

Logarithmic tensor category theory, IV: Constructions of tensor product bifunctors and the compatibility conditions

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Abstract

This is the fourth part in a series of papers in which we introduce and develop a natural, general tensor category theory for suitable module categories for a vertex (operator) algebra. In this paper (Part IV), we give constructions of the $P(z)$ - and $Q(z)$ -tensor product bifunctors using what we call “compatibility conditions” and certain other conditions.

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In this paper, Part IV of a series of eight papers on logarithmic tensor category theory, we give constructions of the $P(z)$ - and $Q(z)$ -tensor product bifunctors using what we call “compatibility conditions” and certain other conditions. The sections, equations, theorems and so on are numbered globally in the series of papers rather than within each paper, so that for example equation (a.b) is the b-th labeled equation in Section a, which is contained in the paper indicated as follows: In Part I [HLZ1], which contains Sections 1 and 2, we give a detailed overview of our theory, state our main results and introduce the basic objects that we shall study in this work. We include a brief discussion of some of the recent applications of this theory, and also a discussion of some recent literature. In Part II [HLZ2], which contains Section 3, we develop logarithmic formal calculus and study logarithmic intertwining

operators. In Part III [HLZ3], which contains Section 4, we introduce and study intertwining maps and tensor product bifunctors. The present paper, Part IV, contains Sections 5 and 6. In Part V [HLZ4], which contains Sections 7 and 8, we study products and iterates of intertwining maps and of logarithmic intertwining operators and we begin the development of our analytic approach. In Part VI [HLZ5], which contains Sections 9 and 10, we construct the appropriate natural associativity isomorphisms between triple tensor product functors. In Part VII [HLZ6], which contains Section 11, we give sufficient conditions for the existence of the associativity isomorphisms. In Part VIII [HLZ7], which contains Section 12, we construct braided tensor category structure.

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5 Constructions of the $P(z)$ - and $Q(z)$ -tensor product bifunctors; the compatibility conditions

In Section 4 we defined and studied the $P(z)$ - and $Q(z)$ -tensor product bifunctors. But in order to prove substantial results about these bifunctors, notably, that under suitable conditions, they give rise to braided tensor category structure, we will need to give a useful, general construction of (models for) these bifunctors, when they in fact exist. This section is devoted to such constructions. Our constructions hinge on subtle conditions, including “compatibility conditions,” on elements of certain dual spaces.

The results in this section are generalizations to the setting of the present work of the constructions of the $P(z)$ - and $Q(z)$ -tensor product bifunctors in [HL1]–[HL3]. In the earlier work [HL1]–[HL3] of the first two authors, the $Q(z)$ -tensor product of two modules was studied and developed first, in [HL1] and [HL2]. The $P(z)$ -tensor product was then studied systematically in [HL3], and many proofs for the $P(z)$ case were given by using the results established for the $Q(z)$ case in [HL1] and [HL2], rather than by carrying out the subtle arguments in the $P(z)$ case itself, arguments that are similar to (but different from) those in the $Q(z)$ case. In the present section and the next section, instead of following this approach of [HL1]–[HL3], we shall construct the $P(z)$ -tensor product and $Q(z)$ -tensor product of two (generalized) modules independently. In particular, even for the finitely reductive case carried out in [HL1]–[HL3], some of the present results and proofs of the main theorems are completely new. One new result is Proposition 5.9 below, which was not stated or proved (or needed) in the finitely reductive case in [HL3]. This is proved below, by a direct argument in the $P(z)$ setting, rather than by the use of the $Q(z)$ structure. Theorems 5.44, 5.45, 5.76 and 5.77 formulated below will be proved in the next section. The proofs of Theorems 5.44 and 5.45 are new, even in the finitely reductive case.

Recall the setup and conventions in Assumption 4.1, including the categories $\mathcal{M}_{sg}$ and

$\mathcal{GM}_{sg}$.

We continue to let z be a fixed nonzero complex number.

Later in this section (Assumption 5.30), we shall take $\mathcal{C}$ to be a full subcategory of $\mathcal{M}_{sg}$ or $\mathcal{GM}_{sg}$, with certain additional properties.

5.1 Affinizations of vertex algebras and the opposite-operator map

Just as in [HL1]–[HL3], we shall use the Jacobi identity as a motivation to construct tensor products of (generalized) V -modules in a suitable category. To do this, we need to study various “affinizations” of a vertex algebra with respect to certain algebras and vector spaces of formal Laurent series and formal rational functions. The treatment of these matters below is very similar to that in [HL1], but here we must take into account the gradings by A and $\tilde{A}$. Here, as in Section 2 above, we are replacing the symbol $*$ for the “opposite-operator map” in [HL1] by o .

In Sections 5.2 and 5.3 below, we will be using the material in this section to construct certain actions $\tau_{P(z)}$ and $\tau_{Q(z)}$ (see Definitions 5.3 and 5.51 as well as (5.110) and (5.174)), in order to construct $P(z)$ - and $Q(z)$ -tensor products. These actions will be used to reformulate the notions of $P(z)$ - and $Q(z)$ -intertwining maps (see Propositions 5.24 and 5.60), and these reformulations will enable us to correspondingly reformulate the notions of $P(z)$ - and $Q(z)$ -tensor products (see Corollaries 5.28 and 5.64). This in turn will lead to the desired constructions of $P(z)$ - and $Q(z)$ -tensor products when they in fact exist (see Propositions 5.37 and 5.69 and Theorems 5.50 and 5.80).

Let (W, Y_W) be a generalized V -module. We adjoin the formal variable t to our list of commuting formal variables. This variable will play a special role. Consider the vector spaces

$$V[t, t^{-1}] = V \otimes \mathbb{C}[t, t^{-1}] \subset V \otimes \mathbb{C}((t)) \subset V \otimes \mathbb{C}[[t, t^{-1}]] \subset V[[t, t^{-1}]]$$

(note carefully the distinction between the last two, since V is typically infinite-dimensional) and $W \otimes \mathbb{C}\{t\} \subset W\{t\}$ (recall (2.1)). The linear map

$$\begin{aligned} \tau_W : V[t, t^{-1}] &\rightarrow \text{End } W \\ v \otimes t^n &\mapsto v_n \end{aligned} \tag{5.1}$$

($v \in V, n \in \mathbb{Z}$) extends canonically to

$$\begin{aligned} \tau_W : V \otimes \mathbb{C}((t)) &\rightarrow \text{End } W \\ v \otimes \sum_{n > N} a_n t^n &\mapsto \sum_{n > N} a_n v_n \end{aligned} \tag{5.2}$$

(but not to $V((t))$), in view of (2.49) and Assumption 4.1. It further extends canonically to

$$\tau_W : (V \otimes \mathbb{C}((t)))[[x, x^{-1}]] \rightarrow (\text{End } W)[[x, x^{-1}]], \tag{5.3}$$

where of course $(V \otimes \mathbb{C}((t)))[[x, x^{-1}]]$ can be viewed as the subspace of $V[[t, t^{-1}, x, x^{-1}]]$ such that the coefficient of each power of x lies in $V \otimes \mathbb{C}((t))$.

Let $v \in V$ and define the “generic vertex operator”

$$Y_t(v, x) = \sum_{n \in \mathbb{Z}} (v \otimes t^n) x^{-n-1} \in (V \otimes \mathbb{C}[t, t^{-1}])[[x, x^{-1}]]. \quad (5.4)$$

Then

$$\begin{aligned} Y_t(v, x) &= v \otimes x^{-1} \delta\left(\frac{t}{x}\right) \\ &= v \otimes t^{-1} \delta\left(\frac{x}{t}\right) \\ &\in V \otimes \mathbb{C}[[t, t^{-1}, x, x^{-1}]] \\ &(\subset V[[t, t^{-1}, x, x^{-1}]]), \end{aligned} \quad (5.5)$$

and the linear map

$$\begin{aligned} V &\rightarrow V \otimes \mathbb{C}[[t, t^{-1}, x, x^{-1}]] \\ v &\mapsto Y_t(v, x) \end{aligned} \quad (5.6)$$

is simply the map given by tensoring by the “universal element” $x^{-1} \delta\left(\frac{t}{x}\right)$. We have

$$\tau_W(Y_t(v, x)) = Y_W(v, x). \quad (5.7)$$

For all $f(x) \in \mathbb{C}[[x, x^{-1}]]$, $f(x)Y_t(v, x)$ is defined and

$$f(x)Y_t(v, x) = f(t)Y_t(v, x); \quad (5.8)$$

it is crucial to keep in mind the delta-function substitution principles (2.5) and (2.11), which we will be using regularly.

In case $f(x) \in \mathbb{C}((x))$, then $\tau_W(f(x)Y_t(v, x))$ is also defined, and

$$f(x)Y_W(v, x) = f(x)\tau_W(Y_t(v, x)) = \tau_W(f(x)Y_t(v, x)) = \tau_W(f(t)Y_t(v, x)). \quad (5.9)$$

The expansion coefficients, in powers of x , of $Y_t(v, x)$ span $v \otimes \mathbb{C}[t, t^{-1}]$, the x -expansion coefficients of $Y_W(v, x)$ span $\tau_W(v \otimes \mathbb{C}[t, t^{-1}])$ and for $f(x) \in \mathbb{C}[[x, x^{-1}]]$, the x -expansion coefficients of $f(x)Y_t(v, x)$ span $v \otimes f(t)\mathbb{C}[t, t^{-1}]$. In case $f(x) \in \mathbb{C}((x))$, the x -expansion coefficients of $f(x)Y_W(v, x)$ span $\tau_W(v \otimes f(t)\mathbb{C}[t, t^{-1}])$.

Using this viewpoint, we shall examine each of the three terms in the Jacobi identity (3.26) in the definition of logarithmic intertwining operator. First we consider the formal Laurent series in x_0, x_1, x_2 and t given by

$$\begin{aligned} x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y_t(v, x_0) &= x_1^{-1} \delta\left(\frac{x_2 + x_0}{x_1}\right) Y_t(v, x_0) \\ &= v \otimes x_1^{-1} \delta\left(\frac{x_2 + t}{x_1}\right) x_0^{-1} \delta\left(\frac{t}{x_0}\right) \end{aligned} \quad (5.10)$$

(cf. the right-hand side of (3.26)). The expansion coefficients in powers of x_0 , x_1 and x_2 of (5.10) span just the space $v \otimes \mathbb{C}[t, t^{-1}]$. However, the expansion coefficients in x_0 and x_1 only (but not in x_2) of

$$\begin{aligned} x_1^{-1} \delta \left(\frac{x_2 + x_0}{x_1} \right) Y_t(v, x_0) &= v \otimes x_1^{-1} \delta \left(\frac{x_2 + t}{x_1} \right) x_0^{-1} \delta \left(\frac{t}{x_0} \right) \\ &= v \otimes \left(\sum_{m \in \mathbb{Z}} (x_2 + t)^m x_1^{-m-1} \right) \left(\sum_{n \in \mathbb{Z}} t^n x_0^{-n-1} \right) \end{aligned} \quad (5.11)$$

span

$$v \otimes \iota_{x_2, t} \mathbb{C}[t, t^{-1}, x_2 + t, (x_2 + t)^{-1}] \subset v \otimes \mathbb{C}[x_2, x_2^{-1}](t),$$

where $\iota_{x_2, t}$ is the operation of expanding a formal rational function in the indicated algebra as a formal Laurent series involving only finitely many negative powers of t (cf. the notation ι_{12} , etc., considered at the end of Section 2). We shall use similar ι -notations below. Specifically, the coefficient of $x_0^{-n-1} x_1^{-m-1}$ ($m, n \in \mathbb{Z}$) in (5.11) is $v \otimes (x_2 + t)^m t^n$.

We may specialize $x_2 \mapsto z \in \mathbb{C}^\times$, and (5.11) becomes

$$\begin{aligned} z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) &= x_1^{-1} \delta \left(\frac{z + x_0}{x_1} \right) Y_t(v, x_0) \\ &= v \otimes x_1^{-1} \delta \left(\frac{z + t}{x_1} \right) x_0^{-1} \delta \left(\frac{t}{x_0} \right) \\ &= v \otimes \left(\sum_{m \in \mathbb{Z}} (z + t)^m x_1^{-m-1} \right) \left(\sum_{n \in \mathbb{Z}} t^n x_0^{-n-1} \right). \end{aligned} \quad (5.12)$$

The coefficient of $x_0^{-n-1} x_1^{-m-1}$ ($m, n \in \mathbb{Z}$) in (5.12) is $v \otimes (z + t)^m t^n \in V \otimes \mathbb{C}((t))$, and these coefficients span

$$v \otimes \mathbb{C}[t, t^{-1}, (z + t)^{-1}] \subset v \otimes \mathbb{C}((t)). \quad (5.13)$$

Our $Q(z)$ -tensor product construction in Section 5.3 below will be based on a certain action of the space $V \otimes \mathbb{C}[t, t^{-1}, (z + t)^{-1}]$, and the description of this space as the span of the coefficients of the expression (5.12) (as $v \in V$ varies) will be very useful.

Now consider

$$\begin{aligned} x_0^{-1} \delta \left(\frac{-x_2 + x_1}{x_0} \right) Y_t(v, x_1) &= v \otimes x_0^{-1} \delta \left(\frac{-x_2 + t}{x_0} \right) x_1^{-1} \delta \left(\frac{t}{x_1} \right) \\ &= v \otimes \left(\sum_{n \in \mathbb{Z}} (-x_2 + t)^n x_0^{-n-1} \right) \left(\sum_{m \in \mathbb{Z}} t^m x_1^{-m-1} \right) \end{aligned} \quad (5.14)$$

(cf. the second term on the left-hand side of (3.26)). The expansion coefficients in powers of x_0 and x_1 (but not x_2) span

$$v \otimes \iota_{x_2, t} \mathbb{C}[t, t^{-1}, -x_2 + t, (-x_2 + t)^{-1}],$$

and in fact the coefficient of $x_0^{-n-1}x_1^{-m-1}$ ($m, n \in \mathbb{Z}$) in (5.14) is $v \otimes (-x_2 + t)^n t^m$. Again specializing $x_2 \mapsto z \in \mathbb{C}^\times$, we obtain

$$\begin{aligned} x_0^{-1} \delta \left(\frac{-z + x_1}{x_0} \right) Y_t(v, x_1) &= v \otimes x_0^{-1} \delta \left(\frac{-z + t}{x_0} \right) x_1^{-1} \delta \left(\frac{t}{x_1} \right) \\ &= v \otimes \left(\sum_{n \in \mathbb{Z}} (-z + t)^n x_0^{-n-1} \right) \left(\sum_{m \in \mathbb{Z}} t^m x_1^{-m-1} \right). \end{aligned} \quad (5.15)$$

The coefficient of $x_0^{-n-1}x_1^{-m-1}$ ($m, n \in \mathbb{Z}$) in (5.15) is $v \otimes (-z + t)^n t^m$, and these coefficients span

$$v \otimes \mathbb{C}[t, t^{-1}, (-z + t)^{-1}] \subset v \otimes \mathbb{C}((t)). \quad (5.16)$$

Finally, consider

$$x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_t(v, x_1) = v \otimes x_0^{-1} \delta \left(\frac{t - x_2}{x_0} \right) x_1^{-1} \delta \left(\frac{t}{x_1} \right) \quad (5.17)$$

(cf. the first term on the left-hand side of (3.26)). The coefficient of $x_0^{-n-1}x_1^{-m-1}$ ($m, n \in \mathbb{Z}$) is $v \otimes (t - x_2)^n t^m$, and these expansion coefficients span

$$v \otimes \iota_{t, x_2} \mathbb{C}[t, t^{-1}, t - x_2, (t - x_2)^{-1}].$$

If we again specialize $x_2 \mapsto z$, we get

$$x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) Y_t(v, x_1) = v \otimes x_0^{-1} \delta \left(\frac{t - z}{x_0} \right) x_1^{-1} \delta \left(\frac{t}{x_1} \right), \quad (5.18)$$

whose coefficient of $x_0^{-n-1}x_1^{-m-1}$ is $v \otimes (t - z)^n t^m$. These coefficients span

$$v \otimes \mathbb{C}[t, t^{-1}, (t - z)^{-1}] \subset v \otimes \mathbb{C}((t^{-1})) \quad (5.19)$$

(cf. (5.13), (5.16)).

In the construction of $P(z)$ -tensor products in Section 5.2, we shall also need the following expression, which is slightly different from what we have just analyzed:

$$x_0^{-1} \delta \left(\frac{x_1^{-1} - x_2}{x_0} \right) Y_t(v, x_1) = v \otimes x_0^{-1} \delta \left(\frac{t^{-1} - x_2}{x_0} \right) x_1^{-1} \delta \left(\frac{t}{x_1} \right). \quad (5.20)$$

The coefficient of $x_0^{-n-1}x_1^{-m-1}$ ($m, n \in \mathbb{Z}$) is $v \otimes (t^{-1} - x_2)^n t^m$, and these expansion coefficients span

$$v \otimes \iota_{t^{-1}, x_2} \mathbb{C}[t, t^{-1}, t^{-1} - x_2, (t^{-1} - x_2)^{-1}].$$

If we again specialize $x_2 \mapsto z$, we get

$$x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) = v \otimes x_0^{-1} \delta \left(\frac{t^{-1} - z}{x_0} \right) x_1^{-1} \delta \left(\frac{t}{x_1} \right), \quad (5.21)$$

whose coefficient of $x_0^{-n-1}x_1^{-m-1}$ is $v \otimes (t^{-1} - z)^n t^m$. These coefficients span

$$v \otimes \mathbb{C}[t, t^{-1}, (t^{-1} - z)^{-1}] = v \otimes \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}] \subset v \otimes \mathbb{C}((t)). \quad (5.22)$$

Our $P(z)$ -tensor product construction in Section 5.2 below will be based on a certain action of the space $V \otimes \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ (see Definition 5.3 below).

In fact, we shall be evaluating the Jacobi identity (3.26), or more specifically, (4.4), on the elements of the contragredient module W'_3 , and this will in particular allow us to convert the expansion (5.19) into an expansion in positive powers of t , as in the previous paragraph; see (5.81) and Definition 5.3. For describing the action given in Definition 5.3, it will be useful to examine the notions of opposite and contragredient vertex operators Y^o and Y' more closely (recall Section 2, in particular, (2.57), (2.74) and Theorem 2.34).

We shall interpret the opposite vertex operator map Y_W^o by means of an operation on $V \otimes \mathbb{C}[[t, t^{-1}]]$ that will convert vertex operators into their opposites. We shall write this “opposite-operator” map, in various contexts, as “ o .” The operation o will be an involution. We proceed as follows: First we generalize Y^o in the following way: Recall that by Assumption 4.1, $L(1)$ acts nilpotently on any element $v \in V$. In particular, $e^{xL(1)}v$ is a polynomial in the formal variable x . Given any vector space U and any linear map

$$\begin{aligned} Z(\cdot, x) : V &\rightarrow U[[x, x^{-1}]] \quad \left(= \prod_{n \in \mathbb{Z}} U \otimes x^n \right) \\ v &\mapsto Z(v, x) \end{aligned} \quad (5.23)$$

from V into $U[[x, x^{-1}]]$ (i.e., given any family of linear maps from V into the spaces $U \otimes x^n$), we define $Z^o(\cdot, x) : V \rightarrow U[[x, x^{-1}]]$ by

$$Z^o(v, x) = Z(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}), \quad (5.24)$$

where we use the obvious linear map $Z(\cdot, x^{-1}) : V \rightarrow U[[x, x^{-1}]]$, and where we extend $Z(\cdot, x^{-1})$ canonically to a linear map $Z(\cdot, x^{-1}) : V[x, x^{-1}] \rightarrow U[[x, x^{-1}]]$. Then by formula (5.3.1) in [FHL] (the proof of Proposition 5.3.1), we have

$$\begin{aligned} Z^{oo}(v, x) &= Z^o(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) \\ &= Z(e^{x^{-1}L(1)}(-x^2)^{L(0)}e^{xL(1)}(-x^{-2})^{L(0)}v, x) \\ &= Z(v, x). \end{aligned} \quad (5.25)$$

That is,

$$Z^{oo}(\cdot, x) = Z(\cdot, x). \quad (5.26)$$

Moreover, if $Z(v, x) \in U((x))$, then $Z^o(v, x) \in U((x^{-1}))$ and vice versa.

Now we expand $Z(v, x)$ and $Z^o(v, x)$ in components. Write

$$Z(v, x) = \sum_{n \in \mathbb{Z}} v_{(n)} x^{-n-1}, \quad (5.27)$$

where for all $n \in \mathbb{Z}$,

$$\begin{aligned} V &\rightarrow U \\ v &\mapsto v_{(n)} \end{aligned} \quad (5.28)$$

is a linear map depending on $Z(\cdot, x)$ (and in fact, as $Z(\cdot, x)$ varies, these linear maps are arbitrary). Also write

$$Z^o(v, x) = \sum_{n \in \mathbb{Z}} v_{(n)}^o x^{-n-1} \quad (5.29)$$

where

$$\begin{aligned} V &\rightarrow U \\ v &\mapsto v_{(n)}^o \end{aligned} \quad (5.30)$$

is a linear map depending on $Z(\cdot, x)$. We shall compute $v_{(n)}^o$. First note that

$$\sum_{n \in \mathbb{Z}} v_{(n)}^o x^{-n-1} = \sum_{n \in \mathbb{Z}} (e^{xL(1)}(-x^{-2})^{L(0)}v)_{(n)} x^{n+1}. \quad (5.31)$$

For convenience, suppose that $v \in V_{(h)}$, for $h \in \mathbb{Z}$. Then the right-hand side of (5.31) is equal to

$$\begin{aligned} &(-1)^h \sum_{n \in \mathbb{Z}} (e^{xL(1)}v)_{(-n)} x^{-n+1-2h} \\ &= (-1)^h \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \frac{1}{m!} (L(1)^m v)_{(-n)} x^{m-n+1-2h} \\ &= (-1)^h \sum_{m \in \mathbb{N}} \frac{1}{m!} \sum_{n \in \mathbb{Z}} (L(1)^m v)_{(-n-m-2+2h)} x^{-n-1}, \end{aligned} \quad (5.32)$$

that is,

$$v_{(n)}^o = (-1)^h \sum_{m \in \mathbb{N}} \frac{1}{m!} (L(1)^m v)_{(-n-m-2+2h)}. \quad (5.33)$$

(Recall that by Assumption 4.1, $L(1)^m v = 0$ when m is sufficiently large, so that these expressions are well defined.) For $v \in V$ not necessarily homogeneous, $v_{(n)}^o$ is given by the appropriate sum of such expressions.

Now consider the special case where $U = V \otimes \mathbb{C}[t, t^{-1}]$ and where $Z(\cdot, x)$ is the “generic” linear map

$$\begin{aligned} Y_t(\cdot, x) : V &\rightarrow (V \otimes \mathbb{C}[t, t^{-1}])[[x, x^{-1}]] \\ v &\mapsto Y_t(v, x) = \sum_{n \in \mathbb{Z}} (v \otimes t^n) x^{-n-1} \end{aligned} \quad (5.34)$$

(recall (5.4)), i.e.,

$$v_{(n)} = v \otimes t^n. \quad (5.35)$$

Then for $v \in V_{(h)}$,

$$v_{(n)}^o = (-1)^h \sum_{m \in \mathbb{N}} \frac{1}{m!} ((L(1))^m v) \otimes t^{-n-m-2+2h} \quad (5.36)$$

in this case.

This motivates defining an o -operation on $V \otimes \mathbb{C}[t, t^{-1}]$ as follows: For any $n, h \in \mathbb{Z}$ and $v \in V_{(h)}$, define

$$(v \otimes t^n)^o = (-1)^h \sum_{m \in \mathbb{N}} \frac{1}{m!} (L(1)^m v) \otimes t^{-n-m-2+2h} \in V \otimes \mathbb{C}[t, t^{-1}], \quad (5.37)$$

and extend by linearity to $V \otimes \mathbb{C}[t, t^{-1}]$. That is, $(v \otimes t^n)^o = v_{(n)}^o$ for the special case $Z(\cdot, x) = Y_t(\cdot, x)$ discussed above. (Note that for general Z , we cannot expect to be able to define an analogous o -operation on U .) Also consider the map

$$\begin{aligned} Y_t^o(\cdot, x) &= (Y_t(\cdot, x))^o : V \rightarrow (V \otimes \mathbb{C}[t, t^{-1}])[[x, x^{-1}]] \\ v &\mapsto Y_t^o(v, x) = \sum_{n \in \mathbb{Z}} (v \otimes t^n)^o x^{-n-1}. \end{aligned} \quad (5.38)$$

Then for general $Z(\cdot, x)$ as above, we can define a linear map

$$\begin{aligned} \varepsilon_Z : V \otimes \mathbb{C}[t, t^{-1}] &\rightarrow U \\ v \otimes t^n &\mapsto v_{(n)} \end{aligned} \quad (5.39)$$

(“evaluation with respect to Z ”), i.e.,

$$\varepsilon_Z : Y_t(v, x) \mapsto Z(v, x), \quad (5.40)$$

and a linear map

$$\begin{aligned} \varepsilon_Z^o : V \otimes \mathbb{C}[t, t^{-1}] &\rightarrow U \\ v \otimes t^n &\mapsto v_{(n)}^o, \end{aligned} \quad (5.41)$$

i.e.,

$$\varepsilon_Z^o : Y_t^o(v, x) \mapsto Z^o(v, x). \quad (5.42)$$

Then

$$\varepsilon_Z^o = \varepsilon_Z \circ o, \quad (5.43)$$

that is,

$$\varepsilon_Z(Y_t^o(v, x)) = Z^o(v, x), \quad (5.44)$$

or equivalently, the diagram

$$\begin{array}{ccc} Y_t(v, x) & \xrightarrow{\varepsilon_Z} & Z(v, x) \\ o \downarrow & & \downarrow (Z(\cdot, x) \mapsto Z^o(\cdot, x)) \\ Y_t^o(v, x) & \xrightarrow{\varepsilon_Z} & Z^o(v, x) \end{array} \quad (5.45)$$

commutes. Note that the components $v_{(n)}^o$ of $Z^o(v, x)$ depend on all the components $v_{(n)}$ of $Z(v, x)$ (for arbitrary v), whereas the component $(v \otimes t^n)^o$ of $Y_t^o(v, x)$ can be defined generically and abstractly; $(v \otimes t^n)^o$ depends linearly on $v \in V$ alone.

Since in general $Z^{oo}(v, x) = Z(v, x)$, we know that

$$Y_t^{oo}(v, x) = Y_t(v, x) \quad (5.46)$$

as a special case, and in particular (and equivalently),

$$(v \otimes t^n)^{oo} = v \otimes t^n \quad (5.47)$$

for all $v \in V$ and $n \in \mathbb{Z}$. Thus o is an involution of $V \otimes \mathbb{C}[t, t^{-1}]$.

Furthermore, the involution o of $V \otimes \mathbb{C}[t, t^{-1}]$ extends canonically to a linear map

$$V \otimes \mathbb{C}[[t, t^{-1}]] \xrightarrow{o} V \otimes \mathbb{C}[[t, t^{-1}]].$$

In fact, consider the restriction of o to $V = V \otimes t^0$:

$$\begin{aligned} V &\xrightarrow{o} V \otimes \mathbb{C}[t, t^{-1}] \\ v &\mapsto v^o = (-1)^h \sum_{m \in \mathbb{N}} \frac{1}{m!} (L(1)^m v) \otimes t^{-m-2+2h}, \end{aligned} \quad (5.48)$$

extended by linearity from $V_{(h)}$ to V . Then for $v \in V$, we may write

$$v^o = e^{t^{-1}L(1)}(-t^2)^{L(0)}vt^{-2}. \quad (5.49)$$

Also, for $v \in V$ and $n \in \mathbb{Z}$,

$$(v \otimes t^n)^o = v^o t^{-n}, \quad (5.50)$$

and it is clear that o extends to $V \otimes \mathbb{C}[[t, t^{-1}]]$: For $f(t) \in \mathbb{C}[[t, t^{-1}]]$,

$$(v \otimes f(t))^o = v^o f(t^{-1}). \quad (5.51)$$

To see that o is an involution of this larger space, first note that

$$v^{oo} = v \quad (5.52)$$

(although $v^o \notin V$ in general). (This could of course alternatively be proved by direct calculation using formula (5.37).) Also, for $g(t) \in \mathbb{C}[t, t^{-1}]$ and $f(t) \in \mathbb{C}[[t, t^{-1}]]$,

$$(v \otimes g(t)f(t))^o = v^o g(t^{-1})f(t^{-1}) = (v \otimes g(t))^o f(t^{-1}). \quad (5.53)$$

Thus for all $u \in V \otimes \mathbb{C}[t, t^{-1}]$ and $f(t) \in \mathbb{C}[[t, t^{-1}]]$,

$$(uf(t))^o = u^o f(t^{-1}). \quad (5.54)$$

It follows that

$$\begin{aligned}
(v \otimes f(t))^{oo} &= (v^o f(t^{-1}))^o \\
&= v^{oo} f(t) \\
&= v f(t) \\
&= v \otimes f(t),
\end{aligned} \tag{5.55}$$

and we have shown that o is an involution of $V \otimes \mathbb{C}[[t, t^{-1}]]$. We have

$$o : V \otimes \mathbb{C}((t)) \leftrightarrow V \otimes \mathbb{C}((t^{-1})). \tag{5.56}$$

Note that

$$\begin{aligned}
Y_t^o(v, x) &= \sum_{n \in \mathbb{Z}} (v \otimes t^n)^o x^{-n-1} \\
&= v^o \sum_{n \in \mathbb{Z}} t^{-n} x^{-n-1} \\
&= v^o x^{-1} \delta(tx) \\
&= v^o t \delta(tx) \\
&\in V \otimes \mathbb{C}[[t, t^{-1}, x, x^{-1}]].
\end{aligned} \tag{5.57}$$

Thus the map $v \mapsto Y_t^o(v, x)$ is the linear map given by multiplying v^o by the “universal element” $t\delta(tx)$ (cf. the comment following (5.6)). By (5.49), we also have

$$\begin{aligned}
Y_t^o(v, x) &= e^{t^{-1}L(1)}(-t^2)^{L(0)} v t^{-1} \delta(tx) \\
&= e^{xL(1)}(-x^{-2})^{L(0)} v \otimes x \delta(tx).
\end{aligned} \tag{5.58}$$

For all $f(x) \in \mathbb{C}[[x, x^{-1}]]$, $f(x)Y_t^o(v, x)$ is defined and

$$\begin{aligned}
f(x)Y_t^o(v, x) &= f(t^{-1})Y_t^o(v, x) \\
&= v^o f(t^{-1})t\delta(tx).
\end{aligned} \tag{5.59}$$

Now we return to the starting point—the original special case: $U = \text{End } W$ and $Z(\cdot, x) = Y_W(\cdot, x) : V \rightarrow (\text{End } W)[[x, x^{-1}]]$. The corresponding map

$$\begin{aligned}
\varepsilon_Z = \varepsilon_{Y_W} : V[t, t^{-1}] &\rightarrow \text{End } W \\
v \otimes t^n &\mapsto v_{(n)}
\end{aligned} \tag{5.60}$$

(recall (5.39)) is just the map $\tau_W : v \otimes t^n \mapsto v_n$ (recall (5.1)), i.e., $v_{(n)} = v_n$ in this case. Recall that this map extends canonically to $V \otimes \mathbb{C}((t))$. The map ε_Z^o (see above) is

$$\tau_W^o = \tau_W \circ o : V \otimes \mathbb{C}[t, t^{-1}] \rightarrow \text{End } W,$$

and this map extends canonically to $V \otimes \mathbb{C}((t^{-1}))$. In addition to (5.7), we have

$$\tau_W(Y_t^o(v, x)) = Y_W^o(v, x) \tag{5.61}$$

$(v_{(n)}^o = v_n^o$ in this case; recall (2.58)). In case $f(x) \in \mathbb{C}((x^{-1}))$,

$$f(x)Y_W^o(v, x) = \tau_W(f(x)Y_t^o(v, x))$$

is defined and is equal to $\tau_W(f(t^{-1})Y_t^o(v, z))$ (which is also defined).

The x -expansion coefficients of $f(x)Y_t^o(v, x)$, for $f(x) \in \mathbb{C}[[x, x^{-1}]]$, span

$$\begin{aligned} v^o f(t^{-1})\mathbb{C}[t, t^{-1}] &= (v\mathbb{C}[t, t^{-1}])^o f(t^{-1}) \\ &= (vf(t)\mathbb{C}[t, t^{-1}])^o. \end{aligned} \quad (5.62)$$

The x -expansion coefficients of $Y_W^o(v, x)$ span

$$\begin{aligned} \tau_W(v^o\mathbb{C}[t, t^{-1}]) &= \tau_W((v \otimes \mathbb{C}[t, t^{-1}])^o) \\ &= \tau_W^o(v \otimes \mathbb{C}[t, t^{-1}]). \end{aligned} \quad (5.63)$$

In case $f(x) \in \mathbb{C}((x^{-1}))$, the x -expansion coefficients of $f(x)Y_W^o(v, x)$ span

$$\tau_W(v^o f(t^{-1})\mathbb{C}[t, t^{-1}]) = \tau_W^o(vf(t)\mathbb{C}[t, t^{-1}]).$$

(Cf. the comments after (5.9).)

We shall need spaces of the forms $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ and $V \otimes \iota_- \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$, where we use the notations

$$\begin{aligned} \iota_+ : \mathbb{C}(t) &\hookrightarrow \mathbb{C}((t)) \subset \mathbb{C}[[t, t^{-1}]] \\ \iota_- : \mathbb{C}(t) &\hookrightarrow \mathbb{C}((t^{-1})) \subset \mathbb{C}[[t, t^{-1}]] \end{aligned} \quad (5.64)$$

to denote the operations of expanding a rational function of the formal variable t in the indicated directions (as in Section 8.1 of [FLM]). We shall also need certain translation operations, as well as the o -operation. For $a \in \mathbb{C}$, we define the translation isomorphism

$$\begin{aligned} T_a : \mathbb{C}(t) &\xrightarrow{\sim} \mathbb{C}(t) \\ f(t) &\mapsto f(t+a) \end{aligned} \quad (5.65)$$

and (for our use below) we also set

$$T_a^\pm = \iota_\pm \circ T_a \circ \iota_\pm^{-1} : \iota_\pm \mathbb{C}(t) \hookrightarrow \mathbb{C}((t^\pm)). \quad (5.66)$$

(Note that the domains of these maps consist of certain series expansions of formal rational functions rather than of formal rational functions themselves.) The following lemma will be needed for our action $\tau_{P(z)}$ in Section 5.2 below (we shall sometimes write $o(Y_t(v, x_1))$ for $Y_t^o(v, x_1)$, etc.):

Lemma 5.1 *Let $z \in \mathbb{C}^\times$. Then*

$$o \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) = x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t^o(v, x_1), \quad (5.67)$$

$$(\iota_+ \circ \iota_-^{-1} \circ o) \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) = x_0^{-1} \delta \left(\frac{z - x_1^{-1}}{-x_0} \right) Y_t^o(v, x_1), \quad (5.68)$$

$$\begin{aligned} & (\iota_+ \circ T_z \circ \iota_-^{-1} \circ o) \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \\ &= z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) Y_t(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} v, x_0). \end{aligned} \quad (5.69)$$

Proof Formula (5.67) is immediate from the definition of the map o (recall (5.37)). By (5.67), (5.57) and (2.5), we have

$$\begin{aligned} & (\iota_+ \circ \iota_-^{-1} \circ o) \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \\ &= (\iota_+ \circ \iota_-^{-1}) \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t^o(v, x_1) \right) \\ &= (\iota_+ \circ \iota_-^{-1}) \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) v^o t \delta(tx_1) \right) \\ &= (\iota_+ \circ \iota_-^{-1}) \left(x_0^{-1} \delta \left(\frac{t - z}{x_0} \right) v^o t \delta(tx_1) \right) \\ &= x_0^{-1} \delta \left(\frac{z - t}{-x_0} \right) v^o t \delta(tx_1) \\ &= x_0^{-1} \delta \left(\frac{z - x_1^{-1}}{-x_0} \right) v^o t \delta(tx_1) \\ &= x_0^{-1} \delta \left(\frac{z - x_1^{-1}}{-x_0} \right) Y_t^o(v, x_1), \end{aligned} \quad (5.70)$$

proving (5.68). For (5.69), note that by (5.58), the coefficient of x_0^{-n-1} in the right-hand side of (5.67) is

$$\begin{aligned} & (x_1^{-1} - z)^n \left(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} v \otimes x_1 \delta \left(\frac{t}{x_1^{-1}} \right) \right) \\ &= (t - z)^n \left(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} v \otimes x_1 \delta \left(\frac{t}{x_1^{-1}} \right) \right). \end{aligned}$$

Acted on by $\iota_+ \circ T_z \circ \iota_-^{-1}$, this becomes

$$\begin{aligned} & t^n \left(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} v \otimes x_1 \delta \left(\frac{z + t}{x_1^{-1}} \right) \right) \\ &= z^{-1} \delta \left(\frac{x_1^{-1} - t}{z} \right) (e^{x_1 L(1)}(-x_1^{-2})^{L(0)} v \otimes t^n), \end{aligned}$$

which by (5.5) is the coefficient of x_0^{-n-1} in the right-hand side of (5.69). $\square$

We shall be interested in

$$T_{-z}^{\pm} : \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}] \hookrightarrow \mathbb{C}((t^{\pm 1})), \quad (5.71)$$

where z is an arbitrary nonzero complex number, as above. The images of these two maps are $\iota_{\pm} \mathbb{C}[t, t^{-1}, (z-t)^{-1}]$.

Extend the maps $T_{-z}^{\pm}$ to linear isomorphisms

$$T_{-z}^{\pm} : V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}] \xrightarrow{\sim} V \otimes \iota_{\pm} \mathbb{C}[t, t^{-1}, (z-t)^{-1}] \quad (5.72)$$

given by $1 \otimes T_{-z}^{\pm}$ with $T_{-z}^{\pm}$ as defined above. Note that the domain of these two maps is described by (5.12)–(5.13), that the image of the map T_{-z}^+ is described by (5.15)–(5.16) and that the image of the map T_{-z}^- is described by (5.18)–(5.19).

We have the two mutually inverse maps

$$\begin{aligned} V \otimes \iota_- \mathbb{C}[t, t^{-1}, (z-t)^{-1}] &\xrightarrow{o} V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1}-t)^{-1}] \\ v \otimes f(t) &\mapsto v^o f(t^{-1}) \end{aligned} \quad (5.73)$$

and

$$\begin{aligned} V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1}-t)^{-1}] &\xrightarrow{o} V \otimes \iota_- \mathbb{C}[t, t^{-1}, (z-t)^{-1}] \\ v \otimes f(t) &\mapsto v^o f(t^{-1}), \end{aligned} \quad (5.74)$$

which are both isomorphisms. We form the composition

$$T_{-z}^o = o \circ T_{-z}^- \quad (5.75)$$

to obtain another isomorphism

$$T_{-z}^o : V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}] \xrightarrow{\sim} V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1}-t)^{-1}].$$

The maps T_{-z}^+ and T_{-z}^o will be the main ingredients of our action $\tau_{Q(z)}$ (see Section 5.3 below). The following result asserts that T_{-z}^+ , T_{-z}^- and T_{-z}^o transform the expression (5.12) into (5.15), (5.18) and the o -transform of (5.18), respectively:

Lemma 5.2 *We have*

$$T_{-z}^+ \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) = x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) Y_t(v, x_1), \quad (5.76)$$

$$T_{-z}^- \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) = x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) Y_t(v, x_1), \quad (5.77)$$

$$T_{-z}^o \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) = x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) Y_t^o(v, x_1). \quad (5.78)$$

Proof We prove (5.76): From (5.12), the coefficient of $x_0^{-n-1}x_1^{-m-1}$ in the left-hand side of (5.76) is $T_{-z}^+(v \otimes (z+t)^m t^n)$. By the definitions,

$$T_{-z}^+(v \otimes (z+t)^m t^n) = v \otimes t^m(-(z-t))^n. \quad (5.79)$$

On the other hand, the right-hand side of (5.76) can be written as

$$v \otimes x_0^{-1} \delta\left(\frac{z-x_1}{-x_0}\right) x_1^{-1} \delta\left(\frac{t}{x_1}\right) = v \otimes x_0^{-1} \delta\left(\frac{z-t}{-x_0}\right) x_1^{-1} \delta\left(\frac{t}{x_1}\right), \quad (5.80)$$

where we have used (5.5) and the fundamental property (2.5) of the formal δ -function. The coefficient of $x_0^{-n-1}x_1^{-m-1}$ in the right-hand side of (5.80) is also $v \otimes t^m(-(z-t))^n$, proving (5.76). Formula (5.77) is proved similarly, and (5.78) is obtained from (5.77) by the application of the map o . $\square$

5.2 Constructions of $P(z)$ -tensor products

We proceed to the construction of $P(z)$ -tensor products. While one can certainly consider categories $\mathcal{C}$ in Remark 4.25 that are not closed under the contragredient functor, it is most natural to consider such categories $\mathcal{C}$ that are indeed closed under this functor (recall Notation 2.36). Our constructions of $P(z)$ -tensor products will in fact use the contragredient functor; the $P(z)$ -tensor product of (generalized) modules W_1 and W_2 will arise as the contragredient module of a certain subspace of the vector space dual $(W_1 \otimes W_2)^*$. We now present this “double-dual” approach to the construction of $P(z)$ -tensor products, generalizing the double-dual approach carried out in [HL1]–[HL3]. At first, we need not fix any subcategory $\mathcal{C}$ of $\mathcal{M}_{sg}$ or $\mathcal{GM}_{sg}$. As usual, we take $z \in \mathbb{C}^\times$.

We shall be constructing an action of the space $V \otimes \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ on the space $(W_1 \otimes W_2)^*$, given generalized V -modules W_1 and W_2 . This action will be based on the translation operations and on the o -operation discussed in the preceding section. More precisely, it is the space $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ whose action we shall define.

Let I be a $P(z)$ -intertwining map of type $\binom{W_3}{W_1 W_2}$, as in Definition 4.2. Consider the contragredient generalized V -module (W'_3, Y'_3) , recall the opposite vertex operator (2.57) and formula (2.73), and recall why the ingredients of formula (4.4) are well defined. For $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$, applying $w'_{(3)}$ to (4.4), replacing x_1 by x_1^{-1} in the resulting formula and then replacing v by $e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v$, we get:

$$\begin{aligned} & \left\langle x_0^{-1} \delta\left(\frac{x_1^{-1} - z}{x_0}\right) Y'_3(v, x_1) w'_{(3)}, I(w_{(1)} \otimes w_{(2)}) \right\rangle \\ &= \left\langle w'_{(3)}, z^{-1} \delta\left(\frac{x_1^{-1} - x_0}{z}\right) I(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v, x_0) w_{(1)} \otimes w_{(2)}) \right\rangle \\ &+ \left\langle w'_{(3)}, x_0^{-1} \delta\left(\frac{z - x_1^{-1}}{-x_0}\right) I(w_{(1)} \otimes Y_2^o(v, x_1) w_{(2)}) \right\rangle. \end{aligned} \quad (5.81)$$

We shall use this to motivate our action.

As we discussed in the preceding section (see (5.21) and (5.22)), in the left-hand side of (5.81), the coefficients of

$$x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y'_3(v, x_1) \quad (5.82)$$

in powers of x_0 and x_1 , for all $v \in V$, span

$$\tau_{W'_3}(V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]) \quad (5.83)$$

(recall (5.2) and (5.7)). Let us now define an action of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ on $(W_1 \otimes W_2)^*$.

Definition 5.3 Define the linear action $\tau_{P(z)}$ of

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$$

on $(W_1 \otimes W_2)^*$ by

$$\begin{aligned} (\tau_{P(z)}(\xi)\lambda)(w_{(1)} \otimes w_{(2)}) &= \lambda(\tau_{W_1}((\iota_+ \circ T_z \circ \iota_-^{-1} \circ o)\xi)w_{(1)} \otimes w_{(2)}) \\ &\quad + \lambda(w_{(1)} \otimes \tau_{W_2}((\iota_+ \circ \iota_-^{-1} \circ o)\xi)w_{(2)}) \end{aligned} \quad (5.84)$$

for $\xi \in V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. (The fact that the right-hand side is well defined follows immediately from the generating-function reformulation of (5.84) given in (5.86) below.) Denote by $Y'_{P(z)}$ the action of $V \otimes \mathbb{C}[t, t^{-1}]$ on $(W_1 \otimes W_2)^*$ thus defined, that is,

$$Y'_{P(z)}(v, x) = \tau_{P(z)}(Y_t(v, x)) \quad (5.85)$$

for $v \in V$.

By Lemma 5.1, (5.7) and (5.61), we see that formula (5.84) can be written in terms of generating functions as

$$\begin{aligned} & \left(\tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\ &= z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\ &\quad + x_0^{-1} \delta \left(\frac{z - x_1^{-1}}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2^o(v, x_1)w_{(2)}) \end{aligned} \quad (5.86)$$

for $v \in V$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$; note that by (5.21)–(5.22), the expansion coefficients in x_0 and x_1 of the left-hand side span the space of elements in the left-hand side of (5.84). Compare formula (5.86) with the motivating formula (5.81). The generating function form of the action $Y'_{P(z)}$ can be obtained by taking Res_{x_0} of both sides of (5.86), that is,

$$\begin{aligned} (Y'_{P(z)}(v, x_1)\lambda)(w_{(1)} \otimes w_{(2)}) &= \lambda(w_{(1)} \otimes Y_2^o(v, x_1)w_{(2)}) \\ &\quad + \text{Res}_{x_0} z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}). \end{aligned} \quad (5.87)$$

Remark 5.4 Using the actions $\tau_{W'_3}$ and $\tau_{P(z)}$, we can write (5.81) as

$$\left(x_0^{-1}\delta\left(\frac{x_1^{-1}-z}{x_0}\right)Y'_3(v, x_1)w'_{(3)}\right) \circ I = \tau_{P(z)}\left(x_0^{-1}\delta\left(\frac{x_1^{-1}-z}{x_0}\right)Y_t(v, x_1)\right)(w'_{(3)} \circ I)$$

or equivalently, as

$$\left(\tau_{W'_3}\left(x_0^{-1}\delta\left(\frac{x_1^{-1}-z}{x_0}\right)Y_t(v, x_1)\right)w'_{(3)}\right) \circ I = \tau_{P(z)}\left(x_0^{-1}\delta\left(\frac{x_1^{-1}-z}{x_0}\right)Y_t(v, x_1)\right)(w'_{(3)} \circ I).$$

In the spirit of the discussion related to Lemma 4.41, we find it natural to introduce subspaces of $(W_1 \otimes W_2)^*$ homogeneous with respect to $\tilde{A}$. Since W_1 and W_2 are $\tilde{A}$ -graded, $W_1 \otimes W_2$ also has a natural $\tilde{A}$ -grading—the tensor product grading, and we shall write $(W_1 \otimes W_2)^{(\beta)}$ for the homogeneous subspace of degree $\beta \in \tilde{A}$ of $W_1 \otimes W_2$. For $\beta \in \tilde{A}$, define

$$((W_1 \otimes W_2)^*)^{(\beta)} = \{\lambda \in (W_1 \otimes W_2)^* \mid \lambda(\tilde{w}) = 0 \text{ for } \tilde{w} \in (W_1 \otimes W_2)^{(\gamma)} \text{ with } \gamma \neq -\beta\} \quad (5.88)$$

(cf. (2.97) and note the minus sign). Of course, the full space $(W_1 \otimes W_2)^*$ is not $\tilde{A}$ -graded since it is not a direct sum of subspaces homogeneous with respect to $\tilde{A}$.

The space $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ also has an A -grading, induced from the A -grading on V : For $\alpha \in A$,

$$(V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}])^{(\alpha)} = V^{(\alpha)} \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]. \quad (5.89)$$

Using these gradings, we formulate:

Definition 5.5 We call a linear action τ of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ on $(W_1 \otimes W_2)^*$ $\tilde{A}$ -compatible if for $\alpha \in A$, $\beta \in \tilde{A}$, $\xi \in (V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}])^{(\alpha)}$ and $\lambda \in ((W_1 \otimes W_2)^*)^{(\beta)}$,

$$\tau(\xi)\lambda \in ((W_1 \otimes W_2)^*)^{(\alpha+\beta)}.$$

From (5.84) or (5.86), we have:

Proposition 5.6 *The action $\tau_{P(z)}$ is $\tilde{A}$ -compatible.* $\square$

Remark 5.7 Notice that Proposition 5.6 is analogous to the condition (2.87) in the definition of the notion of (generalized) module. We now proceed to establish several more of the module-action properties for our action $\tau_{P(z)}$ on $(W_1 \otimes W_2)^*$, in both the conformal and Möbius cases. However, while we will prove the commutator formula for our action (see Proposition 5.9 below), we will *not* be able to prove the Jacobi identity on an element $\lambda \in (W_1 \otimes W_2)^*$ until we assume the “ $P(z)$ -compatibility condition” for the element λ (see Theorem 5.44 below). We shall be constructing a certain subspace $W_1 \boxtimes_{P(z)} W_2$ of $(W_1 \otimes W_2)^*$ which under suitable conditions will be a generalized V -module and whose contragredient module will be $W_1 \boxtimes_{P(z)} W_2$ (see Remark 5.29 and Proposition 5.37), and we shall use the $P(z)$ -compatibility condition to describe this subspace (see Theorem 5.50).

We have the following result generalizing Proposition 13.3 in [HL3]:

Proposition 5.8 *The action $Y'_{P(z)}$ has the property*

$$Y'_{P(z)}(\mathbf{1}, x) = 1,$$

where $\mathbf{1}$ on the right-hand side is the identity map of $(W_1 \otimes W_2)^*$. It also has the $L(-1)$ -derivative property

$$\frac{d}{dx} Y'_{P(z)}(v, x) = Y'_{P(z)}(L(-1)v, x)$$

for $v \in V$.

Proof The first statement follows directly from the definition. We prove the $L(-1)$ -derivative property. From (5.87), we obtain, using (2.62),

$$\begin{aligned} & \left(\frac{d}{dx} Y'_{P(z)}(v, x) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\ &= \frac{d}{dx} \lambda(w_{(1)} \otimes Y_2^o(v, x) w_{(2)}) \\ & \quad + \frac{d}{dx} \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0) w_{(1)} \otimes w_{(2)}) \\ &= \lambda \left(w_{(1)} \otimes \frac{d}{dx} Y_2^o(v, x) w_{(2)} \right) \\ & \quad + \text{Res}_{x_0} \left(\frac{d}{dx} \left(z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \right) \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0) w_{(1)} \otimes w_{(2)}) \\ & \quad + \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \frac{d}{dx} \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0) w_{(1)} \otimes w_{(2)}) \\ &= \lambda(w_{(1)} \otimes Y_2^o(L(-1)v, x) w_{(2)}) \\ & \quad + \text{Res}_{x_0} \left(\frac{d}{dx} \left(z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \right) \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0) w_{(1)} \otimes w_{(2)}) \\ & \quad + \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}L(1)(-x^{-2})^{L(0)}v, x_0) w_{(1)} \otimes w_{(2)}) \\ & \quad - 2\text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}L(0)x^{-1}(-x^{-2})^{L(0)}v, x_0) w_{(1)} \otimes w_{(2)}). \quad (5.90) \end{aligned}$$

The second term on the right-hand side of (5.90) is equal to

$$\begin{aligned} & -\text{Res}_{x_0} x^{-2} \left(\frac{d}{dx^{-1}} \left(z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \right) \right) \cdot \\ & \quad \cdot \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0) w_{(1)} \otimes w_{(2)}) \\ &= \text{Res}_{x_0} x^{-2} \left(\frac{d}{dx_0} \left(z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \right) \right) \cdot \end{aligned}$$

$$\begin{aligned}
& \cdot \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
&= -\text{Res}_{x_0} x^{-2} z^{-1} \delta\left(\frac{x^{-1} - x_0}{z}\right) \cdot \\
& \quad \cdot \frac{d}{dx_0} \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
&= -\text{Res}_{x_0} x^{-2} z^{-1} \delta\left(\frac{x^{-1} - x_0}{z}\right) \cdot \\
& \quad \cdot \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}). \tag{5.91}
\end{aligned}$$

By (5.91), (3.74) and (3.66), the right-hand side of (5.90) is equal to

$$\begin{aligned}
& \lambda(w_{(1)} \otimes Y_2^o(L(-1)v, x)w_{(2)}) \\
& \quad + \text{Res}_{x_0} z^{-1} \delta\left(\frac{x^{-1} - x_0}{z}\right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}L(-1)v, x_0)w_{(1)} \otimes w_{(2)}) \\
&= (Y'_{P(z)}(L(-1)v, x)\lambda)(w_{(1)} \otimes w_{(2)}),
\end{aligned}$$

proving the $L(-1)$ -derivative property. $\square$

Proposition 5.9 *The action $Y'_{P(z)}$ satisfies the commutator formula for vertex operators, that is, on $(W_1 \otimes W_2)^*$,*

$$\begin{aligned}
& [Y'_{P(z)}(v_1, x_1), Y'_{P(z)}(v_2, x_2)] \\
&= \text{Res}_{x_0} x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y'_{P(z)}(Y(v_1, x_0)v_2, x_2)
\end{aligned}$$

for $v_1, v_2 \in V$.

Proof In the following proof, the reader should note the well-definedness of each expression and the justifiability of each use of a δ -function property.

Let $\lambda \in (W_1 \otimes W_2)^*$, $v_1, v_2 \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. By (5.87),

$$\begin{aligned}
& (Y'_{P(z)}(v_1, x_1)Y'_{P(z)}(v_2, x_2)\lambda)(w_{(1)} \otimes w_{(2)}) \\
&= (Y'_{P(z)}(v_2, x_2)\lambda)(w_{(1)} \otimes Y_2^o(v_1, x_1)w_{(2)}) \\
& \quad + \text{Res}_{y_1} z^{-1} \delta\left(\frac{x_1^{-1} - y_1}{z}\right) (Y'_{P(z)}(v_2, x_2)\lambda)(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v_1, y_1)w_{(1)} \otimes w_{(2)}) \\
&= \lambda(w_{(1)} \otimes Y_2^o(v_2, x_2)Y_2^o(v_1, x_1)w_{(2)}) \\
& \quad + \text{Res}_{y_2} z^{-1} \delta\left(\frac{x_2^{-1} - y_2}{z}\right) \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)}v_2, y_2)w_{(1)} \otimes Y_2^o(v_1, x_1)w_{(2)}) \\
& \quad + \text{Res}_{y_1} z^{-1} \delta\left(\frac{x_1^{-1} - y_1}{z}\right) \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v_1, y_1)w_{(1)} \otimes Y_2^o(v_2, x_2)w_{(2)}) \\
& \quad + \text{Res}_{y_1} \text{Res}_{y_2} z^{-1} \delta\left(\frac{x_1^{-1} - y_1}{z}\right) z^{-1} \delta\left(\frac{x_2^{-1} - y_2}{z}\right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)}v_2, y_2)Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v_1, y_1)w_{(1)} \otimes w_{(2)}). \tag{5.92}
\end{aligned}$$

Transposing the subscripts 1 and 2 of the symbols v , x and y , we also have

$$\begin{aligned}
& (Y'_{P(z)}(v_2, x_2)Y'_{P(z)}(v_1, x_1)\lambda)(w_{(1)} \otimes w_{(2)}) \\
&= \lambda(w_{(1)} \otimes Y_2^o(v_1, x_1)Y_2^o(v_2, x_2)w_{(2)}) \\
&+ \text{Res}_{y_1} z^{-1} \delta\left(\frac{x_1^{-1} - y_1}{z}\right) \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v_1, y_1)w_{(1)} \otimes Y_2^o(v_2, x_2)w_{(2)}) \\
&+ \text{Res}_{y_2} z^{-1} \delta\left(\frac{x_2^{-1} - y_2}{z}\right) \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)}v_2, y_2)w_{(1)} \otimes Y_2^o(v_1, x_1)w_{(2)}) \\
&+ \text{Res}_{y_2} \text{Res}_{y_1} z^{-1} \delta\left(\frac{x_2^{-1} - y_2}{z}\right) z^{-1} \delta\left(\frac{x_1^{-1} - y_1}{z}\right) \cdot \\
&\quad \cdot \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v_1, y_1)Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)}v_2, y_2)w_{(1)} \otimes w_{(2)}). \quad (5.93)
\end{aligned}$$

The equalities (5.92) and (5.93) give

$$\begin{aligned}
& ([Y'_{P(z)}(v_1, x_1), Y'_{P(z)}(v_2, x_2)]\lambda)(w_{(1)} \otimes w_{(2)}) \\
&= \lambda(w_{(1)} \otimes [Y_2^o(v_2, x_2), Y_2^o(v_1, x_1)]w_{(2)}) \\
&\quad - \text{Res}_{y_1} \text{Res}_{y_2} z^{-1} \delta\left(\frac{x_1^{-1} - y_1}{z}\right) z^{-1} \delta\left(\frac{x_2^{-1} - y_2}{z}\right) \cdot \\
&\quad \cdot \lambda([Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v_1, y_1), Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)}v_2, y_2)]w_{(1)} \otimes w_{(2)}) \\
&= \text{Res}_{x_0} x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) \lambda(w_{(1)} \otimes Y_2^o(Y(v_1, x_0)v_2, x_2)w_{(2)}) \\
&\quad - \text{Res}_{y_1} \text{Res}_{y_2} \text{Res}_{x_0} z^{-1} \delta\left(\frac{x_1^{-1} - y_1}{z}\right) z^{-1} \delta\left(\frac{x_2^{-1} - y_2}{z}\right) y_2^{-1} \delta\left(\frac{y_1 - x_0}{y_2}\right) \cdot \\
&\quad \cdot \lambda(Y_1(Y(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v_1, x_0)e^{x_2 L(1)}(-x_2^{-2})^{L(0)}v_2, y_2)w_{(1)} \otimes w_{(2)}) \quad (5.94)
\end{aligned}$$

(recall (2.61)).

But we have

$$\begin{aligned}
& z^{-1} \delta\left(\frac{x_1^{-1} - y_1}{z}\right) z^{-1} \delta\left(\frac{x_2^{-1} - y_2}{z}\right) y_2^{-1} \delta\left(\frac{y_1 - x_0}{y_2}\right) \\
&= \left(\sum_{m, n \in \mathbb{Z}} \frac{(x_1^{-1} - y_1)^m}{z^{m+1}} \frac{(x_2^{-1} - y_2)^n}{z^{n+1}} \right) y_2^{-1} \delta\left(\frac{y_1 - x_0}{y_2}\right) \\
&= \left(\sum_{m, n \in \mathbb{Z}} (x_2^{-1} - y_2)^{-1} \left(\frac{x_1^{-1} - y_1}{x_2^{-1} - y_2}\right)^m \frac{(x_2^{-1} - y_2)^{m+n+1}}{z^{m+n+2}} \right) y_2^{-1} \delta\left(\frac{y_1 - x_0}{y_2}\right) \\
&= \left(\sum_{m, k \in \mathbb{Z}} (x_2^{-1} - y_2)^{-1} \left(\frac{x_1^{-1} - y_1}{x_2^{-1} - y_2}\right)^m z^{-1} \left(\frac{x_2^{-1} - y_2}{z}\right)^k \right) y_2^{-1} \delta\left(\frac{y_1 - x_0}{y_2}\right) \\
&= (x_2^{-1} - y_2)^{-1} \delta\left(\frac{x_1^{-1} - y_1}{x_2^{-1} - y_2}\right) z^{-1} \delta\left(\frac{x_2^{-1} - y_2}{z}\right) y_2^{-1} \delta\left(\frac{y_1 - x_0}{y_2}\right) \\
&= x_2 \delta\left(\frac{x_1^{-1} - (y_1 - y_2)}{x_2^{-1}}\right) z^{-1} \delta\left(\frac{x_2^{-1} - y_2}{z}\right) y_1^{-1} \delta\left(\frac{y_2 + x_0}{y_1}\right)
\end{aligned}$$

$$= x_2 \delta \left(\frac{x_1^{-1} - x_0}{x_2^{-1}} \right) z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) y_1^{-1} \delta \left(\frac{y_2 + x_0}{y_1} \right). \quad (5.95)$$

By (3.61), (3.62) and (3.67), we also have

$$\begin{aligned} & Y(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} v_1, x_0) e^{x_2 L(1)}(-x_2^{-2})^{L(0)} \\ &= e^{x_2 L(1)} Y \left(e^{-x_2(1+x_0 x_2) L(1)} (1+x_0 x_2)^{-2L(0)} e^{x_1 L(1)}(-x_1^{-2})^{L(0)} v_1, \frac{x_0}{1+x_0 x_2} \right) (-x_2^{-2})^{L(0)} \\ &= e^{x_2 L(1)}(-x_2^{-2})^{L(0)} Y \left((-x_2^2)^{L(0)} e^{-x_2(1+x_0 x_2) L(1)} \cdot \right. \\ &\quad \left. \cdot (1+x_0 x_2)^{-2L(0)} e^{x_1 L(1)}(-x_1^{-2})^{L(0)} v_1, -\frac{x_0 x_2^2}{1+x_0 x_2} \right) \\ &= e^{x_2 L(1)}(-x_2^{-2})^{L(0)} Y \left(e^{-x_2(1+x_0 x_2)(-x_2^{-2}) L(1)} (-x_2^2)^{L(0)} \cdot \right. \\ &\quad \left. \cdot (1+x_0 x_2)^{-2L(0)} e^{x_1 L(1)}(-x_1^{-2})^{L(0)} v_1, -\frac{x_0 x_2^2}{1+x_0 x_2} \right) \\ &= e^{x_2 L(1)}(-x_2^{-2})^{L(0)} Y \left(e^{(x_2^{-1}+x_0) L(1)} (-(x_2^{-1}+x_0)^2)^{-L(0)} e^{x_1 L(1)}(-x_1^{-2})^{L(0)} v_1, -\frac{x_0 x_2}{x_2^{-1}+x_0} \right) \\ &= e^{x_2 L(1)}(-x_2^{-2})^{L(0)} Y \left(e^{(x_2^{-1}+x_0) L(1)} e^{-x_1(x_2^{-1}+x_0)^2 L(1)} \cdot \right. \\ &\quad \left. \cdot (-(x_2^{-1}+x_0)^2)^{-L(0)} (-x_1^{-2})^{L(0)} v_1, -\frac{x_0 x_2}{x_2^{-1}+x_0} \right) \\ &= e^{x_2 L(1)}(-x_2^{-2})^{L(0)} Y \left(e^{(x_2^{-1}+x_0) L(1)} e^{-x_1(x_2^{-1}+x_0)^2 L(1)} ((x_2^{-1}+x_0)x_1)^{-2L(0)} v_1, -\frac{x_0 x_2}{x_2^{-1}+x_0} \right). \end{aligned} \quad (5.96)$$

Using (5.95), (5.96) and the basic properties of the formal delta function, we see that (5.94) becomes

$$\begin{aligned} & ([Y'_{P(z)}(v_1, x_1), Y'_{P(z)}(v_2, x_2)] \lambda)(w_{(1)} \otimes w_{(2)}) \\ &= \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \lambda(w_{(1)} \otimes Y_2^o(Y(v_1, x_0) v_2, x_2) w_{(2)}) \\ &\quad - \text{Res}_{y_1} \text{Res}_{y_2} \text{Res}_{x_0} x_2 \delta \left(\frac{x_1^{-1} - x_0}{x_2^{-1}} \right) z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) y_1^{-1} \delta \left(\frac{y_2 + x_0}{y_1} \right) \cdot \\ &\quad \cdot \lambda \left(Y_1 \left(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} Y \left(e^{(x_2^{-1}+x_0) L(1)} e^{-x_1(x_2^{-1}+x_0)^2 L(1)} \cdot \right. \right. \right. \\ &\quad \left. \left. \left. \cdot ((x_2^{-1}+x_0)x_1)^{-2L(0)} v_1, -\frac{x_0 x_2}{x_2^{-1}+x_0} \right) v_2, y_2 \right) w_{(1)} \otimes w_{(2)} \right) \\ &= \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \lambda(w_{(1)} \otimes Y_2^o(Y(v_1, x_0) v_2, x_2) w_{(2)}) \end{aligned}$$

$$\begin{aligned}
& -\text{Res}_{x_0} \text{Res}_{y_2} x_2 \delta \left(\frac{x_1^{-1} - x_0}{x_2^{-1}} \right) z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} Y(e^{x_1^{-1} L(1)} e^{-x_1^{-1} L(1)} \cdot \\
& \quad \cdot (x_1^{-1} x_1)^{-2L(0)} v_1, -x_0 x_1 x_2) v_2, y_2) w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \lambda(w_{(1)} \otimes Y_2^o(Y(v_1, x_0) v_2, x_2) w_{(2)}) \\
& \quad - \text{Res}_{x_0} x_2 \delta \left(\frac{x_2 + (-x_0 x_1 x_2)}{x_1} \right) \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} Y(v_1, -x_0 x_1 x_2) v_2, y_2) w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \lambda(w_{(1)} \otimes Y_2^o(Y(v_1, x_0) v_2, x_2) w_{(2)}) \\
& \quad + \text{Res}_{y_0} x_1^{-1} \delta \left(\frac{x_2 + y_0}{x_1} \right) \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} Y(v_1, y_0) v_2, y_2) w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \lambda(w_{(1)} \otimes Y_2^o(Y(v_1, x_0) v_2, x_2) w_{(2)}) \\
& \quad + \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} Y(v_1, x_0) v_2, y_2) w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) (Y'_{P(z)}(Y(v_1, x_0) v_2, x_2) \lambda)(w_{(1)} \otimes w_{(2)}). \tag{5.97}
\end{aligned}$$

Since λ , $w_{(1)}$ and $w_{(2)}$ are arbitrary, this equality gives the commutator formula for $Y'_{P(z)}$. $\square$

The following observations are analogous to those in Remark 8.1 of [HL2] (concerning the case of $Q(z)$ rather than $P(z)$):

Remark 5.10 The proof of Proposition 5.9 suggests the following: Using the definitions (5.84) and (5.86) as motivation, we define a (linear) action $\sigma_{P(z)}$ of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ on the vector space $W_1 \otimes W_2$ (as opposed to $(W_1 \otimes W_2)^*$) as follows:

$$\sigma_{P(z)}(\xi)(w_{(1)} \otimes w_{(2)}) = \tau_{W_1}((\iota_+ \circ T_z \circ \iota_-^{-1} \circ o)\xi) w_{(1)} \otimes w_{(2)} + w_{(1)} \otimes \tau_{W_2}((\iota_+ \circ \iota_-^{-1} \circ o)\xi) w_{(2)} \tag{5.98}$$

for $\xi \in V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, or equivalently,

$$\begin{aligned}
& \left(\sigma_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \right) (w_{(1)} \otimes w_{(2)}) \\
& = z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)} \\
& \quad + x_0^{-1} \delta \left(\frac{z - x_1^{-1}}{-x_0} \right) w_{(1)} \otimes Y_2^o(v, x_1) w_{(2)}. \tag{5.99}
\end{aligned}$$

That is, the operators $\sigma_{P(z)}(\xi)$ and $\tau_{P(z)}(\xi)$ are mutually adjoint:

$$(\tau_{P(z)}(\xi)\lambda)(w_{(1)} \otimes w_{(2)}) = \lambda(\sigma_{P(z)}(\xi)(w_{(1)} \otimes w_{(2)})). \quad (5.100)$$

While this action on $W_1 \otimes W_2$ is not very useful, it has the following three properties:

$$\sigma_{P(z)}(Y_t(\mathbf{1}, x)) = 1, \quad (5.101)$$

$$\frac{d}{dx} \sigma_{P(z)}(Y_t(v, x)) = \sigma_{P(z)}(Y_t(L(-1)v, x)) \quad (5.102)$$

for $v \in V$,

$$\begin{aligned} & [\sigma_{P(z)}(Y_t(v_2, x_2)), \sigma_{P(z)}(Y_t(v_1, x_1))] \\ &= \text{Res}_{x_0} x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) \sigma_{P(z)}(Y_t(Y(v_1, x_0)v_2, x_2)) \end{aligned} \quad (5.103)$$

for $v_1, v_2 \in V$ (the opposite commutator formula). These follow either from the assertions of Propositions 5.8 and 5.9 or, better, from the fact that it was actually (5.101)–(5.103) that the proofs of these propositions were proving.

Remark 5.11 Taking Res_{x_0} of (5.99), we obtain

$$\begin{aligned} & (\sigma_{P(z)}(Y_t(v, x_1)))(w_{(1)} \otimes w_{(2)}) \\ &= \text{Res}_{x_0} z^{-1} \delta\left(\frac{x_1^{-1} - x_0}{z}\right) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)} \\ & \quad + w_{(1)} \otimes Y_2^o(v, x_1)w_{(2)}. \end{aligned} \quad (5.104)$$

Substituting first $(-x_1^{-2})^{-L(0)}e^{-x_1 L(1)}v$ for v in (5.104) and then x_1^{-1} for x_1 in the same formula and using (3.67), (5.5) and (5.58), we obtain

$$\begin{aligned} (\sigma_{P(z)}(Y_t^o(v, x_1)))(w_{(1)} \otimes w_{(2)}) &= \text{Res}_{x_0} z^{-1} \delta\left(\frac{x_1 - x_0}{z}\right) Y_1(v, x_0)w_{(1)} \otimes w_{(2)} \\ & \quad + w_{(1)} \otimes Y_2(v, x_1)w_{(2)}. \end{aligned} \quad (5.105)$$

Using this, we see that $\sigma_{P(z)}$ can actually be viewed as a map from $V[t, t^{-1}]$ to $V((t)) \otimes V[t, t^{-1}]$ defined by

$$\begin{aligned} \sigma_{P(z)}(Y_t^o(v, x_1)) &= \text{Res}_{x_0} z^{-1} \delta\left(\frac{x_1 - x_0}{z}\right) Y_t(v, x_0) \otimes \mathbf{1} \\ & \quad + \mathbf{1} \otimes Y_t(v, x_1). \end{aligned} \quad (5.106)$$

Let $\Delta_{P(z)} = \sigma_{P(z)} \circ o$. Then (5.106) becomes

$$\begin{aligned} \Delta_{P(z)}(Y_t(v, x_1)) &= \text{Res}_{x_0} z^{-1} \delta\left(\frac{x_1 - x_0}{z}\right) Y_t(v, x_0) \otimes \mathbf{1} \\ & \quad + \mathbf{1} \otimes Y_t(v, x_1), \end{aligned} \quad (5.107)$$

which again can be viewed as a map

$$\Delta_{P(z)} : V[t, t^{-1}] \rightarrow V((t)) \otimes V[t, t^{-1}]. \quad (5.108)$$

In formula (2.4) of [MS], Moore and Seiberg introduced a map $\Delta_{z,0}$ which in fact corresponds exactly to the map $\Delta_{P(z)}$ defined by (5.107). They proposed to define a V -module structure (called “a representation of $\mathcal{A}$ ” in [MS], where $\mathcal{A}$ corresponds to our vertex algebra V) on $W_1 \otimes W_2$ by using this map, which can be viewed as a sort of analogue of a coproduct, but they acknowledged that E. Witten pointed out “subtleties in this definition which are related to the fact that [a] representation of $\mathcal{A}$ obtained this way is not always a highest weight representation.” In fact, it is these subtleties that make it impossible to work with $W_1 \otimes W_2$; in virtually all interesting cases, $W_1 \otimes W_2$ does not have a natural (generalized) V -module structure. This is exactly the reason why we had to use a completely different approach to construct our $P(z)$ -tensor product of W_1 and W_2 .

When V is in fact a conformal (rather than Möbius) vertex algebra, we will write

$$Y'_{P(z)}(\omega, x) = \sum_{n \in \mathbb{Z}} L'_{P(z)}(n) x^{-n-2}. \quad (5.109)$$

Then from the last two propositions we see that the coefficient operators of $Y'_{P(z)}(\omega, x)$ satisfy the Virasoro algebra commutator relations, that is,

$$[L'_{P(z)}(m), L'_{P(z)}(n)] = (m - n) L'_{P(z)}(m + n) + \frac{1}{12} (m^3 - m) \delta_{m+n,0} c,$$

with $c \in \mathbb{C}$ the central charge of V (recall Definition 2.2). Moreover, in this case, by setting $v = \omega$ in (5.87) and taking the coefficient of x_1^{-j-2} for $j = -1, 0, 1$, we find that

$$\begin{aligned} & (L'_{P(z)}(j)\lambda)(w_{(1)} \otimes w_{(2)}) \\ &= \lambda \left(w_{(1)} \otimes L(-j)w_{(2)} + \left(\sum_{i=0}^{1-j} \binom{1-j}{i} z^i L(-j-i) \right) w_{(1)} \otimes w_{(2)} \right), \end{aligned} \quad (5.110)$$

by (2.63). If V is just a Möbius vertex algebra, we define the actions $L'_{P(z)}(j)$ on $(W_1 \otimes W_2)^*$ by (5.110) for $j = -1, 0$ and 1 .

Remark 5.12 In view of the action $L'_{P(z)}(j)$, the $\mathfrak{sl}(2)$ -bracket relations (4.5) for a $P(z)$ -intertwining map I , with notation as in Definition 4.2, can be written as

$$(L'(j)w'_{(3)}) \circ I = L'_{P(z)}(j)(w'_{(3)} \circ I) \quad (5.111)$$

for $w'_{(3)} \in W'_3$ and $j = -1, 0$ and 1 (cf. (5.81) and Remark 5.4).

Remark 5.13 We have

$$L'_{P(z)}(j)((W_1 \otimes W_2)^*)^{(\beta)} \subset ((W_1 \otimes W_2)^*)^{(\beta)}$$

for $j = -1, 0, 1$ and $\beta \in \tilde{A}$ (cf. Proposition 5.6).

When V is a conformal vertex algebra, from the commutator formula for $Y'_{P(z)}(\omega, x)$, we see that $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$ realize the actions of L_{-1} , L_0 and L_1 in $\mathfrak{sl}(2)$ (recall (2.27)) on $(W_1 \otimes W_2)^*$. When V is just a Möbius vertex algebra, the same conclusion still holds but a proof is needed. We state this as a proposition:

Proposition 5.14 *Let V be a Möbius vertex algebra and let W_1 and W_2 be generalized V -modules. Then the operators $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$ realize the actions of L_{-1} , L_0 and L_1 in $\mathfrak{sl}(2)$ on $(W_1 \otimes W_2)^*$, according to (2.27).*

Proof For $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $j, k = -1, 0, 1$, we have

$$\begin{aligned}
& (L'_{P(z)}(j)L'_{P(z)}(k)\lambda)(w_{(1)} \otimes w_{(2)}) \\
&= (L'_{P(z)}(k)\lambda) \left(w_{(1)} \otimes L(-j)w_{(2)} + \left(\sum_{i=0}^{1-j} \binom{1-j}{i} z^i L(-j-i) \right) w_{(1)} \otimes w_{(2)} \right) \\
&= (L'_{P(z)}(k)\lambda)(w_{(1)} \otimes L(-j)w_{(2)}) \\
&\quad + (L'_{P(z)}(k)\lambda) \left(\left(\sum_{i=0}^{1-j} \binom{1-j}{i} z^i L(-j-i) \right) w_{(1)} \otimes w_{(2)} \right) \\
&= \lambda \left(w_{(1)} \otimes L(-k)L(-j)w_{(2)} + \left(\sum_{l=0}^{1-k} \binom{1-k}{l} z^l L(-k-l) \right) w_{(1)} \otimes L(-j)w_{(2)} \right) \\
&\quad + \lambda \left(\left(\sum_{i=0}^{1-j} \binom{1-j}{i} z^i L(-j-i) \right) w_{(1)} \otimes L(-k)w_{(2)} \right. \\
&\quad \left. + \left(\sum_{l=0}^{1-k} \binom{1-k}{l} z^l L(-k-l) \right) \left(\sum_{i=0}^{1-j} \binom{1-j}{i} z^i L(-j-i) \right) w_{(1)} \otimes w_{(2)} \right). \tag{5.112}
\end{aligned}$$

From formula (5.112) we obtain

$$\begin{aligned}
& ([L'_{P(z)}(j), L'_{P(z)}(k)]\lambda)(w_{(1)} \otimes w_{(2)}) \\
&= \lambda \left(w_{(1)} \otimes [L(-k), L(-j)]w_{(2)} \right. \\
&\quad \left. + \left(\sum_{l=0}^{1-k} \sum_{i=0}^{1-j} \binom{1-k}{l} \binom{1-j}{i} z^{l+i} [L(-k-l), L(-j-i)] \right) w_{(1)} \otimes w_{(2)} \right) \\
&= \lambda \left(w_{(1)} \otimes (j-k)L(-k-j)w_{(2)} \right. \\
&\quad \left. + \left(\sum_{l=0}^{1-k} \sum_{i=0}^{1-j} \binom{1-k}{l} \binom{1-j}{i} z^{l+i} (j+i-k-l)L(-k-l-j-i) \right) w_{(1)} \otimes w_{(2)} \right) \tag{5.113}
\end{aligned}$$

Taking $j = 1$ and $k = -1, 0$ in (5.113), we obtain

$$\begin{aligned}
& ([L'_{P(z)}(1), L'_{P(z)}(k)]\lambda)(w_{(1)} \otimes w_{(2)}) \\
&= \lambda \left(w_{(1)} \otimes (1-k)L(-k-1)w_{(2)} \right. \\
&\quad \left. + \left(\sum_{l=0}^{1-k} \binom{1-k}{l} z^l (1-k-l)L(-k-l-1) \right) w_{(1)} \otimes w_{(2)} \right) \\
&= \lambda \left(w_{(1)} \otimes (1-k)L(-k-1)w_{(2)} \right. \\
&\quad \left. + \left(\sum_{l=0}^{1-(k+1)} \binom{1-k}{l} z^l (1-k-l)L(-k-l-1) \right) w_{(1)} \otimes w_{(2)} \right) \\
&= (1-k)\lambda \left(w_{(1)} \otimes L(-1-k)w_{(2)} \right. \\
&\quad \left. + \left(\sum_{l=0}^{1-(k+1)} \binom{1-(k+1)}{l} z^l L(-(k+1)-l) \right) w_{(1)} \otimes w_{(2)} \right) \\
&= ((1-k)L'_{P(z)}(1+k)\lambda)(w_{(1)} \otimes w_{(2)}),
\end{aligned}$$

proving the commutator formula

$$[L'_{P(z)}(1), L'_{P(z)}(k)] = (1-k)L'_{P(z)}(1+k)$$

for $k = -1, 0$.

Taking $j = 0$ and $k = -1$ in (5.113), we obtain

$$\begin{aligned}
& ([L'_{P(z)}(0), L'_{P(z)}(-1)]\lambda)(w_{(1)} \otimes w_{(2)}) \\
&= \lambda \left(w_{(1)} \otimes L(1)w_{(2)} + \left(\sum_{l=0}^2 \sum_{i=0}^1 \binom{2}{l} z^{l+i} (i+1-l)L(1-l-i) \right) w_{(1)} \otimes w_{(2)} \right) \\
&= \lambda(w_{(1)} \otimes L(1)w_{(2)} + (L(1) + 2zL(0) + z^2L(-1))w_{(1)} \otimes w_{(2)}) \\
&= \lambda \left(w_{(1)} \otimes L(1)w_{(2)} + \left(\sum_{m=0}^{1-(-1)} \binom{2}{m} z^m L(-(-1)-m) \right) w_{(1)} \otimes w_{(2)} \right) \\
&= (L'_{P(z)}(-1)\lambda)(w_{(1)} \otimes w_{(2)}),
\end{aligned}$$

proving that

$$[L'_{P(z)}(0), L'_{P(z)}(-1)] = L'_{P(z)}(-1).$$

The three commutator formulas we have proved show that $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$ indeed realize the actions of L_{-1} , L_0 and L_1 in $\mathfrak{sl}(2)$. $\square$

The commutator formulas corresponding to (2.28)–(2.30) (recall Definition 2.11)) also need to be proved in the Möbius case:

Proposition 5.15 *Let V be a Möbius vertex algebra and let W_1 and W_2 be generalized V -modules. Then for $v \in V$, we have the following commutator formulas:*

$$[L(-1), Y'_{P(z)}(v, x)] = Y'_{P(z)}(L(-1)v, x), \quad (5.114)$$

$$[L(0), Y'_{P(z)}(v, x)] = Y'_{P(z)}(L(0)v, x) + xY'_{P(z)}(L(-1)v, x), \quad (5.115)$$

$$[L(1), Y'_{P(z)}(v, x)] = Y'_{P(z)}(L(1)v, x) + 2xY'_{P(z)}(L(0)v, x) + x^2Y'_{P(z)}(L(-1)v, x), \quad (5.116)$$

where for brevity we write $L'_{P(z)}(j)$ acting on $(W_1 \otimes W_2)^*$ as $L(j)$.

Proof Let $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. Using (5.110), (5.87), the commutator formulas for $L(j)$ and $Y_1(v, x_0)$ for $j = -1, 0, 1$ and $v \in V$ (recall Definition 2.11), and the commutator formulas for $L(j)$ and $Y_2^o(v, x)$ for $j = -1, 0, 1$ and $v \in V$ (recall Lemma 2.22), we obtain, for $j = -1, 0, 1$,

$$\begin{aligned} & ([L(j), Y'_{P(z)}(v, x)]\lambda)(w_{(1)} \otimes w_{(2)}) \\ &= (L(j)Y'_{P(z)}(v, x)\lambda)(w_{(1)} \otimes w_{(2)}) - (Y'_{P(z)}(v, x)L(j)\lambda)(w_{(1)} \otimes w_{(2)}) \\ &= (Y'_{P(z)}(v, x)\lambda)(w_{(1)} \otimes L(-j)w_{(2)}) \\ &\quad + \sum_{i=0}^{1-j} \binom{1-j}{i} (Y'_{P(z)}(v, x)\lambda)(z^i L(-j-i)w_{(1)} \otimes w_{(2)}) \\ &\quad - (L(j)\lambda)(w_{(1)} \otimes Y_2^o(v, x)w_{(2)}) \\ &\quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) (L(j)\lambda)(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\ &= \lambda(w_{(1)} \otimes Y_2^o(v, x)L(-j)w_{(2)}) \\ &\quad + \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes L(-j)w_{(2)}) \\ &\quad + \sum_{i=0}^{1-j} \binom{1-j}{i} \lambda(z^i L(-j-i)w_{(1)} \otimes Y_2^o(v, x)w_{(2)}) \\ &\quad + \sum_{i=0}^{1-j} \binom{1-j}{i} \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \cdot \\ &\quad \quad \cdot \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)z^i L(-j-i)w_{(1)} \otimes w_{(2)}) \\ &\quad - \lambda(w_{(1)} \otimes L(-j)Y_2^o(v, x)w_{(2)}) \\ &\quad - \sum_{i=0}^{1-j} \binom{1-j}{i} \lambda(z^i L(-j-i)w_{(1)} \otimes Y_2^o(v, x)w_{(2)}) \\ &\quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes L(-j)w_{(2)}) \\ &\quad - \sum_{i=0}^{1-j} \binom{1-j}{i} \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \cdot \end{aligned}$$

$$\begin{aligned}
& \cdot \lambda(z^i L(-j-i) Y_1(e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \\
& = \lambda(w_{(1)} \otimes [Y_2^o(v, x), L(-j)] w_{(2)}) \\
& \quad - \sum_{i=0}^{1-j} \binom{1-j}{i} \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \cdot \\
& \quad \cdot \lambda([z^i L(-j-i), Y_1(e^{xL(1)}(-x^{-2})^{L(0)} v, x_0)] w_{(1)} \otimes w_{(2)}) \\
& = \sum_{k=0}^{j+1} \binom{j+1}{k} x^{j+1-k} \lambda(w_{(1)} \otimes Y_2^o(L(k-1)v, x) w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \sum_{i=0}^{1-j} \sum_{k=0}^{-j-i+1} \binom{1-j}{i} \binom{-j-i+1}{k} z^i x_0^{-j-i+1-k} \cdot \\
& \quad \cdot \lambda(Y_1(L(k-1)e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}). \tag{5.117}
\end{aligned}$$

Using (2.6) and (2.11) when necessary, we see that the second term on the right-hand side of (5.117) is equal to the following expressions for $j = 1, 0$ and -1 , respectively:

$$\begin{aligned}
& -\text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}), \tag{5.118} \\
& -\text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \sum_{i=0}^1 \sum_{k=0}^{-i+1} \binom{1}{i} \binom{1-i}{k} z^i x_0^{1-i-k} \cdot \\
& \quad \cdot \lambda(Y_1(L(k-1)e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \\
& = -\text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) x_0 \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(L(0)e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) z \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \\
& = -\text{Res}_{x_0} x \delta \left(\frac{z + x_0}{x^{-1}} \right) (x_0 + z) \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(L(0)e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \\
& = -\text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) x^{-1} \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(L(0)e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \tag{5.119}
\end{aligned}$$

and

$$\begin{aligned}
& -\text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \sum_{i=0}^2 \sum_{k=0}^{2-i} \binom{2}{i} \binom{2-i}{k} z^i x_0^{2-i-k} \\
& \quad \cdot \lambda(Y_1(L(k-1)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& = -\text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) x_0^2 \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) 2x_0 \lambda(Y_1(L(0)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(L(1)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) 2zx_0 \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) 2z \lambda(Y_1(L(0)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) z^2 \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& = -\text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) x^{-2} \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) 2x^{-1} \lambda(Y_1(L(0)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(L(1)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}).
\end{aligned} \tag{5.120}$$

Using (3.74) and (3.66), we see that (5.118), (5.119) and (5.120) are respectively equal to

$$\begin{aligned}
& -\text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \cdot \lambda(Y_1(e^{xL(1)}(x^2L(1) - 2xL(0) + L(-1))(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \cdot \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}(L(1) + 2xL(0) + x^2L(-1))v, x_0)w_{(1)} \otimes w_{(2)}),
\end{aligned} \tag{5.121}$$

$$\begin{aligned}
& -\text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) x^{-1} \cdot \lambda(Y_1(e^{xL(1)}(x^2L(1) - 2xL(0) + L(-1))(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)})
\end{aligned}$$

$$\begin{aligned}
& -\text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-xL(1) + L(0))(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) x^{-1} \cdot \\
& \quad \cdot \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}(L(1) + 2xL(0) + x^2L(-1))v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad + \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}(-x^{-1}L(1) - L(0))v, x_0)w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}(L(0) + xL(-1))v, x_0)w_{(1)} \otimes w_{(2)})
\end{aligned} \tag{5.122}$$

and

$$\begin{aligned}
& -\text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) x^{-2} \cdot \\
& \quad \cdot \lambda(Y_1(e^{xL(1)}(x^2L(1) - 2xL(0) + L(-1))(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) 2x^{-1} \lambda(Y_1(e^{xL(1)}(-xL(1) + L(0))(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}L(1)(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) x^{-2} \cdot \\
& \quad \cdot \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}(L(1) + 2xL(0) + x^2L(-1))v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad + \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) 2x^{-1} \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}(-x^{-1}L(1) - L(0))v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad + \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}x^{-2}L(1)v, x_0)w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}L(-1)v, x_0)w_{(1)} \otimes w_{(2)}).
\end{aligned} \tag{5.123}$$

The right-hand sides of (5.121), (5.122) and (5.123) can be written as

$$\sum_{k=0}^{j+1} \binom{j+1}{k} x^{j+1-k} \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}L(k-1)v, x_0)w_{(1)} \otimes w_{(2)}),$$

for $j = 1, 0, -1$, respectively. Thus the right-hand side of (5.117) is equal to

$$\sum_{k=0}^{j+1} \binom{j+1}{k} x^{j+1-k} \lambda(w_{(1)} \otimes Y_2^o(L(k-1)v, x)w_{(2)})$$

$$\begin{aligned}
& + \sum_{k=0}^{j+1} \binom{j+1}{k} x^{j+1-k} \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}L(k-1)v, x_0)w_{(1)} \otimes w_{(2)}) \\
& = \sum_{k=0}^{j+1} \binom{j+1}{k} x^{j+1-k} (Y'_{P(z)}(L(k-1)v, x)\lambda)(w_{(1)} \otimes w_{(2)}), \tag{5.124}
\end{aligned}$$

proving the proposition. $\square$

We have seen in (5.2), (5.82) and (5.83) that for a generalized V -module (W, Y_W) , the space $V \otimes \mathbb{C}((t))$, and in particular, the space $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$, acts naturally on W via the action τ_W , in view of (2.49) and Assumption 4.1; recall that $v \otimes t^n$ ($v \in V$, $n \in \mathbb{Z}$) acts as the component v_n of $Y_W(v, x)$, and that more generally,

$$\tau_W \left(v \otimes \sum_{n>N} a_n t^n \right) = \sum_{n>N} a_n v_n \tag{5.125}$$

for $a_n \in \mathbb{C}$. For generalized V -modules W_1 , W_2 and W_3 , we shall next relate the $P(z)$ -intertwining maps of type $\binom{W_3}{W_1 W_2}$ to certain linear maps from W'_3 to $(W_1 \otimes W_2)^*$ intertwining the actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ and of $\mathfrak{sl}(2)$ on W'_3 and on $(W_1 \otimes W_2)^*$ (see Proposition 5.24 and Notation 5.25 below). For this, as is suggested by Lemma 4.41 and Proposition 5.6, we need to consider $\tilde{A}$ -compatibility for linear maps from W_3 to $(W_1 \otimes W_2)^*$:

Definition 5.16 We call a map $J \in \text{Hom}(W_3, (W_1 \otimes W_2)^*)$ $\tilde{A}$ -compatible if

$$J((W_3)^{(\beta)}) \subset ((W_1 \otimes W_2)^*)^{(\beta)} \tag{5.126}$$

for $\beta \in \tilde{A}$.

As in the discussion preceding Lemma 4.41, we see that an element λ of $(W_1 \otimes W_2 \otimes W_3)^*$ amounts exactly to a linear map

$$J_\lambda : W_3 \rightarrow (W_1 \otimes W_2)^*.$$

If λ is $\tilde{A}$ -compatible (see that discussion), then for $w_{(1)} \in W_1^{(\beta)}$, $w_{(2)} \in W_2^{(\gamma)}$ and $w_{(3)} \in W_3^{(\delta)}$ such that $\beta + \gamma + \delta \neq 0$,

$$J_\lambda(w_{(3)})(w_{(1)} \otimes w_{(2)}) = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) = 0,$$

so that

$$J_\lambda(w_{(3)}) \in ((W_1 \otimes W_2)^*)^{(\delta)},$$

and so J_λ is $\tilde{A}$ -compatible. Similarly, if J_λ is $\tilde{A}$ -compatible, then so is λ . Thus using Lemma 4.41 we have:

Lemma 5.17 *The linear functional $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ is $\tilde{A}$ -compatible if and only if J_λ is $\tilde{A}$ -compatible. The map given by $\lambda \mapsto J_\lambda$ is the unique linear isomorphism from the space of $\tilde{A}$ -compatible elements of $(W_1 \otimes W_2 \otimes W_3)^*$ to the space of $\tilde{A}$ -compatible linear maps from W_3 to $(W_1 \otimes W_2)^*$ such that*

$$J_\lambda(w_{(3)})(w_{(1)} \otimes w_{(2)}) = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. In particular, the correspondence $I_\lambda \mapsto J_\lambda$ defines a (unique) linear isomorphism from the space of $\tilde{A}$ -compatible linear maps

$$I = I_\lambda : W_1 \otimes W_2 \rightarrow \overline{W'_3}$$

to the space of $\tilde{A}$ -compatible linear maps

$$J = J_\lambda : W_3 \rightarrow (W_1 \otimes W_2)^*$$

such that

$$\langle w_{(3)}, I(w_{(1)} \otimes w_{(2)}) \rangle = J(w_{(3)})(w_{(1)} \otimes w_{(2)})$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. $\square$

Remark 5.18 From Lemma 5.17 (with W_3 replaced by W'_3) we have a canonical isomorphism from the space of $\tilde{A}$ -compatible linear maps

$$I : W_1 \otimes W_2 \rightarrow \overline{W'_3}$$

to the space of $\tilde{A}$ -compatible linear maps

$$J : W'_3 \rightarrow (W_1 \otimes W_2)^*,$$

determined by:

$$\langle w'_{(3)}, I(w_{(1)} \otimes w_{(2)}) \rangle = J(w'_{(3)})(w_{(1)} \otimes w_{(2)}) \quad (5.127)$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$, or equivalently,

$$w'_{(3)} \circ I = J(w'_{(3)}) \quad (5.128)$$

for $w'_{(3)} \in W'_3$.

We introduce another notion, corresponding to the lower truncation condition (4.3) for $P(z)$ -intertwining maps:

Definition 5.19 We call a map $J \in \text{Hom}(W_3, (W_1 \otimes W_2)^*)$ *grading restricted* if for $n \in \mathbb{C}$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$,

$$J((W_3)_{[n-m]})(w_{(1)} \otimes w_{(2)}) = 0 \quad \text{for } m \in \mathbb{N} \text{ sufficiently large.} \quad (5.129)$$

Remark 5.20 If $J \in \text{Hom}(W_3, (W_1 \otimes W_2)^*)$ is $\tilde{A}$ -compatible, then J is also grading restricted, as we see using (2.85).

Remark 5.21 If in addition W_3 is lower bounded (recall (2.89)), then the stronger condition

$$J((W_3)_{[n]})(w_{(1)} \otimes w_{(2)}) = 0 \quad \text{for } \Re(n) \text{ sufficiently negative} \quad (5.130)$$

holds.

Remark 5.22 Under the natural isomorphism given in Remark 5.18 (see (5.127)) in the $\tilde{A}$ -compatible setting, the map $J : W'_3 \rightarrow (W_1 \otimes W_2)^*$ is grading restricted (recall Definition 5.19) if and only if the map $I : W_1 \otimes W_2 \rightarrow \overline{W_3}$ satisfies the lower truncation condition (4.3). But notice also that in this $\tilde{A}$ -compatible setting, we have seen that both I and J automatically have these properties.

Remark 5.23 Analogous comments hold for the stronger conditions (4.7) and (5.21) in case W_3 is lower bounded.

Using the above together with Remarks 5.4 and 5.12, we now have the following result, generalizing Proposition 13.1 in [HL3]:

Proposition 5.24 *Let W_1 , W_2 and W_3 be generalized V -modules. Under the natural isomorphism described in Remark 5.18 between the space of $\tilde{A}$ -compatible linear maps*

$$I : W_1 \otimes W_2 \rightarrow \overline{W_3}$$

and the space of $\tilde{A}$ -compatible linear maps

$$J : W'_3 \rightarrow (W_1 \otimes W_2)^*$$

determined by (5.127), the $P(z)$ -intertwining maps I of type $\binom{W_3}{W_1 W_2}$ correspond exactly to the (grading restricted) $\tilde{A}$ -compatible maps J that intertwine the actions of both

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$$

and $\mathfrak{sl}(2)$ on W'_3 and on $(W_1 \otimes W_2)^$. If W_3 is lower bounded, we may replace the grading restrictions by (4.7) and (5.130).*

Proof In view of (5.128), Remark 5.4 asserts that (5.81), or equivalently, (4.4), is equivalent to the condition

$$J \left(\tau_{W'_3} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) w'_{(3)} \right) = \tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) J(w'_{(3)}), \quad (5.131)$$

that is, the condition that J intertwines the actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ on W'_3 and on $(W_1 \otimes W_2)^*$ (recall (5.21)–(5.22)). Similarly, Remark 5.12 asserts that (4.5) is equivalent to the condition

$$J(L'(j)w'_{(3)}) = L'_{P(z)}(j)J(w'_{(3)}) \quad (5.132)$$

for $j = -1, 0, 1$, that is, the condition that J intertwines the actions of $\mathfrak{sl}(2)$ on W'_3 and on $(W_1 \otimes W_2)^*$. $\square$

Notation 5.25 Given generalized V -modules W_1 , W_2 and W_3 , we shall write $\mathcal{N}[P(z)]_{W'_3}^{(W_1 \otimes W_2)^*}$, or $\mathcal{N}_{W'_3}^{(W_1 \otimes W_2)^*}$ if there is no ambiguity, for the space of (grading restricted) $\tilde{A}$ -compatible linear maps

$$J : W'_3 \rightarrow (W_1 \otimes W_2)^*$$

that intertwine the actions of both

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$$

and $\mathfrak{sl}(2)$ on W'_3 and on $(W_1 \otimes W_2)^*$. Note that Proposition 5.24 gives a natural linear isomorphism

$$\begin{array}{ccc} \mathcal{M}[P(z)]_{W_1 W_2}^{W_3} = \mathcal{M}_{W_1 W_2}^{W_3} & \xrightarrow{\sim} & \mathcal{N}_{W'_3}^{(W_1 \otimes W_2)^*} \\ I & \mapsto & J \end{array}$$

(recall from Definition 4.2 the notations for the space of $P(z)$ -intertwining maps), and if W_3 is lower bounded, the spaces satisfy the stronger grading restrictions (4.7) and (5.130). Let us use the symbol “prime” to denote this isomorphism in both directions:

$$\begin{array}{ccc} \mathcal{M}_{W_1 W_2}^{W_3} & \xrightarrow{\sim} & \mathcal{N}_{W'_3}^{(W_1 \otimes W_2)^*} \\ I & \mapsto & I' \\ J' & \leftarrow & J, \end{array}$$

so that in particular,

$$I'' = I \quad \text{and} \quad J'' = J$$

for $I \in \mathcal{M}_{W_1 W_2}^{W_3}$ and $J \in \mathcal{N}_{W'_3}^{(W_1 \otimes W_2)^*}$, and the relation between I and I' is determined by

$$\langle w'_{(3)}, I(w_{(1)} \otimes w_{(2)}) \rangle = I'(w'_{(3)})(w_{(1)} \otimes w_{(2)})$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$, or equivalently,

$$w'_{(3)} \circ I = I'(w'_{(3)}).$$

Remark 5.26 Combining Proposition 5.24 with Proposition 4.8, we see that for any integer p , we also have a natural linear isomorphism

$$\mathcal{N}_{W'_3}^{(W_1 \otimes W_2)^*} \xrightarrow{\sim} \mathcal{V}_{W_1 W_2}^{W_3}$$

from $\mathcal{N}_{W'_3}^{(W_1 \otimes W_2)^*}$ to the space of logarithmic intertwining operators of type $\left(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix} \right)$. In particular, given a logarithmic intertwining operator $\mathcal{Y}$ of type $\left(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix} \right)$, the map

$$I'_{\mathcal{Y}, p} : W'_3 \rightarrow (W_1 \otimes W_2)^*$$

defined by

$$I'_{\mathcal{Y}, p}(w'_{(3)})(w_{(1)} \otimes w_{(2)}) = \langle w'_{(3)}, \mathcal{Y}(w_{(1)}, e^{lp(z)})w_{(2)} \rangle_{W_3}$$

is $\tilde{A}$ -compatible and intertwines both actions on both spaces. If W_3 is lower bounded, we have the stronger grading restrictions (recall Remark 4.9).

Recall that we have formulated the notions of $P(z)$ -product and $P(z)$ -tensor product using $P(z)$ -intertwining maps (Definitions 4.13 and 4.15). Since we now know that $P(z)$ -intertwining maps can be interpreted as in Proposition 5.24 (and Notation 5.25), we can easily reformulate the notions of $P(z)$ -product and $P(z)$ -tensor product correspondingly:

Proposition 5.27 *Let $\mathcal{C}_1$ be either of the categories $\mathcal{M}_{sg}$ or $\mathcal{GM}_{sg}$, as in Definition 4.13. For $W_1, W_2 \in \text{ob } \mathcal{C}_1$, a $P(z)$ -product $(W_3; I_3)$ of W_1 and W_2 (recall Definition 4.13) amounts to an object (W_3, Y_3) of $\mathcal{C}_1$ equipped with a map $I'_3 \in \mathcal{N}_{W'_3}^{(W_1 \otimes W_2)^*}$, that is, equipped with an $\tilde{A}$ -compatible map*

$$I'_3 : W'_3 \rightarrow (W_1 \otimes W_2)^*$$

that intertwines the two actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ and of $\mathfrak{sl}(2)$. The map I'_3 corresponds to the $P(z)$ -intertwining map

$$I_3 : W_1 \otimes W_2 \rightarrow \overline{W_3}$$

as above:

$$I'_3(w'_{(3)}) = w'_{(3)} \circ I_3$$

for $w'_{(3)} \in W'_3$ (recall 5.128)). Denoting this structure by $(W_3, Y_3; I'_3)$ or simply by $(W_3; I'_3)$, let $(W_4; I'_4)$ be another such structure. Then a morphism of $P(z)$ -products from W_3 to W_4 amounts to a module map $\eta : W_3 \rightarrow W_4$ such that the diagram

$$\begin{array}{ccc} & (W_1 \otimes W_2)^* & \\ I'_4 \nearrow & & \nwarrow I'_3 \\ W'_4 & \xrightarrow{\eta'} & W'_3 \end{array}$$

commutes, where η' is the natural map given by (2.102).

Proof All we need to check is that the diagram in Definition 4.13 commutes if and only if the diagram above commutes. But this follows from the definitions and the fact that for $\overline{w_{(3)}} \in \overline{W_3}$ and $w'_{(4)} \in W'_4$,

$$\langle \eta'(w'_{(4)}), \overline{w_{(3)}} \rangle = \langle w'_{(4)}, \overline{\eta(w_{(3)})} \rangle,$$

which in turn follows from (2.102). $\square$

Corollary 5.28 *Let $\mathcal{C}$ be a full subcategory of either $\mathcal{M}_{sg}$ or $\mathcal{GM}_{sg}$, as in Definition 4.15. For $W_1, W_2 \in \text{ob } \mathcal{C}$, a $P(z)$ -tensor product $(W_0; I_0)$ of W_1 and W_2 in $\mathcal{C}$, if it exists, amounts to an object $W_0 = W_1 \boxtimes_{P(z)} W_2$ of $\mathcal{C}$ and a structure $(W_0 = W_1 \boxtimes_{P(z)} W_2; I'_0)$ as in Proposition 5.27, with*

$$I'_0 : (W_1 \boxtimes_{P(z)} W_2)' \longrightarrow (W_1 \otimes W_2)^*$$

in $\mathcal{N}_{(W_1 \boxtimes_{P(z)} W_2)'}^{(W_1 \otimes W_2)^}$, such that for any such pair $(W; I')$ ($W \in \text{ob } \mathcal{C}$), with*

$$I' : W' \longrightarrow (W_1 \otimes W_2)^*$$

in $\mathcal{N}_{W'}^{(W_1 \otimes W_2)^*}$, there is a unique module map

$$\chi : W' \longrightarrow (W_1 \boxtimes_{P(z)} W_2)'$$

such that the diagram

$$\begin{array}{ccc} & (W_1 \otimes W_2)^* & \\ I' \nearrow & & \nwarrow I'_0 \\ W' & \xrightarrow{\chi} & (W_1 \boxtimes_{P(z)} W_2)' \end{array}$$

commutes. Here $\chi = \eta'$, where η is a correspondingly unique module map

$$\eta : W_1 \boxtimes_{P(z)} W_2 \longrightarrow W.$$

Also, the map I'_0 , which is $\tilde{A}$ -compatible and which intertwines the two actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ and of $\mathfrak{sl}(2)$, is related to the $P(z)$ -intertwining map

$$I_0 = \boxtimes_{P(z)} : W_1 \otimes W_2 \longrightarrow \overline{W_1 \boxtimes_{P(z)} W_2}$$

by

$$I'_0(w') = w' \circ \boxtimes_{P(z)}$$

for $w' \in (W_1 \boxtimes_{P(z)} W_2)'$, that is,

$$I'_0(w')(w_{(1)} \otimes w_{(2)}) = \langle w', w_{(1)} \boxtimes_{P(z)} w_{(2)} \rangle$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, using the notation (4.31). $\square$

Remark 5.29 From Corollary 5.28 we see that it is natural to try to construct $W_1 \boxtimes_{P(z)} W_2$, when it exists, as the contragredient of a suitable natural substructure of $(W_1 \otimes W_2)^*$. We shall now proceed to do this. Under suitable assumptions, we shall in fact construct a module-like structure

$$W_1 \boxdot_{P(z)} W_2 \subset (W_1 \otimes W_2)^*$$

for $W_1, W_2 \in \text{ob } \mathcal{C}$, and we will show that $W_1 \boxdot_{P(z)} W_2$ is an object of $\mathcal{C}$ if and only if $W_1 \boxtimes_{P(z)} W_2$ exists in $\mathcal{C}$, in which case we will have

$$W_1 \boxtimes_{P(z)} W_2 = (W_1 \boxdot_{P(z)} W_2)'$$

(observe the notation

$$\boxtimes = \boxdot',$$

as in the special cases studied in [HL1]–[HL3]). It is important to keep in mind that the space $W_1 \boxdot_{P(z)} W_2$ will depend on the category $\mathcal{C}$.

We formalize certain of the properties of the category $\mathcal{C}$ that we have been using, and some new ones, as follows:

Assumption 5.30 *Throughout the remainder of this work, we shall assume that $\mathcal{C}$ is a full subcategory of the category $\mathcal{M}_{sg}$ or $\mathcal{GM}_{sg}$ closed under the contragredient functor (recall Notation 2.36; for now, we are not assuming that $V \in \text{ob } \mathcal{C}$). We shall also assume that $\mathcal{C}$ is closed under taking finite direct sums.*

Definition 5.31 For $W_1, W_2 \in \text{ob } \mathcal{C}$, define the subset

$$W_1 \boxtimes_{P(z)} W_2 \subset (W_1 \otimes W_2)^*$$

of $(W_1 \otimes W_2)^*$ to be the union of the images

$$I'(W') \subset (W_1 \otimes W_2)^*$$

as $(W; I)$ ranges through all the $P(z)$ -products of W_1 and W_2 with $W \in \text{ob } \mathcal{C}$. Equivalently, $W_1 \boxtimes_{P(z)} W_2$ is the union of the images $I'(W')$ as W (or W') ranges through $\text{ob } \mathcal{C}$ and I' ranges through $\mathcal{N}_{W'}^{(W_1 \otimes W_2)^*}$ —the space of $\tilde{A}$ -compatible linear maps

$$W' \rightarrow (W_1 \otimes W_2)^*$$

intertwining the actions of both

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$$

and $\mathfrak{sl}(2)$ on both spaces.

Remark 5.32 Since $\mathcal{C}$ is closed under finite direct sums (Assumption 5.30), it is clear that $W_1 \boxtimes_{P(z)} W_2$ is in fact a linear subspace of $(W_1 \otimes W_2)^*$, and in particular, it can be defined alternatively as the sum of all the images $I'(W')$:

$$W_1 \boxtimes_{P(z)} W_2 = \sum I'(W') = \bigcup I'(W') \subset (W_1 \otimes W_2)^*, \quad (5.133)$$

where the sum and union both range over $W \in \text{ob } \mathcal{C}$, $I \in \mathcal{M}_{W_1 W_2}^W$.

For any generalized V -modules W_1 and W_2 , using the operator $L'_{P(z)}(0)$ (recall (5.110)) on $(W_1 \otimes W_2)^*$ we define the generalized $L'_{P(z)}(0)$ -eigenspaces $((W_1 \otimes W_2)^*)_{[n]}$ for $n \in \mathbb{C}$ in the usual way:

$$((W_1 \otimes W_2)^*)_{[n]} = \{w \in (W_1 \otimes W_2)^* \mid (L'_{P(z)}(0) - n)^m w = 0 \text{ for } m \in \mathbb{N} \text{ sufficiently large}\}. \quad (5.134)$$

Then we have the (proper) subspace

$$\prod_{n \in \mathbb{C}} ((W_1 \otimes W_2)^*)_{[n]} \subset (W_1 \otimes W_2)^*. \quad (5.135)$$

We also define the ordinary $L'_{P(z)}(0)$ -eigenspaces $((W_1 \otimes W_2)^*)_{(n)}$ in the usual way:

$$((W_1 \otimes W_2)^*)_{(n)} = \{w \in (W_1 \otimes W_2)^* \mid L'_{P(z)}(0)w = nw\}. \quad (5.136)$$

Then we have the (proper) subspace

$$\prod_{n \in \mathbb{C}} ((W_1 \otimes W_2)^*)_{(n)} \subset (W_1 \otimes W_2)^*. \quad (5.137)$$

Proposition 5.33 *Let $W_1, W_2 \in \text{ob } \mathcal{C}$.*

(a) *The elements of $W_1 \boxtimes_{P(z)} W_2$ are exactly the linear functionals on $W_1 \otimes W_2$ of the form $w' \circ I(\cdot \otimes \cdot)$ for some $P(z)$ -intertwining map I of type $\binom{W}{W_1 W_2}$ and some $w' \in W'$, $W \in \text{ob } \mathcal{C}$.*

(b) *Let $(W; I)$ be any $P(z)$ -product of W_1 and W_2 , with W any generalized V -module. Then for $n \in \mathbb{C}$,*

$$I'(W'_{[n]}) \subset ((W_1 \otimes W_2)^*)_{[n]}$$

and

$$I'(W'_{(n)}) \subset ((W_1 \otimes W_2)^*)_{(n)}.$$

(c) *The structure $(W_1 \boxtimes_{P(z)} W_2, Y'_{P(z)})$ (recall (5.85)) satisfies all the axioms in the definition of (strongly $\tilde{A}$ -graded) generalized V -module except perhaps for the two grading conditions (2.85) and (2.86).*

(d) *Suppose that the objects of the category $\mathcal{C}$ consist only of (strongly $\tilde{A}$ -graded) ordinary, as opposed to generalized, V -modules. Then the structure $(W_1 \boxtimes_{P(z)} W_2, Y'_{P(z)})$ satisfies all the axioms in the definition of (strongly $\tilde{A}$ -graded ordinary) V -module except perhaps for (2.85) and (2.86).*

Proof Part (a) is clear from the definition of $W_1 \boxtimes_{P(z)} W_2$, and (b) follows from (5.132) with $j = 0$.

For (c), let $(W; I)$ be any any $P(z)$ -product of W_1 and W_2 , with W any generalized V -module. Then $(I'(W'), Y'_{P(z)})$ satisfies all the conditions in the definition of (strongly $\tilde{A}$ -graded) generalized V -module since I' is $\tilde{A}$ -compatible and intertwines the actions of $V \otimes \mathbb{C}[t, t^{-1}]$ and of $\mathfrak{sl}(2)$; the $\mathbb{C}$ -grading follows from Part (b). Note that

$$I' : W' \rightarrow I'(W') \tag{5.138}$$

is a map of generalized V -modules. Since $W_1 \boxtimes_{P(z)} W_2$ is the sum of these structures $I'(W')$ over $W \in \text{ob } \mathcal{C}$ (recall (5.133)), we see that $(W_1 \boxtimes_{P(z)} W_2, Y'_{P(z)})$ satisfies all the conditions in the definition of generalized module except perhaps for (2.85) and (2.86).

Finally, Part (d) is proved by the same argument as for (c). In fact, for $(W; I)$ any $P(z)$ -product of possibly generalized V -modules W_1 and W_2 , with W any ordinary V -module, $(I'(W'), Y'_{P(z)})$ satisfies all the conditions in the definition of (strongly $\tilde{A}$ -graded) ordinary V -module; the $\mathbb{C}$ -grading (this time, by ordinary $L'_{P(z)}(0)$ -eigenspaces) again follows from Part (b). $\square$

Remark 5.34 Later, it will be convenient to introduce the term “doubly graded” generalized module (or module) for the module structures in Parts (c) and (d) (see Definition 9.14 below).

The following terminology will be useful:

Definition 5.35 The category $\mathcal{C}$ is *closed under images* if, given an object M of $\mathcal{C}$, a generalized V -module N , and a map $\phi : M \rightarrow N$ of generalized V -modules, the image $\phi(M)$ is an object of $\mathcal{C}$.

Under this assumption, we have the following alternative description of $W_1 \boxtimes_{P(z)} W_2$:

Proposition 5.36 *Suppose that $\mathcal{C}$ is closed under images (as well as under contragredients and finite direct sums (Assumption 5.30)). Let $W_1, W_2 \in \text{ob } \mathcal{C}$. Then the subspace $W_1 \boxtimes_{P(z)} W_2$ of $(W_1 \otimes W_2)^*$ is equal to the union and also to the sum of the objects of $\mathcal{C}$ lying in $(W_1 \otimes W_2)^*$:*

$$W_1 \boxtimes_{P(z)} W_2 = \bigcup W = \sum W \subset (W_1 \otimes W_2)^*,$$

where, in the union and in the sum, W ranges through the subspaces of $(W_1 \otimes W_2)^*$ that are objects of $\mathcal{C}$ when equipped with the action $Y'_{P(z)}(\cdot, x)$ of V and the corresponding action of $\mathfrak{sl}(2)$ on $(W_1 \otimes W_2)^*$. In particular, every object of $\mathcal{C}$ lying in $(W_1 \otimes W_2)^*$ is a subspace of $W_1 \boxtimes_{P(z)} W_2$ (and for this assertion, the assumption that $\mathcal{C}$ is closed under images is not needed).

Proof In view of Remark 5.32, it suffices to show that the subspaces $I'(W')$ of $(W_1 \otimes W_2)^*$, as W ranges through $\text{ob } \mathcal{C}$ and I' ranges through the maps indicated in Definition 5.31, coincide with the objects of $\mathcal{C}$ lying in $(W_1 \otimes W_2)^*$. But each $I'(W') \in \text{ob } \mathcal{C}$ because $\mathcal{C}$ is closed under images and contragredients, and $I' : W' \rightarrow I'(W')$ is a map of generalized V -modules (recall (5.138)). Conversely, consider an arbitrary object of $\mathcal{C}$ lying in $(W_1 \otimes W_2)^*$ and write it as W' for a suitable $W \in \text{ob } \mathcal{C}$. Then since the embedding

$$J : W' \hookrightarrow (W_1 \otimes W_2)^*$$

has the properties indicated in Notation 5.25, it can be written as I' (with $I = J'$). $\square$

The next result characterizes $W_1 \boxtimes_{P(z)} W_2$, including its existence, in terms of $W_1 \boxtimes_{P(z)} W_2$; this result generalizes Proposition 13.7 in [HL3]:

Proposition 5.37 *Let $W_1, W_2 \in \text{ob } \mathcal{C}$. If $(W_1 \boxtimes_{P(z)} W_2, Y'_{P(z)})$ is an object of $\mathcal{C}$, denote by $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)})$ its contragredient module:*

$$W_1 \boxtimes_{P(z)} W_2 = (W_1 \boxtimes_{P(z)} W_2)'$$

Then the $P(z)$ -tensor product of W_1 and W_2 in $\mathcal{C}$ exists and is

$$(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; i'),$$

where i is the natural inclusion from $W_1 \boxtimes_{P(z)} W_2$ to $(W_1 \otimes W_2)^*$ (recall Notation 5.25). Conversely, let us assume that $\mathcal{C}$ is closed under images. If the $P(z)$ -tensor product of W_1 and W_2 in $\mathcal{C}$ exists, then $(W_1 \boxtimes_{P(z)} W_2, Y'_{P(z)})$ is an object of $\mathcal{C}$.

Proof Suppose that $(W_1 \boxtimes_{P(z)} W_2, Y'_{P(z)})$ is an object of $\mathcal{C}$ and take $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)})$ and the map i as indicated. Then

$$i \in \mathcal{N}_{W_1 \boxtimes_{P(z)} W_2}^{(W_1 \otimes W_2)^*},$$

and

$$i' \in \mathcal{M}_{W_1 W_2}^{W_1 \boxtimes_{P(z)} W_2}.$$

In the notation of Corollary 5.28, we take $I_0 = i'$, $I'_0 = i$. For any pair $(W; I')$ as in Corollary 5.28, we have

$$I'(W') \subset W_1 \boxtimes_{P(z)} W_2$$

(which is the union of all such images), so that there is certainly a unique module map

$$\chi : W' \rightarrow W_1 \boxtimes_{P(z)} W_2$$

such that

$$i \circ \chi = I',$$

namely, I' itself, viewed as a module map. Thus by Corollary 5.28, $W_1 \boxtimes_{P(z)} W_2$ exists as indicated.

Conversely, if the $P(z)$ -tensor product of W_1 and W_2 in $\mathcal{C}$ exists and is $(W_0; I_0)$, then for any $P(z)$ -product $(W; I)$ with $W \in \text{ob } \mathcal{C}$, we have a unique module map $\chi : W' \rightarrow W'_0$ as in Corollary 5.28 such that $I' = I'_0 \circ \chi$, so that $I'(W') \subset I'_0(W'_0)$, proving that

$$W_1 \boxtimes_{P(z)} W_2 \subset I'_0(W'_0).$$

On the other hand, $(W_0; I_0)$ is itself a $P(z)$ -product, so that

$$I'_0(W'_0) \subset W_1 \boxtimes_{P(z)} W_2.$$

Thus

$$W_1 \boxtimes_{P(z)} W_2 = I'_0(W'_0),$$

and so $W_1 \boxtimes_{P(z)} W_2$ is a generalized V -module and is the image of the module map

$$I'_0 : W'_0 \rightarrow W_1 \boxtimes_{P(z)} W_2.$$

Since $\mathcal{C}$ is closed under images by assumption, we have that $W_1 \boxtimes_{P(z)} W_2 \in \text{ob } \mathcal{C}$. $\square$

Remark 5.38 Suppose that $W_1 \boxtimes_{P(z)} W_2$ is an object of $\mathcal{C}$. From Corollary 5.28 and Proposition 5.37 we see that

$$\langle \lambda, w_{(1)} \boxtimes_{P(z)} w_{(2)} \rangle_{W_1 \boxtimes_{P(z)} W_2} = \lambda(w_{(1)} \otimes w_{(2)}) \quad (5.139)$$

for $\lambda \in W_1 \boxtimes_{P(z)} W_2 \subset (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$.

Our next goal is to present a crucial alternative description of the subspace $W_1 \boxtimes_{P(z)} W_2$ of $(W_1 \otimes W_2)^*$. The main ingredient of this description will be the “ $P(z)$ -compatibility condition,” which was a cornerstone of the development of tensor product theory in the special cases treated in [HL1]–[HL3] and [H].

Assume now that W_1 and W_2 are arbitrary generalized V -modules. Let $(W; I)$ (W a generalized V -module) be a $P(z)$ -product of W_1 and W_2 and let $w' \in W'$. Then from (5.131), Proposition 5.27, (5.125), (5.7) and (5.85), we have, for all $v \in V$,

$$\begin{aligned}
& \tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) I'(w') \\
&= I' \left(\tau_{W'} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) w' \right) \\
&= I' \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_{W'}(v, x_1) w' \right) \\
&= x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) I'(Y_{W'}(v, x_1) w') \\
&= x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) I'(\tau_{W'}(Y_t(v, x_1)) w') \\
&= x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) \tau_{P(z)}(Y_t(v, x_1)) I'(w') \\
&= x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y'_{P(z)}(v, x_1) I'(w'). \tag{5.140}
\end{aligned}$$

That is, $I'(w')$ satisfies the following nontrivial and subtle condition on

$$\lambda \in (W_1 \otimes W_2)^* :$$

The $P(z)$ -compatibility condition

- (a) The $P(z)$ -lower truncation condition: For all $v \in V$, the formal Laurent series $Y'_{P(z)}(v, x)\lambda$ involves only finitely many negative powers of x .
- (b) The following formula holds:

$$\begin{aligned}
& \tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \lambda \\
&= x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y'_{P(z)}(v, x_1) \lambda \quad \text{for all } v \in V. \tag{5.141}
\end{aligned}$$

(Note that the two sides of (5.141) are not *a priori* equal for general $\lambda \in (W_1 \otimes W_2)^*$. Note also that Condition (a) insures that the right-hand side in Condition (b) is well defined.)

Notation 5.39 Note that the set of elements of $(W_1 \otimes W_2)^*$ satisfying either the full $P(z)$ -compatibility condition or Part (a) of this condition forms a subspace. We shall denote the space of elements of $(W_1 \otimes W_2)^*$ satisfying the $P(z)$ -compatibility condition by

$$\text{COMP}_{P(z)}((W_1 \otimes W_2)^*).$$

Recall from (5.88) that for each $\beta \in \tilde{A}$ we have the subspace $((W_1 \otimes W_2)^*)^{(\beta)}$ of $(W_1 \otimes W_2)^*$. The sum of these subspaces is of course direct, and we denote it by $((W_1 \otimes W_2)^*)^{(\tilde{A})}$:

$$((W_1 \otimes W_2)^*)^{(\tilde{A})} = \sum_{\beta \in \tilde{A}} ((W_1 \otimes W_2)^*)^{(\beta)} = \prod_{\beta \in \tilde{A}} ((W_1 \otimes W_2)^*)^{(\beta)}.$$

Each space $((W_1 \otimes W_2)^*)^{(\beta)}$ is $L'_{P(z)}(0)$ -stable (recall Proposition 5.6 and Remark 5.13), so that we may consider the subspaces

$$\prod_{n \in \mathbb{C}} ((W_1 \otimes W_2)^*)_{[n]}^{(\beta)} \subset ((W_1 \otimes W_2)^*)^{(\beta)}$$

and

$$\prod_{n \in \mathbb{C}} ((W_1 \otimes W_2)^*)_{(n)}^{(\beta)} \subset ((W_1 \otimes W_2)^*)^{(\beta)}$$

(recall Remark 2.13). We now define the two subspaces

$$((W_1 \otimes W_2)^*)_{[\mathbb{C}]}^{(\tilde{A})} = \prod_{n \in \mathbb{C}} \prod_{\beta \in \tilde{A}} ((W_1 \otimes W_2)^*)_{[n]}^{(\beta)} \subset ((W_1 \otimes W_2)^*)^{(\tilde{A})} \subset (W_1 \otimes W_2)^* \quad (5.142)$$

and

$$((W_1 \otimes W_2)^*)_{(\mathbb{C})}^{(\tilde{A})} = \prod_{n \in \mathbb{C}} \prod_{\beta \in \tilde{A}} ((W_1 \otimes W_2)^*)_{(n)}^{(\beta)} \subset ((W_1 \otimes W_2)^*)^{(\tilde{A})} \subset (W_1 \otimes W_2)^*. \quad (5.143)$$

Remark 5.40 Any $L'_{P(z)}(0)$ -stable subspace of $((W_1 \otimes W_2)^*)_{[\mathbb{C}]}^{(\tilde{A})}$ is graded by generalized eigenspaces (again recall Remark 2.13), and if such a subspace is also $\tilde{A}$ -graded, then it is doubly graded; similarly for subspaces of $((W_1 \otimes W_2)^*)_{(\mathbb{C})}^{(\tilde{A})}$.

We have:

Lemma 5.41 *Suppose that $\lambda \in ((W_1 \otimes W_2)^*)_{[\mathbb{C}]}^{(\tilde{A})}$ satisfies the $P(z)$ -compatibility condition. Then every $\tilde{A}$ -homogeneous component of λ also satisfies this condition. In particular, the space*

$$(\text{COMP}_{P(z)}((W_1 \otimes W_2)^*)) \cap ((W_1 \otimes W_2)^*)^{(\tilde{A})}$$

is $\tilde{A}$ -graded.

Proof When $v \in V$ is $\tilde{A}$ -homogeneous,

$$\tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \quad \text{and} \quad x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y'_{P(z)}(v, x_1)$$

are both $\tilde{A}$ -homogeneous as operators, in the obvious sense. By comparing the $\tilde{A}$ -homogeneous components of both sides of (5.141), we see that the $\tilde{A}$ -homogeneous components of λ also satisfy the $P(z)$ -compatibility condition. $\square$

Remark 5.42 Both the spaces $((W_1 \otimes W_2)^*)_{[\mathbb{C}]}^{(\tilde{A})}$ and $((W_1 \otimes W_2)^*)_{(\mathbb{C})}^{(\tilde{A})}$ are stable under the component operators $\tau_{P(z)}(v \otimes t^m)$ of the operators $Y'_{P(z)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$. For the $\tilde{A}$ -grading, this follows from Proposition 5.6 and Remark 5.13, and for the $\mathbb{C}$ -gradings, we simply follow the proof of Proposition 2.19, using Propositions 5.8 and 5.9 together with (5.115).

Again let $(W; I)$ (W a generalized V -module) be a $P(z)$ -product of W_1 and W_2 and let $w' \in W'$. Since I' in particular intertwines the actions of $V \otimes \mathbb{C}[t, t^{-1}]$ and of $\mathfrak{sl}(2)$, and is $\tilde{A}$ -compatible, $I'(W')$ is a generalized V -module, as we have seen in the proof of Proposition 5.33. Therefore, for each $w' \in W'$, $I'(w')$ also satisfies the following condition on

$$\lambda \in (W_1 \otimes W_2)^* :$$

The $P(z)$ -local grading restriction condition

(a) The $P(z)$ -grading condition: λ is a (finite) sum of generalized eigenvectors for the operator $L'_{P(z)}(0)$ on $(W_1 \otimes W_2)^*$ that are also homogeneous with respect to $\tilde{A}$, that is,

$$\lambda \in ((W_1 \otimes W_2)^*)_{[\mathbb{C}]}^{(\tilde{A})}.$$

(b) Let W_λ be the smallest doubly graded (or equivalently, $\tilde{A}$ -graded; recall Remark 5.40) subspace of $((W_1 \otimes W_2)^*)_{[\mathbb{C}]}^{(\tilde{A})}$ containing λ and stable under the component operators $\tau_{P(z)}(v \otimes t^m)$ of the operators $Y'_{P(z)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$. (In view of Remark 5.42, W_λ indeed exists.) Then W_λ has the properties

$$\dim(W_\lambda)_{[n]}^{(\beta)} < \infty, \tag{5.144}$$

$$(W_\lambda)_{[n+k]}^{(\beta)} = 0 \quad \text{for } k \in \mathbb{Z} \text{ sufficiently negative,} \tag{5.145}$$

for any $n \in \mathbb{C}$ and $\beta \in \tilde{A}$, where as usual the subscripts denote the $\mathbb{C}$ -grading and the superscripts denote the $\tilde{A}$ -grading.

In the case that W is an (ordinary) V -module and $w' \in W'$, $I'(w')$ also satisfies the following $L(0)$ -semisimple version of this condition on $\lambda \in (W_1 \otimes W_2)^*$:

The $L(0)$ -semisimple $P(z)$ -local grading restriction condition

(a) The $L(0)$ -semisimple $P(z)$ -grading condition: λ is a (finite) sum of eigenvectors for the operator $L'_{P(z)}(0)$ on $(W_1 \otimes W_2)^*$ that are also homogeneous with respect to $\tilde{A}$, that is,

$$\lambda \in ((W_1 \otimes W_2)^*)_{(\mathbb{C})}^{(\tilde{A})}.$$

(b) Consider W_λ as above, which in this case is in fact the smallest doubly graded (or equivalently, $\tilde{A}$ -graded) subspace of $((W_1 \otimes W_2)^*)_{(\mathbb{C})}^{(\tilde{A})}$ containing λ and stable under the component operators $\tau_{P(z)}(v \otimes t^m)$ of the operators $Y'_{P(z)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$. Then W_λ has the properties

$$\dim(W_\lambda)_{(n)}^{(\beta)} < \infty, \quad (5.146)$$

$$(W_\lambda)_{(n+k)}^{(\beta)} = 0 \quad \text{for } k \in \mathbb{Z} \text{ sufficiently negative,} \quad (5.147)$$

for any $n \in \mathbb{C}$ and $\beta \in \tilde{A}$, where the subscripts denote the $\mathbb{C}$ -grading and the superscripts denote the $\tilde{A}$ -grading.

Notation 5.43 Note that the set of elements of $(W_1 \otimes W_2)^*$ satisfying either of these two $P(z)$ -local grading restriction conditions, or either of the Part (a)'s in these conditions, forms a subspace. We shall denote the space of elements of $(W_1 \otimes W_2)^*$ satisfying the $P(z)$ -local grading restriction condition and the $L(0)$ -semisimple $P(z)$ -local grading restriction condition by

$$\text{LGR}_{[\mathbb{C}];P(z)}((W_1 \otimes W_2)^*)$$

and

$$\text{LGR}_{(\mathbb{C});P(z)}((W_1 \otimes W_2)^*),$$

respectively.

The following theorems are among the most important in this work. Note that even in the finitely reductive case studied in [HL3], they are stronger and more general than (the last assertion of) Theorem 13.9 in [HL3]. The proofs of these two theorems will be given in the next section.

Theorem 5.44 *Let λ be an element of $(W_1 \otimes W_2)^*$ satisfying the $P(z)$ -compatibility condition. Then when acting on λ , the Jacobi identity for $Y'_{P(z)}$ holds, that is,*

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y'_{P(z)}(u, x_1) Y'_{P(z)}(v, x_2) \lambda \\ & - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y'_{P(z)}(v, x_2) Y'_{P(z)}(u, x_1) \lambda \\ & = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y'_{P(z)}(Y(u, x_0)v, x_2) \lambda \end{aligned} \quad (5.148)$$

for $u, v \in V$.

Theorem 5.45 *The subspace $\text{COMP}_{P(z)}((W_1 \otimes W_2)^*)$ of $(W_1 \otimes W_2)^*$ is stable under the operators $\tau_{P(z)}(v \otimes t^n)$ for $v \in V$ and $n \in \mathbb{Z}$, and in the Möbius case, also under the operators $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$; similarly for the subspaces $\text{LGR}_{[\mathbb{C}];P(z)}((W_1 \otimes W_2)^*)$ and $\text{LGR}_{(\mathbb{C});P(z)}((W_1 \otimes W_2)^*)$.*

Remark 5.46 The converse of Theorem 5.44 is not true. One can see this in the tensor product theory of the “trivial” case where V is a vertex operator algebra associated with a finite-dimensional unital commutative associative algebra $(A, \cdot, 1)$ with derivation $D = 0$ (cf. Remark 2.3). In this case, $(V, Y, \mathbf{1}, \omega) = (A, \cdot, 1, 0)$ and the Jacobi identity for a V -module W reduces to

$$\begin{aligned} x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) u \cdot (v \cdot w) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) v \cdot (u \cdot w) \\ = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) (u \cdot v) \cdot w \end{aligned}$$

for $u, v \in A$ and $w \in W$, where we also use “ $\cdot$ ” to denote the action of A on its modules. In particular, a V -module is just a finite-dimensional module for the associative algebra A . Given V -modules W_1 and W_2 , the action $Y'_{P(z)}$ given in (5.87) now becomes

$$(Y'_{P(z)}(v, x)\lambda)(w_{(1)} \otimes w_{(2)}) = \lambda(w_{(1)} \otimes v \cdot w_{(2)}).$$

From this it is clear that (5.148) holds for any element $\lambda \in (W_1 \otimes W_2)^*$. However, the $P(z)$ -compatibility condition (5.141) in this case reduces to

$$\lambda(v \cdot w_{(1)} \otimes w_{(2)}) = \lambda(w_{(1)} \otimes v \cdot w_{(2)})$$

for all $v \in A$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, which is not necessarily true for every λ . This example is discussed further in Remark 2.20 of [HLLZ], which treats a range of issues related to the compatibility condition, intertwining operators, and tensor product theory.

We now generalize the notion of “weak module” for a vertex operator algebra to our Möbius or conformal vertex algebra V :

Definition 5.47 A *weak module* for V (or *weak V -module*) is a vector space W equipped with a vertex operator map

$$Y_W : V \otimes W \rightarrow W[[x, x^{-1}]]$$

satisfying (only) the axioms (2.35), (2.36), (2.37) and (2.40) in Definition 2.9 (note that there is no grading given on W) and in case V is Möbius, also the existence of a representation of $\mathfrak{sl}(2)$ on W , as in Definition 2.11, satisfying the conditions (2.28)–(2.30).

Then we have:

Theorem 5.48 *The space $\text{COMP}_{P(z)}((W_1 \otimes W_2)^*)$, equipped with the vertex operator map $Y'_{P(z)}$ and, in case V is Möbius, also equipped with the operators $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$, is a weak V -module; similarly for the spaces*

$$(\text{COMP}_{P(z)}((W_1 \otimes W_2)^*)) \cap (\text{LGR}_{[\mathbb{C}]; P(z)}((W_1 \otimes W_2)^*)) \quad (5.149)$$

and

$$(\text{COMP}_{P(z)}((W_1 \otimes W_2)^*)) \cap (\text{LGR}_{(\mathbb{C}); P(z)}((W_1 \otimes W_2)^*)). \quad (5.150)$$

Proof By Theorem 5.45, $Y'_{P(z)}$ is a map from the tensor product of V with any of these three subspaces to the space of formal Laurent series with elements of the subspace as coefficients. By Proposition 5.8 and Theorem 5.44 and, in the case that V is a Möbius vertex algebra, also by Propositions 5.14 and 5.15, we see that all the axioms for weak V -module are satisfied. $\square$

We also have:

Theorem 5.49 *Let*

$$\lambda \in \text{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{[\mathbb{C}]; P(z)}((W_1 \otimes W_2)^*).$$

Then W_λ (recall Part (b) of the $P(z)$ -local grading restriction condition) equipped with the vertex operator map $Y'_{P(z)}$ and, in case V is Möbius, also equipped with the operators $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$, is a (strongly-graded) generalized V -module. If in addition

$$\lambda \in \text{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{(\mathbb{C}); P(z)}((W_1 \otimes W_2)^*),$$

that is, λ is a sum of eigenvectors of $L'_{P(z)}(0)$, then $W_\lambda \subset ((W_1 \otimes W_2)^)_{(\mathbb{C})}^{(\tilde{A})}$ is a (strongly-graded) V -module.*

Proof Decompose λ as

$$\lambda = \sum_{\beta \in \tilde{A}} \lambda^{(\beta)}$$

(finite sum), where

$$\lambda^{(\beta)} \in ((W_1 \otimes W_2)^*)^{(\beta)}.$$

By Lemma 5.41, each $\lambda^{(\beta)}$ satisfies the $P(z)$ -compatibility condition. Also, each $\lambda^{(\beta)}$ satisfies the $P(z)$ -grading condition (and in the semisimple case, the $L(0)$ -semisimple $P(z)$ -grading condition), and each $W_{\lambda^{(\beta)}}$ is simply the smallest subspace containing $\lambda^{(\beta)}$ and stable under the operators listed above (without the $\tilde{A}$ -gradedness condition). Moreover, each $W_{\lambda^{(\beta)}} \subset W_\lambda$ and in fact

$$W_\lambda = \sum_{\beta \in \tilde{A}} W_{\lambda^{(\beta)}}.$$

Thus each $\lambda^{(\beta)}$ lies in the space (5.149) (or (5.150)). (Note that we have reduced Theorem 5.49 to the $\tilde{A}$ -homogeneous case.) By Theorem 5.48, each $W_{\lambda^{(\beta)}}$ is a weak submodule of the weak module (5.149) (or (5.150)), and hence is a (strongly-graded) generalized module (or module). Thus W_λ has the same properties. $\square$

Now we can give an alternative description of $W_{1 \boxtimes P(z)} W_2$ by characterizing the elements of $W_{1 \boxtimes P(z)} W_2$ using the $P(z)$ -compatibility condition and the $P(z)$ -local grading restriction conditions, generalizing Theorem 13.10 in [HL3]. This description combines the results we have just presented; these results will be crucial in later sections, especially in the construction of the associativity isomorphisms, specifically, in the proof of Theorem 9.17 below.

Theorem 5.50 *Suppose that for every element*

$$\lambda \in \text{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{[\mathbb{C}]; P(z)}((W_1 \otimes W_2)^*)$$

the (strongly-graded) generalized module W_λ given in Theorem 5.49 is a generalized submodule of some object of $\mathcal{C}$ included in $(W_1 \otimes W_2)^$ (this of course holds in particular if $\mathcal{C} = \mathcal{GM}_{sg}$). Then*

$$W_1 \boxtimes_{P(z)} W_2 = \text{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{[\mathbb{C}]; P(z)}((W_1 \otimes W_2)^*).$$

Suppose that $\mathcal{C}$ is a category of strongly-graded V -modules (that is, $\mathcal{C} \subset \mathcal{M}_{sg}$) and that for every element

$$\lambda \in \text{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{(\mathbb{C}); P(z)}((W_1 \otimes W_2)^*)$$

the (strongly-graded) V -module W_λ given in Theorem 5.49 is a submodule of some object of $\mathcal{C}$ included in $(W_1 \otimes W_2)^$ (which of course holds in particular if $\mathcal{C} = \mathcal{M}_{sg}$). Then*

$$W_1 \boxtimes_{P(z)} W_2 = \text{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{(\mathbb{C}); P(z)}((W_1 \otimes W_2)^*).$$

Proof We have seen that

$$W_1 \boxtimes_{P(z)} W_2 \subset \text{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{[\mathbb{C}]; P(z)}((W_1 \otimes W_2)^*)$$

and, in case $\mathcal{C} \subset \mathcal{M}_{sg}$,

$$W_1 \boxtimes_{P(z)} W_2 \subset \text{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{(\mathbb{C}); P(z)}((W_1 \otimes W_2)^*).$$

On the other hand, by the assumptions, every element λ of

$$\text{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{[\mathbb{C}]; P(z)}((W_1 \otimes W_2)^*)$$

and, in case $\mathcal{C} \subset \mathcal{M}_{sg}$, every element λ of

$$\text{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{(\mathbb{C}); P(z)}((W_1 \otimes W_2)^*),$$

is contained in W_λ and thus in some object of $\mathcal{C}$, and for any such (generalized) module, the inclusion map into $(W_1 \otimes W_2)^*$ satisfies the intertwining conditions in Proposition 5.24. Thus λ lies in $W_1 \boxtimes_{P(z)} W_2$, proving the desired inclusion. $\square$

5.3 Constructions of $Q(z)$ -tensor products

We now give the construction of $Q(z)$ -tensor products. It is analogous to that of $P(z)$ -tensor products, and the formulations, results and proofs in this section largely parallel those in Section 5.2. As usual, $z \in \mathbb{C}^\times$.

Given generalized V -modules W_1 and W_2 , we shall be constructing an action of the space $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ on the space $(W_1 \otimes W_2)^*$.

Let I be a $Q(z)$ -intertwining map of type $\binom{W_3}{W_1 W_2}$, as in Definition 4.36. Consider the contragredient generalized V -module (W'_3, Y'_3) , recall the opposite vertex operator (2.57) and formula (2.73), and recall why the ingredients of formula (4.73) are well defined. For $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$, applying $w'_{(3)}$ to (4.73) we obtain

$$\begin{aligned} & \left\langle z^{-1} \delta\left(\frac{x_1 - x_0}{z}\right) Y'_3(v, x_0) w'_{(3)}, I(w_{(1)} \otimes w_{(2)}) \right\rangle \\ &= \left\langle w'_{(3)}, x_0^{-1} \delta\left(\frac{x_1 - z}{x_0}\right) I(Y_1^o(v, x_1) w_{(1)} \otimes w_{(2)}) \right\rangle \\ & \quad - \left\langle w'_{(3)}, x_0^{-1} \delta\left(\frac{z - x_1}{-x_0}\right) I(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}) \right\rangle. \end{aligned} \quad (5.151)$$

We shall use this to motivate our action.

As we discussed in Section 5.1 (see (5.12) and (5.13)), in the left-hand side of (5.151), the coefficients of

$$z^{-1} \delta\left(\frac{x_1 - x_0}{z}\right) Y'_3(v, x_0) \quad (5.152)$$

in powers of x_0 and x_1 , for all $v \in V$, span

$$\tau_{W'_3}(V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]) \quad (5.153)$$

(recall (5.2) and (5.7)). We now define a linear action of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ on $(W_1 \otimes W_2)^*$, that is, a linear map

$$\tau_{Q(z)} : V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}] \rightarrow \text{End}(W_1 \otimes W_2)^*.$$

Recall the notations T_{-z}^+ and T_{-z}^o from Section 5.1 ((5.72) and (5.75)).

Definition 5.51 We define the linear action $\tau_{Q(z)}$ of

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$$

on $(W_1 \otimes W_2)^*$ by

$$(\tau_{Q(z)}(\xi)\lambda)(w_{(1)} \otimes w_{(2)}) = \lambda(\tau_{W_1}(T_{-z}^o \xi) w_{(1)} \otimes w_{(2)}) - \lambda(w_{(1)} \otimes \tau_{W_2}(T_{-z}^+ \xi) w_{(2)}) \quad (5.154)$$

for $\xi \in V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, and denote by $Y'_{Q(z)}$ the action of $V \otimes \mathbb{C}[t, t^{-1}]$ on $(W_1 \otimes W_2)^*$ thus defined, that is,

$$Y'_{Q(z)}(v, x) = \tau_{Q(z)}(Y_t(v, x)) \quad (5.155)$$

for $v \in V$.

Using Lemma 5.2, (5.7) and (5.61), we see that (5.154) can be written using generating functions as

$$\begin{aligned}
& \left(\tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\
&= x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) \lambda(Y_1^o(v, x_1) w_{(1)} \otimes w_{(2)}) \\
&\quad - x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2(v, x_1) w_{(2)})
\end{aligned} \tag{5.156}$$

for $v \in V$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$; compare this with (5.151). The generating function form of the action $Y'_{Q(z)}$ can be obtained by taking Res_{x_1} of both sides of (5.156):

$$\begin{aligned}
& (Y'_{Q(z)}(v, x_0) \lambda)(w_{(1)} \otimes w_{(2)}) \\
&= \text{Res}_{x_1} x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) \lambda(Y_1^o(v, x_1) w_{(1)} \otimes w_{(2)}) \\
&\quad - \text{Res}_{x_1} x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}) \\
&= \lambda(Y_1^o(v, x_0 + z) w_{(1)} \otimes w_{(2)}) \\
&\quad - \text{Res}_{x_1} x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}).
\end{aligned} \tag{5.157}$$

Remark 5.52 Using the actions $\tau_{W'_3}$ and $\tau_{Q(z)}$, we can write (5.151) as

$$\left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y'_3(v, x_0) w'_{(3)} \right) \circ I = \tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) (w'_{(3)} \circ I)$$

or equivalently, as

$$\left(\tau_{W'_3} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) w'_{(3)} \right) \circ I = \tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) (w'_{(3)} \circ I).$$

Recall the $\tilde{A}$ -grading on $(W_1 \otimes W_2)^*$ and the A -grading on $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$. Similarly, we also have an A -grading on $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z + t)^{-1}]$. Definition 5.5 also applies to a linear action of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z + t)^{-1}]$ on $(W_1 \otimes W_2)^*$. From (5.154) or (5.156), we have:

Proposition 5.53 *The action $\tau_{Q(z)}$ is $\tilde{A}$ -compatible.* $\square$

We also have:

Proposition 5.54 *The action $Y'_{Q(z)}$ has the property*

$$Y'_{Q(z)}(\mathbf{1}, x) = 1 \tag{5.158}$$

and the $L(-1)$ -derivative property

$$\frac{d}{dx}Y'_{Q(z)}(v, x) = Y'_{Q(z)}(L(-1)v, x) \quad (5.159)$$

for $v \in V$.

Proof From (5.157), (2.57) and (2.7),

$$\begin{aligned} (Y'_{Q(z)}(\mathbf{1}, x)\lambda)(w_{(1)} \otimes w_{(2)}) &= \text{Res}_{x_1} x^{-1} \delta \left(\frac{x_1 - z}{x} \right) \lambda(w_{(1)} \otimes w_{(2)}) \\ &\quad - \text{Res}_{x_1} x^{-1} \delta \left(\frac{z - x_1}{-x} \right) \lambda(w_{(1)} \otimes w_{(2)}) \\ &= \text{Res}_{x_1} x_1^{-1} \delta \left(\frac{z + x}{x_1} \right) \lambda(w_{(1)} \otimes w_{(2)}) \\ &= \lambda(w_{(1)} \otimes w_{(2)}), \end{aligned} \quad (5.160)$$

proving (5.158). To prove the $L(-1)$ -derivative property, observe that from (5.157),

$$\begin{aligned} &\left(\left(\frac{d}{dx} Y'_{Q(z)}(v, x) \right) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\ &= \frac{d}{dx} \lambda(Y_1^o(v, x + z)w_{(1)} \otimes w_{(2)}) \\ &\quad + \text{Res}_{x_1} \left(\frac{d}{dx} z^{-1} \delta \left(\frac{-x + x_1}{z} \right) \right) \lambda(w_{(1)} \otimes Y_2(v, x_1)w_{(2)}). \end{aligned} \quad (5.161)$$

But for any formal Laurent series $f(x)$, we have

$$\frac{d}{dx} f \left(\frac{-x + x_1}{z} \right) = -\frac{d}{dx_1} f \left(\frac{-x + x_1}{z} \right) \quad (5.162)$$

and if $f(x)$ involves only finitely many negative powers of x ,

$$\text{Res}_{x_1} \left(\frac{d}{dx_1} z^{-1} \delta \left(\frac{-x + x_1}{z} \right) \right) f(x_1) = -\text{Res}_{x_1} z^{-1} \delta \left(\frac{-x + x_1}{z} \right) \frac{d}{dx_1} f(x_1) \quad (5.163)$$

(since the residue of a derivative is 0). We also have the $L(-1)$ -derivative property (2.62) for Y^o . Thus the right-hand side of (5.161) equals

$$\begin{aligned} &\lambda(Y_1^o(L(-1)v, x + z)w_{(1)} \otimes w_{(2)}) \\ &\quad + \text{Res}_{x_1} z^{-1} \delta \left(\frac{-x + x_1}{z} \right) \frac{d}{dx_1} \lambda(w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \\ &= \lambda(Y_1^o(L(-1)v, x + z)w_{(1)} \otimes w_{(2)}) \\ &\quad + \text{Res}_{x_1} z^{-1} \delta \left(\frac{-x + x_1}{z} \right) \lambda(w_{(1)} \otimes Y_2(L(-1)v, x_1)w_{(2)}) \\ &= (Y'_{Q(z)}(L(-1)v, x)\lambda)(w_{(1)} \otimes w_{(2)}), \end{aligned} \quad (5.164)$$

proving (5.159). $\square$

Proposition 5.55 *The action $Y'_{Q(z)}$ satisfies the commutator formula for vertex operators: On $(W_1 \otimes W_2)^*$,*

$$\begin{aligned} & [Y'_{Q(z)}(v_1, x_1), Y'_{Q(z)}(v_2, x_2)] \\ &= \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y'_{Q(z)}(Y(v_1, x_0)v_2, x_2) \end{aligned} \quad (5.165)$$

for $v_1, v_2 \in V$.

Proof As usual, the reader should note the well-definedness of each expression and the justifiability of each use of a δ -function property in the argument that follows. This argument is the same as the proof of Proposition 5.2 of [HL1], given in Section 8 of [HL2]. Let $\lambda \in (W_1 \otimes W_2)^*$, $v_1, v_2 \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. By (5.157),

$$\begin{aligned} & (Y'_{Q(z)}(v_1, x_1)Y'_{Q(z)}(v_2, x_2)\lambda)(w_{(1)} \otimes w_{(2)}) \\ &= \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{y_1 - z}{x_1} \right) (Y'_{Q(z)}(v_2, x_2)\lambda)(Y_1^o(v_1, y_1)w_{(1)} \otimes w_{(2)}) \\ & \quad - \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) (Y'_{Q(z)}(v_2, x_2)\lambda)(w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\ &= \text{Res}_{y_1} \text{Res}_{y_2} x_1^{-1} \delta \left(\frac{y_1 - z}{x_1} \right) x_2^{-1} \delta \left(\frac{y_2 - z}{x_2} \right) \cdot \\ & \quad \cdot \lambda(Y_1^o(v_2, y_2)Y_1^o(v_1, y_1)w_{(1)} \otimes w_{(2)}) \\ & \quad - \text{Res}_{y_1} \text{Res}_{y_2} x_1^{-1} \delta \left(\frac{y_1 - z}{x_1} \right) x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \cdot \\ & \quad \cdot \lambda(Y_1^o(v_1, y_1)w_{(1)} \otimes Y_2(v_2, y_2)w_{(2)}) \\ & \quad - \text{Res}_{y_1} \text{Res}_{y_2} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) x_2^{-1} \delta \left(\frac{y_2 - z}{x_2} \right) \cdot \\ & \quad \cdot \lambda(Y_1^o(v_2, y_2)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\ & \quad + \text{Res}_{y_1} \text{Res}_{y_2} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \cdot \\ & \quad \cdot \lambda(w_{(1)} \otimes Y_2(v_2, y_2)Y_2(v_1, y_1)w_{(2)}). \end{aligned} \quad (5.166)$$

Transposing the subscripts 1 and 2 of the symbols v , x and y , we have

$$\begin{aligned} & (Y'_{Q(z)}(v_2, x_2)Y'_{Q(z)}(v_1, x_1)\lambda)(w_{(1)} \otimes w_{(2)}) \\ &= \text{Res}_{y_2} \text{Res}_{y_1} x_2^{-1} \delta \left(\frac{y_2 - z}{x_2} \right) x_1^{-1} \delta \left(\frac{y_1 - z}{x_1} \right) \cdot \\ & \quad \cdot \lambda(Y_1^o(v_1, y_1)Y_1^o(v_2, y_2)w_{(1)} \otimes w_{(2)}) \\ & \quad - \text{Res}_{y_2} \text{Res}_{y_1} x_2^{-1} \delta \left(\frac{y_2 - z}{x_2} \right) x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \cdot \\ & \quad \cdot \lambda(Y_1^o(v_2, y_2)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \end{aligned}$$

$$\begin{aligned}
& -\text{Res}_{y_2} \text{Res}_{y_1} x_2^{-1} \delta \left(\frac{z-y_2}{-x_2} \right) x_1^{-1} \delta \left(\frac{y_1-z}{x_1} \right) \cdot \\
& \quad \cdot \lambda(Y_1^o(v_1, y_1)w_{(1)} \otimes Y_2(v_2, y_2)w_{(2)}) \\
& + \text{Res}_{y_2} \text{Res}_{y_1} x_2^{-1} \delta \left(\frac{z-y_2}{-x_2} \right) x_1^{-1} \delta \left(\frac{z-y_1}{-x_1} \right) \cdot \\
& \quad \cdot \lambda(w_{(1)} \otimes Y_2(v_1, y_1)Y_2(v_2, y_2)w_{(2)}).
\end{aligned} \tag{5.167}$$

Formulas (5.166) and (5.167) give

$$\begin{aligned}
& ([Y'_{Q(z)}(v_1, x_1), Y'_{Q(z)}(v_2, x_2)]\lambda)(w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{y_2} \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{y_1-z}{x_1} \right) x_2^{-1} \delta \left(\frac{y_2-z}{x_2} \right) \cdot \\
& \quad \cdot \lambda([Y_1^o(v_2, y_2), Y_1^o(v_1, y_1)]w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{y_2} \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z-y_1}{-x_1} \right) x_2^{-1} \delta \left(\frac{z-y_2}{-x_2} \right) \cdot \\
& \quad \cdot \lambda(w_{(1)} \otimes [Y_2(v_1, y_1), Y_2(v_2, y_2)]w_{(2)}) \\
& = \text{Res}_{y_2} \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{y_1-z}{x_1} \right) x_2^{-1} \delta \left(\frac{y_2-z}{x_2} \right) \cdot \\
& \quad \cdot \lambda \left(\text{Res}_{x_0} y_2^{-1} \delta \left(\frac{y_1-x_0}{y_2} \right) Y_1^o(Y(v_1, x_0)v_2, y_2)w_{(1)} \otimes w_{(2)} \right) \\
& \quad - \text{Res}_{y_2} \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z-y_1}{-x_1} \right) x_2^{-1} \delta \left(\frac{z-y_2}{-x_2} \right) \cdot \\
& \quad \cdot \lambda \left(w_{(1)} \otimes \text{Res}_{x_0} y_2^{-1} \delta \left(\frac{y_1-x_0}{y_2} \right) Y_2(Y(v_1, x_0)v_2, y_2)w_{(2)} \right) \\
& = \text{Res}_{x_0} \text{Res}_{y_2} \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{y_1-z}{x_1} \right) x_2^{-1} \delta \left(\frac{y_2-z}{x_2} \right) y_2^{-1} \delta \left(\frac{y_1-x_0}{y_2} \right) \cdot \\
& \quad \cdot \lambda(Y_1^o(Y(v_1, x_0)v_2, y_2)w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} \text{Res}_{y_2} \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z-y_1}{-x_1} \right) x_2^{-1} \delta \left(\frac{z-y_2}{-x_2} \right) y_2^{-1} \delta \left(\frac{y_1-x_0}{y_2} \right) \cdot \\
& \quad \cdot \lambda(w_{(1)} \otimes Y_2(Y(v_1, x_0)v_2, y_2)w_{(2)}).
\end{aligned} \tag{5.168}$$

But

$$\begin{aligned}
& x_1^{-1} \delta \left(\frac{y_1-z}{x_1} \right) x_2^{-1} \delta \left(\frac{y_2-z}{x_2} \right) y_2^{-1} \delta \left(\frac{y_1-x_0}{y_2} \right) \\
& = y_1^{-1} \delta \left(\frac{x_1+z}{y_1} \right) y_2^{-1} \delta \left(\frac{x_2+z}{y_2} \right) (x_2+z)^{-1} \delta \left(\frac{(x_1+z)-x_0}{x_2+z} \right) \\
& = y_1^{-1} \delta \left(\frac{x_1+z}{y_1} \right) y_2^{-1} \delta \left(\frac{x_2+z}{y_2} \right) x_2^{-1} \delta \left(\frac{x_1-x_0}{x_2} \right) \\
& = y_1^{-1} \delta \left(\frac{x_1+z}{y_1} \right) x_2^{-1} \delta \left(\frac{y_2-z}{x_2} \right) x_2^{-1} \delta \left(\frac{x_1-x_0}{x_2} \right)
\end{aligned} \tag{5.169}$$

and

$$\begin{aligned}
& x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) y_2^{-1} \delta \left(\frac{y_1 - x_0}{y_2} \right) \\
&= z^{-1} \delta \left(\frac{-x_1 + y_1}{z} \right) z^{-1} \delta \left(\frac{-x_2 + y_2}{z} \right) y_2^{-1} \delta \left(\frac{y_1 - x_0}{y_2} \right) \\
&= \left(\sum_{m, n \in \mathbb{Z}} \frac{(-x_1 + y_1)^m}{z^{m+1}} \frac{(-x_2 + y_2)^n}{z^{n+1}} \right) y_2^{-1} \delta \left(\frac{y_1 - x_0}{y_2} \right) \\
&= \left(\sum_{m, n \in \mathbb{Z}} (-x_2 + y_2)^{-1} \left(\frac{-x_1 + y_1}{-x_2 + y_2} \right)^m \frac{(-x_2 + y_2)^{m+n+1}}{z^{m+n+2}} \right) y_2^{-1} \delta \left(\frac{y_1 - x_0}{y_2} \right) \\
&= \left(\sum_{m, k \in \mathbb{Z}} (-x_2 + y_2)^{-1} \left(\frac{-x_1 + y_1}{-x_2 + y_2} \right)^m z^{-1} \left(\frac{-x_2 + y_2}{z} \right)^k \right) y_2^{-1} \delta \left(\frac{y_1 - x_0}{y_2} \right) \\
&= (-x_2 + y_2)^{-1} \delta \left(\frac{-x_1 + y_1}{-x_2 + y_2} \right) z^{-1} \delta \left(\frac{-x_2 + y_2}{z} \right) y_2^{-1} \delta \left(\frac{y_1 - x_0}{y_2} \right) \\
&= (-x_2)^{-1} \delta \left(\frac{x_1 - (y_1 - y_2)}{x_2} \right) z^{-1} \delta \left(\frac{-x_2 + y_2}{z} \right) y_1^{-1} \delta \left(\frac{y_2 + x_0}{y_1} \right) \\
&= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) y_1^{-1} \delta \left(\frac{y_2 + x_0}{y_1} \right). \tag{5.170}
\end{aligned}$$

Thus (5.168) becomes

$$\begin{aligned}
& ([Y'_{Q(z)}(v_1, x_1), Y'_{Q(z)}(v_2, x_2)] \lambda)(w_{(1)} \otimes w_{(2)}) \\
&= \text{Res}_{x_0} \text{Res}_{y_2} \text{Res}_{y_1} y_1^{-1} \delta \left(\frac{x_1 + z}{y_1} \right) x_2^{-1} \delta \left(\frac{y_2 - z}{x_2} \right) x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \cdot \\
&\quad \cdot \lambda(Y_1^o(Y(v_1, x_0)v_2, y_2)w_{(1)} \otimes w_{(2)}) \\
&\quad - \text{Res}_{x_0} \text{Res}_{y_2} \text{Res}_{y_1} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) y_1^{-1} \delta \left(\frac{y_2 + x_0}{y_1} \right) \cdot \\
&\quad \cdot \lambda(w_{(1)} \otimes Y_2(Y(v_1, x_0)v_2, y_2)w_{(2)}) \\
&= \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \cdot \\
&\quad \cdot \left(\text{Res}_{y_2} x_2^{-1} \delta \left(\frac{y_2 - z}{x_2} \right) \lambda(Y_1^o(Y(v_1, x_0)v_2, y_2)w_{(1)} \otimes w_{(2)}) \right. \\
&\quad \left. - \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \lambda(w_{(1)} \otimes Y_2(Y(v_1, x_0)v_2, y_2)w_{(2)}) \right) \\
&= \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) (Y'_{Q(z)}(Y(v_1, x_0)v_2, x_2) \lambda)(w_{(1)} \otimes w_{(2)}). \tag{5.171}
\end{aligned}$$

Since λ , $w_{(1)}$ and $w_{(2)}$ are arbitrary, this equality gives the commutator formula (5.165) for $Y'_{Q(z)}$. $\square$

When V is in fact a conformal vertex algebra, we write

$$Y'_{Q(z)}(\omega, x) = \sum_{n \in \mathbb{Z}} L'_{Q(z)}(n) x^{-n-2}. \quad (5.172)$$

Then from the last two propositions we see that the coefficient operators of $Y'_{Q(z)}(\omega, x)$ satisfy the Virasoro algebra commutator relations:

$$[L'_{Q(z)}(m), L'_{Q(z)}(n)] = (m - n)L'_{Q(z)}(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c. \quad (5.173)$$

Moreover, in this case, by setting $v = \omega$ in (5.157) and taking $\text{Res}_{x_0} x_0^{j+1}$ for $j = -1, 0, 1$, we see that

$$\begin{aligned} & (L'_{Q(z)}(j)\lambda)(w_{(1)} \otimes w_{(2)}) \\ &= \text{Res}_{x_1} (x_1 - z)^{j+1} \lambda(Y_1^o(\omega, x_1)w_{(1)} \otimes w_{(2)}) \\ & \quad - \text{Res}_{x_1} (-z + x_1)^{j+1} \lambda(w_{(1)} \otimes Y_2(\omega, x_1)w_{(2)}) \\ &= \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i \lambda(L(i-j)w_{(1)} \otimes w_{(2)}) \\ & \quad - \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i \lambda(w_{(1)} \otimes L(j-i)w_{(2)}) \end{aligned} \quad (5.174)$$

for $j = -1, 0, 1$. If V is just a Möbius vertex algebra, we define the actions $L'_{Q(z)}(j)$ on $(W_1 \otimes W_2)^*$ by the right-hand side of (5.174) for $j = -1, 0$ and 1 .

Remark 5.56 In view of the action $L'_{Q(z)}(j)$, the $\mathfrak{sl}(2)$ -bracket relations (4.74) for a $Q(z)$ -intertwining map can be written as

$$(L'(j)w'_{(3)}) \circ I = L'_{Q(z)}(j)(w'_{(3)} \circ I) \quad (5.175)$$

for $w'_{(3)} \in W'_3$ and $j = -1, 0$, and 1 .

Remark 5.57 We have

$$L'_{Q(z)}(j)((W_1 \otimes W_2)^*)^{(\beta)} \subset ((W_1 \otimes W_2)^*)^{(\beta)}$$

for $j = -1, 0, 1$ and $\beta \in \tilde{A}$ (cf. Proposition 5.53).

In the case that V is a conformal vertex algebra, $L'_{Q(z)}(-1)$, $L'_{Q(z)}(0)$ and $L'_{Q(z)}(1)$ realize the actions of L_{-1} , L_0 and L_1 in $\mathfrak{sl}(2)$ (cf. (2.27)) on $(W_1 \otimes W_2)^*$. In the case that V is just a Möbius vertex algebra, we now state this fact as a proposition. This proposition is needed in the proof of Theorem 5.78 and therefore also for Theorems 5.79 and 5.80, but neither this proposition nor any of these three theorems are needed anywhere else in this work, so we omit the proof of this proposition. Of course, however, the proof is straightforward, as is the case with all the $\mathfrak{sl}(2)$ formulas.

Proposition 5.58 *Let V be a Möbius vertex algebra and let W_1 and W_2 be generalized V -modules. Then the operators $L'_{Q(z)}(-1)$, $L'_{Q(z)}(0)$ and $L'_{Q(z)}(1)$ realize the actions of L_{-1} , L_0 and L_1 in $\mathfrak{sl}(2)$ on $(W_1 \otimes W_2)^*$.*

We also have:

Proposition 5.59 *Let V be a Möbius vertex algebra and let W_1 and W_2 be generalized V -modules. Then for $v \in V$,*

$$[L(-1), Y'_{Q(z)}(v, x)] = Y'_{Q(z)}(L(-1)v, x), \quad (5.176)$$

$$[L(0), Y'_{Q(z)}(v, x)] = Y'_{Q(z)}(L(0)v, x) + xY'_{Q(z)}(L(-1)v, x), \quad (5.177)$$

$$[L(1), Y'_{Q(z)}(v, x)] = Y'_{Q(z)}(L(1)v, x) + 2xY'_{Q(z)}(L(0)v, x) + x^2Y'_{Q(z)}(L(-1)v, x), \quad (5.178)$$

where for brevity we write $L'_{Q(z)}(j)$ acting on $(W_1 \otimes W_2)^*$ as $L(j)$.

Proof We prove only (5.177) since it is needed for Remark 5.74 and in Section 6. We omit the proofs of (5.176) and (5.178) for the same reasons as above; they are used only for Theorems 5.78–5.80.

Let $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. Using (5.174), (5.157), the commutator formulas for $L(j)$ and $Y_1(v, x_0)$ for $j = -1, 0, 1$ and $v \in V$ (recall Definition 2.11), and the commutator formulas for $L(j)$ and $Y_2^o(v, x)$ for $j = -1, 0, 1$ and $v \in V$ (recall Lemma 2.22), we obtain

$$\begin{aligned} & ([L(0), Y'_{Q(z)}(v, x)]\lambda)(w_{(1)} \otimes w_{(2)}) \\ &= (Y'_{Q(z)}(v, x)\lambda)(L(0)w_{(1)} \otimes w_{(2)}) \\ &\quad - z(Y'_{Q(z)}(v, x)\lambda)(L(1)w_{(1)} \otimes w_{(2)}) \\ &\quad - (Y'_{Q(z)}(v, x)\lambda)(w_{(1)} \otimes L(0)w_{(2)}) \\ &\quad + z(Y'_{Q(z)}(v, x)\lambda)(w_{(1)} \otimes L(-1)w_{(2)}) \\ &\quad - (L(0)\lambda)(Y_1^o(v, x+z)w_{(1)} \otimes w_{(2)}) \\ &\quad + \text{Res}_{x_1} x^{-1} \delta \left(\frac{z-x_1}{-x} \right) (L(0)\lambda)(w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \\ &= \lambda(Y_1^o(v, x+z)L(0)w_{(1)} \otimes w_{(2)}) \\ &\quad - \text{Res}_{x_1} x^{-1} \delta \left(\frac{z-x_1}{-x} \right) \lambda(L(0)w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \\ &\quad - z\lambda(Y_1^o(v, x+z)L(1)w_{(1)} \otimes w_{(2)}) \\ &\quad + z\text{Res}_{x_1} x^{-1} \delta \left(\frac{z-x_1}{-x} \right) \lambda(L(1)w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \\ &\quad - \lambda(Y_1^o(v, x+z)w_{(1)} \otimes L(0)w_{(2)}) \\ &\quad + \text{Res}_{x_1} x^{-1} \delta \left(\frac{z-x_1}{-x} \right) \lambda(w_{(1)} \otimes Y_2(v, x_1)L(0)w_{(2)}) \end{aligned}$$

$$\begin{aligned}
& +z\lambda(Y_1^o(v, x+z)w_{(1)} \otimes L(-1)w_{(2)}) \\
& -z\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(w_{(1)} \otimes Y_2(v, x_1)L(-1)w_{(2)}) \\
& -\lambda(L(0)Y_1^o(v, x+z)w_{(1)} \otimes w_{(2)}) \\
& +z\lambda(L(1)Y_1^o(v, x+z)w_{(1)} \otimes w_{(2)}) \\
& +\lambda(Y_1^o(v, x+z)w_{(1)} \otimes L(0)w_{(2)}) \\
& -z\lambda(Y_1^o(v, x+z)w_{(1)} \otimes L(-1)w_{(2)}) \\
& +\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(L(0)w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \\
& -z\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(L(1)w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \\
& -\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(w_{(1)} \otimes L(0)Y_2(v, x_1)w_{(2)}) \\
& +z\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(w_{(1)} \otimes L(-1)Y_2(v, x_1)w_{(2)}) \\
& = \lambda([Y_1^o(v, x+z), L(0)]w_{(1)} \otimes w_{(2)}) \\
& -z\lambda([Y_1^o(v, x+z), L(1)]w_{(1)} \otimes w_{(2)}) \\
& +z\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(w_{(1)} \otimes [L(-1), Y_2(v, x_1)]w_{(2)}) \\
& -\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(w_{(1)} \otimes [L(0), Y_2(v, x_1)]w_{(2)}) \\
& = \lambda(Y_1^o((L(0) + (x+z)L(-1))v, x+z)w_{(1)} \otimes w_{(2)}) \\
& -z\lambda(Y_1^o(L(-1)v, x+z)w_{(1)} \otimes w_{(2)}) \\
& +z\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(w_{(1)} \otimes Y_2(L(-1)v, x_1)w_{(2)}) \\
& -\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(w_{(1)} \otimes Y_2((L(0) + x_1L(-1))v, x_1)w_{(2)}) \\
& = \lambda(Y_1^o((L(0) + xL(-1))v, x+z)w_{(1)} \otimes w_{(2)}) \\
& -\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(w_{(1)} \otimes Y_2((L(0) + (x-z)L(-1))v, x_1)w_{(2)}) \\
& = \lambda(Y_1^o((L(0) + xL(-1))v, x+z)w_{(1)} \otimes w_{(2)}) \\
& -\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(w_{(1)} \otimes Y_2((L(0) + xL(-1))v, x_1)w_{(2)}) \\
& = (Y'_{Q(z)}((L(0) + xL(-1))v, x)\lambda)(w_{(1)} \otimes w_{(2)}) \\
& = (Y'_{Q(z)}(L(0)v, x)\lambda)(w_{(1)} \otimes w_{(2)}) + (xY'_{Q(z)}(L(-1)v, x)\lambda)(w_{(1)} \otimes w_{(2)}),
\end{aligned}$$

proving (5.177). $\square$

Let W_3 also be an object of $\mathcal{C}$. Note that $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ acts on W'_3 in the natural way. The following result provides further motivation for the definition of our action (5.156) on $(W_1 \otimes W_2)^*$; recall the discussion preceding Proposition 5.24:

Proposition 5.60 *Let W_1 , W_2 and W_3 be generalized V -modules. Under the natural isomorphism described in Remark 5.18 between the space of $\tilde{A}$ -compatible linear maps*

$$I : W_1 \otimes W_2 \rightarrow \overline{W_3}$$

and the space of $\tilde{A}$ -compatible linear maps

$$J : W'_3 \rightarrow (W_1 \otimes W_2)^*$$

determined by (5.127), the $Q(z)$ -intertwining maps I of type $\binom{W_3}{W_1 W_2}$ correspond exactly to the (grading restricted) $\tilde{A}$ -compatible maps J that intertwine the actions of both

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$$

and $\mathfrak{sl}(2)$ on W'_3 and on $(W_1 \otimes W_2)^$. If W_3 is lower bounded, we may replace the grading restrictions by (4.76) and (5.130).*

Proof In view of (5.128), Remark 5.52 asserts that (5.151), or equivalently, (4.73), is equivalent to the condition

$$J \left(\tau_{W'_3} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) w'_{(3)} \right) = \tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) J(w'_{(3)}), \quad (5.179)$$

that is, the condition that J intertwines the actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ on W'_3 and on $(W_1 \otimes W_2)^*$ (recall (5.12)–(5.13)). Similarly, Remark 5.56 asserts that (4.74) is equivalent to the condition

$$J(L'(j)w'_{(3)}) = L'_{Q(z)}(j)J(w'_{(3)}) \quad (5.180)$$

for $j = -1, 0, 1$, that is, the condition that J intertwines the actions of $\mathfrak{sl}(2)$ on W'_3 and on $(W_1 \otimes W_2)^*$. $\square$

Notation 5.61 Given generalized V -modules W_1 , W_2 and W_3 , we shall write $\mathcal{N}[Q(z)]_{W'_3}^{(W_1 \otimes W_2)^*}$ for the space of (grading restricted) $\tilde{A}$ -compatible linear maps

$$J : W'_3 \rightarrow (W_1 \otimes W_2)^*$$

that intertwine the actions of both

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$$

and $\mathfrak{sl}(2)$ on W'_3 and on $(W_1 \otimes W_2)^*$. Note that Proposition 5.60 gives a natural linear isomorphism

$$\begin{array}{ccc} \mathcal{M}[Q(z)]_{W_1 W_2}^{W_3} & \xrightarrow{\sim} & \mathcal{N}[Q(z)]_{W'_3}^{(W_1 \otimes W_2)^*} \\ I & \mapsto & J \end{array}$$

(recall from Definition 4.36 the notation for the space of $Q(z)$ -intertwining maps), and if W_3 is lower bounded, the spaces satisfy the stronger grading restrictions (4.76) and (5.130). As in Notation 5.25, we still use the symbol “prime” to denote this isomorphism in both directions:

$$\begin{array}{ccc} \mathcal{M}[Q(z)]_{W_1 W_2}^{W_3} & \xrightarrow{\sim} & \mathcal{N}[Q(z)]_{W'_3}^{(W_1 \otimes W_2)^*} \\ I & \mapsto & I' \\ J' & \leftarrow & J, \end{array}$$

so that in particular,

$$I'' = I \quad \text{and} \quad J'' = J$$

for $I \in \mathcal{M}[Q(z)]_{W_1 W_2}^{W_3}$ and $J \in \mathcal{N}[Q(z)]_{W'_3}^{(W_1 \otimes W_2)^*}$, and the relation between I and I' is determined by

$$\langle w'_{(3)}, I(w_{(1)} \otimes w_{(2)}) \rangle = I'(w'_{(3)})(w_{(1)} \otimes w_{(2)})$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$, or equivalently,

$$w'_{(3)} \circ I = I'(w'_{(3)}).$$

Remark 5.62 Combining Proposition 5.60 with Proposition 4.44, we see that for any integer p , we also have a natural linear isomorphism

$$\mathcal{N}[Q(z)]_{W'_3}^{(W_1 \otimes W_2)^*} \xrightarrow{\sim} \mathcal{V}_{W'_3 W_2}^{W'_1}$$

from $\mathcal{N}[Q(z)]_{W'_3}^{(W_1 \otimes W_2)^*}$ to the space of logarithmic intertwining operators of type $(\begin{smallmatrix} W'_1 \\ W'_3 W_2 \end{smallmatrix})$. In particular, given any such logarithmic intertwining operator $\mathcal{Y}$ and integer p , the map

$$(I_{\mathcal{Y}, p}^{Q(z)})' : W'_3 \rightarrow (W_1 \otimes W_2)^*$$

defined by

$$(I_{\mathcal{Y}, p}^{Q(z)})'(w'_{(3)})(w_{(1)} \otimes w_{(2)}) = \langle w_{(1)}, \mathcal{Y}(w'_{(3)}, e^{l_p(z)} w_{(2)}) \rangle_{W'_1}$$

is $\tilde{A}$ -compatible and intertwines both actions on both spaces. If the modules involved are lower bounded, we have the stronger grading restrictions (cf. Remark 5.26).

We have formulated the notions of $Q(z)$ -product and $Q(z)$ -tensor product using $Q(z)$ -intertwining maps (Definitions 4.46 and 4.47). Now that we know that $Q(z)$ -intertwining maps can be interpreted as in Proposition 5.60 (and Notation 5.61), we can reformulate the notions of $Q(z)$ -product and $Q(z)$ -tensor product correspondingly (the proof of the next result is the same as that of Proposition 5.27):

Proposition 5.63 *Let $\mathcal{C}_1$ be either of the categories $\mathcal{M}_{sg}$ or $\mathcal{GM}_{sg}$, as in Definition 4.46. For $W_1, W_2 \in \text{ob } \mathcal{C}_1$, a $Q(z)$ -product $(W_3; I_3)$ of W_1 and W_2 (recall Definition 4.46) amounts to an object (W_3, Y_3) of $\mathcal{C}_1$ equipped with a map $I'_3 \in \mathcal{N}[Q(z)]_{W'_3}^{(W_1 \otimes W_2)^*}$, that is, equipped with an $\tilde{A}$ -compatible map*

$$I'_3 : W'_3 \rightarrow (W_1 \otimes W_2)^*$$

that intertwines the two actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ and of $\mathfrak{sl}(2)$. The map I'_3 corresponds to the $Q(z)$ -intertwining map

$$I_3 : W_1 \otimes W_2 \rightarrow \overline{W_3}$$

as above:

$$I'_3(w'_{(3)}) = w'_{(3)} \circ I_3$$

for $w'_{(3)} \in W'_3$ (recall 5.128)). Denoting this structure by $(W_3, Y_3; I'_3)$ or simply by $(W_3; I'_3)$, let $(W_4; I'_4)$ be another such structure. Then a morphism of $Q(z)$ -products from W_3 to W_4 amounts to a module map $\eta : W_3 \rightarrow W_4$ such that the diagram

$$\begin{array}{ccc} & (W_1 \otimes W_2)^* & \\ I'_4 \nearrow & & \nwarrow I'_3 \\ W'_4 & \xrightarrow{\eta'} & W'_3 \end{array}$$

commutes, where η' is the natural map given by (2.102). $\square$

Corollary 5.64 *Let $\mathcal{C}$ be a full subcategory of either $\mathcal{M}_{sg}$ or $\mathcal{GM}_{sg}$, as in Definition 4.47. For $W_1, W_2 \in \text{ob } \mathcal{C}$, a $Q(z)$ -tensor product $(W_0; I_0)$ of W_1 and W_2 in $\mathcal{C}$, if it exists, amounts to an object $W_0 = W_1 \boxtimes_{Q(z)} W_2$ of $\mathcal{C}$ and a structure $(W_0 = W_1 \boxtimes_{Q(z)} W_2; I'_0)$ as in Proposition 5.63, with*

$$I'_0 : (W_1 \boxtimes_{Q(z)} W_2)' \longrightarrow (W_1 \otimes W_2)^*$$

in $\mathcal{N}[Q(z)]_{(W_1 \boxtimes_{Q(z)} W_2)'}^{(W_1 \otimes W_2)^}$, such that for any such pair $(W; I')$ ($W \in \text{ob } \mathcal{C}$), with*

$$I' : W' \longrightarrow (W_1 \otimes W_2)^*$$

in $\mathcal{N}[Q(z)]_{W'}^{(W_1 \otimes W_2)^}$, there is a unique module map*

$$\chi : W' \longrightarrow (W_1 \boxtimes_{Q(z)} W_2)'$$

such that the diagram

$$\begin{array}{ccc} & (W_1 \otimes W_2)^* & \\ I' \nearrow & & \nwarrow I'_0 \\ W' & \xrightarrow{\chi} & (W_1 \boxtimes_{Q(z)} W_2)' \end{array}$$

commutes. Here $\chi = \eta'$, where η is a correspondingly unique module map

$$\eta : W_1 \boxtimes_{Q(z)} W_2 \longrightarrow W.$$

Also, the map I'_0 , which is $\tilde{A}$ -compatible and which intertwines the two actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ and of $\mathfrak{sl}(2)$, is related to the $Q(z)$ -intertwining map

$$I_0 = \boxtimes_{Q(z)} : W_1 \otimes W_2 \longrightarrow \overline{W_1 \boxtimes_{Q(z)} W_2}$$

by

$$I'_0(w') = w' \circ \boxtimes_{Q(z)}$$

for $w' \in (W_1 \boxtimes_{Q(z)} W_2)'$, that is,

$$I'_0(w')(w_{(1)} \otimes w_{(2)}) = \langle w', w_{(1)} \boxtimes_{Q(z)} w_{(2)} \rangle$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, using the notation (4.83). $\square$

Definition 5.65 For $W_1, W_2 \in \text{ob } \mathcal{C}$, define the subset

$$W_1 \boxtimes_{Q(z)} W_2 \subset (W_1 \otimes W_2)^*$$

of $(W_1 \otimes W_2)^*$ to be the union of the images

$$I'(W') \subset (W_1 \otimes W_2)^*$$

as $(W; I)$ ranges through all the $Q(z)$ -products of W_1 and W_2 with $W \in \text{ob } \mathcal{C}$. Equivalently, $W_1 \boxtimes_{P(z)} W_2$ is the union of the images $I'(W')$ as W (or W') ranges through $\text{ob } \mathcal{C}$ and I' ranges through $\mathcal{N}[Q(z)]_{W'}^{(W_1 \otimes W_2)^*}$ —the space of $\tilde{A}$ -compatible linear maps

$$W' \rightarrow (W_1 \otimes W_2)^*$$

intertwining the actions of both

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$$

and $\mathfrak{sl}(2)$ on both spaces.

Remark 5.66 Since $\mathcal{C}$ is closed under finite direct sums (Assumption 5.30), it is clear that $W_1 \boxtimes_{Q(z)} W_2$ is in fact a linear subspace of $(W_1 \otimes W_2)^*$, and in particular, it can be defined alternatively as the sum of all the images $I'(W')$:

$$W_1 \boxtimes_{Q(z)} W_2 = \sum I'(W') = \bigcup I'(W') \subset (W_1 \otimes W_2)^*, \quad (5.181)$$

where the sum and union both range over $W \in \text{ob } \mathcal{C}$, $I \in \mathcal{M}[Q(z)]_{W_1 W_2}^W$.

For any generalized V -modules W_1 and W_2 , using the operator $L'_{Q(z)}(0)$ (recall (5.174)) on $(W_1 \otimes W_2)^*$ we define the generalized $L'_{Q(z)}(0)$ -eigenspaces $((W_1 \otimes W_2)^*)_{[n];Q(z)}$ for $n \in \mathbb{C}$ in the usual way:

$$((W_1 \otimes W_2)^*)_{[n];Q(z)} = \{w \in (W_1 \otimes W_2)^* \mid (L'_{Q(z)}(0) - n)^m w = 0 \text{ for } m \in \mathbb{N} \text{ sufficiently large}\}. \quad (5.182)$$

Then we have the (proper) subspace

$$\coprod_{n \in \mathbb{C}} ((W_1 \otimes W_2)^*)_{[n];Q(z)} \subset (W_1 \otimes W_2)^*. \quad (5.183)$$

We also define the ordinary $L'_{Q(z)}(0)$ -eigenspaces $((W_1 \otimes W_2)^*)_{(n);Q(z)}$ in the usual way:

$$((W_1 \otimes W_2)^*)_{(n);Q(z)} = \{w \in (W_1 \otimes W_2)^* \mid L'_{Q(z)}(0)w = nw\}. \quad (5.184)$$

Then we have the (proper) subspace

$$\coprod_{n \in \mathbb{C}} ((W_1 \otimes W_2)^*)_{(n);Q(z)} \subset (W_1 \otimes W_2)^*. \quad (5.185)$$

Just as in Proposition 5.33, we have:

Proposition 5.67 *Let $W_1, W_2 \in \text{ob } \mathcal{C}$.*

(a) *The elements of $W_1 \boxtimes_{Q(z)} W_2$ are exactly the linear functionals on $W_1 \otimes W_2$ of the form $w' \circ I(\cdot \otimes \cdot)$ for some $Q(z)$ -intertwining map I of type $\binom{W}{W_1 W_2}$ and some $w' \in W'$, $W \in \text{ob } \mathcal{C}$.*

(b) *Let $(W; I)$ be any $Q(z)$ -product of W_1 and W_2 , with W any generalized V -module. Then for $n \in \mathbb{C}$,*

$$I'(W'_{[n]}) \subset ((W_1 \otimes W_2)^*)_{[n];Q(z)}$$

and

$$I'(W'_{(n)}) \subset ((W_1 \otimes W_2)^*)_{(n);Q(z)}.$$

(c) *The structure $(W_1 \boxtimes_{Q(z)} W_2, Y'_{Q(z)})$ (recall (5.155)) satisfies all the axioms in the definition of (strongly $\tilde{A}$ -graded) generalized V -module except perhaps for the two grading conditions (2.85) and (2.86).*

(d) *Suppose that the objects of the category $\mathcal{C}$ consist only of (strongly $\tilde{A}$ -graded) ordinary, as opposed to generalized, V -modules. Then the structure $(W_1 \boxtimes_{Q(z)} W_2, Y'_{Q(z)})$ satisfies all the axioms in the definition of (strongly $\tilde{A}$ -graded ordinary) V -module except perhaps for (2.85) and (2.86).*

Proof Part (a) is clear from the definition of $W_1 \boxtimes_{Q(z)} W_2$, and (b) follows from (5.180) with $j = 0$.

To prove (c), let $(W; I)$ be any $Q(z)$ -product of W_1 and W_2 , with W any generalized V -module. Then $(I'(W'), Y'_{Q(z)})$ satisfies all the conditions in the definition of (strongly

$\tilde{A}$ -graded) generalized V -module since I' is $\tilde{A}$ -compatible and intertwines the actions of $V \otimes \mathbb{C}[t, t^{-1}]$ and of $\mathfrak{sl}(2)$; the $\mathbb{C}$ -grading follows from Part (b). Note that

$$I' : W' \rightarrow I'(W') \quad (5.186)$$

is a map of generalized V -modules. Since $W_1 \boxtimes_{Q(z)} W_2$ is the sum of these structures $I'(W')$ over $W' \in \text{ob } \mathcal{C}$ (recall (5.133)), $(W_1 \boxtimes_{Q(z)} W_2, Y'_{Q(z)})$ satisfies all the conditions in the definition of generalized module except perhaps for (2.85) and (2.86).

Part (d) is proved by the same argument as for (c): For $(W; I)$ any $Q(z)$ -product of possibly generalized V -modules W_1 and W_2 , with W any ordinary V -module, $(I'(W'), Y'_{Q(z)})$ satisfies all the conditions in the definition of (strongly $\tilde{A}$ -graded) ordinary V -module; the $\mathbb{C}$ -grading (by ordinary $L'_{Q(z)}(0)$ -eigenspaces) again follows from Part (b). $\square$

Just as in Proposition 5.36, we have:

Proposition 5.68 *Suppose that $\mathcal{C}$ is closed under images (as well as under contragredients and finite direct sums (Assumption 5.30)). Let $W_1, W_2 \in \text{ob } \mathcal{C}$. Then the subspace $W_1 \boxtimes_{Q(z)} W_2$ of $(W_1 \otimes W_2)^*$ is equal to the union and also to the sum of the objects of $\mathcal{C}$ lying in $(W_1 \otimes W_2)^*$:*

$$W_1 \boxtimes_{Q(z)} W_2 = \bigcup W = \sum W \subset (W_1 \otimes W_2)^*,$$

where in the union and in the sum, W ranges through the subspaces of $(W_1 \otimes W_2)^*$ that are objects of $\mathcal{C}$ when equipped with the action $Y'_{Q(z)}(\cdot, x)$ of V and the corresponding action of $\mathfrak{sl}(2)$ on $(W_1 \otimes W_2)^*$. In particular, every object of $\mathcal{C}$ lying in $(W_1 \otimes W_2)^*$ is a subspace of $W_1 \boxtimes_{Q(z)} W_2$ (and for this assertion, the assumption that $\mathcal{C}$ is closed under images is not needed). $\square$

We also have the following generalization of Proposition 5.8 in [HL1], characterizing $W_1 \boxtimes_{Q(z)} W_2$, including its existence, in terms of $W_1 \boxtimes_{Q(z)} W_2$; the proof is the same as that of Proposition 5.37:

Proposition 5.69 *Let $W_1, W_2 \in \text{ob } \mathcal{C}$. If $(W_1 \boxtimes_{Q(z)} W_2, Y'_{Q(z)})$ is an object of $\mathcal{C}$, denote by $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)})$ its contragredient module:*

$$W_1 \boxtimes_{Q(z)} W_2 = (W_1 \boxtimes_{Q(z)} W_2)'$$

Then the $Q(z)$ -tensor product of W_1 and W_2 in $\mathcal{C}$ exists and is

$$(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; i'),$$

where i is the natural inclusion from $W_1 \boxtimes_{Q(z)} W_2$ to $(W_1 \otimes W_2)^*$ (recall Notation 5.61). Conversely, let us assume that $\mathcal{C}$ is closed under images (recall Definition 5.35). If the $Q(z)$ -tensor product of W_1 and W_2 in $\mathcal{C}$ exists, then $(W_1 \boxtimes_{Q(z)} W_2, Y'_{Q(z)})$ is an object of $\mathcal{C}$. $\square$

Remark 5.70 Suppose that $W_1 \boxtimes_{Q(z)} W_2$ is an object of $\mathcal{C}$. From Corollary 5.64 and Proposition 5.69 we see that

$$\langle \lambda, w_{(1)} \boxtimes_{Q(z)} w_{(2)} \rangle_{W_1 \boxtimes_{Q(z)} W_2} = \lambda(w_{(1)} \otimes w_{(2)}) \quad (5.187)$$

for $\lambda \in W_1 \boxtimes_{Q(z)} W_2 \subset (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$.

As in the $P(z)$ -case, our next goal is to present an alternative description of the subspace $W_1 \boxtimes_{Q(z)} W_2$ of $(W_1 \otimes W_2)^*$. The main ingredient of this description will be the “ $Q(z)$ -compatibility condition,” as was the case in [HL1]–[HL2].

Take W_1 and W_2 to be arbitrary generalized V -modules. Let (W, I) (W a generalized V -module) be a $Q(z)$ -product of W_1 and W_2 and let $w' \in W'$. Then from (5.179), Proposition 5.63, (5.125), (5.7) and (5.155), we have, for all $v \in V$,

$$\begin{aligned} & \tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) I'(w') \\ &= I' \left(\tau_{W'} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) w' \right) \\ &= I' \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_{W'}(v, x_0) w' \right) \\ &= z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) I'(Y_{W'}(v, x_0) w') \\ &= z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) I'(\tau_{W'}(Y_t(v, x_0)) w') \\ &= z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) \tau_{Q(z)}(Y_t(v, x_0)) I'(w') \\ &= z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y'_{Q(z)}(v, x_0) I'(w'). \end{aligned} \quad (5.188)$$

That is, $I'(w')$ satisfies the following nontrivial and subtle condition on

$$\lambda \in (W_1 \otimes W_2)^* :$$

The $Q(z)$ -compatibility condition

(a) The $Q(z)$ -lower truncation condition: For all $v \in V$, the formal Laurent series $Y'_{Q(z)}(v, x)\lambda$ involves only finitely many negative powers of x .

(b) The following formula holds:

$$\begin{aligned} & \tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) \lambda \\ &= z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y'_{Q(z)}(v, x_0) \lambda \quad \text{for all } v \in V. \end{aligned} \quad (5.189)$$

(Note that the two sides of (5.189) are not *a priori* equal for general $\lambda \in (W_1 \otimes W_2)^*$. Note also that Condition (a) insures that the right-hand side in Condition (b) is well defined.)

Notation 5.71 Note that the set of elements of $(W_1 \otimes W_2)^*$ satisfying either the full $Q(z)$ -compatibility condition or Part (a) of this condition forms a subspace. We shall denote the space of elements of $(W_1 \otimes W_2)^*$ satisfying the $Q(z)$ -compatibility condition by

$$\text{COMP}_{Q(z)}((W_1 \otimes W_2)^*).$$

We know that each space $((W_1 \otimes W_2)^*)^{(\beta)}$ is $L'_{Q(z)}(0)$ -stable (recall Proposition 5.53 and Remark 5.57), so that we may consider the subspaces

$$\coprod_{n \in \mathbb{C}} ((W_1 \otimes W_2)^*)_{[n]; Q(z)}^{(\beta)} \subset ((W_1 \otimes W_2)^*)^{(\beta)}$$

and

$$\coprod_{n \in \mathbb{C}} ((W_1 \otimes W_2)^*)_{(n); Q(z)}^{(\beta)} \subset ((W_1 \otimes W_2)^*)^{(\beta)}$$

(recall Remark 2.13). We define the two subspaces

$$((W_1 \otimes W_2)^*)_{[\mathbb{C}]; Q(z)}^{(\tilde{A})} = \coprod_{n \in \mathbb{C}} \coprod_{\beta \in \tilde{A}} ((W_1 \otimes W_2)^*)_{[n]; Q(z)}^{(\beta)} \subset (W_1 \otimes W_2)^* \quad (5.190)$$

and

$$((W_1 \otimes W_2)^*)_{(\mathbb{C}); Q(z)}^{(\tilde{A})} = \coprod_{n \in \mathbb{C}} \coprod_{\beta \in \tilde{A}} ((W_1 \otimes W_2)^*)_{(n); Q(z)}^{(\beta)} \subset (W_1 \otimes W_2)^*. \quad (5.191)$$

Remark 5.72 Any $L'_{Q(z)}(0)$ -stable subspace of $((W_1 \otimes W_2)^*)_{[\mathbb{C}]; Q(z)}^{(\tilde{A})}$ is graded by generalized eigenspaces (again recall Remark 2.13), and if such a subspace is also $\tilde{A}$ -graded, then it is doubly graded; similarly for subspaces of $((W_1 \otimes W_2)^*)_{(\mathbb{C}); Q(z)}^{(\tilde{A})}$.

We have:

Lemma 5.73 *Suppose that $\lambda \in ((W_1 \otimes W_2)^*)_{[\mathbb{C}]; Q(z)}^{(\tilde{A})}$ satisfies the $Q(z)$ -compatibility condition. Then every $\tilde{A}$ -homogeneous component of λ also satisfies this condition.*

Proof When $v \in V$ is $\tilde{A}$ -homogeneous,

$$\tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) \quad \text{and} \quad z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y'_{Q(z)}(v, x_0)$$

are both $\tilde{A}$ -homogeneous as operators. By comparing the $\tilde{A}$ -homogeneous components of both sides of (5.189), we see that the $\tilde{A}$ -homogeneous components of λ also satisfy the $Q(z)$ -compatibility condition. $\square$

Remark 5.74 Just as in Remark 5.42, note that both the spaces $((W_1 \otimes W_2)^*)_{[\mathbb{C}];Q(z)}^{(\tilde{A})}$ and $((W_1 \otimes W_2)^*)_{(\mathbb{C});Q(z)}^{(\tilde{A})}$ are stable under the component operators $\tau_{Q(z)}(v \otimes t^m)$ of the operators $Y'_{Q(z)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{Q(z)}(-1)$, $L'_{Q(z)}(0)$ and $L'_{Q(z)}(1)$; this uses Proposition 5.53, Remark 5.57, Propositions 5.54 and 5.55, and (5.177).

Again let $(W; I)$ (W a generalized V -module) be a $Q(z)$ -product of W_1 and W_2 and let $w' \in W'$. Since I' in particular intertwines the actions of $V \otimes \mathbb{C}[t, t^{-1}]$ and of $\mathfrak{sl}(2)$, and is $\tilde{A}$ -compatible, $I'(W')$ is a generalized V -module (recall the proof of Proposition 5.67). Thus for every $w' \in W'$, $I'(w')$ also satisfies the following condition on

$$\lambda \in (W_1 \otimes W_2)^* :$$

The $Q(z)$ -local grading restriction condition

(a) The $Q(z)$ -grading condition: λ is a (finite) sum of generalized eigenvectors for the operator $L'_{Q(z)}(0)$ on $(W_1 \otimes W_2)^*$ that are also homogeneous with respect to $\tilde{A}$, that is,

$$\lambda \in ((W_1 \otimes W_2)^*)_{[\mathbb{C}];Q(z)}^{(\tilde{A})}.$$

(b) Let $W_{\lambda;Q(z)}$ be the smallest doubly graded (or equivalently, $\tilde{A}$ -graded; recall Remark 5.72) subspace of $((W_1 \otimes W_2)^*)_{[\mathbb{C}];Q(z)}^{(\tilde{A})}$ containing λ and stable under the component operators $\tau_{Q(z)}(v \otimes t^m)$ of the operators $Y'_{Q(z)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{Q(z)}(-1)$, $L'_{Q(z)}(0)$ and $L'_{Q(z)}(1)$. (In view of Remark 5.74, $W_{\lambda;Q(z)}$ indeed exists.) Then $W_{\lambda;Q(z)}$ has the properties

$$\dim(W_{\lambda})_{[n];Q(z)}^{(\beta)} < \infty, \tag{5.192}$$

$$(W_{\lambda})_{[n+k];Q(z)}^{(\beta)} = 0 \quad \text{for } k \in \mathbb{Z} \text{ sufficiently negative,} \tag{5.193}$$

for any $n \in \mathbb{C}$ and $\beta \in \tilde{A}$, where as usual the subscripts denote the $\mathbb{C}$ -grading and the superscripts denote the $\tilde{A}$ -grading.

In the case that W is an (ordinary) V -module and $w' \in W'$, $I'(w')$ also satisfies the following $L(0)$ -semisimple version of this condition on $\lambda \in (W_1 \otimes W_2)^*$:

The $L(0)$ -semisimple $Q(z)$ -local grading restriction condition

(a) The $L(0)$ -semisimple $Q(z)$ -grading condition: λ is a (finite) sum of eigenvectors for the operator $L'_{Q(z)}(0)$ on $(W_1 \otimes W_2)^*$ that are also homogeneous with respect to $\tilde{A}$, that is,

$$\lambda \in ((W_1 \otimes W_2)^*)_{(\mathbb{C});Q(z)}^{(\tilde{A})}.$$

(b) Consider $W_{\lambda;Q(z)}$ as above, which in this case is in fact the smallest doubly graded (or equivalently, $\tilde{A}$ -graded) subspace of $((W_1 \otimes W_2)^*)_{(\mathbb{C});Q(z)}^{(\tilde{A})}$ containing λ and stable under the component operators $\tau_{Q(z)}(v \otimes t^m)$ of the operators $Y'_{Q(z)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{Q(z)}(-1)$, $L'_{Q(z)}(0)$ and $L'_{Q(z)}(1)$. Then $W_{\lambda;Q(z)}$ has the properties

$$\dim(W_{\lambda;Q(z)})_{(n);Q(z)}^{(\beta)} < \infty, \quad (5.194)$$

$$(W_{\lambda})_{(n+k);Q(z)}^{(\beta)} = 0 \quad \text{for } k \in \mathbb{Z} \text{ sufficiently negative,} \quad (5.195)$$

for any $n \in \mathbb{C}$ and $\beta \in \tilde{A}$, where the subscripts denote the $\mathbb{C}$ -grading and the superscripts denote the $\tilde{A}$ -grading.

Notation 5.75 Note that the set of elements of $(W_1 \otimes W_2)^*$ satisfying either of these two $Q(z)$ -local grading restriction conditions, or either of the Part (a)'s in these conditions, forms a subspace. We shall denote the space of elements of $(W_1 \otimes W_2)^*$ satisfying the $Q(z)$ -local grading restriction condition and the $L(0)$ -semisimple $Q(z)$ -local grading restriction condition by

$$\text{LGR}_{[\mathbb{C}];Q(z)}((W_1 \otimes W_2)^*)$$

and

$$\text{LGR}_{(\mathbb{C});Q(z)}((W_1 \otimes W_2)^*),$$

respectively.

We have the following important theorems generalizing the corresponding results stated in [HL1] and proved in [HL2]. The proofs of these theorems will be given in the next section.

Theorem 5.76 *Let λ be an element of $(W_1 \otimes W_2)^*$ satisfying the $Q(z)$ -compatibility condition. Then when acting on λ , the Jacobi identity for $Y'_{Q(z)}$ holds, that is,*

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y'_{Q(z)}(u, x_1) Y'_{Q(z)}(v, x_2) \lambda \\ & - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y'_{Q(z)}(v, x_2) Y'_{Q(z)}(u, x_1) \lambda \\ & = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y'_{Q(z)}(Y(u, x_0)v, x_2) \lambda \end{aligned} \quad (5.196)$$

for $u, v \in V$.

Theorem 5.77 *The subspace $\text{COMP}_{Q(z)}((W_1 \otimes W_2)^*)$ of $(W_1 \otimes W_2)^*$ is stable under the operators $\tau_{Q(z)}(v \otimes t^n)$ for $v \in V$ and $n \in \mathbb{Z}$, and in the Möbius case, also under the operators $L'_{Q(z)}(-1)$, $L'_{Q(z)}(0)$ and $L'_{Q(z)}(1)$; similarly for the subspaces $\text{LGR}_{[\mathbb{C}];Q(z)}((W_1 \otimes W_2)^*$ and $\text{LGR}_{(\mathbb{C});Q(z)}((W_1 \otimes W_2)^*$.*

We have:

Theorem 5.78 *The space $\text{COMP}_{Q(z)}((W_1 \otimes W_2)^*)$, equipped with the vertex operator map $Y'_{Q(z)}$ and, in case V is Möbius, also equipped with the operators $L'_{Q(z)}(-1)$, $L'_{Q(z)}(0)$ and $L'_{Q(z)}(1)$, is a weak V -module; similarly for the spaces*

$$(\text{COMP}_{Q(z)}((W_1 \otimes W_2)^*)) \cap (\text{LGR}_{[\mathbb{C}];Q(z)}((W_1 \otimes W_2)^*))$$

and

$$(\text{COMP}_{Q(z)}((W_1 \otimes W_2)^*)) \cap (\text{LGR}_{(\mathbb{C});Q(z)}((W_1 \otimes W_2)^*)).$$

Proof By Theorem 5.77, $Y'_{Q(z)}$ is a map from the tensor product of V with any of these three subspaces to the space of formal Laurent series with elements of the subspace as coefficients. By Proposition 5.54 and Theorem 5.76 and, in the case that V is Möbius, also by Propositions 5.58 and 5.59, we see that all the axioms for weak V -module are satisfied. $\square$

Moreover, we have the following consequence of Theorem 5.78 and Lemma 5.73, just as in Theorem 5.49:

Theorem 5.79 *Let*

$$\lambda \in \text{COMP}_{Q(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{[\mathbb{C}];Q(z)}((W_1 \otimes W_2)^*).$$

Then $W_{\lambda;Q(z)}$ (recall Part (b) of the $Q(z)$ -local grading restriction condition) equipped with the vertex operator map $Y'_{Q(z)}$ and, in case V is Möbius, also equipped with the operators $L'_{Q(z)}(-1)$, $L'_{Q(z)}(0)$ and $L'_{Q(z)}(1)$, is a (strongly-graded) generalized V -module. If in addition

$$\lambda \in \text{COMP}_{Q(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{(\mathbb{C});Q(z)}((W_1 \otimes W_2)^*),$$

that is, λ is a sum of eigenvectors of $L'_{Q(z)}(0)$, then $W_{\lambda;Q(z)} \subset ((W_1 \otimes W_2)^)_{(\mathbb{C});Q(z)}^{(\tilde{A})}$ is a (strongly-graded) V -module. $\square$*

Finally, as in Theorem 5.50, we can give an alternative description of $W_1 \boxtimes_{Q(z)} W_2$ by characterizing the elements of $W_1 \boxtimes_{Q(z)} W_2$ using the $Q(z)$ -compatibility condition and the $Q(z)$ -local grading restriction conditions, generalizing Theorem 6.3 in [HL1]. The proof of the following theorem is the same as that of Theorem 5.50.

Theorem 5.80 *Suppose that for every element*

$$\lambda \in \text{COMP}_{Q(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{[\mathbb{C}];Q(z)}((W_1 \otimes W_2)^*)$$

the (strongly-graded) generalized module $W_{\lambda;Q(z)}$ given in Theorem 5.79 is a generalized submodule of some object of $\mathcal{C}$ included in $(W_1 \otimes W_2)^$ (this of course holds in particular if $\mathcal{C} = \mathcal{GM}_{sg}$). Then*

$$W_1 \boxtimes_{Q(z)} W_2 = \text{COMP}_{Q(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{[\mathbb{C}];Q(z)}((W_1 \otimes W_2)^*).$$

Suppose that $\mathcal{C}$ is a category of strongly-graded V -modules (that is, $\mathcal{C} \subset \mathcal{M}_{sg}$) and that for every element

$$\lambda \in \text{COMP}_{Q(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{(\mathbb{C});Q(z)}((W_1 \otimes W_2)^*)$$

the (strongly-graded) V -module $W_{\lambda;Q(z)}$ given in Theorem 5.79 is a submodule of some object of $\mathcal{C}$ included in $(W_1 \otimes W_2)^*$ (which of course holds in particular if $\mathcal{C} = \mathcal{M}_{sg}$). Then

$$W_1 \boxtimes_{Q(z)} W_2 = \text{COMP}_{Q(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{(\mathbb{C});Q(z)}((W_1 \otimes W_2)^*). \quad \square$$

6 Proof of the theorems used in the constructions

The primary goal of this section is to prove Theorems 5.44, 5.45, 5.76 and 5.77. In Section 6.1 we prove Theorems 5.44 and 5.45, and in Section 6.2, Theorems 5.76 and 5.77. The proofs in Section 6.1 are new, even for the category of (ordinary) modules for a vertex operator algebra satisfying the finiteness and reductivity conditions treated in [HL1]–[HL3]. In [HL1]–[HL3], for a vertex operator algebra satisfying these conditions, Theorems 5.76 and 5.77, in the $Q(z)$ case, were proved first, and then Theorems 5.44 and 5.45, in the $P(z)$ case, were proved using results from the $Q(z^{-1})$ case and relations between $P(z)$ -tensor products and $Q(z^{-1})$ -tensor products. In Section 6.1, we prove Theorems 5.44 and 5.45 directly, without using any results from the $Q(z)$ case. As usual, the reader should observe the justifiability of each step in the arguments (the well-definedness of the formal series, etc.); again as usual, this is sometimes quite subtle.

We continue to work in the setting Section 5. In particular, we have Assumptions 4.1 and 5.30, and $z \in \mathbb{C}^\times$.

6.1 Proofs of Theorems 5.44 and 5.45

We first prove a formula for vertex operators that will be needed in the proofs of both Theorem 5.44 and Theorem 5.45.

Lemma 6.1 *For $u, v \in V$, we have*

$$\begin{aligned} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y(e^{x_1 L(1)} (-x_1^{-2})^{L(0)} u, -x_0 x_1^{-1} x_2^{-1}) e^{x_2 L(1)} (-x_2^{-2})^{L(0)} v \\ = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) e^{x_2 L(1)} (-x_2^{-2})^{L(0)} Y(u, x_0) v. \end{aligned} \quad (6.1)$$

Proof Using (3.61), (3.62), (3.67) and (2.11), we have

$$\begin{aligned} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y(e^{x_1 L(1)} (-x_1^{-2})^{L(0)} u, -x_0 x_1^{-1} x_2^{-1}) e^{x_2 L(1)} (-x_2^{-2})^{L(0)} \\ = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) e^{x_2 L(1)} Y(e^{-x_2(1-x_0 x_1^{-1}) L(1)} (1 - x_0 x_1^{-1})^{-2L(0)} \cdot \\ \cdot e^{x_1 L(1)} (-x_1^{-2})^{L(0)} u, -x_0 x_1^{-1} x_2^{-1} (1 - x_0 x_1^{-1})^{-1}) (-x_2^{-2})^{L(0)} \\ = x_2^{-1} \delta \left(\frac{x_2^{-1} - x_0 x_1^{-1} x_2^{-1}}{x_1^{-1}} \right) e^{x_2 L(1)} (-x_2^{-2})^{L(0)} \cdot \\ \cdot Y((-x_2^2)^{L(0)} e^{-x_2(1-x_0 x_1^{-1}) L(1)} (1 - x_0 x_1^{-1})^{-2L(0)} \cdot \\ \cdot e^{x_1 L(1)} (-x_1^{-2})^{L(0)} u, x_0 x_1^{-1} (x_2^{-1} - x_0 x_1^{-1} x_2^{-1})^{-1}) \\ = x_2^{-1} \delta \left(\frac{x_2^{-1} - x_0 x_1^{-1} x_2^{-1}}{x_1^{-1}} \right) e^{x_2 L(1)} (-x_2^{-2})^{L(0)} \cdot \\ \cdot Y(e^{x_2^{-1}(1-x_0 x_1^{-1}) L(1)} (-x_2^2)^{L(0)} (1 - x_0 x_1^{-1})^{-2L(0)} e^{x_1 L(1)} (-x_1^{-2})^{L(0)} u, x_0) \end{aligned}$$

$$\begin{aligned}
&= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) e^{x_2 L(1)} (-x_2^{-2})^{L(0)} \cdot \\
&\quad \cdot Y(e^{x_2^{-1}(1-x_0x_1^{-1})L(1)} (-x_2^{-1}(1-x_0x_1^{-1}))^{-2})^{L(0)} e^{x_1 L(1)} (-x_1^{-2})^{L(0)} u, x_0) \\
&= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) e^{x_2 L(1)} (-x_2^{-2})^{L(0)} \cdot \\
&\quad \cdot Y(e^{x_2^{-1}(1-x_0x_1^{-1})L(1)} e^{-x_1 x_2^{-2}(1-x_0x_1^{-1})^2 L(1)} (-x_2^{-1}(1-x_0x_1^{-1}))^{-2})^{L(0)} (-x_1^{-2})^{L(0)} u, x_0) \\
&= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) e^{x_2 L(1)} (-x_2^{-2})^{L(0)} \cdot \\
&\quad \cdot Y(e^{x_2^{-1}(1-x_0x_1^{-1})(1-x_2^{-1}(x_1-x_0))L(1)} (x_2^{-1}(x_1-x_0))^{-2L(0)} u, x_0) \\
&= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) e^{x_2 L(1)} (-x_2^{-2})^{L(0)} Y(u, x_0).
\end{aligned}$$

□

Proof of Theorem 5.44 Let λ be an element of $(W_1 \otimes W_2)^*$ satisfying the $P(z)$ -compatibility condition, that is, satisfying (a) the $P(z)$ -lower truncation condition—for all $v \in V$, the formal Laurent series $Y'_{P(z)}(v, x)\lambda$ involves only finitely many negative powers of x , and (b) formula (5.141) for all $v \in V$.

For $u, v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, by definition,

$$\begin{aligned}
&\left(x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y'_{P(z)}(u, x_1) Y'_{P(z)}(v, x_2) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\
&= x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \left((Y'_{P(z)}(v, x_2) \lambda) (w_{(1)} \otimes Y_2^o(u, x_1) w_{(2)}) \right. \\
&\quad \left. + \text{Res}_{y_1} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) (Y'_{P(z)}(v, x_2) \lambda) (Y_1(e^{x_1 L(1)} (-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes w_{(2)}) \right) \\
&= x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \left(\lambda(w_{(1)} \otimes Y_2^o(v, x_2) Y_2^o(u, x_1) w_{(2)}) \right. \\
&\quad \left. + \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) \lambda(Y_1(e^{x_2 L(1)} (-x_2^{-2})^{L(0)} v, y_2) w_{(1)} \otimes Y_2^o(u, x_1) w_{(2)}) \right. \\
&\quad \left. + \text{Res}_{y_1} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) (Y'_{P(z)}(v, x_2) \lambda) (Y_1(e^{x_1 L(1)} (-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes w_{(2)}) \right) \\
&= x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \lambda(w_{(1)} \otimes Y_2^o(v, x_2) Y_2^o(u, x_1) w_{(2)}) \\
&\quad + x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) \cdot \\
&\quad \cdot \lambda(Y_1(e^{x_2 L(1)} (-x_2^{-2})^{L(0)} v, y_2) w_{(1)} \otimes Y_2^o(u, x_1) w_{(2)}) \\
&\quad + x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \text{Res}_{y_1} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) \cdot
\end{aligned}$$

$$\cdot (Y'_{P(z)}(v, x_2)\lambda)(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, y_1)w_{(1)} \otimes w_{(2)}). \quad (6.2)$$

Using (2.6) and (5.141), we see that the third term on the right-hand side of (6.2) is equal to

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_2^{-1}-x_1^{-1}}{x_0x_1^{-1}x_2^{-1}}\right)\text{Res}_{y_1}z^{-1}\delta\left(\frac{x_1^{-1}-y_1}{z}\right) \cdot \\ & \quad \cdot (Y'_{P(z)}(v, x_2)\lambda)(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, y_1)w_{(1)} \otimes w_{(2)}) \\ & = \text{Res}_{y_1}x_1^{-1}x_2^{-1}(x_0x_1^{-1}x_2^{-1})^{-1}\delta\left(\frac{x_2^{-1}-y_1-z}{x_0x_1^{-1}x_2^{-1}}\right)z^{-1}\delta\left(\frac{x_1^{-1}-y_1}{z}\right) \cdot \\ & \quad \cdot (Y'_{P(z)}(v, x_2)\lambda)(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, y_1)w_{(1)} \otimes w_{(2)}) \\ & = \text{Res}_{y_1}x_1^{-1}x_2^{-1}(x_0x_1^{-1}x_2^{-1}+y_1)^{-1}\delta\left(\frac{x_2^{-1}-z}{x_0x_1^{-1}x_2^{-1}+y_1}\right)z^{-1}\delta\left(\frac{x_1^{-1}-y_1}{z}\right) \cdot \\ & \quad \cdot (Y'_{P(z)}(v, x_2)\lambda)(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, y_1)w_{(1)} \otimes w_{(2)}) \\ & = \text{Res}_{y_1}x_1^{-1}x_2^{-1}z^{-1}\delta\left(\frac{x_1^{-1}-y_1}{z}\right) \cdot \\ & \quad \cdot \left(\tau_{P(z)}\left((x_0x_1^{-1}x_2^{-1}+y_1)^{-1}\delta\left(\frac{x_2^{-1}-z}{x_0x_1^{-1}x_2^{-1}+y_1}\right)Y_t(v, x_2)\right)\lambda\right) \\ & \quad (Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, y_1)w_{(1)} \otimes w_{(2)}) \\ & = \text{Res}_{y_1}x_1^{-1}x_2^{-1}z^{-1}\delta\left(\frac{x_1^{-1}-y_1}{z}\right) \cdot \\ & \quad \cdot \left(z^{-1}\delta\left(\frac{x_2^{-1}-x_0x_1^{-1}x_2^{-1}-y_1}{z}\right) \cdot \right. \\ & \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)}v, x_0x_1^{-1}x_2^{-1}+y_1)Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, y_1)w_{(1)} \otimes w_{(2)}) \\ & \quad \left. + (x_0x_1^{-1}x_2^{-1}+y_1)^{-1}\delta\left(\frac{z-x_2^{-1}}{-x_0x_1^{-1}x_2^{-1}-y_1}\right) \cdot \right. \\ & \quad \left. \cdot \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, y_1)w_{(1)} \otimes Y_2^o(v, x_2)w_{(2)})\right) \\ & = \text{Res}_{y_1}x_2^{-1}\delta\left(\frac{z+y_1}{x_1^{-1}}\right)(z+y_1)^{-1}\delta\left(\frac{x_2^{-1}-x_0x_1^{-1}x_2^{-1}}{z+y_1}\right) \cdot \\ & \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)}v, x_0x_1^{-1}x_2^{-1}+y_1)Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, y_1)w_{(1)} \otimes w_{(2)}) \\ & \quad + \text{Res}_{y_1}x_2^{-1}\delta\left(\frac{z+y_1}{x_1^{-1}}\right)(x_0x_1^{-1}x_2^{-1})^{-1}\delta\left(\frac{z+y_1-x_2^{-1}}{-x_0x_1^{-1}x_2^{-1}}\right) \cdot \\ & \quad \cdot \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, y_1)w_{(1)} \otimes Y_2^o(v, x_2)w_{(2)}). \quad (6.3) \end{aligned}$$

By (2.11) and (2.6), the right-hand side of (6.3) is equal to

$$\text{Res}_{y_1}x_2^{-1}\delta\left(\frac{z+y_1}{x_1^{-1}}\right)x_1\delta\left(\frac{x_2^{-1}-x_0x_1^{-1}x_2^{-1}}{x_1^{-1}}\right).$$

$$\begin{aligned}
& \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)}v, x_0 x_1^{-1} x_2^{-1} + y_1) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, y_1) w_{(1)} \otimes w_{(2)}) \\
& + \text{Res}_{y_1} x_2^{-1} \delta \left(\frac{z + y_1}{x_1^{-1}} \right) (x_0 x_1^{-1} x_2^{-1})^{-1} \delta \left(\frac{x_1^{-1} - x_2^{-1}}{-x_0 x_1^{-1} x_2^{-1}} \right) \cdot \\
& \cdot \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, y_1) w_{(1)} \otimes Y_2^o(v, x_2) w_{(2)}) \\
& = \text{Res}_{y_1} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \cdot \\
& \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)}v, x_0 x_1^{-1} x_2^{-1} + y_1) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, y_1) w_{(1)} \otimes w_{(2)}) \\
& + \text{Res}_{y_1} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \cdot \\
& \cdot \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, y_1) w_{(1)} \otimes Y_2^o(v, x_2) w_{(2)}). \tag{6.4}
\end{aligned}$$

Since

$$\text{Res}_{y_2} y_2^{-1} \delta \left(\frac{x_0 x_1^{-1} x_2^{-1} + y_1}{y_2} \right) = 1,$$

the first term on the right-hand side of (6.4) can be written as

$$\begin{aligned}
& \text{Res}_{y_1} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_2} y_2^{-1} \delta \left(\frac{x_0 x_1^{-1} x_2^{-1} + y_1}{y_2} \right) \cdot \\
& \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)}v, x_0 x_1^{-1} x_2^{-1} + y_1) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, y_1) w_{(1)} \otimes w_{(2)}) \\
& = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_1} \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) (x_0 x_1^{-1} x_2^{-1})^{-1} \delta \left(\frac{y_2 - y_1}{x_0 x_1^{-1} x_2^{-1}} \right) \cdot \\
& \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)}v, y_2) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, y_1) w_{(1)} \otimes w_{(2)}), \tag{6.5}
\end{aligned}$$

where we have also used (2.11) and (2.6). Again using (2.6) and (2.11), we see that the right-hand side of (6.5) is also equal to

$$\begin{aligned}
& x_2^{-1} \delta \left(\frac{x_2^{-1} - x_0 x_1^{-1} x_2^{-1}}{x_1^{-1}} \right) \text{Res}_{y_1} \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) y_2^{-1} \delta \left(\frac{x_0 x_1^{-1} x_2^{-1} + y_1}{y_2} \right) \cdot \\
& \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)}v, y_2) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, y_1) w_{(1)} \otimes w_{(2)}) \\
& = x_2^{-1} \delta \left(\frac{x_2^{-1} - x_0 x_1^{-1} x_2^{-1}}{x_1^{-1}} \right) \text{Res}_{y_1} \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - x_0 x_1^{-1} x_2^{-1} - y_1}{z} \right) \cdot \\
& \cdot y_2^{-1} \delta \left(\frac{x_0 x_1^{-1} x_2^{-1} + y_1}{y_2} \right) \cdot \\
& \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)}v, y_2) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, y_1) w_{(1)} \otimes w_{(2)}) \\
& = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_1} \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) y_2^{-1} \delta \left(\frac{x_0 x_1^{-1} x_2^{-1} + y_1}{y_2} \right) \cdot \\
& \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)}v, y_2) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, y_1) w_{(1)} \otimes w_{(2)}) \\
& = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_1} \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) (x_0 x_1^{-1} x_2^{-1})^{-1} \delta \left(\frac{y_2 - y_1}{x_0 x_1^{-1} x_2^{-1}} \right) \cdot \\
& \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)}v, y_2) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, y_1) w_{(1)} \otimes w_{(2)}). \tag{6.6}
\end{aligned}$$

That is, in the middle delta-function expression in the right-hand side of (6.5), we may replace x_1 by x_2 and y_1 by y_2 .

From (6.2)–(6.6), we obtain

$$\begin{aligned}
& \left(x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y'_{P(z)}(u, x_1) Y'_{P(z)}(v, x_2) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\
&= x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \lambda(w_{(1)} \otimes Y_2^o(v, x_2) Y_2^o(u, x_1) w_{(2)}) \\
&\quad + x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) \cdot \\
&\quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} v, y_2) w_{(1)} \otimes Y_2^o(u, x_1) w_{(2)}) \\
&\quad + x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_1} \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) (x_0 x_1^{-1} x_2^{-1})^{-1} \delta \left(\frac{y_2 - y_1}{x_0 x_1^{-1} x_2^{-1}} \right) \cdot \\
&\quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} v, y_2) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes w_{(2)}) \\
&\quad + x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \text{Res}_{y_1} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) \cdot \\
&\quad \cdot \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes Y_2^o(v, x_2) w_{(2)}) \Big). \tag{6.7}
\end{aligned}$$

From (6.6) and (6.7), replacing u, v, x_1, x_2, x_0 by $v, u, x_2, x_1, -x_0$, respectively, and also using (2.6), we find that

$$\begin{aligned}
& \left(-x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y'_{P(z)}(v, x_2) Y'_{P(z)}(u, x_1) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\
&= -x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2^o(u, x_1) Y_2^o(v, x_2) w_{(2)}) \\
&\quad - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \text{Res}_{y_1} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) \cdot \\
&\quad \cdot \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes Y_2^o(v, x_2) w_{(2)}) \\
&\quad - x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_1} \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) (x_0 x_1^{-1} x_2^{-1})^{-1} \delta \left(\frac{y_1 - y_2}{-x_0 x_1^{-1} x_2^{-1}} \right) \cdot \\
&\quad \cdot \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} v, y_2) w_{(1)} \otimes w_{(2)}) \\
&\quad - x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) \cdot \\
&\quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} v, y_2) w_{(1)} \otimes Y_2^o(u, x_1) w_{(2)}) \Big). \tag{6.8}
\end{aligned}$$

Using (6.7), (6.8), the Jacobi identity, the opposite Jacobi identity (2.61) and (2.6), we obtain

$$\left(x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y'_{P(z)}(u, x_1) Y'_{P(z)}(v, x_2) \lambda \right)$$

$$\begin{aligned}
& -x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y'_{P(z)}(v,x_2)Y'_{P(z)}(u,x_1)\lambda\Big(w_{(1)}\otimes w_{(2)}\Big) \\
& = \lambda\left(w_{(1)}\otimes\left(x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_2^o(v,x_2)Y_2^o(u,x_1)\right.\right. \\
& \quad \left.\left.-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_2^o(u,x_1)Y_2^o(v,x_2)\right)w_{(2)}\right) \\
& \quad +x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}\text{Res}_{y_2}z^{-1}\delta\left(\frac{x_2^{-1}-y_2}{z}\right) \\
& \quad \cdot\lambda\left(\left((x_0x_1^{-1}x_2^{-1})^{-1}\delta\left(\frac{y_2-y_1}{x_0x_1^{-1}x_2^{-1}}\right)\right.\right. \\
& \quad \cdot Y_1(e^{x_2L(1)}(-x_2^{-2})^{L(0)}v,y_2)Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}u,y_1) \\
& \quad \left.\left.-(x_0x_1^{-1}x_2^{-1})^{-1}\delta\left(\frac{y_1-y_2}{-x_0x_1^{-1}x_2^{-1}}\right)\right.\right. \\
& \quad \left.\left.\cdot Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}u,y_1)Y_1(e^{x_2L(1)}(-x_2^{-2})^{L(0)}v,y_2)\right)w_{(1)}\otimes w_{(2)}\right) \\
& = \lambda\left(w_{(1)}\otimes\left(x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_2^o(Y(u,x_0)v,x_2)\right)w_{(2)}\right) \\
& \quad +x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}\text{Res}_{y_2}z^{-1}\delta\left(\frac{x_2^{-1}-y_2}{z}\right) \\
& \quad \cdot\lambda\left(\left(y_2^{-1}\delta\left(\frac{y_1+x_0x_1^{-1}x_2^{-1}}{y_2}\right)\right.\right. \\
& \quad \left.\left.\cdot Y_1(Y(e^{x_1L(1)}(-x_1^{-2})^{L(0)}u,-x_0x_1^{-1}x_2^{-1})e^{x_2L(1)}(-x_2^{-2})^{L(0)}v,y_2)\right)w_{(1)}\otimes w_{(2)}\right) \\
& = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\lambda(w_{(1)}\otimes Y_2^o(Y(u,x_0)v,x_2)w_{(2)}) \\
& \quad +x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}\text{Res}_{y_2}z^{-1}\delta\left(\frac{x_2^{-1}-y_2}{z}\right) \\
& \quad \cdot\lambda\left(\left(y_1^{-1}\delta\left(\frac{y_2-x_0x_1^{-1}x_2^{-1}}{y_1}\right)\right.\right. \\
& \quad \left.\left.\cdot Y_1(Y(e^{x_1L(1)}(-x_1^{-2})^{L(0)}u,-x_0x_1^{-1}x_2^{-1})e^{x_2L(1)}(-x_2^{-2})^{L(0)}v,y_2)\right)w_{(1)}\otimes w_{(2)}\right) \\
& = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\lambda(w_{(1)}\otimes Y_2^o(Y(u,x_0)v,x_2)w_{(2)}) \\
& \quad +x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_2}z^{-1}\delta\left(\frac{x_2^{-1}-y_2}{z}\right) \\
& \quad \cdot\lambda(Y_1(Y(e^{x_1L(1)}(-x_1^{-2})^{L(0)}u,-x_0x_1^{-1}x_2^{-1})e^{x_2L(1)}(-x_2^{-2})^{L(0)}v,y_2)w_{(1)}\otimes w_{(2)})
\end{aligned} \tag{6.9}$$

Finally, from (6.1) we see that the right-hand side of (6.9) becomes

$$\begin{aligned}
& x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \lambda(w_{(1)} \otimes Y_2^o(Y(u, x_0)v, x_2)w_{(2)}) \\
& + x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) \cdot \\
& \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)}Y(u, x_0)v, y_2)w_{(1)} \otimes w_{(2)}) \\
& = \left(x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y'_{P(z)}(Y(u, x_0)v, x_2)\lambda \right) (w_{(1)} \otimes w_{(2)}), \tag{6.10}
\end{aligned}$$

and we have proved the Jacobi identity and hence Theorem 5.44. $\square$

Proof of Theorem 5.45 Let λ be an element of $(W_1 \otimes W_2)^*$ satisfying the $P(z)$ -compatibility condition. We first want to prove that the coefficient of each power of x in $Y'_{P(z)}(u, x_0)Y'_{P(z)}(v, x)\lambda$ is a formal Laurent series involving only finitely many negative powers of x_0 and that

$$\begin{aligned}
& \tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(u, x_1) \right) Y'_{P(z)}(v, x)\lambda \\
& = x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y'_{P(z)}(u, x_1)Y'_{P(z)}(v, x)\lambda \tag{6.11}
\end{aligned}$$

for all $u, v \in V$. Using the commutator formula (Proposition 5.9) for $Y'_{P(z)}$, we have

$$\begin{aligned}
& Y'_{P(z)}(u, x_0)Y'_{P(z)}(v, x)\lambda \\
& = Y'_{P(z)}(v, x)Y'_{P(z)}(u, x_0)\lambda \\
& \quad - \text{Res}_y x_0^{-1} \delta \left(\frac{x - y}{x_0} \right) Y'_{P(z)}(Y(v, y)u, x_0)\lambda. \tag{6.12}
\end{aligned}$$

Each coefficient in x of the right-hand side of (6.12) is a formal Laurent series involving only finitely many negative powers of x_0 since λ satisfies the $P(z)$ -lower truncation condition. Thus the coefficients in x of $Y'_{P(z)}(v, x)\lambda$ satisfy the $P(z)$ -lower truncation condition.

By (5.86) and (5.87), we have

$$\begin{aligned}
& \left(\tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(u, x_1) \right) Y'_{P(z)}(v, x)\lambda \right) (w_{(1)} \otimes w_{(2)}) \\
& = z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) (Y'_{P(z)}(v, x)\lambda)(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad + x_0^{-1} \delta \left(\frac{z - x_1^{-1}}{-x_0} \right) (Y'_{P(z)}(v, x)\lambda)(w_{(1)} \otimes Y_2^o(u, x_1)w_{(2)}) \\
& = z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) \left(\lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, x_0)w_{(1)} \otimes Y_2^o(v, x)w_{(2)}) \right. \\
& \quad \left. + \text{Res}_{x_2} z^{-1} \delta \left(\frac{x^{-1} - x_2}{z} \right) \right).
\end{aligned}$$

$$\begin{aligned}
& \cdot \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_2)Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}u, x_0)w_{(1)} \otimes w_{(2)}) \\
& + x_0^{-1}\delta\left(\frac{z-x_1^{-1}}{-x_0}\right)\left(\lambda(w_{(1)} \otimes Y_2^o(v, x)Y_2^o(u, x_1)w_{(2)})\right. \\
& \left. + \text{Res}_{x_2}z^{-1}\delta\left(\frac{x^{-1}-x_2}{z}\right)\lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_2)w_{(1)} \otimes Y_2^o(u, x_1)w_{(2)})\right).
\end{aligned} \tag{6.13}$$

Now the distributive law applies, giving us four terms. Then using the opposite commutator formula for Y_2^o (recall (2.61)) and the commutator formula for Y_2 , and (5.86), we can write the right-hand side of (6.13) as

$$\begin{aligned}
& z^{-1}\delta\left(\frac{x_1^{-1}-x_0}{z}\right)\lambda(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}u, x_0)w_{(1)} \otimes Y_2^o(v, x)w_{(2)}) \\
& + z^{-1}\delta\left(\frac{x_1^{-1}-x_0}{z}\right)\text{Res}_{x_2}z^{-1}\delta\left(\frac{x^{-1}-x_2}{z}\right) \cdot \\
& \cdot \lambda(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}u, x_0)Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_2)w_{(1)} \otimes w_{(2)}) \\
& + z^{-1}\delta\left(\frac{x_1^{-1}-x_0}{z}\right)\text{Res}_{x_2}z^{-1}\delta\left(\frac{x^{-1}-x_2}{z}\right)\text{Res}_{x_3}x_0^{-1}\delta\left(\frac{x_2-x_3}{x_0}\right) \cdot \\
& \cdot \lambda(Y_1(Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x_3)e^{x_1L(1)}(-x_1^{-2})^{L(0)}u, x_0)w_{(1)} \otimes w_{(2)}) \\
& + x_0^{-1}\delta\left(\frac{z-x_1^{-1}}{-x_0}\right)\lambda(w_{(1)} \otimes Y_2^o(u, x_1)Y_2^o(v, x)w_{(2)}) \\
& - x_0^{-1}\delta\left(\frac{z-x_1^{-1}}{-x_0}\right)\text{Res}_{x_3}x_1^{-1}\delta\left(\frac{x-x_3}{x_1}\right)\lambda(w_{(1)} \otimes Y_2^o(Y(v, x_3)u, x_1)w_{(2)}) \\
& + x_0^{-1}\delta\left(\frac{z-x_1^{-1}}{-x_0}\right)\text{Res}_{x_2}z^{-1}\delta\left(\frac{x^{-1}-x_2}{z}\right) \cdot \\
& \cdot \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_2)w_{(1)} \otimes Y_2^o(u, x_1)w_{(2)}) \\
& = \left(\tau_{P(z)}\left(x_0^{-1}\delta\left(\frac{x_1^{-1}-z}{x_0}\right)Y_t(u, x_1)\right)\lambda\right)(w_{(1)} \otimes Y_2^o(v, x)w_{(2)}) \\
& + \text{Res}_{x_2}z^{-1}\delta\left(\frac{x^{-1}-x_2}{z}\right) \cdot \\
& \cdot \left(\tau_{P(z)}\left(x_0^{-1}\delta\left(\frac{x_1^{-1}-z}{x_0}\right)Y_t(u, x_1)\right)\lambda\right)(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_2)w_{(1)} \otimes w_{(2)}) \\
& + \text{Res}_{x_2}z^{-1}\delta\left(\frac{x^{-1}-x_2}{z}\right)\text{Res}_{x_3}x_0^{-1}\delta\left(\frac{x_2-x_3}{x_0}\right)z^{-1}\delta\left(\frac{x_1^{-1}-x_0}{z}\right) \cdot \\
& \cdot \lambda(Y_1(Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x_3)e^{x_1L(1)}(-x_1^{-2})^{L(0)}u, x_0)w_{(1)} \otimes w_{(2)}) \\
& - \text{Res}_{x_3}x_1^{-1}\delta\left(\frac{x-x_3}{x_1}\right)x_0^{-1}\delta\left(\frac{z-x_1^{-1}}{-x_0}\right)\lambda(w_{(1)} \otimes Y_2^o(Y(v, x_3)u, x_1)w_{(2)}).
\end{aligned} \tag{6.14}$$

Since λ satisfies the $P(z)$ -compatibility condition (5.141), by (5.87) the sum of the first two terms of (6.14) is equal to

$$\begin{aligned} & \left(Y'_{P(z)}(v, x) \tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(u, x_1) \right) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\ &= x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) (Y'_{P(z)}(v, x) Y'_{P(z)}(u, x_1) \lambda) (w_{(1)} \otimes w_{(2)}). \end{aligned} \quad (6.15)$$

Changing the dummy variable x_3 to $-x_3 x^{-1} x_1^{-1}$ where we use x_3 to denote the new dummy variable, using (2.11), (2.6) and (6.1), and then evaluating Res_{x_2} , we see that the third term of (6.14) is equal to

$$\begin{aligned} & -\text{Res}_{x_2} \text{Res}_{x_3} z^{-1} \delta \left(\frac{x^{-1} - x_2}{z} \right) x^{-1} x_1^{-1} x_0^{-1} \delta \left(\frac{x_2 + x_3 x^{-1} x_1^{-1}}{x_0} \right) z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) \cdot \\ & \quad \cdot \lambda(Y_1(Y(e^{xL(1)}(-x^{-2})^{L(0)}v, -x_3 x^{-1} x_1^{-1}) e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, x_0) w_{(1)} \otimes w_{(2)}) \\ &= -\text{Res}_{x_2} z^{-1} \delta \left(\frac{x^{-1} - x_2}{z} \right) \text{Res}_{x_3} x^{-1} x_1^{-1} x_2^{-1} \delta \left(\frac{x_0 - x_3 x^{-1} x_1^{-1}}{x_2} \right) z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) \cdot \\ & \quad \cdot \lambda(Y_1(Y(e^{xL(1)}(-x^{-2})^{L(0)}v, -x_3 x^{-1} x_1^{-1}) e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, x_0) w_{(1)} \otimes w_{(2)}) \\ &= -\text{Res}_{x_3} \text{Res}_{x_2} z^{-1} \delta \left(\frac{x^{-1} - x_0 + x_3 x^{-1} x_1^{-1}}{z} \right) \cdot \\ & \quad \cdot x^{-1} x_1^{-1} x_2^{-1} \delta \left(\frac{x_0 - x_3 x^{-1} x_1^{-1}}{x_2} \right) z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) \cdot \\ & \quad \cdot \lambda(Y_1(Y(e^{xL(1)}(-x^{-2})^{L(0)}v, -x_3 x^{-1} x_1^{-1}) e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, x_0) w_{(1)} \otimes w_{(2)}) \\ &= -\text{Res}_{x_3} \text{Res}_{x_2} (z + x_0)^{-1} \delta \left(\frac{x^{-1} + x_3 x^{-1} x_1^{-1}}{z + x_0} \right) \cdot \\ & \quad \cdot x^{-1} x_1^{-1} x_2^{-1} \delta \left(\frac{x_0 - x_3 x^{-1} x_1^{-1}}{x_2} \right) x_1 \delta \left(\frac{z + x_0}{x_1^{-1}} \right) \cdot \\ & \quad \cdot \lambda(Y_1(Y(e^{xL(1)}(-x^{-2})^{L(0)}v, -x_3 x^{-1} x_1^{-1}) e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, x_0) w_{(1)} \otimes w_{(2)}) \\ &= -\text{Res}_{x_3} \text{Res}_{x_2} x_1 \delta \left(\frac{x^{-1} + x_3 x^{-1} x_1^{-1}}{x_1^{-1}} \right) \cdot \\ & \quad \cdot x^{-1} x_1^{-1} x_2^{-1} \delta \left(\frac{x_0 - x_3 x^{-1} x_1^{-1}}{x_2} \right) x_1 \delta \left(\frac{z + x_0}{x_1^{-1}} \right) \cdot \\ & \quad \cdot \lambda(Y_1(Y(e^{xL(1)}(-x^{-2})^{L(0)}v, -x_3 x^{-1} x_1^{-1}) e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, x_0) w_{(1)} \otimes w_{(2)}) \\ &= -\text{Res}_{x_3} \text{Res}_{x_2} x_1^{-1} \delta \left(\frac{x - x_3}{x_1} \right) \cdot \\ & \quad \cdot x_2^{-1} \delta \left(\frac{x_0 - x_3 x^{-1} x_1^{-1}}{x_2} \right) z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) \cdot \\ & \quad \cdot \lambda(Y_1(Y(e^{xL(1)}(-x^{-2})^{L(0)}v, -x_3 x^{-1} x_1^{-1}) e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, x_0) w_{(1)} \otimes w_{(2)}) \end{aligned}$$

$$\begin{aligned}
&= -\text{Res}_{x_3} \text{Res}_{x_2} x_1^{-1} \delta \left(\frac{x-x_3}{x_1} \right) x_2^{-1} \delta \left(\frac{x_0-x_3x_1^{-1}x_1^{-1}}{x_2} \right) z^{-1} \delta \left(\frac{x_1^{-1}-x_0}{z} \right) \\
&\quad \cdot \lambda(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}Y(v, x_3)u, x_0)w_{(1)} \otimes w_{(2)}) \\
&= -\text{Res}_{x_3} x_1^{-1} \delta \left(\frac{x-x_3}{x_1} \right) z^{-1} \delta \left(\frac{x_1^{-1}-x_0}{z} \right) \\
&\quad \cdot \lambda(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}Y(v, x_3)u, x_0)w_{(1)} \otimes w_{(2)}). \tag{6.16}
\end{aligned}$$

From (6.16), (5.86) and (5.141), the sum of the last two terms of (6.14) becomes

$$\begin{aligned}
&-\text{Res}_{x_3} x_1^{-1} \delta \left(\frac{x-x_3}{x_1} \right) z^{-1} \delta \left(\frac{x_1^{-1}-x_0}{z} \right) \\
&\quad \cdot \lambda(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}Y(v, x_3)u, x_0)w_{(1)} \otimes w_{(2)}) \\
&\quad -\text{Res}_{x_3} x_1^{-1} \delta \left(\frac{x-x_3}{x_1} \right) x_0^{-1} \delta \left(\frac{z-x_1^{-1}}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2^o(Y(v, x_3)u, x_1)w_{(2)}) \\
&= -\text{Res}_{x_3} x_1^{-1} \delta \left(\frac{x-x_3}{x_1} \right) \left(\tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1}-z}{x_0} \right) Y_t(Y(v, x_3)u, x_1) \right) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\
&= -x_0^{-1} \delta \left(\frac{x_1^{-1}-z}{x_0} \right) \text{Res}_{x_3} x_1^{-1} \delta \left(\frac{x-x_3}{x_1} \right) (Y'_{P(z)}(Y(v, x_3)u, x_1)\lambda) (w_{(1)} \otimes w_{(2)}). \tag{6.17}
\end{aligned}$$

Using (6.15), (6.17) and the commutator formula for $Y'_{P(z)}$, we now see that the right-hand side of (6.14) is equal to

$$\begin{aligned}
&x_0^{-1} \delta \left(\frac{x_1^{-1}-z}{x_0} \right) (Y'_{P(z)}(v, x)Y'_{P(z)}(u, x_1)\lambda) (w_{(1)} \otimes w_{(2)}) \\
&\quad -x_0^{-1} \delta \left(\frac{x_1^{-1}-z}{x_0} \right) \text{Res}_{x_3} x_1^{-1} \delta \left(\frac{x-x_3}{x_1} \right) (Y'_{P(z)}(Y(v, x_3)u, x_1)\lambda) (w_{(1)} \otimes w_{(2)}) \\
&= x_0^{-1} \delta \left(\frac{x_1^{-1}-z}{x_0} \right) (Y'_{P(z)}(u, x_1)Y'_{P(z)}(v, x)\lambda) (w_{(1)} \otimes w_{(2)}). \tag{6.18}
\end{aligned}$$

The formulas (6.13), (6.14) and (6.18) together prove (6.11), as desired. For the Möbius case, the corresponding verification for $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$ is straightforward, as usual, and we omit this verification. The first half of Theorem 5.45 holds.

For the second half of Theorem 5.45, suppose that $\lambda \in (W_1 \otimes W_2)^*$ satisfies either the $P(z)$ -local grading restriction condition or the $L(0)$ -semisimple condition. Assume without loss of generality that λ is doubly homogeneous. From Remark 5.42, we see that for $v \in V$ doubly homogeneous, $m \in \mathbb{Z}$ and $j = -1, 0, 1$, the elements $\tau_{P(z)}(v \otimes t^m)\lambda$ and $L'_{P(z)}(j)\lambda$ are also doubly homogeneous. Each such element μ lies in W_λ , and so $W_\mu \subset W_\lambda$. Thus μ satisfies the $P(z)$ -local grading restriction condition (or the $L(0)$ -semisimple condition), and the second half of Theorem 5.45 follows. $\square$

6.2 Proofs of Theorems 5.76 and 5.77

In this section, we follow [HL2]; the arguments given there carry over to our more general situation with very little change. We first prove certain formulas that will be useful later.

Let $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. From (5.174) we have

$$\begin{aligned} & (L'_{Q(z)}(0)\lambda)(w_{(1)} \otimes w_{(2)}) \\ &= \lambda((L(0) - zL(1))w_{(1)} \otimes w_{(2)}) - \lambda(w_{(1)} \otimes (L(0) - zL(-1))w_{(2)}), \end{aligned} \quad (6.19)$$

where (as usual) we have used the same notations $L(0), L(-1), L(1)$ to denote operators on both W_1 and W_2 . For convenience we write $L(-1) = L'_{Q(z)}(-1)$, $L(0) = L'_{Q(z)}(0)$ and $L(1) = L'_{Q(z)}(1)$ in the rest of this section. There will be no confusion since the operators act on different spaces.

Lemma 6.2 *For $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, we have*

$$\begin{aligned} & \left(\left(1 - \frac{y_1}{z}\right)^{L(0)} \lambda \right) (w_{(1)} \otimes w_{(2)}) \\ &= \lambda \left(\left(1 - \frac{y_1}{z}\right)^{L(0)-zL(1)} w_{(1)} \otimes \left(1 - \frac{y_1}{z}\right)^{-(L(0)-zL(-1))} w_{(2)} \right). \end{aligned} \quad (6.20)$$

Proof From (6.19),

$$\begin{aligned} & \left(\left(1 - \frac{y_1}{z}\right)^{L(0)} \lambda \right) (w_{(1)} \otimes w_{(2)}) \\ &= (e^{L(0) \log(1 - \frac{y_1}{z})} \lambda) (w_{(1)} \otimes w_{(2)}) \\ &= \lambda(e^{(L(0)-zL(1)) \log(1 - \frac{y_1}{z})} w_{(1)} \otimes e^{-(L(0)-zL(-1)) \log(1 - \frac{y_1}{z})} w_{(2)}) \\ &= \lambda \left(\left(1 - \frac{y_1}{z}\right)^{L(0)-zL(1)} w_{(1)} \otimes \left(1 - \frac{y_1}{z}\right)^{-(L(0)-zL(-1))} w_{(2)} \right). \quad \square \end{aligned} \quad (6.21)$$

Lemma 6.3 *For $v \in V$,*

$$Y'_{Q(z)}(v, x) = \left(1 - \frac{y_1}{z}\right)^{L(0)} Y'_{Q(z)} \left(\left(1 - \frac{y_1}{z}\right)^{-L(0)} v, \frac{x}{1 - y_1/z} \right) \left(1 - \frac{y_1}{z}\right)^{-L(0)}. \quad (6.22)$$

This formula also holds for the vertex operators associated with any generalized V -module.

Proof The identity (6.22) will follow from the formula

$$e^{yL(0)} Y'_{Q(z)}(v, x) e^{-yL(0)} = Y'_{Q(z)}(e^{yL(0)} v, e^y x). \quad (6.23)$$

To prove this, assume without loss of generality that $\text{wt } v = h \in \mathbb{Z}$, and use the $L(-1)$ -derivative property (5.159) and the commutator formulas (5.165) and (5.177) to get

$$[L(0), Y'_{Q(z)}(v, x)] = \left(x \frac{d}{dx} + h \right) Y'_{Q(z)}(v, x). \quad (6.24)$$

Formula (6.23) now follows from (an easier version of) the proof of (3.86). $\square$

Lemma 6.4 *Let $L(-1)$ and $L(0)$ be any operators satisfying the commutator relation*

$$[L(0), L(-1)] = L(-1). \quad (6.25)$$

Then

$$\left(1 - \frac{y_1}{x}\right)^{L(0)-xL(-1)} = e^{y_1 L(-1)} \left(1 - \frac{y_1}{x}\right)^{L(0)}. \quad (6.26)$$

Proof We first prove that the derivative with respect to y of

$$(1-y)^{L(0)-xL(-1)}(1-y)^{-L(0)}e^{-xyL(-1)}$$

is 0. Write $A = (1-y)^{L(0)-xL(-1)}$, $B = (1-y)^{-L(0)}$, $C = e^{-xyL(-1)}$. Then

$$\begin{aligned} \frac{d}{dy}(ABC) &= -A(1-y)^{-1}(L(0) - xL(-1))BC \\ &\quad + A(1-y)^{-1}L(0)BC \\ &\quad - xABL(-1)C. \end{aligned} \quad (6.27)$$

Using (3.73) we have

$$\begin{aligned} BL(-1) &= (1-y)^{-L(0)}L(-1) \\ &= e^{(-\log(1-y))L(0)}L(-1) \\ &= L(-1)e^{(-\log(1-y))L(0)}e^{-\log(1-y)} \\ &= L(-1)(1-y)^{-L(0)}(1-y)^{-1} \\ &= (1-y)^{-1}L(-1)B, \end{aligned} \quad (6.28)$$

and substituting (6.28) into (6.27) gives

$$\begin{aligned} \frac{d}{dy}(ABC) &= -A(1-y)^{-1}L(0)BC + xA(1-y)^{-1}L(-1)BC \\ &\quad + A(1-y)^{-1}L(0)BC - xA(1-y)^{-1}L(-1)BC \\ &= 0. \end{aligned}$$

Thus ABC is constant in y , and since $ABC|_{y=0} = 1$, we have $ABC = 1$, which is equivalent to (6.26). $\square$

Proof of Theorem 5.76 As always, the reader should again observe the justifiability of each formal step in the argument.

Let λ be an element of $(W_1 \otimes W_2)^*$ satisfying the $Q(z)$ -compatibility condition, that is, (a) the $Q(z)$ -lower truncation condition—for all $v \in V$, $Y'_{Q(z)}(v, x)\lambda = \tau_{Q(z)}(Y_t(v, x))\lambda$ involves only finitely many negative powers of x , and (b)

$$\begin{aligned} \tau_{Q(z)}\left(z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right)Y_t(v, x_0)\right)\lambda \\ = z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right)Y'_{Q(z)}(v, x_0)\lambda \quad \text{for all } v \in V. \end{aligned} \quad (6.29)$$

By (5.156) and (5.157), (6.29) is equivalent to

$$\begin{aligned}
& x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) \lambda(Y_1^o(v, x_1)w_{(1)} \otimes w_{(2)}) \\
& - x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \\
& = z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) \left(\text{Res}_{y_1} x_0^{-1} \delta \left(\frac{y_1 - z}{x_0} \right) \lambda(Y_1^o(v, y_1)w_{(1)} \otimes w_{(2)}) \right. \\
& \quad \left. - \text{Res}_{y_1} x_0^{-1} \delta \left(\frac{z - y_1}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2(v, y_1)w_{(2)}) \right)
\end{aligned} \tag{6.30}$$

for all $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. It is important to note that on the right-hand side the distributive law is not valid since the two individual products are not defined. One critical feature of the argument that follows is that we must rewrite expressions to allow the application of distributivity.

By (5.157), we have

$$\begin{aligned}
& \left(x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y'_{Q(z)}(v_1, x_1) Y'_{Q(z)}(v_2, x_2) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\
& = x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) (Y'_{Q(z)}(v_1, x_1) Y'_{Q(z)}(v_2, x_2) \lambda) (w_{(1)} \otimes w_{(2)}) \\
& = x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \left(\text{Res}_{y_1} x_1^{-1} \delta \left(\frac{y_1 - z}{x_1} \right) \cdot \right. \\
& \quad \cdot (Y'_{Q(z)}(v_2, x_2) \lambda) (Y_1^o(v_1, y_1)w_{(1)} \otimes w_{(2)}) \\
& \quad \left. - \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) (Y'_{Q(z)}(v_2, x_2) \lambda) (w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \right) \\
& = x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \left(\text{Res}_{y_1} x_1^{-1} \delta \left(\frac{y_1 - z}{x_1} \right) \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{y_2 - z}{x_2} \right) \cdot \right. \\
& \quad \cdot \lambda(Y_1^o(v_2, y_2)Y_1^o(v_1, y_1)w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{y_1 - z}{x_1} \right) \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \cdot \\
& \quad \cdot \lambda(Y_1^o(v_1, y_1)w_{(1)} \otimes Y_2(v_2, y_2)w_{(2)}) \\
& \quad \left. - \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) (Y'_{Q(z)}(v_2, x_2) \lambda) (w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \right).
\end{aligned} \tag{6.31}$$

From the properties of the formal δ -function, we see that the right-hand side of (6.31) is equal to

$$\begin{aligned}
& x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \left(\text{Res}_{y_1} y_1^{-1} \delta \left(\frac{x_1 + z}{y_1} \right) \text{Res}_{y_2} y_2^{-1} \delta \left(\frac{x_2 + z}{y_2} \right) \cdot \right. \\
& \quad \cdot \lambda(Y_1^o(v_2, x_2 + z)Y_1^o(v_1, x_1 + z)w_{(1)} \otimes w_{(2)})
\end{aligned}$$

$$\begin{aligned}
& -\text{Res}_{y_1} y_1^{-1} \delta \left(\frac{x_1 + z}{y_1} \right) \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \cdot \\
& \quad \cdot \lambda(Y_1^o(v_1, x_1 + z)w_{(1)} \otimes Y_2(v_2, y_2)w_{(2)}) \\
& -\text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) (Y'_{Q(z)}(v_2, x_2)\lambda)(w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
& = x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \left(\lambda(Y_1^o(v_2, x_2 + z)Y_1^o(v_1, x_1 + z)w_{(1)} \otimes w_{(2)}) \right. \\
& \quad - \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \lambda(Y_1^o(v_1, x_1 + z)w_{(1)} \otimes Y_2(v_2, y_2)w_{(2)}) \\
& \quad \left. - \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) (Y'_{Q(z)}(v_2, x_2)\lambda)(w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \right). \tag{6.32}
\end{aligned}$$

From the $L(-1)$ -derivative property for Y_1 , the $L(-1)$ -derivative property (2.62) for Y_1^o and the commutator formulas for $L(-1)$, $Y_1(\cdot, x)$ and for $L(1)$, $Y_1^o(\cdot, x)$ (recall Lemma 2.22), we obtain

$$\begin{aligned}
Y_1(v, x + z) &= Y_1(e^{zL(-1)}v, x) \\
&= e^{zL(-1)}Y_1(v, x)e^{-zL(-1)} \\
&= \sum_{n \geq 0} \frac{z^n}{n!} \frac{d^n}{dx^n} Y_1(v, x)
\end{aligned} \tag{6.33}$$

and

$$\begin{aligned}
Y_1^o(v, x + z) &= Y_1^o(e^{zL(-1)}v, x) \\
&= e^{-zL(1)}Y_1^o(v, x)e^{zL(1)} \\
&= \sum_{n \geq 0} \frac{z^n}{n!} \frac{d^n}{dx^n} Y_1^o(v, x).
\end{aligned} \tag{6.34}$$

(Note that all these expressions are in fact defined.) Using (6.34), we see that the right-hand side of (6.32) can be written as

$$\begin{aligned}
& x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \left(\lambda(Y_1^o(e^{zL(-1)}v_2, x_2)Y_1^o(e^{zL(-1)}v_1, x_1)w_{(1)} \otimes w_{(2)}) \right. \\
& \quad - \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \lambda(Y_1^o(e^{zL(-1)}v_1, x_1)w_{(1)} \otimes Y_2(v_2, y_2)w_{(2)}) \\
& \quad \left. - \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) (Y'_{Q(z)}(v_2, x_2)\lambda)(w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \right) \\
& = x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \left(\lambda(e^{-zL(1)}Y_1^o(v_2, x_2)Y_1^o(v_1, x_1)e^{zL(1)}w_{(1)} \otimes w_{(2)}) \right. \\
& \quad - \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \lambda(e^{-zL(1)}Y_1^o(v_1, x_1)e^{zL(1)}w_{(1)} \otimes Y_2(v_2, y_2)w_{(2)}) \\
& \quad \left. - \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) (Y'_{Q(z)}(v_2, x_2)\lambda)(w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \right). \tag{6.35}
\end{aligned}$$

Note that it is easier to verify the well-definedness of the terms on the right-hand side of (6.32) than that of the terms in (6.35), though (6.35) is sometimes easier to use if it is known that every term is well defined. Below we shall write expressions like those on the right-hand side of (6.32) in whichever way suits our needs. The distributive law applies to the right-hand side of (6.32) (or (6.35)) since all three of the following expressions are defined:

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \lambda(Y_1^o(v_2, x_2 + z) Y_1^o(v_1, x_1 + z) w_{(1)} \otimes w_{(2)}), \\ & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \lambda(Y_1^o(v_1, x_1 + z) w_{(1)} \otimes Y_2(v_2, y_2) w_{(2)}), \\ & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) (Y'_{Q(z)}(v_2, x_2) \lambda)(w_{(1)} \otimes Y_2(v_1, y_1) w_{(2)}). \end{aligned}$$

Now we examine the last expression in (6.35). Rewriting the formal δ -functions $x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right)$ and $x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right)$, and using Lemma 6.3 and (2.11), we have:

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \cdot (Y'_{Q(z)}(v_2, x_2) \lambda)(w_{(1)} \otimes Y_2(v_1, y_1) w_{(2)}) \\ &= \left(\frac{x_1}{z} \right)^{-1} \left(\frac{x_0}{x_1/z} \right)^{-1} \delta \left(\frac{z + \left(\frac{x_2}{-x_1/z} \right)}{\frac{x_0}{x_1/z}} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{1 - y_1/z}{-x_1/z} \right) \cdot \\ & \quad \cdot \left(\left(1 - \frac{y_1}{z} \right)^{L(0)} Y'_{Q(z)} \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_2}{1 - y_1/z} \right) \cdot \right. \\ & \quad \cdot \left. \left(1 - \frac{y_1}{z} \right)^{-L(0)} \lambda \right) (w_{(1)} \otimes Y_2(v_1, y_1) w_{(2)}) \\ &= \left(\frac{x_1}{z} \right)^{-1} \left(\frac{x_0}{x_1/z} \right)^{-1} \delta \left(\frac{z + \left(\frac{x_2}{-x_1/z} \right)}{\frac{x_0}{x_1/z}} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{1 - y_1/z}{-x_1/z} \right) \cdot \\ & \quad \cdot \left(\left(1 - \frac{y_1}{z} \right)^{L(0)} Y'_{Q(z)} \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_2}{-x_1/z} \right) \cdot \right. \\ & \quad \cdot \left. \left(1 - \frac{y_1}{z} \right)^{-L(0)} \lambda \right) (w_{(1)} \otimes Y_2(v_1, y_1) w_{(2)}). \end{aligned} \tag{6.36}$$

By Lemma 6.2 and (6.29), the right-hand side of (6.36) is equal to

$$\begin{aligned} & \left(\frac{x_1}{z} \right)^{-1} \left(\frac{x_0}{x_1/z} \right)^{-1} \delta \left(\frac{z + \left(\frac{x_2}{-x_1/z} \right)}{\frac{x_0}{x_1/z}} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{1 - y_1/z}{-x_1/z} \right) \cdot \\ & \quad \cdot \left(Y'_{Q(z)} \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_2}{-x_1/z} \right) \left(1 - \frac{y_1}{z} \right)^{-L(0)} \lambda \right). \end{aligned}$$

$$\begin{aligned}
& \cdot \left(\left(1 - \frac{y_1}{z}\right)^{L(0)-zL(1)} w_{(1)} \otimes \left(1 - \frac{y_1}{z}\right)^{-(L(0)-zL(-1))} Y_2(v_1, y_1) w_{(2)} \right) \\
& = \left(\frac{x_1}{z}\right)^{-1} \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{1 - y_1/z}{-x_1/z} \right) \cdot \\
& \quad \cdot \left(\tau_{Q(z)} \left(\left(\frac{x_0}{x_1/z}\right)^{-1} \delta \left(\frac{z + (\frac{x_2}{-x_1/z})}{\frac{x_0}{x_1/z}} \right) \right) \cdot \right. \\
& \quad \cdot Y_t \left(\left(1 - \frac{y_1}{z}\right)^{-L(0)} v_2, \frac{x_2}{-x_1/z} \right) \left(1 - \frac{y_1}{z}\right)^{-L(0)} \lambda \Big) \cdot \\
& \quad \cdot \left(\left(1 - \frac{y_1}{z}\right)^{L(0)-zL(1)} w_{(1)} \otimes \left(1 - \frac{y_1}{z}\right)^{-(L(0)-zL(-1))} Y_2(v_1, y_1) w_{(2)} \right). \quad (6.37)
\end{aligned}$$

By (5.156), the right-hand side of (6.37) becomes

$$\begin{aligned}
& \left(\frac{x_1}{z}\right)^{-1} \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{1 - y_1/z}{-x_1/z} \right) \left(\left(\frac{x_2}{-x_1/z}\right)^{-1} \delta \left(\frac{\frac{x_0}{x_1/z} - z}{\frac{x_2}{-x_1/z}} \right) \cdot \right. \\
& \quad \cdot \left(\left(1 - \frac{y_1}{z}\right)^{-L(0)} \lambda \right) \left(Y_1^o \left(\left(1 - \frac{y_1}{z}\right)^{-L(0)} v_2, \frac{x_0}{x_1/z} \right) \cdot \right. \\
& \quad \cdot \left(1 - \frac{y_1}{z}\right)^{L(0)-zL(1)} w_{(1)} \otimes \left(1 - \frac{y_1}{z}\right)^{-(L(0)-zL(1))} Y_2(v_1, y_1) w_{(2)} \Big) \\
& \quad - \left(\frac{x_2}{-x_1/z}\right)^{-1} \delta \left(\frac{z - \frac{x_0}{x_1/z}}{-\frac{x_2}{-x_1/z}} \right) \left(\left(1 - \frac{y_1}{z}\right)^{-L(0)} \lambda \right) \left(\left(1 - \frac{y_1}{z}\right)^{L(0)-zL(1)} \cdot \right. \\
& \quad \cdot w_{(1)} \otimes Y_2 \left(\left(1 - \frac{y_1}{z}\right)^{-L(0)} v_2, \frac{x_0}{x_1/z} \right) \cdot \\
& \quad \cdot \left(1 - \frac{y_1}{z}\right)^{-(L(0)-zL(-1))} Y_2(v_1, y_1) w_{(2)} \Big) \Big). \quad (6.38)
\end{aligned}$$

Using Lemma 6.2 again but with $1 - \frac{y_1}{z}$ replaced by $(1 - \frac{y_1}{z})^{-1}$, rewriting formal δ -functions and then using the distributive law, we see that (6.38) is equal to

$$\begin{aligned}
& \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{1 - y_1/z}{-x_1/z} \right) \left(-x_2^{-1} \delta \left(\frac{x_0 - x_1}{-x_2} \right) \cdot \right. \\
& \quad \cdot \lambda \left(\left(1 - \frac{y_1}{z}\right)^{-(L(0)-zL(1))} Y_1^o \left(\left(1 - \frac{y_1}{z}\right)^{-L(0)} v_2, \frac{x_0}{-(1 - y_1/z)} \right) \cdot \right. \\
& \quad \cdot \left(1 - \frac{y_1}{z}\right)^{L(0)-zL(1)} w_{(1)} \otimes Y_2(v_1, y_1) w_{(2)} \Big) \\
& \quad + x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \lambda \left(w_{(1)} \otimes \left(1 - \frac{y_1}{z}\right)^{L(0)-zL(-1)} \cdot \right. \\
& \quad \cdot Y_2 \left(\left(1 - \frac{y_1}{z}\right)^{-L(0)} v_2, \frac{x_0}{-(1 - y_1/z)} \right) \cdot \\
& \quad \cdot \left(1 - \frac{y_1}{z}\right)^{-(L(0)-zL(-1))} Y_2(v_1, y_1) w_{(2)} \Big) \Big)
\end{aligned}$$

$$\begin{aligned}
&= -\text{Res}_{y_1} x_1^{-1} \delta \left(\frac{1 - y_1/z}{-x_1/z} \right) x_2^{-1} \delta \left(\frac{x_0 - x_1}{-x_2} \right) \cdot \\
&\quad \cdot \lambda \left(\left(1 - \frac{y_1}{z} \right)^{-(L(0)-zL(1))} Y_1^o \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_0}{-(1 - y_1/z)} \right) \right) \cdot \\
&\quad \cdot \left(1 - \frac{y_1}{z} \right)^{L(0)-zL(1)} w_{(1)} \otimes Y_2(v_1, y_1) w_{(2)} \Big) \\
&+ \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{1 - y_1/z}{-x_1/z} \right) x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \lambda \left(w_{(1)} \otimes \right. \\
&\quad \otimes \left(1 - \frac{y_1}{z} \right)^{L(0)-zL(-1)} Y_2 \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_0}{-(1 - y_1/z)} \right) \Big) \cdot \\
&\quad \cdot \left(1 - \frac{y_1}{z} \right)^{-(L(0)-zL(-1))} Y_2(v_1, y_1) w_{(2)} \Big). \tag{6.39}
\end{aligned}$$

But by Lemmas 6.4 and 6.3,

$$\begin{aligned}
&\left(1 - \frac{y_1}{z} \right)^{L(0)-zL(-1)} Y_2 \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_0}{-(1 - y_1/z)} \right) \left(1 - \frac{y_1}{z} \right)^{-(L(0)-zL(-1))} \\
&= e^{y_1 L(-1)} \left(1 - \frac{y_1}{z} \right)^{L(0)} Y_2 \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_0}{-(1 - y_1/z)} \right) \cdot \\
&\quad \cdot \left(1 - \frac{y_1}{z} \right)^{-L(0)} e^{-y_1 L(-1)} \\
&= e^{y_1 L(-1)} Y_2(v_2, -x_0) e^{-y_1 L(-1)} \\
&= Y_2(v_2, -x_0 + y_1) \\
&= Y_2(v_2, -x_0 - (z - y_1) + z) \\
&= Y_2(e^{zL(-1)} v_2, -x_0 - (z - y_1)). \tag{6.40}
\end{aligned}$$

We similarly have, using Lemmas 6.4 and 6.3 for $Y_1'(v_2, x)$ and then using (2.73) and Theorem 2.34,

$$\begin{aligned}
&\left(1 - \frac{y_1}{z} \right)^{-(L(0)-zL(1))} Y_1^o \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_0}{-(1 - y_1/z)} \right) \left(1 - \frac{y_1}{z} \right)^{L(0)-zL(1)} \\
&= e^{-y_1 L(1)} Y_1^o(v_2, -x_0) e^{y_1 L(1)} \\
&= Y_1^o(v_2, -x_0 + y_1) \\
&= Y_1^o(v_2, -x_0 - (z - y_1) + z) \\
&= Y_1^o(e^{zL(-1)} v_2, -x_0 - (z - y_1)). \tag{6.41}
\end{aligned}$$

Substituting (6.40) and (6.41) into the right-hand side of (6.39) and then combining with (6.36)–(6.39), we obtain

$$x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) (Y'_{Q(z)}(v_2, x_2) \lambda)(w_{(1)} \otimes Y_2(v_1, y_1) w_{(2)})$$

$$\begin{aligned}
&= -\text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) x_2^{-1} \delta \left(\frac{x_0 - x_1}{-x_2} \right) \cdot \\
&\quad \cdot \lambda(Y_1^o(v_2, -x_0 - (z - y_1) + z)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
&+ \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \cdot \\
&\quad \cdot \lambda(w_{(1)} \otimes Y_2(v_2, -x_0 + y_1)Y_2(v_1, y_1)w_{(2)}). \tag{6.42}
\end{aligned}$$

(We choose the form of the expression from (6.41) in anticipation of the next step.)

By (2.11) and (6.33), the right-hand side of (6.42) is equal to

$$\begin{aligned}
&-x_2^{-1} \delta \left(\frac{-x_0 + x_1}{x_2} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \cdot \\
&\quad \cdot \lambda(Y_1^o(v_2, -x_0 + x_1 + z)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
&+ x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \cdot \\
&\quad \cdot \lambda(w_{(1)} \otimes Y_2(v_2, -x_0 + y_1)Y_2(v_1, y_1)w_{(2)}) \\
&= -x_2^{-1} \delta \left(\frac{-x_0 + x_1}{x_2} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \cdot \\
&\quad \cdot \lambda(Y_1^o(e^{zL(-1)}v_2, x_2)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
&+ x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \cdot \\
&\quad \cdot \lambda(w_{(1)} \otimes Y_2(v_2, -x_0 + y_1)Y_2(v_1, y_1)w_{(2)}). \tag{6.43}
\end{aligned}$$

Since

$$\text{Res}_{y_2} y_2^{-1} \delta \left(\frac{-x_0 + y_1}{y_2} \right) = 1,$$

the right-hand side of (6.43) can be written as

$$\begin{aligned}
&-x_2^{-1} \delta \left(\frac{-x_0 + x_1}{x_2} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \cdot \\
&\quad \cdot \lambda(Y_1^o(e^{zL(-1)}v_2, x_2)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
&+ x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \text{Res}_{y_2} y_2^{-1} \delta \left(\frac{-x_0 + y_1}{y_2} \right) \cdot \\
&\quad \cdot \lambda(w_{(1)} \otimes Y_2(v_2, -x_0 + y_1)Y_2(v_1, y_1)w_{(2)}). \tag{6.44}
\end{aligned}$$

By (2.6) and (2.11), (6.44) becomes

$$\begin{aligned}
&x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \cdot \\
&\quad \cdot \lambda(Y_1^o(e^{zL(-1)}v_2, x_2)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)})
\end{aligned}$$

$$\begin{aligned}
& +x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}x_1^{-1}\delta\left(\frac{z-y_1}{-x_1}\right)\text{Res}_{y_2}y_2^{-1}\delta\left(\frac{-x_0+y_1}{y_2}\right) \\
& \quad \cdot \lambda(w_{(1)} \otimes Y_2(v_2, y_2)Y_2(v_1, y_1)w_{(2)}) \\
& = x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\text{Res}_{y_1}x_1^{-1}\delta\left(\frac{z-y_1}{-x_1}\right) \\
& \quad \cdot \lambda(Y_1^o(e^{zL(-1)}v_2, x_2)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
& \quad + x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}\text{Res}_{y_2}x_1^{-1}\delta\left(\frac{z-y_1}{-x_1}\right)y_2^{-1}\delta\left(\frac{-x_0+y_1}{y_2}\right) \\
& \quad \cdot \lambda(w_{(1)} \otimes Y_2(v_2, y_2)Y_2(v_1, y_1)w_{(2)}) \\
& = x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\text{Res}_{y_1}x_1^{-1}\delta\left(\frac{z-y_1}{-x_1}\right) \\
& \quad \cdot \lambda(Y_1^o(e^{zL(-1)}v_2, x_2)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
& \quad - x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}\text{Res}_{y_2} \cdot \\
& \quad \cdot (x_2+x_0)^{-1}\delta\left(\frac{z-y_1}{-x_2-x_0}\right)x_0^{-1}\delta\left(\frac{y_2-y_1}{-x_0}\right) \\
& \quad \cdot \lambda(w_{(1)} \otimes Y_2(v_2, y_2)Y_2(v_1, y_1)w_{(2)}) \\
& = x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\text{Res}_{y_1}x_1^{-1}\delta\left(\frac{z-y_1}{-x_1}\right) \\
& \quad \cdot \lambda(Y_1^o(e^{zL(-1)}v_2, x_2)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
& \quad - x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}\text{Res}_{y_2}x_2^{-1}\delta\left(\frac{z-y_2}{-x_2}\right)x_0^{-1}\delta\left(\frac{y_2-y_1}{-x_0}\right) \\
& \quad \cdot \lambda(w_{(1)} \otimes Y_2(v_2, y_2)Y_2(v_1, y_1)w_{(2)}). \tag{6.45}
\end{aligned}$$

Substituting (6.42)–(6.45) into (6.35) we obtain

$$\begin{aligned}
& \left(x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y'_{Q(z)}(v_1, x_1)Y'_{Q(z)}(v_2, x_2)\lambda\right)(w_{(1)} \otimes w_{(2)}) \\
& = x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\lambda(Y_1^o(e^{zL(-1)}v_2, x_2)Y_1^o(e^{zL(-1)}v_1, x_1)w_{(1)} \otimes w_{(2)}) \\
& \quad - x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\text{Res}_{y_2}x_2^{-1}\delta\left(\frac{z-y_2}{-x_2}\right) \\
& \quad \cdot \lambda(Y_1^o(e^{zL(-1)}v_1, x_1)w_{(1)} \otimes Y_2(v_2, x_2)w_{(2)}) \\
& \quad - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\text{Res}_{y_1}x_1^{-1}\delta\left(\frac{z-y_1}{-x_1}\right) \\
& \quad \cdot \lambda(Y_1^o(e^{zL(-1)}v_2, x_2)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
& \quad + x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}\text{Res}_{y_2}x_2^{-1}\delta\left(\frac{z-y_2}{-x_2}\right)x_0^{-1}\delta\left(\frac{y_2-y_1}{-x_0}\right) \\
& \quad \cdot \lambda(w_{(1)} \otimes Y_2(v_2, y_2)Y_2(v_1, y_1)w_{(2)}). \tag{6.46}
\end{aligned}$$

Now consider the result of the calculation from (6.42) to (6.45) except for the last two steps in (6.45). Reversing the subscripts 1 and 2 of the symbols v , x and y and replacing x_0 by $-x_0$ in this result and then using (2.6), we have

$$\begin{aligned}
& x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \cdot \\
& \quad \cdot (Y'_{Q(z)}(v_1, x_1) \lambda)(w_{(1)} \otimes Y_2(v_2, y_2) w_{(2)}) \\
& = x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \cdot \\
& \quad \cdot \lambda(Y_1^o(e^{zL(-1)} v_1, x_1) w_{(1)} \otimes Y_2(v_2, x_2) w_{(2)}) \\
& \quad - x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_1} \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) x_0^{-1} \delta \left(\frac{y_1 - y_2}{x_0} \right) \cdot \\
& \quad \cdot \lambda(w_{(1)} \otimes Y_2(v_1, y_1) Y_2(v_2, y_2) w_{(2)}). \tag{6.47}
\end{aligned}$$

From (6.31)–(6.35), again reversing the subscripts 1 and 2 of the symbols v , x and y and replacing x_0 by $-x_0$, and (6.47), we have

$$\begin{aligned}
& \left(-x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y'_{Q(z)}(v_2, x_2) Y'_{Q(z)}(v_1, x_1) \cdot \lambda \right) (w_{(1)} \otimes w_{(2)}) \\
& = -x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \lambda(Y_1^o(e^{zL(-1)} v_1, x_1) Y_1^o(e^{zL(-1)} v_2, x_2) w_{(1)} \otimes w_{(2)}) \\
& \quad + x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \cdot \\
& \quad \cdot \lambda(Y_1^o(e^{zL(-1)} v_2, x_2) w_{(1)} \otimes Y_2(v_1, y_1) w_{(2)}) \\
& \quad + x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \cdot \\
& \quad \cdot \lambda(Y_1^o(e^{zL(-1)} v_1, x_1) w_{(1)} \otimes Y_2(v_2, x_2) w_{(2)}) \\
& \quad - x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_1} \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) x_0^{-1} \delta \left(\frac{y_1 - y_2}{x_0} \right) \cdot \\
& \quad \cdot \lambda(w_{(1)} \otimes Y_2(v_1, y_1) Y_2(v_2, y_2) w_{(2)}). \tag{6.48}
\end{aligned}$$

The formulas (6.46) and (6.48) give:

$$\begin{aligned}
& \left(\left(x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y'_{Q(z)}(v_1, x_1) Y'_{Q(z)}(v_2, x_2) \right. \right. \\
& \quad \left. \left. - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y'_{Q(z)}(v_2, x_2) Y'_{Q(z)}(v_1, x_1) \right) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\
& = \lambda \left(\left(x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_1^o(e^{zL(-1)} v_2, x_2) Y_1^o(e^{zL(-1)} v_1, x_1) \right. \right. \\
& \quad \left. \left. - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_1^o(e^{zL(-1)} v_1, x_1) Y_1^o(e^{zL(-1)} v_2, x_2) \right) w_{(1)} \otimes w_{(2)} \right)
\end{aligned}$$

$$\begin{aligned}
& -x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}\text{Res}_{y_2}x_2^{-1}\delta\left(\frac{z-y_2}{-x_2}\right) \\
& \quad \cdot \lambda\left(w_{(1)}\otimes\left(x_0^{-1}\delta\left(\frac{y_1-y_2}{x_0}\right)Y_2(v_1,y_1)Y_2(v_2,y_2)\right.\right. \\
& \quad \left.\left.-x_0^{-1}\delta\left(\frac{y_2-y_1}{-x_0}\right)Y_2(v_2,y_2)Y_2(v_1,y_1)\right)w_{(2)}\right). \tag{6.49}
\end{aligned}$$

From the Jacobi identities for Y_1^o and Y_2 and (3.60), the right-hand side of (6.49) is equal to

$$\begin{aligned}
& x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\lambda(Y_1^o(Y(e^{zL(-1)}v_1,x_0)e^{zL(-1)}v_2,x_2)w_{(1)}\otimes w_{(2)}) \\
& \quad -x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}\text{Res}_{y_2}x_2^{-1}\delta\left(\frac{z-y_2}{-x_2}\right)y_2^{-1}\delta\left(\frac{y_1-x_0}{y_2}\right) \\
& \quad \quad \cdot \lambda(w_{(1)}\otimes Y_2(Y(v_1,x_0)v_2,y_2)w_{(2)}) \\
& = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\lambda(Y_1^o(e^{zL(-1)}Y(v_1,x_0)v_2,x_2)w_{(1)}\otimes w_{(2)}) \\
& \quad -x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}\text{Res}_{y_2}x_2^{-1}\delta\left(\frac{z-y_2}{-x_2}\right)y_2^{-1}\delta\left(\frac{y_1-x_0}{y_2}\right) \\
& \quad \quad \cdot \lambda(w_{(1)}\otimes Y_2(Y(v_1,x_0)v_2,y_2)w_{(2)}). \tag{6.50}
\end{aligned}$$

Using (6.34), evaluating Res_{y_1} and then using the definition of $Y'_{Q(z)}$ (recall (5.157)), we finally see that the right-hand side of (6.50) is equal to

$$\begin{aligned}
& x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\lambda(Y_1^o(Y(v_1,x_0)v_2,x_2+z)w_{(1)}\otimes w_{(2)}) \\
& \quad -x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_2}x_2^{-1}\delta\left(\frac{z-y_2}{-x_2}\right) \\
& \quad \quad \cdot \lambda(w_{(1)}\otimes Y_2(Y(v_1,x_0)v_2,y_2)w_{(2)}) \\
& = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)(Y'_{Q(z)}(Y(v_1,x_0)v_2,x_2)\lambda)(w_{(1)}\otimes w_{(2)}), \tag{6.51}
\end{aligned}$$

proving Theorem 5.76. $\square$

Proof of Theorem 5.77 Let λ be an element of $(W_1\otimes W_2)^*$ satisfying the $Q(z)$ -compatibility condition. We first want to prove that each coefficient in x of $Y'_{Q(z)}(u,x_0)Y'_{Q(z)}(v,x)\lambda$ is a formal Laurent series involving only finitely many negative powers of x_0 and that

$$\begin{aligned}
& \tau_{Q(z)}\left(z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)Y_t(u,x_0)\right)Y'_{Q(z)}(v,x)\lambda \\
& = z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)Y'_{Q(z)}(u,x_0)Y'_{Q(z)}(v,x)\lambda \tag{6.52}
\end{aligned}$$

for all $u, v \in V$. Using the commutator formula for $Y'_{Q(z)}$, we have

$$\begin{aligned} & Y'_{Q(z)}(u, x_0)Y'_{Q(z)}(v, x)\lambda \\ &= Y'_{Q(z)}(v, x)Y'_{Q(z)}(u, x_0)\lambda \\ & \quad - \text{Res}_y x_0^{-1} \delta \left(\frac{x-y}{x_0} \right) Y'_{Q(z)}(Y(v, y)u, x_0)\lambda. \end{aligned} \quad (6.53)$$

Each coefficient in x of the right-hand side of (6.53) is a formal Laurent series involving only finitely many negative powers of x_0 since λ satisfies the $Q(z)$ -lower truncation condition. Thus the coefficients in x of $Y'_{Q(z)}(v, x)\lambda$ satisfy the $Q(z)$ -lower truncation condition.

By (5.156) and (5.157), we have

$$\begin{aligned} & \left(\tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(u, x_0) \right) Y'_{Q(z)}(v, x)\lambda \right) (w_{(1)} \otimes w_{(2)}) \\ &= x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) (Y'_{Q(z)}(v, x)\lambda) (Y_1^o(u, x_1)w_{(1)} \otimes w_{(2)}) \\ & \quad - x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) (Y'_{Q(z)}(v, x)\lambda) (w_{(1)} \otimes Y_2(u, x_1)w_{(2)}) \\ &= x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) \left(\lambda(Y_1^o(v, x+z)Y_1^o(u, x_1)w_{(1)} \otimes w_{(2)}) \right. \\ & \quad \left. - \text{Res}_{x_2} x^{-1} \delta \left(\frac{z - x_2}{-x} \right) \lambda(Y_1^o(u, x_1)w_{(1)} \otimes Y_2(v, x_2)w_{(2)}) \right) \\ & \quad - x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) \left(\lambda(Y_1^o(e^{zL(-1)}v, x)w_{(1)} \otimes Y_2(u, x_1)w_{(2)}) \right. \\ & \quad \left. - \text{Res}_{x_2} x^{-1} \delta \left(\frac{z - x_2}{-x} \right) \lambda(w_{(1)} \otimes Y_2(v, x_2)Y_2(u, x_1)w_{(2)}) \right). \end{aligned} \quad (6.54)$$

Now the distributive law applies, giving us four terms. Inserting

$$\text{Res}_{x_4} x_4^{-1} \delta \left(\frac{x+z}{x_4} \right) = 1$$

into the first of these terms and correspondingly replacing $x+z$ by x_4 in $Y_1^o(v, x+z)$, we can apply the commutator formula for Y_1^o in the usual way. Also using the commutator formula for Y_2 , (2.6) and (2.11), we write the right-hand side of (6.54) as

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) \lambda(Y_1^o(u, x_1)Y_1^o(v, x+z)w_{(1)} \otimes w_{(2)}) \\ & \quad - x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) \text{Res}_{x_4} \text{Res}_{x_3} x_1^{-1} \delta \left(\frac{x_4 - x_3}{x_1} \right) x_4^{-1} \delta \left(\frac{x+z}{x_4} \right) \\ & \quad \quad \cdot \lambda(Y_1^o(Y(v, x_3)u, x_1)w_{(1)} \otimes w_{(2)}) \\ & \quad - x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) \text{Res}_{x_2} x^{-1} \delta \left(\frac{z - x_2}{-x} \right) \lambda(Y_1^o(u, x_1)w_{(1)} \otimes Y_2(v, x_2)w_{(2)}) \end{aligned}$$

$$\begin{aligned}
& -x_0^{-1}\delta\left(\frac{z-x_1}{-x_0}\right)\lambda(Y_1^o(e^{zL(-1)}v,x)w_{(1)}\otimes Y_2(u,x_1)w_{(2)}) \\
& +x_0^{-1}\delta\left(\frac{z-x_1}{-x_0}\right)\text{Res}_{x_2}x^{-1}\delta\left(\frac{z-x_2}{-x}\right)\lambda(w_{(1)}\otimes Y_2(u,x_1)Y_2(v,x_2)w_{(2)}) \\
& +x_0^{-1}\delta\left(\frac{z-x_1}{-x_0}\right)\text{Res}_{x_2}x^{-1}\delta\left(\frac{z-x_2}{-x}\right)\text{Res}_{x_3}x_1^{-1}\delta\left(\frac{x_2-x_3}{x_1}\right) \\
& \quad \cdot \lambda(w_{(1)}\otimes Y_2(Y(v,x_3)u,x_1)w_{(2)}) \\
& = \left(x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right)\lambda(Y_1^o(u,x_1)Y_1^o(e^{zL(-1)}v,x)w_{(1)}\otimes w_{(2)})\right. \\
& \quad \left.-x_0^{-1}\delta\left(\frac{z-x_1}{-x_0}\right)\lambda(Y_1^o(e^{zL(-1)}v,x)w_{(1)}\otimes Y_2(u,x_1)w_{(2)})\right) \\
& \quad -\text{Res}_{x_2}x^{-1}\delta\left(\frac{z-x_2}{-x}\right)\left(x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right)\right. \\
& \quad \quad \cdot \lambda(Y_1^o(u,x_1)w_{(1)}\otimes Y_2(v,x_2)w_{(2)}) \\
& \quad \quad \left.-x_0^{-1}\delta\left(\frac{z-x_1}{-x_0}\right)\lambda(w_{(1)}\otimes Y_2(u,x_1)Y_2(v,x_2)w_{(2)})\right) \\
& \quad -\left(\text{Res}_{x_4}\text{Res}_{x_3}(x_0+z)^{-1}\delta\left(\frac{x+z-x_3}{x_0+z}\right)\right. \\
& \quad \quad \cdot x_4^{-1}\delta\left(\frac{x+z}{x_4}\right)x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right) \\
& \quad \quad \cdot \lambda(Y_1^o(Y(v,x_3)u,x_1)w_{(1)}\otimes w_{(2)}) \\
& \quad \quad \left.-\text{Res}_{x_2}\text{Res}_{x_3}x^{-1}\delta\left(\frac{z-(x_1+x_3)}{-x}\right)x_2^{-1}\delta\left(\frac{x_1+x_3}{x_2}\right)\right. \\
& \quad \quad \left.\cdot x_0^{-1}\delta\left(\frac{z-x_1}{-x_0}\right)\lambda(w_{(1)}\otimes Y_2(Y(v,x_3)u,x_1)w_{(2)})\right). \tag{6.55}
\end{aligned}$$

Using (5.156) and (2.6) and evaluating suitable residues, we see that the right-hand side of (6.55) is equal to

$$\begin{aligned}
& \left(\tau_{Q(z)}\left(z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)Y_t(u,x_0)\right)\lambda\right)(Y_1^o(e^{zL(-1)}v,x)w_{(1)}\otimes w_{(2)}) \\
& -\text{Res}_{x_2}x^{-1}\delta\left(\frac{z-x_2}{-x}\right) \\
& \quad \cdot \left(\tau_{Q(z)}\left(z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)Y_t(u,x_0)\right)\lambda\right)(w_{(1)}\otimes Y_2(v,x_2)w_{(2)}) \\
& -\left(\text{Res}_{x_3}x_0^{-1}\delta\left(\frac{x-x_3}{x_0}\right)\right. \\
& \quad \cdot x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right)\lambda(Y_1^o(Y(v,x_3)u,x_1)w_{(1)}\otimes w_{(2)})
\end{aligned}$$

$$\begin{aligned}
& -\text{Res}_{x_3} x^{-1} \delta \left(\frac{x_0 + x_3}{x} \right) \cdot \\
& \cdot x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2(Y(v, x_3)u, x_1)w_{(2)})) \Big). \tag{6.56}
\end{aligned}$$

Using (5.156) and (5.157), we find that the right-hand side of (6.56) becomes

$$\begin{aligned}
& \left(Y'_{Q(z)}(v, x) \tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(u, x_0) \right) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\
& - \left(\text{Res}_{x_3} x_0^{-1} \delta \left(\frac{x - x_3}{x_0} \right) \cdot \right. \\
& \cdot \tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(Y(v, x_3)u, x_0) \right) \lambda \Big) (w_{(1)} \otimes w_{(2)}). \tag{6.57}
\end{aligned}$$

By the compatibility condition for λ and the commutator formula for $Y'_{Q(z)}$, the right-hand side of (6.57) is equal to

$$\begin{aligned}
& z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) (Y'_{Q(z)}(v, x) Y'_{Q(z)}(u, x_0) \lambda) (w_{(1)} \otimes w_{(2)}) \\
& - z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) \left(\text{Res}_{x_3} x_0^{-1} \delta \left(\frac{x - x_3}{x_0} \right) \cdot \right. \\
& \quad \cdot Y'_{Q(z)}(Y(v, x_3)u, x_0) \lambda \Big) (w_{(1)} \otimes w_{(2)}) \\
& = z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) \left(\left(Y'_{Q(z)}(v, x) Y'_{Q(z)}(u, x_0) \right. \right. \\
& \quad \left. \left. - \text{Res}_{x_3} x_0^{-1} \delta \left(\frac{x - x_3}{x_0} \right) Y'_{Q(z)}(Y(v, x_3)u, x_0) \right) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\
& = z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) (Y'_{Q(z)}(u, x_0) Y'_{Q(z)}(v, x) \lambda) (w_{(1)} \otimes w_{(2)}). \tag{6.58}
\end{aligned}$$

This proves (6.52). In the Möbius case, the three operators are handled in the usual way. The first part of Theorem 5.77 is established.

The proof of the second half of Theorem 5.77 is exactly like that for Theorem 5.45. $\square$

References

- [FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, preprint, 1989; *Memoirs Amer. Math. Soc.* **104**, 1993.
- [FLM] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Appl. Math., Vol. 134, Academic Press, Boston, 1988.

- [H] Y.-Z. Huang, A theory of tensor products for module categories for a vertex operator algebra, IV, *J. Pure Appl. Alg.* **100** (1995) 173–216.
- [HL1] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, I, *Selecta Mathematica (New Series)* **1** (1995), 699–756.
- [HL2] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, II, *Selecta Mathematica (New Series)* **1** (1995), 757–786.
- [HL3] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, III, *J. Pure Appl. Alg.* **100** (1995) 141–171.
- [HLLZ] Y.-Z. Huang, J. Lepowsky, H. Li and L. Zhang, On the concepts of intertwining operator and tensor product module in vertex operator algebra theory, *J. Pure Appl. Algebra* **204** (2006), 507–535.
- [HLZ1] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory for generalized modules for a conformal vertex algebra, I: Introduction and strongly graded algebras and their generalized modules, to appear.
- [HLZ2] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, II: Logarithmic formal calculus and properties of logarithmic intertwining operators, to appear.
- [HLZ3] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, III: Intertwining maps and tensor product bifunctors, to appear.
- [HLZ4] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, V: Convergence condition for intertwining maps and the corresponding compatibility condition, to appear.
- [HLZ5] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, VI: Expansion condition, associativity of logarithmic intertwining operators, and the associativity isomorphisms, to appear.
- [HLZ6] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, VII: Convergence and extension properties and applications to expansion for intertwining maps, to appear.
- [HLZ7] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, VIII: Braided tensor category structure on categories of generalized modules for a conformal vertex algebra, to appear.
- [MS] G. Moore and N. Seiberg, Classical and quantum conformal field theory, *Comm. Math. Phys.* **123** (1989), 177–254.

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