x_1, x_2, x_2^{-1}]],$$

so if $|z_2| > |z_0| > 0$ we can substitute z_2, z_0 for y_2, y_0 and obtain (8.5).

To prove (8.6), we see that

$$\begin{aligned} x_1^{-1}\delta\Big(\frac{-y_1 + x_0}{x_1}\Big)y_2^{-1}\delta\Big(\frac{x_0 - x_2}{y_2}\Big) &= x_1^{-1}\delta\Big(\frac{-y_1 + x_0}{x_1}\Big)x_0^{-1}\delta\Big(\frac{y_2 + x_2}{x_0}\Big) \\ &= x_1^{-1}\delta\Big(\frac{-y_1 + y_2 + x_2}{x_1}\Big)x_0^{-1}\delta\Big(\frac{y_2 + x_2}{x_0}\Big), \end{aligned}$$

and the right-hand side and hence both sides lie in

$$\mathbb{C}[y_2, y_2^{-1}, (y_1 - y_2), (y_1 - y_2)^{-1}][[x_0, x_0^{-1}, x_1, x_1^{-1}, x_2]],$$

so that when $|z_1| > |z_2| > 0$ we can substitute z_1, z_2 for y_1, y_2 and obtain (8.6). Finally, we have

$$\begin{aligned} x_2^{-1}\delta\Big(\frac{-y_2 + x_0}{x_2}\Big)x_1^{-1}\delta\Big(\frac{x_2 - y_0}{x_1}\Big) &= x_2^{-1}\delta\Big(\frac{-y_2 + x_0}{x_2}\Big)x_1^{-1}\delta\Big(\frac{-y_2 + x_0 - y_0}{x_1}\Big) \\ &= x_2^{-1}\delta\Big(\frac{-y_2 + x_0}{x_2}\Big)x_1^{-1}\delta\Big(\frac{-(y_2 + y_0) + x_0}{x_1}\Big), \end{aligned}$$

and from the right-hand side we see that both sides lie in

$$\mathbb{C}[(y_2 + y_0), (y_2 + y_0)^{-1}, y_2, y_2^{-1}][[x_0, x_1, x_1^{-1}, x_2, x_2^{-1}]],$$

so that when $|z_2| > |z_0|$ we can substitute z_2, z_0 for y_2, y_0 and obtain the identity (8.7). □

If we assume that the convergence condition for intertwining maps in $\mathcal{C}$ holds and that our generalized modules are objects of $\mathcal{C}$, in the setting of Section 7, then after pairing with an element $w'_{(4)} \in W'_4$, we can specialize the formal variables y_1, y_2 to complex numbers z_1, z_2 in (8.1) whenever $|z_1| > |z_2| > 0$, and we can specialize y_2, y_0 to complex numbers z_2, z_0 in (8.2) whenever $|z_2| > |z_0| > 0$, using Lemma 8.1:

Proposition 8.2 *Assume that the convergence condition for intertwining maps in $\mathcal{C}$ holds and that the generalized modules entering into (8.1) and (8.2) are objects of $\mathcal{C}$. Continuing to use the notation of (8.1) and (8.2), also let $w'_{(4)} \in W'_4$. Let z_1, z_2 be complex numbers* *satisfying* $|z_1| > |z_2| > 0$*. Then for a* $P(z_1)$*-intertwining map* I_1 *of type* $\binom{W_4}{W_1 M_1}$ *and a* $P(z_2)$*-intertwining map* I_2 *of type* $\binom{M_1}{W_2 W_3}$*, the following expressions are absolutely convergent, and the following formula for the product*

$$I_1 \circ (1_{W_1} \otimes I_2)$$

of I_1 *and* I_2 *holds:*

$$\begin{aligned}
&\left\langle w'_{(4)}, x_1^{-1}\delta\Big(\frac{x_0 - z_1}{x_1}\Big) x_2^{-1}\delta\Big(\frac{x_0 - z_2}{x_2}\Big) Y_4(v, x_0)(I_1 \circ (1_{W_1} \otimes I_2))(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})\right\rangle \\
&\quad = \left\langle w'_{(4)}, z_1^{-1}\delta\Big(\frac{x_0 - x_1}{z_1}\Big) x_2^{-1}\delta\Big(\frac{x_0 - z_2}{x_2}\Big) \cdot \right. \\
&\qquad\qquad\qquad \left. \cdot (I_1 \circ (1_{W_1} \otimes I_2))(Y_1(v, x_1)w_{(1)} \otimes w_{(2)} \otimes w_{(3)})\right\rangle \\
&\qquad + \left\langle w'_{(4)}, x_1^{-1}\delta\Big(\frac{-z_1 + x_0}{x_1}\Big) z_2^{-1}\delta\Big(\frac{x_0 - x_2}{z_2}\Big) \cdot \right. \\
&\qquad\qquad\qquad \left. \cdot (I_1 \circ (1_{W_1} \otimes I_2))(w_{(1)} \otimes Y_2(v, x_2)w_{(2)} \otimes w_{(3)})\right\rangle \\
&\qquad + \left\langle w'_{(4)}, x_1^{-1}\delta\Big(\frac{-z_1 + x_0}{x_1}\Big) x_2^{-1}\delta\Big(\frac{-z_2 + x_0}{x_2}\Big) \cdot \right. \\
&\qquad\qquad\qquad \left. \cdot (I_1 \circ (1_{W_1} \otimes I_2))(w_{(1)} \otimes w_{(2)} \otimes Y_3(v, x_0)w_{(3)})\right\rangle .
\end{aligned} \tag{8.9}$$

Moreover, let z_2*,* z_0 *be complex numbers satisfying* $|z_2| > |z_0| > 0$*. Then for a* $P(z_2)$*-intertwining map* I^1 *of type* $\binom{W_4}{M_2 W_3}$ *and a* $P(z_0)$*-intertwining map* I^2 *of type* $\binom{M_2}{W_1 W_2}$*, the following expressions are absolutely convergent, and the following formula for the iterate*

$$I^1 \circ (I^2 \otimes 1_{W_3})$$

of I^1 *and* I^2 *holds:*

$$\begin{aligned}
&\left\langle w'_{(4)}, x_2^{-1}\delta\Big(\frac{x_0 - z_2}{x_2}\Big) x_1^{-1}\delta\Big(\frac{x_2 - z_0}{x_1}\Big) Y_4(v, x_0)(I^1 \circ (I^2 \otimes 1_{W_3}))(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})\right\rangle \\
&\quad = \left\langle w'_{(4)}, z_2^{-1}\delta\Big(\frac{x_0 - x_2}{z_2}\Big) z_0^{-1}\delta\Big(\frac{x_2 - x_1}{z_0}\Big) \cdot \right. \\
&\qquad\qquad\qquad \left. \cdot (I^1 \circ (I^2 \otimes 1_{W_3}))(Y_1(v, x_1)w_{(1)} \otimes w_{(2)} \otimes w_{(3)})\right\rangle \\
&\qquad + \left\langle w'_{(4)}, z_2^{-1}\delta\Big(\frac{x_0 - x_2}{z_2}\Big) x_1^{-1}\delta\Big(\frac{-z_0 + x_2}{x_1}\Big) \cdot \right. \\
&\qquad\qquad\qquad \left. \cdot (I^1 \circ (I^2 \otimes 1_{W_3}))(w_{(1)} \otimes Y_2(v, x_2)w_{(2)} \otimes w_{(3)})\right\rangle
\end{aligned}$$

$$+\Big\langle w'_{(4)}, x_2^{-1}\delta\Big(\frac{-z_2+x_0}{x_2}\Big)x_1^{-1}\delta\Big(\frac{x_2-z_0}{x_1}\Big)\cdot$$
$$\cdot(I^1\circ(I^2\otimes 1_{W_3}))(w_{(1)}\otimes w_{(2)}\otimes Y_3(v,x_0)w_{(3)})\Big\rangle. \tag{8.10}$$

Proof When y_1 and y_2 are specialized to z_1 and z_2, respectively, the product of the two delta-function expressions on the left-hand side of (8.1) and the three products of pairs of delta-function expressions on the right-hand side of (8.1) all converge absolutely in the domain $|z_1| > |z_2| > 0$, by Lemma 8.1; note that for the last of the three products of pairs of delta-function expressions on the right-hand side of (8.1), the convergence is immediate. Analogously, from Lemma 8.1 we see that the corresponding statements also hold for (8.2), when y_0 and y_2 are specialized to z_0 and z_2, respectively, in the domain $|z_1| > |z_2| > 0$. Recalling the notations (7.5), (7.8), (7.15) and (7.16), we see that the result follows from the convergence condition. □

Considering the $\mathfrak{sl}(2)$-action instead of the V-action, by (3.28) we have

$$\begin{aligned}
&L(j)\mathcal{Y}_1(w_{(1)},y_1)\mathcal{Y}_2(w_{(2)},y_2)w_{(3)}\\
&\quad=\sum_{i=0}^{j+1}\binom{j+1}{i}y_1^i\mathcal{Y}_1(L(j-i)w_{(1)},y_1)\mathcal{Y}_2(w_{(2)},y_2)w_{(3)}\\
&\quad\quad+\mathcal{Y}_1(w_{(1)},y_1)L(j)\mathcal{Y}_2(w_{(2)},y_2)w_{(3)}\\
&\quad=\sum_{i=0}^{j+1}\binom{j+1}{i}y_1^i\mathcal{Y}_1(L(j-i)w_{(1)},y_1)\mathcal{Y}_2(w_{(2)},y_2)w_{(3)}\\
&\quad\quad+\mathcal{Y}_1(w_{(1)},y_1)\sum_{k=0}^{j+1}\binom{j+1}{k}y_2^k\mathcal{Y}_2(L(j-k)w_{(2)},y_2)w_{(3)}\\
&\quad\quad+\mathcal{Y}_1(w_{(1)},y_1)\mathcal{Y}_2(w_{(2)},y_2)L(j)w_{(3)}
\end{aligned}\tag{8.11}$$

for $j=-1,0$ and 1. In the setting of Proposition 8.2, if $|z_1|>|z_2|>0$ we can substitute z_1, z_2 for y_1, y_2, respectively, and we obtain, setting $z_0=z_1-z_2$,

$$\begin{aligned}
&\langle w'_{(4)}, L(j)(I_1\circ(1_{W_1}\otimes I_2))(w_{(1)}\otimes w_{(2)}\otimes w_{(3)})\rangle\\
&\quad=\Big\langle w'_{(4)}, \sum_{i=0}^{j+1}\binom{j+1}{i}(z_2+z_0)^i(I_1\circ(1_{W_1}\otimes I_2))(L(j-i)w_{(1)}\otimes w_{(2)}\otimes w_{(3)})\Big\rangle\\
&\quad\quad+\Big\langle w'_{(4)}, \sum_{k=0}^{j+1}\binom{j+1}{k}z_2^k(I_1\circ(1_{W_1}\otimes I_2))(w_{(1)}\otimes L(j-k)w_{(2)}\otimes w_{(3)})\Big\rangle\\
&\quad\quad+\langle w'_{(4)}, (I_1\circ(1_{W_1}\otimes I_2))(w_{(1)}\otimes w_{(2)}\otimes L(j)w_{(3)})\rangle
\end{aligned}\tag{8.12}$$

for $j=-1,0$ and 1.

On the other hand, by (3.28) we also have

$$
\begin{aligned}
&L(j)\mathcal{Y}^1(\mathcal{Y}^2(w_{(1)},y_0)w_{(2)},y_2)w_{(3)}\\
&\quad=\sum_{i=0}^{j+1}\binom{j+1}{i}y_2^i\mathcal{Y}^1(L(j-i)\mathcal{Y}^2(w_{(1)},y_0)w_{(2)},y_2)w_{(3)}\\
&\quad\quad+\mathcal{Y}^1(\mathcal{Y}^2(w_{(1)},y_0)w_{(2)},y_2)L(j)w_{(3)}\\
&\quad=\sum_{i=0}^{j+1}\binom{j+1}{i}y_2^i\mathcal{Y}^1\Bigg(\sum_{k=0}^{j-i+1}\binom{j-i+1}{k}y_0^k\mathcal{Y}^2(L(j-i-k)w_{(1)},y_0)w_{(2)},y_2\Bigg)w_{(3)}\\
&\quad\quad+\sum_{i=0}^{j+1}\binom{j+1}{i}y_2^i\mathcal{Y}^1(\mathcal{Y}^2(w_{(1)},y_0)L(j-i)w_{(2)},y_2)w_{(3)}\\
&\quad\quad+\mathcal{Y}^1(\mathcal{Y}^2(w_{(1)},y_0)w_{(2)},y_2)L(j)w_{(3)} \qquad (8.13)
\end{aligned}
$$

for $j=-1,0$ and 1, The first term of the right-hand side is

$$
\begin{aligned}
&\sum_{i=0}^{j+1}\binom{j+1}{i}y_2^i\sum_{k=0}^{j-i+1}\binom{j-i+1}{k}y_0^k\mathcal{Y}^1(\mathcal{Y}^2(L(j-i-k)w_{(1)},y_0)w_{(2)},y_2)w_{(3)}\\
&\quad=\sum_{t=0}^{j+1}\sum_{k=0}^{t}\binom{j+1}{t-k}\binom{j+1-t+k}{k}y_2^{t-k}y_0^k\mathcal{Y}^1(\mathcal{Y}^2(L(j-t)w_{(1)},y_0)w_{(2)},y_2)w_{(3)}\\
&\quad=\sum_{t=0}^{j+1}\binom{j+1}{t}(y_2+y_0)^t\mathcal{Y}^1(\mathcal{Y}^2(L(j-t)w_{(1)},y_0)w_{(2)},y_2)w_{(3)},
\end{aligned}
$$

where we have used the identity $\binom{j+1}{t-k}\binom{j+1-t+k}{k}=\binom{j+1}{t}\binom{t}{k}$ in the last step. Thus in the setting of Proposition 8.2, if $|z_2|>|z_0|>0$ we can substitute z_2, z_0 for y_2, y_0, respectively, in (8.13), and we obtain

$$
\begin{aligned}
&\langle w'_{(4)},L(j)(I^1\circ(I^2\otimes 1_{W_3}))(w_{(1)}\otimes w_{(2)}\otimes w_{(3)})\rangle\\
&\quad=\Bigg\langle w'_{(4)},\sum_{t=0}^{j+1}\binom{j+1}{t}(z_2+z_0)^t(I^1\circ(I^2\otimes 1_{W_3}))(L(j-t)w_{(1)}\otimes w_{(2)}\otimes w_{(3)})\Bigg\rangle\\
&\quad\quad+\Bigg\langle w'_{(4)},\sum_{i=0}^{j+1}\binom{j+1}{i}z_2^i(I^1\circ(I^2\otimes 1_{W_3}))(w_{(1)}\otimes L(j-i)w_{(2)}\otimes w_{(3)})\Bigg\rangle\\
&\quad\quad+\langle w'_{(4)},(I^1\circ(I^2\otimes 1_{W_3}))(w_{(1)}\otimes w_{(2)}\otimes L(j)w_{(3)})\rangle \qquad (8.14)
\end{aligned}
$$

for $j=-1,0$ and 1.

Of course, in case V is a conformal vertex algebra, these formulas follow from the earlier computation for the V-action (Proposition 8.2), by setting $v=\omega$ and taking $\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\mathrm{Res}_{x_0}x_0^{j+1}$, $j=-1,0,1$.

Lemma 8.1, Proposition 8.2, (8.12), (8.14) and Remark 7.2 motivate the following definition, which is analogous to the definition of the notion of $P(z)$-intertwining map (Definition 4.2):

Definition 8.3 Let $z_0, z_1, z_2 \in \mathbb{C}^\times$ with $z_0 = z_1 - z_2$ (so that in particular $z_1 \neq z_2$, $z_0 \neq z_1$ and $z_0 \neq -z_2$). Let (W_1, Y_1), (W_2, Y_2), (W_3, Y_3) and (W_4, Y_4) be generalized modules for a Möbius (or conformal) vertex algebra V. A $P(z_1, z_2)$-*intertwining map* is a linear map

$$F : W_1 \otimes W_2 \otimes W_3 \to \overline{W}_4$$

such that the following conditions are satisfied: the *grading compatibility condition*: For $\beta, \gamma, \delta \in \tilde{A}$ and $w_{(1)} \in W_1^{(\beta)}$, $w_{(2)} \in W_2^{(\gamma)}$, $w_{(3)} \in W_3^{(\delta)}$,

$$F(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \in \overline{W_4^{(\beta+\gamma+\delta)}}; \tag{8.15}$$

the *lower truncation condition*: for any elements $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$, and any $n \in \mathbb{C}$,

$$\pi_{n-m} F(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) = 0 \quad \text{for} \quad m \in \mathbb{N} \ \text{ sufficiently large} \tag{8.16}$$

(which follows from (8.15), in view of the grading restriction condition (2.85)); the *composite Jacobi identity*:

$$\begin{aligned}
&x_1^{-1}\delta\Big(\frac{x_0 - z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0 - z_2}{x_2}\Big)Y_4(v, x_0)F(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})\\
&\quad = x_0^{-1}\delta\Big(\frac{z_1 + x_1}{x_0}\Big)x_2^{-1}\delta\Big(\frac{z_0 + x_1}{x_2}\Big)F(Y_1(v, x_1)w_{(1)} \otimes w_{(2)} \otimes w_{(3)})\\
&\quad\quad + x_0^{-1}\delta\Big(\frac{z_2 + x_2}{x_0}\Big)x_1^{-1}\delta\Big(\frac{-z_0 + x_2}{x_1}\Big)F(w_{(1)} \otimes Y_2(v, x_2)w_{(2)} \otimes w_{(3)})\\
&\quad\quad + x_1^{-1}\delta\Big(\frac{-z_1 + x_0}{x_1}\Big)x_2^{-1}\delta\Big(\frac{-z_2 + x_0}{x_2}\Big)F(w_{(1)} \otimes w_{(2)} \otimes Y_3(v, x_0)w_{(3)})
\end{aligned} \tag{8.17}$$

for $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$ (note that all the expressions in the right-hand side of (8.17) are well defined, that none of the products of delta-function expressions require restricted domains, and that the left-hand side of (8.17) is meaningful because any infinite linear combination of v_n $(n \in \mathbb{Z})$ of the form $\sum_{n<N} a_n v_n$ $(a_n \in \mathbb{C})$ acts in a well-defined way on any $F(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})$, in view of (8.16)); and the $\mathfrak{sl}(2)$-*bracket relations*: for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$,

$$\begin{aligned}
&L(j)F(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})\\
&\quad = \sum_{i=0}^{j+1} \binom{j+1}{i} z_1^i F(L(j-i)w_{(1)} \otimes w_{(2)} \otimes w_{(3)})\\
&\quad\quad + \sum_{k=0}^{j+1} \binom{j+1}{k} z_2^k F(w_{(1)} \otimes L(j-k)w_{(2)} \otimes w_{(3)})\\
&\quad\quad + F(w_{(1)} \otimes w_{(2)} \otimes L(j)w_{(3)})
\end{aligned} \tag{8.18}$$

for $j=-1,0$ and 1 (again, in case V is a conformal vertex algebra, this follows from (8.17) by setting $v=\omega$ and taking $\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\mathrm{Res}_{x_0}x_0^{j+1}$).

Remark 8.4 (cf. Remark 4.5) If W_4 in Definition 8.3 is lower bounded, then (8.16) can be strengthened to:

$$\pi_n F(w_{(1)}\otimes w_{(2)}\otimes w_{(3)})=0 \quad \text{for } \Re(n) \text{ sufficiently negative.} \tag{8.19}$$

We emphasize that every term in (8.17) and (8.18) in this definition is purely algebraic; that is, no convergence is involved.

From Lemma 8.1, Proposition 8.2, (8.12), (8.14) and Remark 7.2, we have the following:

Proposition 8.5 *In the setting of Proposition 8.2, for intertwining maps I_1, I_2, I^1 and I^2 as indicated, when $|z_1|>|z_2|>0$, $I_1\circ(1_{W_1}\otimes I_2)$ is a $P(z_1,z_2)$-intertwining map and when $|z_2|>|z_0|>0$, $I^1\circ(I^2\otimes 1_{W_3})$ is a $P(z_2+z_0,z_2)$-intertwining map.* $\square$

Now we consider $P(z_1,z_2)$-intertwining maps from a "dual" viewpoint, and we use this to motivate an analogue $\tau_{P(z_1,z_2)}$ of the action $\tau_{P(z)}$ introduced in Section 5.2. Fix any $w'_{(4)}\in W'_4$. Then (8.17) implies:

$$\begin{aligned}
&\left\langle w'_{(4)}, x_1^{-1}\delta\Big(\frac{x_0-z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0-z_2}{x_2}\Big)Y_4(v,x_0)F(w_{(1)}\otimes w_{(2)}\otimes w_{(3)})\right\rangle\\
&\quad=\left\langle w'_{(4)}, x_0^{-1}\delta\Big(\frac{z_1+x_1}{x_0}\Big)x_2^{-1}\delta\Big(\frac{z_0+x_1}{x_2}\Big)F(Y_1(v,x_1)w_{(1)}\otimes w_{(2)}\otimes w_{(3)})\right\rangle\\
&\qquad+\left\langle w'_{(4)}, x_0^{-1}\delta\Big(\frac{z_2+x_2}{x_0}\Big)x_1^{-1}\delta\Big(\frac{-z_0+x_2}{x_1}\Big)F(w_{(1)}\otimes Y_2(v,x_2)w_{(2)}\otimes w_{(3)})\right\rangle\\
&\qquad+\left\langle w'_{(4)}, x_1^{-1}\delta\Big(\frac{-z_1+x_0}{x_1}\Big)x_2^{-1}\delta\Big(\frac{-z_2+x_0}{x_2}\Big)F(w_{(1)}\otimes w_{(2)}\otimes Y_3(v,x_0)w_{(3)})\right\rangle.
\end{aligned} \tag{8.20}$$

The left-hand side can be written as

$$\left\langle x_1^{-1}\delta\Big(\frac{x_0-z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0-z_2}{x_2}\Big)Y'_4(e^{x_0L(1)}(-x_0^2)^{-L(0)}v,x_0^{-1})w'_{(4)}, F(w_{(1)}\otimes w_{(2)}\otimes w_{(3)})\right\rangle,$$

and so by replacing v by $(-x_0^2)^{L(0)}e^{-x_0L(1)}v$ and then replacing x_0 by x_0^{-1} in both sides of (8.20) we see that

$$\begin{aligned}
&\left\langle x_1^{-1}\delta\Big(\frac{x_0^{-1}-z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y'_4(v,x_0)w'_{(4)}, F(w_{(1)}\otimes w_{(2)}\otimes w_{(3)})\right\rangle\\
&\quad=\left\langle w'_{(4)}, x_0\delta\Big(\frac{z_1+x_1}{x_0^{-1}}\Big)x_2^{-1}\delta\Big(\frac{z_0+x_1}{x_2}\Big)\cdot\right.\\
&\qquad\qquad\left. F(Y_1((-x_0^{-2})^{L(0)}e^{-x_0^{-1}L(1)}v,x_1)w_{(1)}\otimes w_{(2)}\otimes w_{(3)})\right\rangle
\end{aligned}$$

$$+\Big\langle w'_{(4)}, x_0\delta\Big(\frac{z_2+x_2}{x_0^{-1}}\Big)x_1^{-1}\delta\Big(\frac{-z_0+x_2}{x_1}\Big)\cdot$$
$$F(w_{(1)}\otimes Y_2((-x_0^{-2})^{L(0)}e^{-x_0^{-1}L(1)}v, x_2)w_{(2)}\otimes w_{(3)})\Big\rangle$$
$$+\Big\langle w'_{(4)}, x_1^{-1}\delta\Big(\frac{-z_1+x_0^{-1}}{x_1}\Big)x_2^{-1}\delta\Big(\frac{-z_2+x_0^{-1}}{x_2}\Big)\cdot$$
$$F(w_{(1)}\otimes w_{(2)}\otimes Y_3((-x_0^{-2})^{L(0)}e^{-x_0^{-1}L(1)}v, x_0^{-1})w_{(3)})\Big\rangle. \tag{8.21}$$

Arguing just as in (5.20)–(5.22), we note that in the left-hand side of (8.21), the coefficients of

$$x_1^{-1}\delta\Big(\frac{x_0^{-1}-z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y'_4(v,x_0)$$

in powers of x_0, x_1 and x_2, for all $v\in V$, span

$$\tau_{W'_4}(V\otimes\iota_+\mathbb{C}[t,t^{-1},(z_1^{-1}-t)^{-1},(z_2^{-1}-t)^{-1}])$$

(recall the notation τ_W from (5.1), (5.2), (5.7) and the notation $\iota_\pm$ from (5.64)). By analogy with the case of $P(z)$-intertwining maps, we shall define an action of

$$V\otimes\iota_+\mathbb{C}[t,t^{-1},(z_1^{-1}-t)^{-1},(z_2^{-1}-t)^{-1}]$$

on $(W_1\otimes W_2\otimes W_3)^*$. We shall need the following analogue of Lemma 5.1, where we use the notations Y_t, T_z and o introduced in Section 5.1, and where we recall that $z_0=z_1-z_2$:

Lemma 8.6 *We have*

$$o\Big(x_1^{-1}\delta\Big(\frac{x_0^{-1}-z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y_t(v,x_0)\Big)$$
$$=x_1^{-1}\delta\Big(\frac{x_0^{-1}-z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y_t^o(v,x_0), \tag{8.22}$$
$$(\iota_+\circ\iota_-^{-1}\circ o)\Big(x_1^{-1}\delta\Big(\frac{x_0^{-1}-z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y_t(v,x_0)\Big)$$
$$=x_1^{-1}\delta\Big(\frac{-z_1+x_0^{-1}}{x_1}\Big)x_2^{-1}\delta\Big(\frac{-z_2+x_0^{-1}}{x_2}\Big)Y_t^o(v,x_0), \tag{8.23}$$
$$(\iota_+\circ T_{z_1}\circ\iota_-^{-1}\circ o)\Big(x_1^{-1}\delta\Big(\frac{x_0^{-1}-z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y_t(v,x_0)\Big)$$
$$=x_0\delta\Big(\frac{z_1+x_1}{x_0^{-1}}\Big)x_2^{-1}\delta\Big(\frac{z_0+x_1}{x_2}\Big)Y_t((-x_0^{-2})^{L(0)}e^{-x_0^{-1}L(1)}v,x_1), \tag{8.24}$$
$$(\iota_+\circ T_{z_2}\circ\iota_-^{-1}\circ o)\Big(x_1^{-1}\delta\Big(\frac{x_0^{-1}-z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y_t(v,x_0)\Big)$$
$$=x_0\delta\Big(\frac{z_2+x_2}{x_0^{-1}}\Big)x_1^{-1}\delta\Big(\frac{-z_0+x_2}{x_1}\Big)Y_t((-x_0^{-2})^{L(0)}e^{-x_0^{-1}L(1)}v,x_2). \tag{8.25}$$

Proof The identity (8.22) immediately follows from (5.37), and (8.23) follows from (8.22), as in the proof of (5.68). For (8.24), note that by (5.58), the coefficient of $x_1^{-m-1}x_2^{-n-1}$ in the right-hand side of (8.22) is

$$\begin{aligned}(x_0^{-1} - z_1)^m (x_0^{-1} - z_2)^n \Big(e^{x_0 L(1)}(-x_0^{-2})^{L(0)} v \otimes x_0 \delta\Big(\frac{t}{x_0^{-1}}\Big)\Big) \\ = (t - z_1)^m (t - z_2)^n \Big(e^{x_0 L(1)}(-x_0^{-2})^{L(0)} v \otimes x_0 \delta\Big(\frac{t}{x_0^{-1}}\Big)\Big).\end{aligned}$$

Acted on by $\iota_+ \circ T_{z_1} \circ \iota_-^{-1}$, this becomes

$$\begin{aligned}t^m (z_0 + t)^n \Big(e^{x_0 L(1)}(-x_0^{-2})^{L(0)} v \otimes x_0 \delta\Big(\frac{z_1 + t}{x_0^{-1}}\Big)\Big) \\ = x_0 \delta\Big(\frac{z_1 + t}{x_0^{-1}}\Big)(z_0 + t)^n \Big(e^{x_0 L(1)}(-x_0^{-2})^{L(0)} v \otimes t^m\Big) \\ = x_0 \delta\Big(\frac{z_1 + t}{x_0^{-1}}\Big)(z_0 + t)^n \Big((-x_0^{-2})^{L(0)} e^{-x_0^{-1} L(1)} v \otimes t^m\Big),\end{aligned}$$

by formula (5.3.1) in [FHL], and using (5.5), we see that this is the coefficient of $x_1^{-m-1}x_2^{-n-1}$ in the right-hand side of (8.24). The analogous identity (8.25) is proved similarly. □

Our analogue of Definition 5.3 is:

Definition 8.7 Let $z_1, z_2 \in \mathbb{C}^\times$, $z_1 \neq z_2$. We define a linear action $\tau_{P(z_1,z_2)}$ of the space

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z_1^{-1} - t)^{-1}, (z_2^{-1} - t)^{-1}] \tag{8.26}$$

on $(W_1 \otimes W_2 \otimes W_3)^*$ by

$$\begin{aligned}(\tau_{P(z_1,z_2)}(\xi)\lambda)(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\ = \lambda(\tau_{W_1}((\iota_+ \circ T_{z_1} \circ \iota_-^{-1} \circ o)\xi) w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\ + \lambda(w_{(1)} \otimes \tau_{W_2}((\iota_+ \circ T_{z_2} \circ \iota_-^{-1} \circ o)\xi) w_{(2)} \otimes w_{(3)}) \\ + \lambda(w_{(1)} \otimes w_{(2)} \otimes \tau_{W_3}((\iota_+ \circ \iota_-^{-1} \circ o)\xi) w_{(3)})\end{aligned} \tag{8.27}$$

for

$$\xi \in V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z_1^{-1} - t)^{-1}, (z_2^{-1} - t)^{-1}],$$
$$\lambda \in (W_1 \otimes W_2 \otimes W_3)^*,$$

$w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. (The fact that the right-hand side is in fact well defined follows immediately from the generating function reformulation of (8.27) given in (8.29) below.) Denote by $Y'_{P(z_1,z_2)}$ the action of $V \otimes \mathbb{C}[t, t^{-1}]$ on $(W_1 \otimes W_2 \otimes W_3)^*$ thus defined, that is,

$$Y'_{P(z_1,z_2)}(v, x) = \tau_{P(z_1,z_2)}(Y_t(v, x)). \tag{8.28}$$

By Lemma 8.6, (5.7) and (5.61), we see that (8.27) can be written in terms of generating functions as

$$\begin{aligned}
&\left(\tau_{P(z_1,z_2)}\left(x_1^{-1}\delta\Big(\frac{x_0^{-1}-z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y_t(v,x_0)\right)\lambda\right)(w_{(1)}\otimes w_{(2)}\otimes w_{(3)})\\
&= x_0\delta\Big(\frac{z_1+x_1}{x_0^{-1}}\Big)x_2^{-1}\delta\Big(\frac{z_0+x_1}{x_2}\Big)\cdot\\
&\qquad\qquad \lambda(Y_1((-x_0^{-2})^{L(0)}e^{-x_0^{-1}L(1)}v,x_1)w_{(1)}\otimes w_{(2)}\otimes w_{(3)})\\
&\quad +x_0\delta\Big(\frac{z_2+x_2}{x_0^{-1}}\Big)x_1^{-1}\delta\Big(\frac{-z_0+x_2}{x_1}\Big)\cdot\\
&\qquad\qquad \lambda(w_{(1)}\otimes Y_2((-x_0^{-2})^{L(0)}e^{-x_0^{-1}L(1)}v,x_2)w_{(2)}\otimes w_{(3)})\\
&\quad +x_1^{-1}\delta\Big(\frac{-z_1+x_0^{-1}}{x_1}\Big)x_2^{-1}\delta\Big(\frac{-z_2+x_0^{-1}}{x_2}\Big)\lambda(w_{(1)}\otimes w_{(2)}\otimes Y_3^o(v,x_0)w_{(3)})
\end{aligned} \tag{8.29}$$

for $v\in V$, $\lambda\in(W_1\otimes W_2\otimes W_3)^*$, $w_{(1)}\in W_1$, $w_{(2)}\in W_2$ and $w_{(3)}\in W_3$; the expansion coefficients in x_0, x_1 and x_2 of the left-hand side span the space of elements in the left-hand side of (8.27). Compare this with the motivating formula (8.21). The generating function form (8.28) of the action $Y'_{P(z_1,z_2)}$ (8.28) can be obtained by taking $\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}$ of both sides of (8.29).

Remark 8.8 The action $\tau_{P(z_1,z_2)}$ of

$$V\otimes\iota_+\mathbb{C}[t,t^{-1},(z_1^{-1}-t)^{-1},(z_2^{-1}-t)^{-1}]$$

on $(W_1\otimes W_2\otimes W_3)^*$, defined for all $z_1,z_2\in\mathbb{C}^\times$ with $z_1\neq z_2$, coincides with the action $\tau^{(1)}_{P(z_1,z_2)}$ when $|z_1|>|z_2|>0$, and coincides with the action $\tau^{(2)}_{P(z_1,z_2)}$ when $|z_2|>|z_1-z_2|>0$, where $\tau^{(1)}_{P(z_1,z_2)}$ and $\tau^{(2)}_{P(z_1,z_2)}$ are the two actions defined in Section 14 of [H2]. The action $\tau_{P(z_1,z_2)}$ and the related notion of $P(z_1,z_2)$-intertwining map extend the corresponding considerations in [H2] in a natural way.

Remark 8.9 (cf. Remark 5.4) Using the action $\tau_{P(z_1,z_2)}$, we can write the equality (8.21) as

$$\begin{aligned}
&\left(x_1^{-1}\delta\Big(\frac{x_0^{-1}-z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y'_4(v,x_0)w'_{(4)}\right)\circ F\\
&\qquad =\tau_{P(z_1,z_2)}\left(x_1^{-1}\delta\Big(\frac{x_0^{-1}-z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y_t(v,x_0)\right)(w'_{(4)}\circ F).
\end{aligned} \tag{8.30}$$

Furthermore, using the action of $V\otimes\iota_+\mathbb{C}[t,t^{-1},(z_1^{-1}-t)^{-1},(z_2^{-1}-t)^{-1}]$ on W'_4 (recall (5.1), (5.2) and (5.7)), we can also write (8.30) as

$$\begin{aligned}
&\left(\tau_{W'_4}\left(x_1^{-1}\delta\Big(\frac{x_0^{-1}-z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y_t(v,x_0)\right)w'_{(4)}\right)\circ F\\
&\qquad =\tau_{P(z_1,z_2)}\left(x_1^{-1}\delta\Big(\frac{x_0^{-1}-z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y_t(v,x_0)\right)(w'_{(4)}\circ F).
\end{aligned} \tag{8.31}$$

As in Section 5, we need to consider gradings by A and $\tilde{A}$.

The space $W_1 \otimes W_2 \otimes W_3$ is naturally $\tilde{A}$-graded, and this gives us naturally-defined subspaces $((W_1 \otimes W_2 \otimes W_3)^*)^{(\beta)}$ for $\beta \in \tilde{A}$, as in the discussion after Remark 5.4.

The space (8.26) is naturally A-graded, from the A-grading on V: For $\alpha \in A$,

$$(V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z_1^{-1}-t)^{-1}, (z_2^{-1}-t)^{-1}])^{(\alpha)} = V^{(\alpha)} \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z_1^{-1}-t)^{-1}, (z_2^{-1}-t)^{-1}]. \quad (8.32)$$

Definition 8.10 We call a linear action τ of

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z_1^{-1} - t)^{-1}, (z_2^{-1} - t)^{-1}]$$

on $(W_1 \otimes W_2 \otimes W_3)^*$ *$\tilde{A}$-compatible* if for $\alpha \in A$, $\beta \in \tilde{A}$,

$$\xi \in (V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z_1^{-1} - t)^{-1}, (z_2^{-1} - t)^{-1}])^{(\alpha)}$$

and $\lambda \in ((W_1 \otimes W_2 \otimes W_3)^*)^{(\beta)}$,

$$\tau(\xi)\lambda \in ((W_1 \otimes W_2 \otimes W_3)^*)^{(\alpha+\beta)}.$$

From (8.27) or (8.29), we have:

Proposition 8.11 *The action $\tau_{P(z_1,z_2)}$ is $\tilde{A}$-compatible.* □

Again as in Section 5, when V is a conformal vertex algebra, we write

$$Y'_{P(z_1,z_2)}(\omega, x) = \sum_{n \in \mathbb{Z}} L'_{P(z_1,z_2)}(n) x^{-n-2}.$$

In this case, by setting $v = \omega$ in (8.29) and taking $\mathrm{Res}_{x_0} x_0{}^{j+1} \mathrm{Res}_{x_1} \mathrm{Res}_{x_2}$ for $j = -1, 0, 1$, we see that

$$\begin{aligned}
&(L'_{P(z_1,z_2)}(j)\lambda)(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
&\quad = \lambda\bigg(\bigg(\sum_{i=0}^{1-j} \binom{1-j}{i} z_1^i L(-j-i)\bigg) w_{(1)} \otimes w_{(2)} \otimes w_{(3)} \\
&\qquad + \sum_{i=0}^{1-j} \binom{1-j}{i} z_2^i w_{(1)} \otimes L(-j-i) w_{(2)} \otimes w_{(3)} \\
&\qquad + w_{(1)} \otimes w_{(2)} \otimes L(-j) w_{(3)}\bigg).
\end{aligned} \quad (8.33)$$

If V is a Möbius vertex algebra, we define the actions $L'_{P(z_1,z_2)}(j)$ on $(W_1 \otimes W_2 \otimes W_3)^*$ by (8.33) for $j = -1, 0$ and 1. Using these notations, the $\mathfrak{sl}(2)$-bracket relations (8.18) for a $P(z_1, z_2)$-intertwining map F can be written as

$$(L'(j)w'_{(4)}) \circ F = L'_{P(z_1,z_2)}(j)(w'_{(4)} \circ F) \quad (8.34)$$

for $w'_{(4)} \in W'_4$, $j = -1, 0, 1$ (cf. Remarks 5.12 and 8.9). We have

$$L'_{P(z_1,z_2)}(j)((W_1 \otimes W_2 \otimes W_3)^*)^{(\beta)} \subset ((W_1 \otimes W_2 \otimes W_3)^*)^{(\beta)}$$

for $j = -1, 0, 1$ and $\beta \in \tilde{A}$ (cf. Remark 5.13 and Proposition 8.11).

For the natural analogue of Proposition 5.24 (see Proposition 8.16 below), we shall use the following analogues of the relevant notions in Sections 4 and 5: A map

$$F \in \mathrm{Hom}(W_1 \otimes W_2 \otimes W_3, (W'_4)^*)$$

is $\tilde{A}$-*compatible* if

$$F \in \mathrm{Hom}(W_1 \otimes W_2 \otimes W_3, \overline{W}_4)$$

and if F satisfies the natural analogue of the condition in (4.80), as in (8.15). A map

$$G \in \mathrm{Hom}(W'_4, (W_1 \otimes W_2 \otimes W_3)^*)$$

is $\tilde{A}$-*compatible* if G satisfies the analogue of (5.126). Then just as in Lemma 5.17 and Remark 5.18:

Remark 8.12 We have a canonical isomorphism from the space of $\tilde{A}$-compatible linear maps

$$F : W_1 \otimes W_2 \otimes W_3 \to \overline{W}_4$$

to the space of $\tilde{A}$-compatible linear maps

$$G : W'_4 \to (W_1 \otimes W_2 \otimes W_3)^*,$$

determined by:

$$\langle w'_{(4)}, F(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})\rangle = G(w'_{(4)})(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \tag{8.35}$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$, or equivalently,

$$w'_{(4)} \circ F = G(w'_{(4)}) \tag{8.36}$$

for $w'_{(4)} \in W'_4$.

We also have the natural analogues of Definition 5.19 and Remarks 5.20 and 5.21:

Definition 8.13 A map $G \in \mathrm{Hom}(W'_4, (W_1 \otimes W_2 \otimes W_3)^*)$ is *grading restricted* if for $n \in \mathbb{C}$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$,

$$G((W'_4)_{[n-m]})(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) = 0 \quad \text{for } \; m \in \mathbb{N} \; \text{ sufficiently large.} \tag{8.37}$$

Remark 8.14 If $G \in \mathrm{Hom}(W'_4, (W_1 \otimes W_2 \otimes W_3)^*)$ is $\tilde{A}$-compatible, then G is also grading restricted.

Remark 8.15 If in addition W_4 (and W'_4) are lower bounded, then the stronger condition

$$G((W'_4)_{[n]})(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) = 0 \quad \text{for} \;\; \Re(n) \;\; \text{sufficiently negative} \tag{8.38}$$

holds.

As in Proposition 5.24 we now have:

Proposition 8.16 *Let $z_1, z_2 \in \mathbb{C}^\times$, $z_1 \neq z_2$. Let W_1, W_2, W_3 and W_4 be generalized V-modules. Then under the canonical isomorphism described in Remark 8.12, the $P(z_1, z_2)$-intertwining maps F correspond exactly to the (grading restricted) $\tilde{A}$-compatible maps G that intertwine the actions of*

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z_1^{-1} - t)^{-1}, (z_2^{-1} - t)^{-1}]$$

and of $L'(j)$ and $L'_{P(z_1,z_2)}(j)$, $j = -1, 0, 1$, on W'_4 and on $(W_1 \otimes W_2 \otimes W_3)^*$. If W_4 is lower bounded, we may replace the grading restrictions by (8.19) and (8.38).

Proof By (8.36), Remark 8.9 asserts that (8.21), or equivalently, (8.17), is equivalent to the condition

$$\begin{aligned} G&\left(\tau_{W'_4}\left(x_1^{-1}\delta\Big(\frac{x_0^{-1} - z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1} - z_2}{x_2}\Big)Y_t(v, x_0)\right) w'_{(4)}\right) \\ &= \tau_{P(z_1,z_2)}\left(x_1^{-1}\delta\Big(\frac{x_0^{-1} - z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1} - z_2}{x_2}\Big)Y_t(v, x_0)\right) G(w'_{(4)}), \end{aligned} \tag{8.39}$$

that is, the condition that G intertwines the actions of

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z_1^{-1} - t)^{-1}, (z_2^{-1} - t)^{-1}]$$

on W'_4 and on $(W_1 \otimes W_2 \otimes W_3)^*$. Analogously, from (8.34) we see that (8.18) is equivalent to the condition

$$G(L'(j)w'_{(4)}) = L'_{P(z_1,z_2)}(j)G(w'_{(4)}) \tag{8.40}$$

for $j = -1, 0, 1$, that is, the condition that G intertwines the actions of $L'(j)$ and $L'_{P(z_1,z_2)}(j)$.
□

Let W_1, W_2 and W_3 be generalized V-modules. By analogy with (5.142) and (5.143), we have the spaces

$$((W_1 \otimes W_2 \otimes W_3)^*)_{[\mathbb{C}]}^{(\tilde{A})} = \coprod_{n \in \mathbb{C}} \coprod_{\beta \in \tilde{A}} ((W_1 \otimes W_2 \otimes W_3)^*)_{[n]}^{(\beta)} \subset (W_1 \otimes W_2 \otimes W_3)^* \tag{8.41}$$

and

$$((W_1 \otimes W_2 \otimes W_3)^*)_{(\mathbb{C})}^{(\tilde{A})} = \coprod_{n \in \mathbb{C}} \coprod_{\beta \in \tilde{A}} ((W_1 \otimes W_2 \otimes W_3)^*)_{(n)}^{(\beta)} \subset (W_1 \otimes W_2 \otimes W_3)^*, \tag{8.42}$$

defined by means of the operator $L'_{P(z_1,z_2)}(0)$. Each space

$$((W_1 \otimes W_2 \otimes W_3)^*)^{(\beta)} \tag{8.43}$$

is defined by analogy with (5.88).

Again by analogy with the situation in Section 5, consider the following conditions for elements

$$\lambda \in (W_1 \otimes W_2 \otimes W_3)^* :$$

The $P(z_1, z_2)$-compatibility condition

(a) The $P(z_1, z_2)$-*lower truncation condition*: For all $v \in V$, the formal Laurent series $Y'_{P(z_1,z_2)}(v, x)\lambda$ involves only finitely many negative powers of x.

(b) The following formula holds for all $v \in V$:

$$\begin{aligned}
&\tau_{P(z_1,z_2)}\Big(x_1^{-1}\delta\Big(\frac{x_0^{-1}-z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y_t(v,x_0)\Big)\lambda\\
&\quad = x_1^{-1}\delta\Big(\frac{x_0^{-1}-z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y'_{P(z_1,z_2)}(v,x_0)\lambda.
\end{aligned} \tag{8.44}$$

(Note that the two sides of (8.44) are not *a priori* equal for general $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$. Condition (a) implies that the right-hand side in Condition (b) is well defined.)

The $P(z_1, z_2)$-local grading restriction condition

(a) The $P(z_1, z_2)$-*grading condition*: There exists a doubly graded subspace of the space (8.41) containing λ and stable under the component operators $\tau_{P(z_1,z_2)}(v \otimes t^m)$ of the operators $Y'_{P(z_1,z_2)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{P(z_1,z_2)}(-1)$, $L'_{P(z_1,z_2)}(0)$ and $L'_{P(z_1,z_2)}(1)$. In particular, λ is a (finite) sum of generalized eigenvectors for $L'_{P(z_1,z_2)}(0)$ that are also homogeneous with respect to $\tilde{A}$.

(b) Let $W_{\lambda;P(z_1,z_2)}$ be the smallest doubly graded (or equivalently, $\tilde{A}$-graded) subspace of the space (8.41) containing λ and stable under the component operators $\tau_{P(z_1,z_2)}(v \otimes t^m)$ of the operators $Y'_{P(z_1,z_2)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{P(z_1,z_2)}(-1)$, $L'_{P(z_1,z_2)}(0)$ and $L'_{P(z_1,z_2)}(1)$ (the existence being guaranteed by Condition (a)). Then $W_{\lambda;P(z_1,z_2)}$ has the properties

$$\dim(W_{\lambda;P(z_1,z_2)})^{(\beta)}_{[n]} < \infty, \tag{8.45}$$

$$(W_{\lambda;P(z_1,z_2)})^{(\beta)}_{[n+k]} = 0 \quad \text{for} \quad k \in \mathbb{Z} \quad \text{sufficiently negative,} \tag{8.46}$$

for any $n \in \mathbb{C}$ and $\beta \in \tilde{A}$, where the subscripts denote the $\mathbb{C}$-grading by (generalized) $L'_{P(z_1,z_2)}(0)$-eigenvalues and the superscripts denote the $\tilde{A}$-grading.

The $L(0)$-semisimple $P(z_1, z_2)$-local grading restriction condition

(a) The $L(0)$*-semisimple* $P(z_1, z_2)$*-grading condition*: There exists a doubly graded subspace of the space (8.42) containing λ and stable under the component operators $\tau_{P(z_1,z_2)}(v \otimes t^m)$ of the operators $Y'_{P(z_1,z_2)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{P(z_1,z_2)}(-1)$, $L'_{P(z_1,z_2)}(0)$ and $L'_{P(z_1,z_2)}(1)$. In particular, λ is a (finite) sum of eigenvectors for $L'_{P(z_1,z_2)}(0)$ that are also homogeneous with respect to $\tilde{A}$.

(b) Consider $W_{\lambda;P(z_1,z_2)}$ as above, which in this case is in fact the smallest doubly graded subspace of the space (8.42) containing λ and stable under the component operators $\tau_{P(z_1,z_2)}(v \otimes t^m)$ of the operators $Y'_{P(z_1,z_2)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{P(z_1,z_2)}(-1)$, $L'_{P(z_1,z_2)}(0)$ and $L'_{P(z_1,z_2)}(1)$. Then $W_{\lambda;P(z_1,z_2)}$ has the properties

$$\dim(W_{\lambda;P(z_1,z_2)})^{(\beta)}_{(n)} < \infty, \tag{8.47}$$

$$(W_{\lambda;P(z_1,z_2)})^{(\beta)}_{(n+k)} = 0 \quad \text{for } \ k \in \mathbb{Z} \ \text{ sufficiently negative,} \tag{8.48}$$

for any $n \in \mathbb{C}$ and $\beta \in \tilde{A}$, where the subscripts denote the $\mathbb{C}$-grading by $L'_{P(z_1,z_2)}(0)$-eigenvalues and the superscripts denote the $\tilde{A}$-grading.

Then we have the following, by analogy with the comments preceding the statement of the $P(z)$-compatibility condition (recall (5.140)) and the $P(z)$-local grading restriction conditions:

Proposition 8.17 *Suppose that* $G \in \mathrm{Hom}(W'_4, (W_1 \otimes W_2 \otimes W_3)^*)$ *corresponds to a* $P(z_1, z_2)$*-intertwining map as in Proposition 8.16. Then for any* $w'_{(4)} \in W'_4$*,* $G(w'_{(4)})$ *satisfies the* $P(z_1, z_2)$*-compatibility condition and the* $P(z_1, z_2)$*-local grading restriction condition. If* W_4 *is an ordinary* V*-module, then* $G(w'_{(4)})$ *satisfies the* $L(0)$*-semisimple* $P(z_1, z_2)$*-local grading restriction condition.*

Proof For any $w'_{(4)} \in W'_4$, the fact that $G(w'_{(4)})$ satisfies the $P(z_1, z_2)$-compatibility condition follows from (8.39), just as in (5.140). Since G in particular intertwines the actions of $V \otimes \mathbb{C}[t, t^{-1}]$ and of the $L(j)$-operators and is $\tilde{A}$-compatible, $G(W'_4)$ is a generalized V-module and thus $G(w'_{(4)})$ satisfies the $P(z_1, z_2)$-local grading restriction condition, and if W_4 is an ordinary V-module, then $G(W'_4)$ must also be an ordinary V-module and thus $G(w'_{(4)})$ satisfies the $L(0)$-semisimple $P(z_1, z_2)$-local grading restriction condition, just as in the comments preceding the statement of the $P(z)$-local grading restriction conditions. □

Remark 8.18 In the next section we will use the following: Assume the $P(z_1, z_2)$-compatibility condition. By (8.3) (a "purely algebraic" identity, involving no convergence issues), (8.44) can be written as

$$\tau_{P(z_1,z_2)}\Big(x_1^{-1}\delta\Big(\frac{x_2 - z_0}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1} - z_2}{x_2}\Big)Y_t(v, x_0)\Big)\lambda$$

$$= x_1^{-1}\delta\Big(\frac{x_2-z_0}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y'_{P(z_1,z_2)}(v,x_0)\lambda$$
$$= x_1^{-1}\delta\Big(\frac{x_2-z_0}{x_1}\Big)\Big(x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y'_{P(z_1,z_2)}(v,x_0)\lambda\Big), \tag{8.49}$$

and all of the indicated products exist; the definition (5.5) of $Y_t(v,x_0)$ makes it clear that the triple product in parentheses on the left-hand side exists, and the simplest way to see that the triple product in the middle expression exists is to repeat the proof (8.8) of (8.3) (with the formal variables y_1 and y_2), multiplying each step by $Y'_{P(z_1,z_2)}(v,x_0)\lambda$, whose powers of x_0 are truncated from below. We can take Res_{x_1} of (8.49) to obtain

$$\tau_{P(z_1,z_2)}\Big(x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y_t(v,x_0)\Big)\lambda = x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y'_{P(z_1,z_2)}(v,x_0)\lambda, \tag{8.50}$$

which is reminiscent of the $P(z)$-compatibility condition (5.141) for $z = z_2$. Now we can multiply both sides by $x_1^{-1}\delta\Big(\frac{x_2-z_0}{x_1}\Big)$, giving

$$x_1^{-1}\delta\Big(\frac{x_2-z_0}{x_1}\Big)\tau_{P(z_1,z_2)}\Big(x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y_t(v,x_0)\Big)\lambda$$
$$= x_1^{-1}\delta\Big(\frac{x_2-z_0}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y'_{P(z_1,z_2)}(v,x_0)\lambda,$$

and as we have seen, these products exist. Thus by (8.44) and (8.49),

$$\tau_{P(z_1,z_2)}\Big(x_1^{-1}\delta\Big(\frac{x_0^{-1}-z_1}{x_1}\Big)x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y_t(v,x_0)\Big)\lambda$$
$$= x_1^{-1}\delta\Big(\frac{x_2-z_0}{x_1}\Big)\tau_{P(z_1,z_2)}\Big(x_2^{-1}\delta\Big(\frac{x_0^{-1}-z_2}{x_2}\Big)Y_t(v,x_0)\Big)\lambda. \tag{8.51}$$

Under the assumption that tensor products exist, we can replace products and iterates of intertwining maps by corresponding products and iterates for which the intermediate module is a tensor product, and in a unique way:

Proposition 8.19 *Assume that the convergence condition for intertwining maps in $\mathcal{C}$ holds. Let W_1, W_2, W_3, W_4 and M_1 be objects of $\mathcal{C}$ and let $z_1, z_2 \in \mathbb{C}$ such that $|z_1| > |z_2| > 0$. Let $I_1 \in \mathcal{M}[P(z_1)]^{W_4}_{W_1M_1}$ and $I_2 \in \mathcal{M}[P(z_2)]^{M_1}_{W_2W_3}$, and assume that $W_2 \boxtimes_{P(z_2)} W_3$ exists (in $\mathcal{C}$). Then there exists a unique*

$$\widetilde{I}_1 \in \mathcal{M}[P(z_1)]^{W_4}_{W_1\,(W_2\boxtimes_{P(z_2)}W_3)}$$

such that

$$I_1 \circ (1_{W_1} \otimes I_2) = \widetilde{I}_1 \circ (1_{W_1} \otimes \boxtimes_{P(z_2)}).$$

Analogously, let W_1, W_2, W_3, W_4 and M_2 be objects of $\mathcal{C}$, and let $z_2, z_0 \in \mathbb{C}$ such that $|z_2| > |z_0| > 0$. Let $I^1 \in \mathcal{M}[P(z_2)]^{W_4}_{M_2W_3}$ and $I^2 \in \mathcal{M}[P(z_0)]^{M_2}_{W_1W_2}$, and assume that $W_1 \boxtimes_{P(z_0)} W_2$ exists. Then there exists a unique

$$\widetilde{I}^1 \in \mathcal{M}[P(z_2)]^{W_4}_{(W_1\boxtimes_{P(z_0)}W_2)\,W_3}$$

such that

$$I^1 \circ (I^2 \otimes 1_{W_3}) = \widetilde{I}^1 \circ (\boxtimes_{P(z_0)} \otimes 1_{W_3}).$$

Proof We prove only the first part; the second part is proved analogously.

By Proposition 4.17, I_2 corresponds naturally to an element η of $\mathrm{Hom}(W_2 \boxtimes_{P(z_2)} W_3, M_1)$ such that $I_2 = \overline{\eta} \circ \boxtimes_{P(z_2)}$. Let

$$\widetilde{I}_1 = I_1 \circ (1_{W_1} \otimes \eta).$$

Then $\widetilde{I}_1$ is a $P(z_1)$-intertwining map of type $\binom{W_4}{W_1\ (W_2 \boxtimes_{P(z_2)} W_3)}$ and we have

$$\begin{array}{rcl} I_1 \circ (1_{W_1} \otimes I_2) & = & I_1 \circ (1_{W_1} \otimes (\overline{\eta} \circ \boxtimes_{P(z_2)})) \\ & = & (I_1 \circ (1_{W_1} \otimes \eta)) \circ (1_{W_1} \otimes \boxtimes_{P(z_2)}) \\ & = & \widetilde{I}_1 \circ (1_{W_1} \otimes \boxtimes_{P(z_2)}), \end{array}$$

where these expressions are understood in the sense of Definition 7.1.

The equality

$$\langle w'_{(4)}, I_1(w_{(1)} \otimes I_2(w_{(2)} \otimes w_{(3)}))\rangle = \langle w'_{(4)}, \widetilde{I}_1(w_{(1)} \otimes (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}))\rangle$$

for all $w_{(j)} \in W_j$ and $w'_{(4)} \in W'_4$ determines the $P(z_1)$-intertwining map $\widetilde{I}_1$ uniquely. Indeed, By Proposition 7.16, this assertion uniquely determines

$$\langle w'_{(4)}, \widetilde{I}_1(w_{(1)} \otimes \pi_n(w_{(2)} \boxtimes_{P(z_2)} w_{(3)}))\rangle$$

for all $n \in \mathbb{R}$ and for all homogeneous vectors and hence for all vectors, and since the components $\pi_n(w_{(2)} \boxtimes_{P(z_2)} w_{(3)})$ span $W_2 \boxtimes_{P(z_2)} W_3$ by Proposition 4.23, $\widetilde{I}_1$ is uniquely determined. □

From Proposition 4.8, in which we take $p = 0$, we obtain the corresponding result for logarithmic intertwining operators:

Corollary 8.20 *Under the assumptions of Proposition 8.19, let* $\mathcal{Y}_1 \in \mathcal{V}^{W_4}_{W_1 M_1}$ *and* $\mathcal{Y}_2 \in \mathcal{V}^{M_1}_{W_2 W_3}$. *Then there exists a unique*

$$\widetilde{\mathcal{Y}}_1 \in \mathcal{V}^{W_4}_{W_1\ (W_2 \boxtimes_{P(z_2)} W_3)}$$

such that for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ *and* $w'_{(4)} \in W'_4$,

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, z_1)\mathcal{Y}_2(w_{(2)}, z_2)w_{(3)}\rangle = \langle w'_{(4)}, \widetilde{\mathcal{Y}}_1(w_{(1)}, z_1)\mathcal{Y}_{\boxtimes_{P(z_2)},0}(w_{(2)}, z_2)w_{(3)}\rangle$$

(recall (4.15), (4.18) and (7.16)). Analogously, let $\mathcal{Y}^1 \in \mathcal{V}^{W_4}_{M_2 W_3}$ *and* $\mathcal{Y}^2 \in \mathcal{V}^{M_2}_{W_1 W_2}$. *Then there exists a unique*

$$\widetilde{\mathcal{Y}}^1 \in \mathcal{V}^{W_4}_{(W_1 \boxtimes_{P(z_0)} W_2)\ W_3}$$

such that for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ *and* $w'_{(4)} \in W'_4$,

$$\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, z_0)w_{(2)}, z_2)w_{(3)}\rangle = \langle w'_{(4)}, \widetilde{\mathcal{Y}}^1(\mathcal{Y}_{\boxtimes_{P(z_0)},0}(w_{(1)}, z_0)w_{(2)}, z_2)w_{(3)}\rangle.$$

(recall (7.15)). □

Remark 8.21 The first half of Proposition 8.19 in fact states that the product of I_1 and I_2 can be rewritten as a new product of intertwining maps such that the intermediate object of the new product is the tensor product generalized module $W_2 \boxtimes_{P(z_2)} W_3$ and the $P(z_2)$-intertwining map is $\boxtimes_{P(z_2)}$. The second half of the proposition can be stated analogously, for iterates of intertwining maps. Corollary 8.20 states that a product or an iterate of logarithmic intertwining operators, evaluated at suitable points, can be expressed as a new product or iterate for which the intermediate object is the relevant tensor product and the second intertwining operator corresponds to the intertwining map defining the tensor product. Thus these results can be viewed as saying that the product of I_1 and I_2, or of $\mathcal{Y}_1$ and $\mathcal{Y}_2$, uniquely "factors through" $W_2 \boxtimes_{P(z_2)} W_3$ and that the iterate of I^1 and I^2, or of $\mathcal{Y}^1$ and $\mathcal{Y}^2$, uniquely "factors through" $W_1 \boxtimes_{P(z_0)} W_2$.

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