Logarithmic tensor category theory, III: Intertwining maps and tensor product bifunctors

Yi-Zhi Huang, James Lepowsky and Lin Zhang

Abstract

This is the third part in a series of papers in which we introduce and develop a natural, general tensor category theory for suitable module categories for a vertex (operator) algebra. In this paper (Part III), we introduce and study intertwining maps and tensor product bifunctors.

Contents

In this paper, Part III of a series of eight papers on logarithmic tensor category theory, we introduce and study intertwining maps and tensor product bifunctors. The sections, equations, theorems and so on are numbered globally in the series of papers rather than within each paper, so that for example equation (a.b) is the b-th labeled equation in Section a, which is contained in the paper indicated as follows: In Part I [HLZ1], which contains Sections 1 and 2, we give a detailed overview of our theory, state our main results and introduce the basic objects that we shall study in this work. We include a brief discussion of some of the recent applications of this theory, and also a discussion of some recent literature. In Part II [HLZ2], which contains Section 3, we develop logarithmic formal calculus and study logarithmic intertwining operators. The present paper, Part III, contains Section 4. In Part IV [HLZ3], which contains Sections 5 and 6, we give constructions of the $P(z)$- and $Q(z)$-tensor product bifunctors using what we call "compatibility conditions" and certain other conditions. In Part V [HLZ4], which contains Sections 7 and 8, we study products and iterates of intertwining maps and of logarithmic intertwining operators and we begin the development of our analytic approach. In Part VI [HLZ5], which contains Sections

9 and 10, we construct the appropriate natural associativity isomorphisms between triple tensor product functors. In Part VII [HLZ6], which contains Section 11, we give sufficient conditions for the existence of the associativity isomorphisms. In Part VIII [HLZ7], which contains Section 12, we construct braided tensor category structure.

Acknowledgments The authors gratefully acknowledge partial support from NSF grants DMS-0070800 and DMS-0401302. Y.-Z. H. is also grateful for partial support from NSF grant PHY-0901237 and for the hospitality of Institut des Hautes Études Scientifiques in the fall of 2007.

4 $P(z)$- and $Q(z)$-intertwining maps and the $P(z)$- and $Q(z)$-tensor product bifunctors

We now generalize to the setting of the present work the notions of $P(z)$- and $Q(z)$-tensor product of modules, for $z \in \mathbb{C}^\times$, introduced in [HL1], [HL2] and [HL3]. The symbols $P(z)$ and $Q(z)$ refer to moduli space elements described in Remarks 4.3 and 4.37, respectively. We introduce the notions of $P(z)$- and $Q(z)$-intertwining map among strongly $\tilde{A}$-graded generalized modules for a strongly A-graded Möbius or conformal vertex algebra V and establish the relationship between such intertwining maps and grading-compatible logarithmic intertwining operators. We define the $P(z)$- and $Q(z)$-tensor product bifunctors for pairs of strongly $\tilde{A}$-graded generalized V-modules using these intertwining maps and natural universal properties. As examples, for a strongly $\tilde{A}$-graded generalized module W, we construct and describe the $P(z)$-tensor products of V and W and also of W and V; the underlying strongly $\tilde{A}$-graded generalized modules of the tensor product structures are W itself, in both of these cases. In the case in which V is a finitely reductive vertex operator algebra (recall the Introduction), we construct and describe the $P(z)$- and $Q(z)$-tensor products of arbitrary V-modules, and we use this structure to motivate the construction of associativity isomorphisms that we will carry out in later sections. At the end of this section we relate the $P(z)$- and $Q(z)$-tensor products.

We emphasize an important issue: Even though, as we have just mentioned, we construct the $P(z)$- and $Q(z)$-tensor product bifunctors in some cases, we do not give any *general* construction of (models for) these bifunctors in this section. But for our deeper results, we will crucially need a suitable general construction of these bifunctors, and indeed, for both $P(z)$ and $Q(z)$, we will construct a useful, particular bifunctor (when it exists) in Section 5. We will use this construction in order to construct the required natural associativity isomorphisms among triple tensor products, leading to braided tensor category structure, under suitable conditions.

In view of the results in Sections 2 and 3 involving contragredient modules, it is natural for us to work in the strongly-graded setting from now on:

Assumption 4.1 *Throughout this section and the remainder of this work, we shall assume the following, unless other assumptions are explicitly made: A is an abelian group and $\tilde{A}$ is* *an abelian group containing A as a subgroup; V is a strongly A-graded Möbius or conformal vertex algebra; all V-modules and generalized V-modules considered are strongly $\tilde{A}$-graded; and all intertwining operators and logarithmic intertwining operators considered are grading-compatible. (Recall Definitions 2.23, 2.25, 3.10 and 3.14.)*

We shall be working with full subcategories $\mathcal{C}$ of the category $\mathcal{M}_{sg}$ of strongly $\tilde{A}$-graded (ordinary) V-modules or the category $\mathcal{GM}_{sg}$ of strongly $\tilde{A}$-graded generalized V-modules (recall Notation 2.36).

In this section, z will be a fixed nonzero complex number.

4.1 $P(z)$-intertwining maps and the notion of $P(z)$-tensor product

We first generalize the notion of $P(z)$-intertwining map given in Section 4 of [HL1]; our $P(z)$-intertwining maps will automatically be grading-compatible by definition. We use the notations given in Definition 2.18. The main part of the following definition, the Jacobi identity (4.4), was previewed in the Introduction (formula (1.19)). It should be compared with the corresponding formula (1.1) in the Lie algebra setting, and with the Jacobi identity (3.26) in the definition of the notion of logarithmic intertwining operator; note that the formal variable x_2 in that Jacobi identity is specialized here to the nonzero complex number z. Also, the $\mathfrak{sl}(2)$-bracket relations (4.5) should be compared with the corresponding relations (3.28). There is no $L(-1)$-derivative formula for intertwining maps; as we shall see, the $P(z)$-intertwining maps are obtained from logarithmic intertwining operators by a process of specialization of the formal variable to the complex variable z.

Definition 4.2 Let (W_1, Y_1), (W_2, Y_2) and (W_3, Y_3) be generalized V-modules. A *$P(z)$-intertwining map of type $\binom{W_3}{W_1\, W_2}$* is a linear map

$$I : W_1 \otimes W_2 \to \overline{W}_3 \tag{4.1}$$

(recall from Definition 2.18 that $\overline{W}_3$ is the formal completion of W_3 with respect to the $\mathbb{C}$-grading) such that the following conditions are satisfied: the *grading compatibility condition*: for $\beta, \gamma \in \tilde{A}$ and $w_{(1)} \in W_1^{(\beta)}$, $w_{(2)} \in W_2^{(\gamma)}$,

$$I(w_{(1)} \otimes w_{(2)}) \in \overline{W_3^{(\beta+\gamma)}}; \tag{4.2}$$

the *lower truncation condition*: for any elements $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, and any $n \in \mathbb{C}$,

$$\pi_{n-m} I(w_{(1)} \otimes w_{(2)}) = 0 \quad \text{for} \;\; m \in \mathbb{N} \;\; \text{sufficiently large} \tag{4.3}$$

(which follows from (4.2), in view of the grading restriction condition (2.85); recall the notation π_n from Definition 2.18); the *Jacobi identity*:

$$\begin{aligned} x_0^{-1}\delta\Big(\frac{x_1 - z}{x_0}\Big) & Y_3(v, x_1) I(w_{(1)} \otimes w_{(2)}) \\ &= z^{-1}\delta\Big(\frac{x_1 - x_0}{z}\Big) I(Y_1(v, x_0)w_{(1)} \otimes w_{(2)}) \\ &\quad + x_0^{-1}\delta\Big(\frac{z - x_1}{-x_0}\Big) I(w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \end{aligned} \tag{4.4}$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ (note that all the expressions in the right-hand side of (4.4) are well defined, and that the left-hand side of (4.4) is meaningful because any infinite linear combination of v_n ($n \in \mathbb{Z}$) of the form $\sum_{n<N} a_n v_n$ ($a_n \in \mathbb{C}$) acts in a well-defined way on any $I(w_{(1)} \otimes w_{(2)})$, in view of (4.3)); and the $\mathfrak{sl}(2)$-*bracket relations*: for any $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$,

$$L(j)I(w_{(1)} \otimes w_{(2)}) = I(w_{(1)} \otimes L(j)w_{(2)}) + \sum_{i=0}^{j+1} \binom{j+1}{i} z^i I((L(j-i)w_{(1)}) \otimes w_{(2)}) \tag{4.5}$$

for $j = -1, 0$ and 1 (note that if V is in fact a conformal vertex algebra, this follows automatically from (4.4) by setting $v = \omega$ and taking $\mathrm{Res}_{x_0}\mathrm{Res}_{x_1} x_1^{j+1}$). The vector space of $P(z)$-intertwining maps of type $\binom{W_3}{W_1 W_2}$ is denoted by

$$\mathcal{M}[P(z)]_{W_1 W_2}^{W_3},$$

or simply by

$$\mathcal{M}_{W_1 W_2}^{W_3}$$

if there is no ambiguity.

Remark 4.3 As we mentioned in the Introduction, $P(z)$ is the Riemann sphere $\hat{\mathbb{C}}$ with one negatively oriented puncture at ∞ and two ordered positively oriented punctures at z and 0, with local coordinates $1/w$, $w - z$ and w, respectively, vanishing at these three punctures. The geometry underlying the notion of $P(z)$-intertwining map and the notions of $P(z)$-product and $P(z)$-tensor product (see below) is determined by $P(z)$.

Remark 4.4 In the case of $\mathbb{C}$-graded ordinary modules for a vertex operator algebra, where the grading restriction condition (2.90) for a module W is replaced by the (more restrictive) condition

$$W_{(n)} = 0 \quad \text{for} \;\; n \in \mathbb{C} \;\; \text{with sufficiently negative real part} \tag{4.6}$$

as in [HL1] (and where, in our context, the abelian groups A and $\tilde{A}$ are trivial), the notion of $P(z)$-intertwining map above agrees with the earlier one introduced in [HL1]; in this case, the conditions (4.2) and (4.3) are automatic.

Remark 4.5 If W_3 in Definition 4.2 is lower bounded, as in Remark 3.25, then (4.3) can be strengthened to:

$$\pi_n I(w_{(1)} \otimes w_{(2)}) = 0 \quad \text{for} \;\; \Re(n) \;\; \text{sufficiently negative} \tag{4.7}$$

($n \in \mathbb{C}$).

Remark 4.6 As in Remark 3.42, it is clear that the $\mathfrak{sl}(2)$-bracket relations (4.5) can equivalently be written as

$$\begin{aligned} I(L(j)w_{(1)} \otimes w_{(2)}) &= \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i L(j-i) I(w_{(1)} \otimes w_{(2)}) \\ &\quad - \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i I(w_{(1)} \otimes L(j-i) w_{(2)}) \end{aligned} \tag{4.8}$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $j = -1, 0$ and 1.

Following [HL1] we will choose the branch of $\log z$ (and of $\arg z$) such that

$$0 \leq \Im(\log z) = \arg z < 2\pi \tag{4.9}$$

(despite the fact that we happened to have used a different branch in (3.12) in the proof of Theorem 3.6), so that

$$\log z = \log |z| + i \arg z.$$

We will also use the notation

$$l_p(z) = \log z + 2\pi i p, \ \ p \in \mathbb{Z}, \tag{4.10}$$

as in [HL1], for arbitrary values of the log function. For a formal expression $f(x)$ as in (3.2), but involving only nonnegative integral powers of $\log x$, and $\zeta \in \mathbb{C}$, whenever

$$f(x)\Big|_{x^n = e^{\zeta n},\ (\log x)^m = \zeta^m,\ n \in \mathbb{C},\ m \in \mathbb{N}} \tag{4.11}$$

exists algebraically, we will write (4.11) simply as $f(x)\Big|_{x=e^\zeta}$ or $f(e^\zeta)$, and we will call this "substituting e^ζ for x in $f(x)$," even though, in general, it depends on ζ, not just on e^ζ. (See also (3.76).) In addition, for a fixed integer p, we will sometimes write

$$f(x)\Big|_{x=z} \ \text{ or } \ f(z) \tag{4.12}$$

instead of $f(x)\Big|_{x=e^{l_p(z)}}$ or $f(e^{l_p(z)})$. We will sometimes say that "$f(e^\zeta)$ exists" or that "$f(z)$ exists."

Remark 4.7 A very important example of an $f(z)$ existing in this sense occurs when

$$f(x) = \mathcal{Y}(w_{(1)}, x) w_{(2)} \ \ (\in W_3[\log x]\{x\})$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and a logarithmic intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1\, W_2}$, in the notation of Definition 3.10; note that (4.11) exists (as an element of $\overline{W}_3$) in this case because of Proposition 3.20(b). Note also that in particular, $\mathcal{Y}(w_{(1)}, e^\zeta)$ (or $\mathcal{Y}(w_{(1)}, z)$) exists as a linear map from W_2 to $\overline{W}_3$, and that $\mathcal{Y}(\cdot, z)\cdot$ exists as a linear map

$$\begin{aligned} W_1 \otimes W_2 &\to \overline{W}_3 \\ w_{(1)} \otimes w_{(2)} &\mapsto \mathcal{Y}(w_{(1)}, z) w_{(2)}. \end{aligned} \tag{4.13}$$

Now we use these considerations to construct correspondences between (grading-compatible) logarithmic intertwining operators and $P(z)$-intertwining maps. Fix an integer p. Let $\mathcal{Y}$ be a logarithmic intertwining operator of type $\binom{W_3}{W_1\,W_2}$. Then we have a linear map

$$I_{\mathcal{Y},p} : W_1 \otimes W_2 \to \overline{W}_3 \tag{4.14}$$

defined by

$$I_{\mathcal{Y},p}(w_{(1)} \otimes w_{(2)}) = \mathcal{Y}(w_{(1)}, e^{l_p(z)})w_{(2)} \tag{4.15}$$

for all $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. The grading-compatibility condition (3.31) yields the grading-compatibility condition (4.2) for $I_{\mathcal{Y},p}$, and (4.3) follows. By substituting $e^{l_p(z)}$ for x_2 in (3.26) and for x in (3.28), we see that $I_{\mathcal{Y},p}$ satisfies the Jacobi identity (4.4) and the $\mathfrak{sl}(2)$-bracket relations (4.5). Hence $I_{\mathcal{Y},p}$ is a $P(z)$-intertwining map. (Note that the $L(-1)$-derivative property (3.27) is not used here, so that, for example, each $\mathcal{Y}^{(k)}$ in Remark 3.26 produces $P(z)$-intertwining maps in this way. But the $L(-1)$-derivative property is indeed needed for the recovery of $\mathcal{Y}$ from $I_{\mathcal{Y},p}$, as we shall now see.)

On the other hand, we note that (3.61) (whose proof uses the $L(-1)$-derivative property of $\mathcal{Y}$) is equivalent to

$$\langle y^{L'(0)}w'_{(3)}, \mathcal{Y}(y^{-L(0)}w_{(1)}, x)y^{-L(0)}w_{(2)}\rangle_{W_3} = \langle w'_{(3)}, \mathcal{Y}(w_{(1)}, xy)w_{(2)}\rangle_{W_3} \tag{4.16}$$

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$, where we are using the pairing between the contragredient module W'_3 and W_3 or $\overline{W}_3$ (recall Definition 2.32, Theorem 2.34, (2.75), (2.101) and (3.55)). Substituting $e^{l_p(z)}$ for x and then $e^{-l_p(z)}x$ for y, we obtain

$$\begin{aligned}\langle y^{L'(0)}x^{L'(0)}w'_{(3)}, \mathcal{Y}(y^{-L(0)}x^{-L(0)}w_{(1)}, e^{l_p(z)})y^{-L(0)}x^{-L(0)}w_{(2)}\rangle_{W_3}\Big|_{y=e^{-l_p(z)}}\\ = \langle w'_{(3)}, \mathcal{Y}(w_{(1)}, x)w_{(2)}\rangle_{W_3},\end{aligned}$$

or equivalently, using the notation (4.15),

$$\begin{aligned}\langle w'_{(3)}, y^{L(0)}x^{L(0)}I_{\mathcal{Y},p}(y^{-L(0)}x^{-L(0)}w_{(1)} \otimes y^{-L(0)}x^{-L(0)}w_{(2)})\rangle_{W_3}\Big|_{y=e^{-l_p(z)}}\\ = \langle w'_{(3)}, \mathcal{Y}(w_{(1)}, x)w_{(2)}\rangle_{W_3}.\end{aligned}$$

Thus we have recovered $\mathcal{Y}$ from $I_{\mathcal{Y},p}$ (with (3.27) having been used in the proof).

This motivates the following definition: Given a $P(z)$-intertwining map I and an integer p, we define a linear map

$$\mathcal{Y}_{I,p} : W_1 \otimes W_2 \to W_3[\log x]\{x\} \tag{4.17}$$

by

$$\begin{aligned}&\mathcal{Y}_{I,p}(w_{(1)}, x)w_{(2)}\\ &\quad = y^{L(0)}x^{L(0)}I(y^{-L(0)}x^{-L(0)}w_{(1)} \otimes y^{-L(0)}x^{-L(0)}w_{(2)})\Big|_{y=e^{-l_p(z)}}\end{aligned} \tag{4.18}$$

for any $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ (this is well defined and indeed maps to $W_3[\log x]\{x\}$, in view of (3.55)). We will also use the notation $w_{(1)}{}^{I,p}_{n;k} w_{(2)} \in W_3$ defined by

$$\mathcal{Y}_{I,p}(w_{(1)}, x)w_{(2)} = \sum_{n\in\mathbb{C}} \sum_{k\in\mathbb{N}} w_{(1)}{}^{I,p}_{n;k} w_{(2)} x^{-n-1}(\log x)^k. \tag{4.19}$$

Observe that since the operator $x^{\pm L(0)}$ always increases the power of x in an expression homogeneous of generalized weight n by $\pm n$, we see from (4.18) that

$$w_{(1)}{}^{I,p}_{n;k} w_{(2)} \in (W_3)_{[n_1+n_2-n-1]} \tag{4.20}$$

for $w_{(1)} \in (W_1)_{[n_1]}$ and $w_{(2)} \in (W_2)_{[n_2]}$. Moreover, for $I = I_{\mathcal{Y},p}$, we have $\mathcal{Y}_{I,p} = \mathcal{Y}$ (from the above), and for $\mathcal{Y} = \mathcal{Y}_{I,p}$, we have $I_{\mathcal{Y},p} = I$.

We can now prove the following proposition generalizing Proposition 12.2 in [HL3].

Proposition 4.8 *For $p \in \mathbb{Z}$, the correspondence*

$$\mathcal{Y} \mapsto I_{\mathcal{Y},p}$$

is a linear isomorphism from the space $\mathcal{V}^{W_3}_{W_1W_2}$ of (grading-compatible) logarithmic intertwining operators of type $\binom{W_3}{W_1\, W_2}$ to the space $\mathcal{M}^{W_3}_{W_1W_2}$ of $P(z)$-intertwining maps of the same type. Its inverse map is given by

$$I \mapsto \mathcal{Y}_{I,p}.$$

Proof We need only show that for any $P(z)$-intertwining map I of type $\binom{W_3}{W_1\, W_2}$, $\mathcal{Y}_{I,p}$ is a logarithmic intertwining operator of the same type. The lower truncation condition (4.3) implies that the lower truncation condition (3.25) for logarithmic intertwining operator holds for $\mathcal{Y}_{I,p}$; for this, (4.20) can be used. Let us now prove the Jacobi identity for $\mathcal{Y}_{I,p}$.

Changing the formal variables x_0 and x_1 to $x_0 e^{l_p(z)} x_2^{-1}$ and $x_1 e^{l_p(z)} x_2^{-1}$, respectively, in the Jacobi identity (4.4) for I, and then changing v to $y^{-L(0)} x_2^{-L(0)} v\Big|_{y=e^{-l_p(z)}}$ we obtain (noting that at first, $e^{l_p(z)}$ could be written simply as z because only integral powers occur)

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) & Y_3(y^{-L(0)}x_2^{-L(0)}v, x_1 y^{-1} x_2^{-1}) I(w_{(1)} \otimes w_{(2)})\Big|_{y=e^{-l_p(z)}} \\ &= x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right) I(Y_1(y^{-L(0)}x_2^{-L(0)}v, x_0 y^{-1} x_2^{-1}) w_{(1)} \otimes w_{(2)})\Big|_{y=e^{-l_p(z)}} \\ &\quad + x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right) I(w_{(1)} \otimes Y_2(y^{-L(0)}x_2^{-L(0)}v, x_1 y^{-1} x_2^{-1}) w_{(2)})\Big|_{y=e^{-l_p(z)}}. \end{aligned}$$

Using the formula

$$Y_3(y^{-L(0)}x_2^{-L(0)}v, x_1 y^{-1} x_2^{-1}) = y^{-L(0)} x_2^{-L(0)} Y_3(v, x_1) y^{L(0)} x_2^{L(0)},$$

which holds on the generalized module W_3, by (3.61), and the similar formulas for Y_1 and Y_2, we get

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)y^{-L(0)}x_2^{-L(0)}Y_3(v,x_1)y^{L(0)}x_2^{L(0)}I(w_{(1)}\otimes w_{(2)})\Big|_{y=e^{-l_p(z)}}\\
&\quad=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)I(y^{-L(0)}x_2^{-L(0)}Y_1(v,x_0)y^{L(0)}x_2^{L(0)}w_{(1)}\otimes w_{(2)})\Big|_{y=e^{-l_p(z)}}\\
&\qquad+x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)I(w_{(1)}\otimes y^{-L(0)}x_2^{-L(0)}Y_2(v,x_1)y^{L(0)}x_2^{L(0)}w_{(2)})\Big|_{y=e^{-l_p(z)}}.
\end{aligned}$$

Replacing $w_{(1)}$ by $y^{-L(0)}x_2^{-L(0)}w_{(1)}\Big|_{y=e^{-l_p(z)}}$ and $w_{(2)}$ by $y^{-L(0)}x_2^{-L(0)}w_{(2)}\Big|_{y=e^{-l_p(z)}}$, and then applying $y^{L(0)}x_2^{L(0)}\Big|_{y=e^{-l_p(z)}}$ to the whole equation, we obtain

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_3(v,x_1)y^{L(0)}x_2^{L(0)}\cdot\\
&\qquad\cdot I(y^{-L(0)}x_2^{-L(0)}w_{(1)}\otimes y^{-L(0)}x_2^{-L(0)}w_{(2)})\Big|_{y=e^{-l_p(z)}}\\
&\quad=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)y^{L(0)}x_2^{L(0)}\cdot\\
&\qquad\cdot I(y^{-L(0)}x_2^{-L(0)}Y_1(v,x_0)w_{(1)}\otimes y^{-L(0)}x_2^{-L(0)}w_{(2)})\Big|_{y=e^{-l_p(z)}}\\
&\qquad+x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)y^{L(0)}x_2^{L(0)}\cdot\\
&\qquad\cdot I(y^{-L(0)}x_2^{-L(0)}w_{(1)}\otimes y^{-L(0)}x_2^{-L(0)}Y_2(v,x_1)w_{(2)})\Big|_{y=e^{-l_p(z)}}.
\end{aligned}$$

But using (4.18), we can write this as

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_3(v,x_1)\mathcal{Y}_{I,p}(w_{(1)},x_2)w_{(2)}\\
&\quad=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}_{I,p}(Y_1(v,x_0)w_{(1)},x_2)w_{(2)}\\
&\qquad+x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\mathcal{Y}_{I,p}(w_{(1)},x_2)Y_2(v,x_1)w_{(2)}.
\end{aligned}$$

That is, the Jacobi identity for $\mathcal{Y}_{I,p}$ holds.

Similar procedures show that the $\mathfrak{sl}(2)$-bracket relations for I imply the $\mathfrak{sl}(2)$-bracket relations for $\mathcal{Y}_{I,p}$, as follows: Let j be -1, 0 or 1. By multiplying (4.5) by $(yx)^j$ and using (3.66) we obtain

$$(yx)^{-L(0)}L(j)(yx)^{L(0)}I(w_{(1)}\otimes w_{(2)})$$

$$= I(w_{(1)} \otimes (yx)^{-L(0)} L(j) (yx)^{L(0)} w_{(2)})$$
$$+ \sum_{i=0}^{j+1} \binom{j+1}{i} z^i (yx)^i I(((yx)^{-L(0)} L(j-i) (yx)^{L(0)} w_{(1)}) \otimes w_{(2)})$$

Replacing $w_{(1)}$ by $(yx)^{-L(0)} w_{(1)}$ and $w_{(2)}$ by $(yx)^{-L(0)} w_{(2)}$, and then applying $(yx)^{L(0)}$ to the whole equation, we obtain

$$\begin{aligned} L(j)(yx)^{L(0)} & I((yx)^{-L(0)} w_{(1)} \otimes (yx)^{-L(0)} w_{(2)}) \\ &= (yx)^{L(0)} I((yx)^{-L(0)} w_{(1)} \otimes (yx)^{-L(0)} L(j) w_{(2)}) \\ &\quad + \sum_{i=0}^{j+1} \binom{j+1}{i} z^i (yx)^i (yx)^{L(0)} I(((yx)^{-L(0)} L(j-i) w_{(1)}) \otimes (yx)^{-L(0)} w_{(2)}). \end{aligned}$$

Evaluating at $y = e^{-l_p(z)}$ and using (4.18) we see that this gives exactly the $\mathfrak{sl}(2)$-bracket relations (3.28) for $\mathcal{Y}_{I,p}$.

Finally, we prove the $L(-1)$-derivative property for $\mathcal{Y}_{I,p}$. This follows from (4.18), (3.57), and the $\mathfrak{sl}(2)$-bracket relation with $j = 0$ for $\mathcal{Y}_{I,p}$, namely,

$$[L(0), \mathcal{Y}_{I,p}(w_{(1)}, x)] = \mathcal{Y}_{I,p}(L(0) w_{(1)}, x) + x \mathcal{Y}_{I,p}(L(-1) w_{(1)}, x),$$

as follows:

$$\begin{aligned} \frac{d}{dx} & \mathcal{Y}_{I,p}(w_{(1)}, x) w_{(2)} \\ &= \frac{d}{dx} e^{-l_p(z)L(0)} x^{L(0)} I(e^{l_p(z)L(0)} x^{-L(0)} w_{(1)} \otimes e^{l_p(z)L(0)} x^{-L(0)} w_{(2)}) \\ &= e^{-l_p(z)L(0)} x^{-1} x^{L(0)} L(0) I(e^{l_p(z)L(0)} x^{-L(0)} w_{(1)} \otimes e^{l_p(z)L(0)} x^{-L(0)} w_{(2)}) \\ &\quad - e^{-l_p(z)L(0)} x^{L(0)} I(e^{l_p(z)L(0)} x^{-1} x^{-L(0)} L(0) w_{(1)} \otimes e^{l_p(z)L(0)} x^{-L(0)} w_{(2)}) \\ &\quad - e^{-l_p(z)L(0)} x^{L(0)} I(e^{l_p(z)L(0)} x^{-L(0)} w_{(1)} \otimes e^{l_p(z)L(0)} x^{-1} x^{-L(0)} L(0) w_{(2)}) \\ &= x^{-1} L(0) \mathcal{Y}_{I,p}(w_{(1)}, x) w_{(2)} - x^{-1} \mathcal{Y}_{I,p}(w_{(1)}, x) L(0) w_{(2)} \\ &\quad - x^{-1} \mathcal{Y}_{I,p}(L(0) w_{(1)}, x) w_{(2)} \\ &= \mathcal{Y}_{I,p}(L(-1) w_{(1)}, x) w_{(2)}. \end{aligned}$$

□

Remark 4.9 From Remarks 3.25 and 4.5, we note that if W_3 is lower bounded, then the spaces of logarithmic intertwining operators and of $P(z)$-intertwining maps in Proposition 4.8 satisfy the stronger conditions (3.43) and (4.7), respectively.

Remark 4.10 Given a generalized V-module (W, Y_W), recall from Remark 3.16 that Y_W is a logarithmic intertwining operator of type $\binom{W}{VW}$ not involving $\log x$ and having only integral powers of x. Then the substitution $x \mapsto z$ in (4.15) is very simple; it is independent of p and $Y_W(\cdot, z)\cdot$ entails only the substitutions $x^n \mapsto z^n$ for $n \in \mathbb{Z}$. As a special case, we can take (W, Y_W) to be (V, Y) itself.

Remark 4.11 Let I be a $P(z)$-intertwining map of type $\binom{W_3}{W_1\,W_2}$ and let $p, p' \in \mathbb{Z}$. From (4.18), we see that the logarithmic intertwining operators $\mathcal{Y}_{I,p}$ and $\mathcal{Y}_{I,p'}$ of this same type differ as follows:

$$\begin{aligned}
&\mathcal{Y}_{I,p'}(w_{(1)}, x)w_{(2)} \\
&\quad = e^{2\pi i(p-p')L(0)}\mathcal{Y}_{I,p}(e^{2\pi i(p'-p)L(0)}w_{(1)}, x)e^{2\pi i(p'-p)L(0)}w_{(2)}
\end{aligned} \tag{4.21}$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. Using the notation in Remark 3.45, we thus have

$$\begin{aligned}
\mathcal{Y}_{I,p'} &= (\mathcal{Y}_{I,p})_{[p-p',p'-p,p'-p]} \\
&= \mathcal{Y}_{I,p}(\cdot, e^{2\pi i(p-p')}\cdot)\,\cdot\,.
\end{aligned} \tag{4.22}$$

Remark 4.12 Let I be a $P(z)$-intertwining map of type $\binom{W_3}{W_1 W_2}$. Then from the correspondence between $P(z)$-intertwining maps and logarithmic intertwining operators in Proposition 4.8, we see that for any nonzero complex number z_1, the linear map I_1 defined by

$$I_1(w_{(1)} \otimes w_{(2)}) = \sum_{n\in\mathbb{C}}\sum_{k\in\mathbb{N}} w_{(1)}{}_{n;k}^{I,p}w_{(2)}e^{l_p(z_1)(-n-1)}(l_p(z_1))^k \tag{4.23}$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ (recall (4.19)) is a $P(z_1)$-intertwining map of the same type. In this sense, $w_{(1)}{}_{n;k}^{I,p}w_{(2)}$ is independent of z. This justifies writing $I(w_{(1)} \otimes w_{(2)})$ alternatively as

$$I(w_{(1)}, z)w_{(2)}, \tag{4.24}$$

indicating that z can be replaced by any nonzero complex number; this notation was sometimes used in [H], although we shall generally not be using it in the present work. However, for a general intertwining map associated to a sphere with punctures not necessarily of type $P(z)$, the corresponding element $w_{(1)}{}_{n;k}^{I,p}w_{(2)}$ will in general be different.

We now proceed to the definition of the $P(z)$-tensor product. As in [HL1], this will be a suitably universal "$P(z)$-product." We generalize these notions from [HL1] using the notations $\mathcal{M}_{sg}$ and $\mathcal{GM}_{sg}$ (the categories of strongly graded V-modules and generalized V-modules, respectively; recall Notation 2.36) as follows:

Definition 4.13 Let $\mathcal{C}_1$ be either of the categories $\mathcal{M}_{sg}$ or $\mathcal{GM}_{sg}$ (recall Notation 2.36). For $W_1, W_2 \in \mathrm{ob}\,\mathcal{C}_1$, a *$P(z)$-product of* W_1 *and* W_2 is an object (W_3, Y_3) of $\mathcal{C}_1$ equipped with a $P(z)$-intertwining map I_3 of type $\binom{W_3}{W_1\,W_2}$. We denote it by $(W_3, Y_3; I_3)$ or simply by $(W_3; I_3)$. Let $(W_4, Y_4; I_4)$ be another $P(z)$-product of W_1 and W_2. A *morphism* from $(W_3, Y_3; I_3)$ to $(W_4, Y_4; I_4)$ is a module map η from W_3 to W_4 such that the diagram

$$\begin{tikzcd}
& W_1 \otimes W_2 \arrow[dl, "I_3"'] \arrow[dr, "I_4"] & \\
\overline{W}_3 \arrow[rr, "\bar{\eta}"] & & \overline{W}_4
\end{tikzcd}$$

commutes, that is,

$$I_4 = \bar{\eta} \circ I_3, \tag{4.25}$$

where

$$\bar{\eta} : \overline{W}_3 \to \overline{W}_4 \tag{4.26}$$

is the natural extension of η. (Note that $\bar{\eta}$ exists because η preserves $\mathbb{C}$-gradings; we shall use the notation $\bar{\eta}$ for any such map η.)

Remark 4.14 In this setting, let η be a morphism from $(W_3, Y_3; I_3)$ to $(W_4, Y_4; I_4)$. We know from (4.17)–(4.19) that for $p \in \mathbb{Z}$, the coefficients $w_{(1)}{}^{I_3,p}_{n;k} w_{(2)}$ and $w_{(1)}{}^{I_4,p}_{n;k} w_{(2)}$ in the formal expansion (4.19) of $\mathcal{Y}_{I_3,p}(w_{(1)}, x)w_{(2)}$ and $\mathcal{Y}_{I_4,p}(w_{(1)}, x)w_{(2)}$, respectively, are determined by I_3 and I_4, and that

$$\eta(w_{(1)}{}^{I_3,p}_{n;k} w_{(2)}) = w_{(1)}{}^{I_4,p}_{n;k} w_{(2)}, \tag{4.27}$$

as we see by applying $\bar{\eta}$ to (4.18).

The notion of $P(z)$-tensor product is now defined by means of a universal property as follows:

Definition 4.15 Let $\mathcal{C}$ be a full subcategory of either $\mathcal{M}_{sg}$ or $\mathcal{GM}_{sg}$. For $W_1, W_2 \in \operatorname{ob}\mathcal{C}$, a *$P(z)$-tensor product of W_1 and W_2 in $\mathcal{C}$* is a $P(z)$-product $(W_0, Y_0; I_0)$ with $W_0 \in \operatorname{ob}\mathcal{C}$ such that for any $P(z)$-product $(W, Y; I)$ with $W \in \operatorname{ob}\mathcal{C}$, there is a unique morphism from $(W_0, Y_0; I_0)$ to $(W, Y; I)$. Clearly, a $P(z)$-tensor product of W_1 and W_2 in $\mathcal{C}$, if it exists, is unique up to unique isomorphism. In this case we will denote it by

$$(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$$

and call the object

$$(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)})$$

the *$P(z)$-tensor product (generalized) module of W_1 and W_2 in $\mathcal{C}$*. We will skip the phrase "in $\mathcal{C}$" if the category $\mathcal{C}$ under consideration is clear in context.

Remark 4.16 Consider the functor from $\mathcal{C}$ to the category **Set** defined by assigning to $W \in \operatorname{ob}\mathcal{C}$ the set $\mathcal{M}^W_{W_1 W_2}$ of all $P(z)$-intertwining maps of type $\binom{W}{W_1\, W_2}$. Then if the $P(z)$-tensor product of W_1 and W_2 exists, it is just the universal element for this functor, and this functor is representable, represented by the $P(z)$-tensor product. (Recall that given a functor f from a category $\mathcal{K}$ to **Set**, a universal element for f, if it exists, is a pair (X, x) where $X \in \operatorname{ob}\mathcal{K}$ and $x \in f(X)$ such that for any pair (Y, y) with $Y \in \operatorname{ob}\mathcal{K}$ and $y \in f(Y)$, there is a unique morphism $\sigma : X \to Y$ such that $f(\sigma)(x) = y$; in this case, f is represented by X.)

Definition 4.15 and Proposition 4.8 immediately give the following result relating the module maps from a $P(z)$-tensor product (generalized) module with the $P(z)$-intertwining maps and the logarithmic intertwining operators:

Proposition 4.17 *Suppose that $W_1 \boxtimes_{P(z)} W_2$ exists. We have a natural isomorphism*

$$\begin{array}{rcl} \mathrm{Hom}_V(W_1 \boxtimes_{P(z)} W_2, W_3) & \stackrel{\sim}{\to} & \mathcal{M}^{W_3}_{W_1 W_2} \\ \eta & \mapsto & \overline{\eta} \circ \boxtimes_{P(z)} \end{array} \tag{4.28}$$

and for $p \in \mathbb{Z}$, a natural isomorphism

$$\begin{array}{rcl} \mathrm{Hom}_V(W_1 \boxtimes_{P(z)} W_2, W_3) & \stackrel{\sim}{\to} & \mathcal{V}^{W_3}_{W_1 W_2} \\ \eta & \mapsto & \mathcal{Y}_{\eta,p} \end{array} \tag{4.29}$$

where $\mathcal{Y}_{\eta,p} = \mathcal{Y}_{I,p}$ with $I = \overline{\eta} \circ \boxtimes_{P(z)}$. $\square$

Suppose that the $P(z)$-tensor product $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ of W_1 and W_2 exists. We will sometimes denote the action of the canonical $P(z)$-intertwining map

$$w_{(1)} \otimes w_{(2)} \mapsto \boxtimes_{P(z)}(w_{(1)} \otimes w_{(2)}) = \boxtimes_{P(z)}(w_{(1)}, z)w_{(2)} \in \overline{W_1 \boxtimes_{P(z)} W_2} \tag{4.30}$$

(recall (4.24)) on elements simply by $w_{(1)} \boxtimes_{P(z)} w_{(2)}$:

$$w_{(1)} \boxtimes_{P(z)} w_{(2)} = \boxtimes_{P(z)}(w_{(1)} \otimes w_{(2)}) = \boxtimes_{P(z)}(w_{(1)}, z)w_{(2)}. \tag{4.31}$$

Remark 4.18 We emphasize that the element $w_{(1)} \boxtimes_{P(z)} w_{(2)}$ defined here is an element of the formal completion $\overline{W_1 \boxtimes_{P(z)} W_2}$, and *not* (in general) of the module $W_1 \boxtimes_{P(z)} W_2$ itself. This is different from the classical case for modules for a Lie algebra (recall Section 1.3), where the tensor product of elements of two modules is an element of the tensor product module.

Remark 4.19 Note that under the natural isomorphism (4.28) for the case $W_3 = W_1 \boxtimes_{P(z)} W_2$, the identity map from $W_1 \boxtimes_{P(z)} W_2$ to itself corresponds to the canonical intertwining map $\boxtimes_{P(z)}$. Furthermore, for $p \in \mathbb{Z}$, the $P(z)$-tensor product of W_1 and W_2 gives rise to a logarithmic intertwining operator $\mathcal{Y}_{\boxtimes_{P(z)},p}$ of type $\binom{W_1 \boxtimes_{P(z)} W_2}{W_1\, W_2}$, according to formula (4.18). If p is changed to $p' \in \mathbb{Z}$, this logarithmic intertwining operator changes according to (4.21). Note that the $P(z)$-intertwining map $\boxtimes_{P(z)}$ is canonical and depends only on z, while a corresponding logarithmic intertwining operator is not; it depends on $p \in \mathbb{Z}$.

Remark 4.20 Sometimes it will be convenient, as in the next proposition, to use the particular isomorphism associated with $p = 0$ (in Proposition 4.8) between the spaces of $P(z)$-intertwining maps and of logarithmic intertwining operators of the same type. In this case, we shall sometimes simplify the notation by dropping the p $(= 0)$ in the notation $w_{(1)}{}^{I,0}_{n;k} w_{(2)}$ (recall (4.19)):

$$w_{(1)}{}^{I}_{n;k} w_{(2)} = w_{(1)}{}^{I,0}_{n;k} w_{(2)}. \tag{4.32}$$

Proposition 4.21 *Suppose that the $P(z)$-tensor product $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ of W_1 and W_2 in $\mathcal{C}$ exists. Then for any complex number $z_1 \neq 0$, the $P(z_1)$-tensor product of W_1 and W_2 in $\mathcal{C}$ also exists, and is given by $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z_1)})$, where the $P(z_1)$-intertwining map $\boxtimes_{P(z_1)}$ is defined by*

$$\boxtimes_{P(z_1)}(w_{(1)} \otimes w_{(2)}) = \sum_{n\in\mathbb{C}} \sum_{k\in\mathbb{N}} w_{(1)}{}^{\boxtimes_{P(z)}}_{n;k} w_{(2)} e^{\log z_1(-n-1)} (\log z_1)^k \tag{4.33}$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$.

Proof By Remark 4.12, (4.33) indeed defines a $P(z_1)$-product. Given any $P(z_1)$-product $(W_3, Y_3; I_1)$ of W_1 and W_2, let I be the $P(z)$-product related to I_1 by formula (4.23) with I_1, I and z_1 in (4.23) replaced by I, I_1 and z, respectively, and with $p = 0$. Then from the definition of $P(z)$-tensor product, there is a unique morphism η from $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ to $(W_3, Y_3; I)$. Thus by (4.27) and (4.33) we see that η is also a morphism from the $P(z_1)$-product $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z_1)})$ to $(W_3, Y_3; I_1)$. The uniqueness of such a morphism follows similarly from the uniqueness of a morphism from $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ to $(W_3, Y_3; I)$. Hence $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z_1)})$ is the $P(z_1)$-tensor product of W_1 and W_2. $\square$

Remark 4.22 In general, it will turn out that the existence of tensor product, and the tensor product (generalized) module itself, do not depend on the geometric data. It is the intertwining map from the two modules to the completion of their tensor product that encodes the geometric information.

Generalizing Lemma 4.9 of [H], we have:

Proposition 4.23 *The generalized module $W_1 \boxtimes_{P(z)} W_2$ (if it exists) is spanned (as a vector space) by the (generalized-) weight components of the elements of $\overline{W_1 \boxtimes_{P(z)} W_2}$ of the form $w_{(1)} \boxtimes_{P(z)} w_{(2)}$, for all $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$.*

Proof Denote by W_0 the vector subspace of $W_1 \boxtimes_{P(z)} W_2$ spanned by all the weight components of all the elements of $\overline{W_1 \boxtimes_{P(z)} W_2}$ of the form $w_{(1)} \boxtimes_{P(z)} w_{(2)}$ for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. For a homogeneous vector $v \in V$ and arbitrary elements $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, equating the $x_0^{-1} x_1^{-m-1}$ coefficients of the Jacobi identity (4.4) gives

$$v_m(w_{(1)} \boxtimes_{P(z)} w_{(2)}) = w_{(1)} \boxtimes_{P(z)} (v_m w_{(2)}) + \sum_{i\in\mathbb{N}} \binom{m}{i} z^{m-i} (v_i w_{(1)}) \boxtimes_{P(z)} w_{(2)} \tag{4.34}$$

for all $m \in \mathbb{Z}$. Note that the summation in the right-hand side of (4.34) is always finite. Hence by taking arbitrary weight components of (4.34) we see that W_0 is closed under the action of V. In case V is Möbius, a similar argument, using (4.5), shows that W_0 is stable under the action of $\mathfrak{sl}(2)$. It is clear that W_0 is $\mathbb{C}$-graded and $\tilde{A}$-graded. Thus W_0 is a submodule of $W_1 \boxtimes_{P(z)} W_2$.

Now consider the quotient module

$$W = (W_1 \boxtimes_{P(z)} W_2)/W_0$$

and let π_W be the canonical map from $W_1 \boxtimes_{P(z)} W_2$ to W. By the definition of W_0, we have

$$\overline{\pi}_W \circ \boxtimes_{P(z)} = 0,$$

using the notation (4.30). The universal property of the $P(z)$-tensor product then demands that $\pi_W = 0$, i.e., that $W_0 = W_1 \boxtimes_{P(z)} W_2$.

(Another argument: The image of the $P(z)$-intertwining map $\boxtimes_{P(z)}$ lies in

$$\overline{W}_0 \subset \overline{W_1 \boxtimes_{P(z)} W_2},$$

so that W_0 is naturally a $P(z)$-product of W_1 and W_2, giving rise to a (unique) V-module map

$$f : W_1 \boxtimes_{P(z)} W_2 \to W_0$$

such that $\overline{f}$ takes each $w_{(1)} \boxtimes_{P(z)} w_{(2)}$ to $w_{(1)} \boxtimes_{P(z)} w_{(2)}$, by the universal property. Writing

$$\iota : W_0 \to W_1 \boxtimes_{P(z)} W_2$$

for the natural injection, we have that $\iota \circ f$ is the identity map on $W_1 \boxtimes_{P(z)} W_2$, by the universal property. Thus ι is surjective (or, alternatively, f is injective and is 1 on W_0), so that $W_0 = W_1 \boxtimes_{P(z)} W_2$.) $\square$

It is clear from Definition 4.15 that the tensor product operation distributes over direct sums in the following sense:

Proposition 4.24 *For $U_1, \dots, U_k$, $W_1, \dots, W_l \in \mathrm{ob}\,\mathcal{C}$, suppose that each $U_i \boxtimes_{P(z)} W_j$ exists. Then $(\coprod_i U_i) \boxtimes_{P(z)} (\coprod_j W_j)$ exists and there is a natural isomorphism*

$$\left(\coprod_i U_i\right) \boxtimes_{P(z)} \left(\coprod_j W_j\right) \xrightarrow{\sim} \coprod_{i,j} U_i \boxtimes_{P(z)} W_j. \qquad \square$$

Remark 4.25 It is of course natural to view the $P(z)$-tensor product as a bifunctor: Suppose that $\mathcal{C}$ is a full subcategory of either $\mathcal{M}_{sg}$ or $\mathcal{GM}_{sg}$ (recall Notation 2.36) such that for all $W_1, W_2 \in \mathrm{ob}\,\mathcal{C}$, the $P(z)$-tensor product of W_1 and W_2 exists in $\mathcal{C}$. Then $\boxtimes_{P(z)}$ provides a (bi)functor

$$\boxtimes_{P(z)} : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \tag{4.35}$$

as follows: For $W_1, W_2 \in \mathrm{ob}\,\mathcal{C}$,

$$\boxtimes_{P(z)}(W_1, W_2) = W_1 \boxtimes_{P(z)} W_2 \in \mathrm{ob}\,\mathcal{C} \tag{4.36}$$

and for V-module maps

$$\sigma_1 : W_1 \to W_3, \tag{4.37}$$

$$\sigma_2 : W_2 \to W_4 \tag{4.38}$$

with $W_3, W_4 \in \mathrm{ob}\,\mathcal{C}$, we have the V-module map, denoted

$$\boxtimes_{P(z)}(\sigma_1, \sigma_2) = \sigma_1 \boxtimes_{P(z)} \sigma_2, \tag{4.39}$$

from $W_1 \boxtimes_{P(z)} W_2$ to $W_3 \boxtimes_{P(z)} W_4$, defined by the universal property of the $P(z)$-tensor product $W_1 \boxtimes_{P(z)} W_2$ and the fact that the composition of $\boxtimes_{P(z)}$ with $\sigma_1 \otimes \sigma_2$ is a $P(z)$-intertwining map

$$\boxtimes_{P(z)} \circ (\sigma_1 \otimes \sigma_2) : W_1 \otimes W_2 \to \overline{W_3 \boxtimes_{P(z)} W_4}. \tag{4.40}$$

Note that it is the effect of this bifunctor on morphisms (rather than on objects) that exhibits the role of the geometric data.

We obtain right exact functors by fixing one of the generalized modules in Remark 4.25[1]:

Proposition 4.26 *In the setting of Remark 4.25, for $W \in \mathrm{ob}\,\mathcal{C}$ the functors $W \boxtimes_{P(z)} \cdot$ and $\cdot \boxtimes_{P(z)} W$ are right exact.*

Proof Let

$$W_1 \xrightarrow{\sigma_1} W_2 \xrightarrow{\sigma_2} W_3 \longrightarrow 0$$

be exact in $\mathcal{C}$. We show that

$$W \boxtimes_{P(z)} W_1 \overset{1_W \boxtimes_{P(z)} \sigma_1}{\longrightarrow} W \boxtimes_{P(z)} W_2 \overset{1_W \boxtimes_{P(z)} \sigma_2}{\longrightarrow} W \boxtimes_{P(z)} W_3 \longrightarrow 0$$

is exact; the proof of right exactness for $\cdot \boxtimes_{P(z)} W$ is completely analogous.

For the surjectivity of $1_W \boxtimes \sigma_2$, we observe that the elements

$$(1_W \boxtimes \sigma_2)(\pi_n(w \boxtimes w_{(2)}))$$

for $w \in W$, $w_{(2)} \in W_2$ and $n \in \mathbb{C}$ span $W \boxtimes W_3$ (we are dropping the subscripts $P(z)$), since this element equals

$$\pi_n(\overline{1_W \boxtimes \sigma_2}(w \boxtimes w_{(2)})) = \pi_n(w \boxtimes \sigma_2(w_{(2)})),$$

and these elements span $W \boxtimes W_3$ by the surjectivity of σ_2 and Proposition 4.23.

Since

$$(1_W \boxtimes \sigma_2)(1_W \boxtimes \sigma_1) = 1_W \boxtimes \sigma_2\sigma_1 = 0,$$

it remains only to show that the natural (surjective) module map

$$\theta : (W_1 \boxtimes W_2)/\mathrm{Im}\,(1_W \boxtimes \sigma_1) \to W_1 \boxtimes W_3$$

is injective. Noting that

$$\overline{(W_1 \boxtimes W_2)/\mathrm{Im}\,(1_W \boxtimes \sigma_1)} = \overline{(W_1 \boxtimes W_2)}/\overline{\mathrm{Im}\,(1_W \boxtimes \sigma_1)},$$

[1]We thank Ingo Runkel for asking us whether our tensor product functors are right exact.

we characterize θ by:

$$\overline{\theta}(w \boxtimes w_{(2)} + \overline{\mathrm{Im}\,(1_W \boxtimes \sigma_1)}) = \overline{1_W \boxtimes \sigma_2}(w \boxtimes w_{(2)}).$$

We construct a $P(z)$-intertwining map

$$I : W \otimes W_3 \to \overline{(W \boxtimes W_2)/\mathrm{Im}\,(1_W \boxtimes \sigma_1)}$$

as follows: For $w \in W$ and $w_{(3)} \in W_3$ set

$$I(w \otimes w_{(3)}) = w \boxtimes w_{(2)} + \overline{\mathrm{Im}\,(1_W \boxtimes \sigma_1)}$$

where $w_{(2)} \in W_2$ is such that

$$\sigma_2(w_{(2)}) = w_{(3)}.$$

Then I is well defined because for $w'_{(2)} \in W_2$ with $\sigma_2(w'_{(2)}) = w_{(3)}$,

$$w \otimes (w_{(2)} - w'_{(2)}) \in w \boxtimes \mathrm{Ker}\,\sigma_2 = w \boxtimes \mathrm{Im}\,\sigma_1 \subset \overline{\mathrm{Im}\,(1_W \otimes \sigma_1)},$$

and it is straightforward to verify that I is in fact a $P(z)$-intertwining map. Thus we have a module map

$$\eta : W \boxtimes W_3 \to (W \boxtimes W_2)/\mathrm{Im}\,(1_W \otimes \sigma_1)$$

such that

$$\overline{\eta}(w \boxtimes w_{(3)}) = w \boxtimes w_{(2)} + \overline{\mathrm{Im}\,(1_W \otimes \sigma_1)},$$

with the elements as above. Then

$$\begin{aligned}\overline{\eta \circ \theta}(w \boxtimes w_{(2)} + \overline{\mathrm{Im}\,(1_W \otimes \sigma_1)}) &= \overline{\eta}(\overline{1_W \otimes \sigma_2}(w \boxtimes w_{(2)})) \\ &= \overline{\eta}(w \boxtimes \sigma_2(w_{(2)})) \\ &= \overline{\eta}(w \boxtimes w_{(3)}) \\ &= w \boxtimes w_{(2)} + \overline{\mathrm{Im}\,(1_W \otimes \sigma_1)},\end{aligned}$$

which shows that $\eta \circ \theta$ is the identity map, and so θ is injective, as desired. $\square$

We now discuss the simplest examples of $P(z)$-tensor products—those in which one or both of W_1 or W_2 is V itself (viewed as a (generalized) V-module); we suppose here that $V \in \mathrm{ob}\,\mathcal{C}$. Since the discussion of the case in which both W_1 and W_2 are V turns out to be no simpler than the case in which $W_1 = V$, we shall discuss only the two more general cases $W_1 = V$ and $W_2 = V$.

Example 4.27 Let (W, Y_W) be an object of $\mathcal{C}$. The vertex operator map Y_W gives a $P(z)$-intertwining map

$$I_{Y_W,p} = Y_W(\cdot, z)\cdot : V \otimes W \to \overline{W}$$

for any fixed $p \in \mathbb{Z}$ (recall Proposition 4.8 and Remark 4.10). We claim that $(W, Y_W; Y_W(\cdot, z)\cdot)$ is the $P(z)$-tensor product of V and W in $\mathcal{C}$. In fact, let $(W_3, Y_3; I)$ be a $P(z)$-product of V and W in $\mathcal{C}$ and suppose that there exists a module map $\eta : W \to W_3$ such that

$$\overline{\eta} \circ (Y_W(\cdot, z)\cdot) = I. \tag{4.41}$$

Then for $w \in W$, we must have

$$\begin{aligned} \eta(w) &= \eta(Y_W(\mathbf{1}, z)w) \\ &= (\overline{\eta} \circ (Y_W(\cdot, z)\cdot))(\mathbf{1} \otimes w) \\ &= I(\mathbf{1} \otimes w), \end{aligned} \tag{4.42}$$

so that η is unique if it exists. We now define $\eta : W \to \overline{W}_3$ using (4.42). We shall show that $\eta(W) \subset W_3$ and that η has the desired properties. Since I is a $P(z)$-intertwining map of type $\binom{W_3}{VW}$, it corresponds to a logarithmic intertwining operator $\mathcal{Y} = \mathcal{Y}_{I,p}$ of the same type, according to Proposition 4.8. Since $L(-1)\mathbf{1} = 0$, we have

$$\frac{d}{dx}\mathcal{Y}(\mathbf{1}, x) = \mathcal{Y}(L(-1)\mathbf{1}, x) = 0.$$

Thus $\mathcal{Y}(\mathbf{1}, x)$ is simply the constant map $\mathbf{1}^{\mathcal{Y}}_{-1;0} : W \to W_3$ (using the notation (3.24)), and this map preserves (generalized) weights, by Proposition 3.20(b). By Proposition 4.8, $I = I_{\mathcal{Y},p}$, so that

$$\begin{aligned} \eta(w) &= I(\mathbf{1} \otimes w) \\ &= I_{\mathcal{Y},p}(\mathbf{1} \otimes w) \\ &= \mathbf{1}^{\mathcal{Y}}_{-1;0} w \end{aligned}$$

for $w \in W$. So $\eta = \mathbf{1}^{\mathcal{Y}}_{-1;0}$ is a linear map from W to W_3 preserving (generalized) weights. Using the Jacobi identity (4.4) for the $P(z)$-intertwining map I and the fact that $Y(u, x_0)\mathbf{1} \in V[[x_0]]$ for $u \in V$, we obtain

$$\begin{aligned} \eta(Y_W(u, x)w) &= I(\mathbf{1} \otimes Y_W(u, x)w) \\ &= Y_3(u, x)I(\mathbf{1} \otimes w) - \mathrm{Res}_{x_0} z^{-1}\delta\left(\frac{x - x_0}{z}\right) I(Y(u, x_0)\mathbf{1} \otimes w) \\ &= Y_3(u, x)I(\mathbf{1} \otimes w) \\ &= Y_3(u, x)\eta(w) \end{aligned}$$

for $u \in V$ and $w \in W$, proving that η is a module map when V is a conformal vertex algebra, and when V is Möbius, η also commutes with the action of $\mathfrak{sl}(2)$, by (4.5). For $w \in W$,

$$\begin{aligned} (\overline{\eta} \circ (Y_W(\cdot, z)\cdot))(\mathbf{1} \otimes w) &= \overline{\eta}(Y_W(\mathbf{1}, z)w) \\ &= \eta(w) \\ &= I(\mathbf{1} \otimes w). \end{aligned} \tag{4.43}$$

Using the Jacobi identity for $P(z)$-intertwining maps, we obtain

$$\begin{aligned} & I(Y(u,x_0)v\otimes w) \\ & \quad = \mathrm{Res}_x x_0^{-1}\delta\left(\frac{x-z}{x_0}\right)Y_3(u,x)I(v\otimes w) - \mathrm{Res}_x x_0^{-1}\delta\left(\frac{z-x}{-x_0}\right)I(v\otimes Y_W(u,x)w) \end{aligned} \tag{4.44}$$

for $u, v \in V$ and $w \in W$. Since η is a module map and $Y_W(\cdot, z)\cdot$ is a $P(z)$-intertwining map of type $\binom{W}{VW}$, $\overline{\eta} \circ Y_W(\cdot, z)\cdot$ is a $P(z)$-intertwining map of type $\binom{W_3}{VW}$. In particular, (4.44) holds when we replace I by $\overline{\eta} \circ Y_W(\cdot, z)\cdot$. Using (4.44) for $v = \mathbf{1}$ together with (4.43), we obtain

$$(\overline{\eta} \circ (Y_W(\cdot, z)\cdot))(u \otimes w) = I(u \otimes w)$$

for $u \in V$ and $w \in W$, proving (4.41), as desired. Thus $(W, Y_W; Y_W(\cdot, z)\cdot)$ is the $P(z)$-tensor product of V and W in $\mathcal{C}$.

Example 4.28 Let (W, Y_W) be an object of $\mathcal{C}$. In order to construct the $P(z)$-tensor product $W \boxtimes_{P(z)} V$, recall from (3.77) and Proposition 3.44 that $\Omega_p(Y_W)$ is a logarithmic intertwining operator of type $\binom{W}{WV}$. It involves only integral powers of the formal variable and no logarithms, and it is independent of p. In fact,

$$\Omega_p(Y_W)(w, x)v = e^{xL(-1)}Y_W(v, -x)w$$

for $v \in V$ and $w \in W$. For $q \in \mathbb{Z}$,

$$I_{\Omega_p(Y_W),q} = \Omega_p(Y_W)(\cdot, z)\cdot : W \otimes V \to \overline{W}$$

is a $P(z)$-intertwining map of the same type and is independent of q. We claim that $(W, Y_W; \Omega_p(Y_W)(\cdot, z)\cdot)$ is the $P(z)$-tensor product of W and V in $\mathcal{C}$. In fact, let $(W_3, Y_3; I)$ be a $P(z)$-product of W and V in $\mathcal{C}$ and suppose that there exists a module map $\eta : W \to W_3$ such that

$$\overline{\eta} \circ \Omega_p(Y_W)(\cdot, z)\cdot = I. \tag{4.45}$$

For $w \in W$, we must have

$$\begin{aligned} \eta(w) &= \eta(Y_W(\mathbf{1}, -z)w) \\ &= e^{-zL(-1)}\overline{\eta}(e^{zL(-1)}Y_W(\mathbf{1}, -z)w) \\ &= e^{-zL(-1)}\overline{\eta}(\Omega_p(Y_W)(w, z)\mathbf{1})) \\ &= e^{-zL(-1)}(\overline{\eta} \circ (\Omega_p(Y_W)(\cdot, z)\cdot))(w \otimes \mathbf{1}) \\ &= e^{-zL(-1)}I(w \otimes \mathbf{1}), \end{aligned} \tag{4.46}$$

and so η is unique if it exists. (Note that the right-hand side of (4.46) is indeed defined, in view of (4.3).) We now define $\eta : W \to \overline{W}_3$ by (4.46). Consider the logarithmic intertwining operator $\mathcal{Y} = \mathcal{Y}_{I,q}$ that corresponds to I by Proposition 4.8. Using Proposition 4.8, (4.9)–(4.11), (3.76) and the equality

$$\begin{aligned} l_q(-z) &= \log|-z| + i(\arg(-z) + 2\pi q) \\ &= \left\{ \begin{array}{ll} \log|z| + i(\arg z + \pi + 2\pi q), & 0 \leq \arg z < \pi \\ \log|z| + i(\arg z - \pi + 2\pi q), & \pi \leq \arg z < 2\pi \end{array} \right. \\ &= \left\{ \begin{array}{ll} l_q(z) + \pi i, & 0 \leq \arg z < \pi \\ l_q(z) - \pi i, & \pi \leq \arg z < 2\pi, \end{array} \right. \end{aligned}$$

we have

$$\begin{aligned} e^{-zL(-1)} I(w \otimes \mathbf{1}) &= e^{-zL(-1)} \mathcal{Y}(w, e^{l_q(z)}) \mathbf{1} \\ &= e^{-xL(-1)} \mathcal{Y}(w, x) \mathbf{1} \big|_{x^n = e^{n l_q(z)},\ (\log x)^m = (l_q(z))^m,\ n \in \mathbb{C},\ m \in \mathbb{N}} \\ &= e^{yL(-1)} \mathcal{Y}(w, e^{\pm \pi i} y) \mathbf{1} \big|_{y^n = e^{n l_q(-z)},\ (\log y)^m = (l_q(-z))^m,\ n \in \mathbb{C},\ m \in \mathbb{N}}, \end{aligned}$$

where $e^{\pm\pi i}$ is $e^{-\pi i}$ when $0 \leq \arg z < \pi$ and is $e^{\pi i}$ when $\pi \leq \arg z < 2\pi$. Then by (3.77), we see that $\eta(w) = e^{-zL(-1)} I(w \otimes \mathbf{1})$ is equal to $\Omega_{-1}(\mathcal{Y})(\mathbf{1}, e^{l_q(-z)})w$ when $0 \leq \arg z < \pi$ and is equal to $\Omega_0(\mathcal{Y})(\mathbf{1}, e^{l_q(-z)})w$ when $\pi \leq \arg z < 2\pi$. By Proposition 3.44, $\Omega_{-1}(\mathcal{Y})$ and $\Omega_0(\mathcal{Y})$ are logarithmic intertwining operators of type $\binom{W_3}{VW}$. As in Example 4.27, we see that $\Omega_{-1}(\mathcal{Y})(\mathbf{1}, y)$ and $\Omega_0(\mathcal{Y})(\mathbf{1}, y)$ are equal to $\mathbf{1}_{-1,0}^{\Omega_{-1}(\mathcal{Y})}$ and $\mathbf{1}_{-1,0}^{\Omega_0(\mathcal{Y})}$, respectively, and these maps preserve (generalized) weights. Therefore η is a linear map from W to W_3 preserving (generalized) weights. Using the Jacobi identity (4.4) for the $P(z)$-intertwining map I and the fact that $Y(u, x_1)\mathbf{1} \in V[[x_1]]$, we have

$$\begin{aligned} \eta(Y_W(u, x_0)w) &= e^{-zL(-1)} I(Y_W(u, x_0)w \otimes \mathbf{1}) \\ &= \mathrm{Res}_{x_1} x_0^{-1} \delta\left(\frac{x_1 - z}{x_0}\right) e^{-zL(-1)} Y_3(u, x_1) I(w \otimes \mathbf{1}) \\ &\quad - \mathrm{Res}_{x_1} x_0^{-1} \delta\left(\frac{z - x_1}{-x_0}\right) e^{-zL(-1)} I(w \otimes Y(u, x_1)\mathbf{1}) \\ &= e^{-zL(-1)} Y_3(u, x_0 + z) I(w \otimes \mathbf{1}) \\ &= Y_3(u, x_0) e^{-zL(-1)} I(w \otimes \mathbf{1}) \\ &= Y_3(u, x_0)\eta(w) \end{aligned}$$

for $u \in V$ and $w \in W$, proving that η is a module map when V is a conformal vertex algebra. As in Example 4.27, when V is Möbius, η also commutes with the action of $\mathfrak{sl}(2)$, this time by (4.8) together with (3.72) with x specialized to $-z$. For $w \in W$,

$$\begin{aligned} (\overline{\eta} \circ (\Omega_p(Y_W)(\cdot, z)\cdot))(w \otimes \mathbf{1}) &= \overline{\eta}(e^{zL(-1)} Y_W(\mathbf{1}, -z)w) \\ &= e^{zL(-1)} \eta(w) \\ &= e^{zL(-1)} e^{-zL(-1)} I(w \otimes \mathbf{1}) \\ &= I(w \otimes \mathbf{1}). \end{aligned} \tag{4.47}$$

Since both $\overline{\eta} \circ (\Omega_p(Y_W)(\cdot, z)\cdot)$ and I are $P(z)$-intertwining maps of type $\binom{W_3}{W V}$, using the Jacobi identity for $P(z)$-intertwining operators and (4.47) (cf. Example 4.27), we have

$$(\overline{\eta} \circ (\Omega_p(Y_W)(\cdot, z)\cdot))(w \otimes v) = I(w \otimes v)$$

for $v \in V$ and $w \in W$, proving (4.45). Thus $(W, Y_W; \Omega_p(Y_W)(\cdot, z)\cdot)$ is the $P(z)$-tensor product of W and V in $\mathcal{C}$.

We discussed the important special class of finitely reductive vertex operator algebras in the Introduction. In case V is a finitely reductive vertex operator algebra, the $P(z)$-tensor product always exists, as we are about to establish (following [HL1] and [HL3]). As in the Introduction, the definition of finite reductivity is:

Definition 4.29 A vertex operator algebra V is *finitely reductive* if

1. Every V-module is completely reducible.
2. There are only finitely many irreducible V-modules (up to equivalence).
3. All the fusion rules (the dimensions of the spaces of intertwining operators among triples of modules) for V are finite.

Remark 4.30 In this case, every V-module is of course a *finite* direct sum of irreducible modules. Also, the third condition holds if the finiteness of the fusion rules among triples of only *irreducible* modules is assumed.

Remark 4.31 We are of course taking the notion of V-module so that the grading restriction conditions are the ones described in Remark 2.27, formulas (2.90) and (2.91); in particular, V-modules are understood to be $\mathbb{C}$-graded. Recall from Remark 2.20 that for an irreducible module, all its weights are congruent to one another modulo $\mathbb{Z}$. Thus for an irreducible module, our grading-truncation condition (2.90) amounts exactly to the condition that the real parts of the weights are bounded from below. In [HL1]–[HL3], boundedness of the real parts of the weights from below was our grading-truncation condition in the definition of the notion of module for a vertex operator algebra. Thus the first two conditions in the notion of finite reductivity are the same whether we use the current grading restriction conditions in the definition of the notion of module or the corresponding conditions in [HL1]–[HL3]. As for intertwining operators, recall from Remark 3.12 and Corollary 3.22 that when the first two conditions are satisfied, the notion of (ordinary, non-logarithmic) intertwining operator here coincides with that in [HL1] because the truncation conditions agree. Also, in this setting, by Remark 3.23, the logarithmic and ordinary intertwining operators are the same, and so the spaces of intertwining operators $\mathcal{V}_{W_1 W_2}^{W_3}$ and fusion rules $N_{W_1 W_2}^{W_3}$ in Definition 3.17 have the same meanings as in [HL1]. Thus the notion of finite reductivity for a vertex operator algebra is the same whether we use the current grading restriction and truncation conditions in the definitions of the notions of module and of intertwining operator or the corresponding conditions in [HL1]–[HL3]. In particular, finite reductivity of V according to Definition 4.29 is equivalent to the corresponding notion, "rationality" (recall the Introduction) in [HL1]–[HL3].

Remark 4.32 For a vertex operator algebra V (in particular, a finitely reductive one), the category $\mathcal{M}$ of V-modules coincides with the category $\mathcal{M}_{sg}$ of strongly graded V-modules; recall Notation 2.36.

For the rest of Section 4.1, let us assume that V is a finitely reductive vertex operator algebra. We shall now show that $P(z)$-tensor products always exist in the category $\mathcal{M} = \mathcal{M}_{sg}$ of V-modules, in the sense of Definition 4.15.

The considerations from here through (4.61) also hold, with natural adjustments, for finite-dimensional modules for a semisimple Lie algebra (even though there are infinitely many irreducible modules up to equivalence) or for a finite group or for a compact group, etc., but in such classical contexts, one does not ordinarily express things in this way because one knows *a priori* that the tensor product functors exist and satisfy natural associativity as in (4.62), (4.63). What we do now shows how to build tensor product functors with knowledge "only" of the spaces of intertwining maps, and uses this to motivate how to approach the problem of constructing appropriate natural associativity isomorphisms, whether or not our vertex algebra V is a finitely reductive vertex operator algebra.

Consider V-modules W_1, W_2 and W_3. We know that

$$N_{W_1 W_2}^{W_3} = \dim \mathcal{V}_{W_1 W_2}^{W_3} < \infty \tag{4.48}$$

and from Proposition 4.8, we also have

$$N_{W_1 W_2}^{W_3} = \dim \mathcal{M}[P(z)]_{W_1 W_2}^{W_3} = \dim \mathcal{M}_{W_1 W_2}^{W_3} < \infty \tag{4.49}$$

(recall Definition 4.2).

The natural evaluation map

$$\begin{array}{rcl} W_1 \otimes W_2 \otimes \mathcal{M}_{W_1 W_2}^{W_3} & \to & \overline{W}_3 \\ w_{(1)} \otimes w_{(2)} \otimes I & \mapsto & I(w_{(1)} \otimes w_{(2)}) \end{array} \tag{4.50}$$

gives a natural map

$$\mathcal{F}[P(z)]_{W_1 W_2}^{W_3} : W_1 \otimes W_2 \to \mathrm{Hom}(\mathcal{M}_{W_1 W_2}^{W_3}, \overline{W}_3) = (\mathcal{M}_{W_1 W_2}^{W_3})^* \otimes \overline{W}_3. \tag{4.51}$$

Since $\dim \mathcal{M}_{W_1 W_2}^{W_3} < \infty$, $(\mathcal{M}_{W_1 W_2}^{W_3})^* \otimes W_3$ is a V-module (with finite-dimensional weight spaces) in the obvious way, and the map $\mathcal{F}[P(z)]_{W_1 W_2}^{W_3}$ is clearly a $P(z)$-intertwining map, where we make the identification

$$(\mathcal{M}_{W_1 W_2}^{W_3})^* \otimes \overline{W}_3 = \overline{(\mathcal{M}_{W_1 W_2}^{W_3})^* \otimes W_3}. \tag{4.52}$$

This gives us a natural $P(z)$-product for the category $\mathcal{M} = \mathcal{M}_{sg}$ (recall Definition 4.13). Moreover, we have a natural linear injection

$$\begin{array}{rcl} i : \mathcal{M}_{W_1 W_2}^{W_3} & \to & \mathrm{Hom}_V((\mathcal{M}_{W_1 W_2}^{W_3})^* \otimes W_3, W_3) \\ I & \mapsto & (f \otimes w_{(3)} \mapsto f(I) w_{(3)}) \end{array} \tag{4.53}$$

which is an isomorphism if W_3 is irreducible, since in this case,

$$\mathrm{Hom}_V(W_3, W_3) \simeq \mathbb{C}$$

(see [FHL], Remark 4.7.1). On the other hand, the natural map

$$\begin{aligned} h : \mathrm{Hom}_V((\mathcal{M}^{W_3}_{W_1 W_2})^* \otimes W_3, W_3) &\rightarrow \mathcal{M}^{W_3}_{W_1 W_2} \\ \eta &\mapsto \overline{\eta} \circ \mathcal{F}[P(z)]^{W_3}_{W_1 W_2} \end{aligned} \tag{4.54}$$

given by composition clearly satisfies the condition that

$$h(i(I)) = I, \tag{4.55}$$

so that if W_3 is irreducible, the maps h and i are mutually inverse isomorphisms and we have the property that for any $I \in \mathcal{M}^{W_3}_{W_1 W_2}$, there exists a unique η such that

$$I = \overline{\eta} \circ \mathcal{F}[P(z)]^{W_3}_{W_1 W_2} \tag{4.56}$$

(cf. Definition 4.15).

Using this, we can now show, in the next result, that $P(z)$-tensor products always exist for the category of modules for a finitely reductive vertex operator algebra, and we shall in fact exhibit the $P(z)$-tensor product. Note that there is no need to assume that W_1 and W_2 are irreducible in the formulation or proof, but by Proposition 4.24, the case in which W_1 and W_2 are irreducible is in fact sufficient, and the tensor product operation is canonically described using only the spaces of intertwining maps among triples of *irreducible* modules.

Proposition 4.33 *Let V be a finitely reductive vertex operator algebra and let W_1 and W_2 be V-modules. Then $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ exists, and in fact*

$$W_1 \boxtimes_{P(z)} W_2 = \coprod_{i=1}^{k} (\mathcal{M}^{M_i}_{W_1 W_2})^* \otimes M_i, \tag{4.57}$$

where $\{M_1, \ldots, M_k\}$ is a set of representatives of the equivalence classes of irreducible V-modules, and the right-hand side of (4.57) is equipped with the V-module and $P(z)$-product structure indicated above. That is,

$$\boxtimes_{P(z)} = \sum_{i=1}^{k} \mathcal{F}[P(z)]^{M_i}_{W_1 W_2}. \tag{4.58}$$

Proof From the comments above and the definitions, it is clear that we have a $P(z)$-product. Let $(W_3, Y_3; I)$ be any $P(z)$-product. Then $W_3 = \coprod_j U_j$ where j ranges through a finite set and each U_j is irreducible. Let $\pi_j : W_3 \to U_j$ denote the j-th projection. A module map $\eta : \coprod_{i=1}^{k} (\mathcal{M}^{M_i}_{W_1 W_2})^* \otimes M_i \to W_3$ amounts to module maps

$$\eta_{ij} : (\mathcal{M}^{M_i}_{W_1 W_2})^* \otimes M_i \to U_j$$

for each i and j such that $U_j \simeq M_i$, and $I = \overline{\eta} \circ \boxtimes_{P(z)}$ if and only if

$$\overline{\pi}_j \circ I = \overline{\eta}_{ij} \circ \mathcal{F}^{M_i}_{W_1 W_2}$$

for each i and j, the bars having the obvious meaning. But $\overline{\pi}_j \circ I$ is a $P(z)$-intertwining map of type $\binom{U_j}{W_1 W_2}$, and so $\bar{\iota} \circ \overline{\pi}_j \circ I \in \mathcal{M}^{M_i}_{W_1 W_2}$, where $\iota : U_j \overset{\sim}{\to} M_i$ is a fixed isomorphism. Denote this map by τ. Thus what we finally want is a unique module map

$$\theta : (\mathcal{M}^{M_i}_{W_1 W_2})^* \otimes M_i \to M_i$$

such that

$$\tau = \overline{\theta} \circ \mathcal{F}[P(z)]^{M_i}_{W_1 W_2}.$$

But we in fact have such a unique θ, by (4.55)–(4.56). $\square$

Remark 4.34 By combining Proposition 4.33 with Proposition 4.8, we can express $W_1 \boxtimes_{P(z)} W_2$ in terms of $\mathcal{V}^{M_i}_{W_1 W_2}$ in place of $\mathcal{M}^{M_i}_{W_1 W_2}$.

Remark 4.35 If we know the fusion rules among triples of irreducible V-modules, then from Proposition 4.33 we know all the $P(z)$-tensor product modules, up to equivalence; that is, we know the multiplicity of each irreducible V-module in each $P(z)$-tensor product module. But recall that the $P(z)$-tensor product structure of $W_1 \boxtimes_{P(z)} W_2$ involves much more than just the V-module structure.

As we discussed in the Introduction, the main theme of this work is to construct natural "associativity" isomorphisms between triple tensor products of the shape $W_1 \boxtimes (W_2 \boxtimes W_3)$ and $(W_1 \boxtimes W_2) \boxtimes W_3$, for (generalized) modules W_1, W_2 and W_3. In the finitely reductive case, let W_1, W_2 and W_3 be V-modules. By Proposition 4.33, we have, as V-modules,

$$\begin{aligned} W_1 \boxtimes_{P(z)} (W_2 \boxtimes_{P(z)} W_3) &= W_1 \boxtimes_{P(z)} \left(\coprod_{i=1}^{k} M_i \otimes (\mathcal{M}^{M_i}_{W_2 W_3})^* \right) \\ &= \coprod_{i=1}^{k} (W_1 \boxtimes_{P(z)} M_i) \otimes (\mathcal{M}^{M_i}_{W_2 W_3})^* \\ &= \coprod_{i=1}^{k} \left(\coprod_{j=1}^{k} (\mathcal{M}^{M_j}_{W_1 M_i})^* \otimes M_j \right) \otimes (\mathcal{M}^{M_i}_{W_2 W_3})^* \\ &= \coprod_{j=1}^{k} \left(\coprod_{i=1}^{k} (\mathcal{M}^{M_j}_{W_1 M_i})^* \otimes (\mathcal{M}^{M_i}_{W_2 W_3})^* \right) \otimes M_j \\ &= \coprod_{j=1}^{k} \left(\coprod_{i=1}^{k} (\mathcal{M}^{M_j}_{W_1 M_i} \otimes \mathcal{M}^{M_i}_{W_2 W_3})^* \right) \otimes M_j \end{aligned} \tag{4.59}$$

and

$$
\begin{aligned}
(W_1 \boxtimes_{P(z)} W_2) \boxtimes_{P(z)} W_3 &= \left(\coprod_{i=1}^{k} M_i \otimes (\mathcal{M}^{M_i}_{W_1 W_2})^* \right) \boxtimes_{P(z)} W_3 \\
&= \coprod_{i=1}^{k} (M_i \boxtimes_{P(z)} W_3) \otimes (\mathcal{M}^{M_i}_{W_1 W_2})^* \\
&= \coprod_{i=1}^{k} \left(\coprod_{j=1}^{k} (\mathcal{M}^{M_j}_{M_i W_3})^* \otimes M_j \right) \otimes (\mathcal{M}^{M_i}_{W_1 W_2})^* \\
&= \coprod_{j=1}^{k} \left(\coprod_{i=1}^{k} (\mathcal{M}^{M_j}_{M_i W_3})^* \otimes (\mathcal{M}^{M_i}_{W_1 W_2})^* \right) \otimes M_j \\
&= \coprod_{j=1}^{k} \left(\coprod_{i=1}^{k} (\mathcal{M}^{M_j}_{M_i W_3} \otimes \mathcal{M}^{M_i}_{W_1 W_2})^* \right) \otimes M_j.
\end{aligned} \tag{4.60}
$$

These two V-modules will be equivalent if for each $j = 1, \dots, k$, their M_j-multiplicities are the same, that is, if

$$
\sum_{i=1}^{k} N^{M_j}_{W_1 M_i} N^{M_i}_{W_2 W_3} = \sum_{i=1}^{k} N^{M_i}_{W_1 W_2} N^{M_j}_{M_i W_3}. \tag{4.61}
$$

However, knowing only that these two V-modules are equivalent (knowing that $\boxtimes$ is "associative" in only a rough sense) is far from enough. What we need is a natural isomorphism between these two modules analogous to the natural isomorphism

$$
\mathcal{W}_1 \otimes (\mathcal{W}_2 \otimes \mathcal{W}_3) \xrightarrow{\sim} (\mathcal{W}_1 \otimes \mathcal{W}_2) \otimes \mathcal{W}_3 \tag{4.62}
$$

of vector spaces $\mathcal{W}_i$ determined by the natural condition

$$
w_{(1)} \otimes (w_{(2)} \otimes w_{(3)}) \mapsto (w_{(1)} \otimes w_{(2)}) \otimes w_{(3)} \tag{4.63}
$$

on elements (recall the Introduction). Suppose that $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$ are finite-dimensional completely reducible modules for some Lie algebra. Then we of course have the analogue of the relation (4.61). But knowing the equality of these multiplicities certainly does not give the natural isomorphism (4.62)–(4.63).

Our intent to construct a natural isomorphism between the spaces (4.59) and (4.60) (under suitable conditions) in fact provides a guide to what we need to do. In (4.59), each space $\mathcal{M}^{M_j}_{W_1 M_i} \otimes \mathcal{M}^{M_i}_{W_2 W_3}$ suggests combining an intertwining map $\mathcal{Y}_1$ of type $\binom{M_j}{W_1 M_i}$ with an intertwining map $\mathcal{Y}_2$ of type $\binom{M_i}{W_2 W_3}$, presumably by composition:

$$
\mathcal{Y}_1(w_{(1)}, z)\mathcal{Y}_2(w_{(2)}, z). \tag{4.64}
$$

But this will not work, since this composition does not exist because the relevant formal series in z does not converge; we must instead take

$$
\mathcal{Y}_1(w_{(1)}, z_1)\mathcal{Y}_2(w_{(2)}, z_2), \tag{4.65}
$$

where the complex numbers z_1 and z_2 are such that

$$|z_1| > |z_2| > 0,$$

by analogy with, and generalizing, the situation in Corollary 2.42. The composition (4.65) must be understood using convergence and "matrix coefficients," again as in Corollary 2.42.

Similarly, in (4.60), each space $\mathcal{M}^{M_j}_{M_i W_3} \otimes \mathcal{M}^{M_i}_{W_1 W_2}$ suggests combining an intertwining map $\mathcal{Y}^1$ of type $\binom{M_j}{M_i W_3}$ with an intertwining map of type $\mathcal{Y}^2$ of type $\binom{M_i}{W_1 W_2}$:

$$\mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, z_1 - z_2)w_{(2)}, z_2),$$

a (convergent) iterate of intertwining maps as in (2.117), with

$$|z_2| > |z_1 - z_2| > 0,$$

not

$$\mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, z)w_{(2)}, z), \tag{4.66}$$

which fails to converge.

The natural way to construct a natural associativity isomorphism between (4.59) and (4.60) will in fact, then, be to implement a correspondence of the type

$$\mathcal{Y}_1(w_{(1)}, z_1)\mathcal{Y}_2(w_{(2)}, z_2) = \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, z_1 - z_2)w_{(2)}, z_2), \tag{4.67}$$

as we have previewed in the Introduction (formula (1.36)) and also in (2.117). Formula (4.67) expresses the existence and associativity of the general nonmeromorphic operator product expansion, as discussed in Remark 2.44. Note that this viewpoint shows that we should not try directly to construct a natural isomorphism

$$W_1 \boxtimes_{P(z)} (W_2 \boxtimes_{P(z)} W_3) \xrightarrow{\sim} (W_1 \boxtimes_{P(z)} W_2) \boxtimes_{P(z)} W_3, \tag{4.68}$$

but rather a natural isomorphism

$$W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \xrightarrow{\sim} (W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3. \tag{4.69}$$

This is what we will actually do in this work, in the general logarithmic, not-necessarily-finitely-reductive case, under suitable conditions. The natural isomorphism (4.69) will act as follows on elements of the completions of the relevant (generalized) modules:

$$w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \mapsto (w_{(1)} \boxtimes_{P(z_1 - z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}, \tag{4.70}$$

implementing the strategy suggested by the classical natural isomorphism (4.62)–(4.63). Recall that we previewed this strategy in the Introduction.

It turns out that in order to carry out this program, including the construction of equalities of the type (4.67) (the existence and associativity of the nonmeromorphic operator product expansion) in general, we cannot use the realization of the $P(z)$-tensor product given in Proposition 4.33, *even when V is a finitely reductive vertex operator algebra.* As in [HL1]–[HL3] and [H], what we do instead is to construct $P(z)$-tensor products in a completely different way (even in the finitely reductive case), a way that allows us to also construct the natural associativity isomorphisms. Section 5 is devoted to this construction of $P(z)$- (and $Q(z)$-)tensor products.

4.2 $Q(z)$-intertwining maps and the notion of $Q(z)$-tensor product

We now generalize the notion of $Q(z)$-tensor product of modules from [HL1] to the setting of the present work, parallel to what we did for the $P(z)$-tensor product above. Here we give only the results that we will need later. Other results similar to those for $P(z)$-tensor products certainly also carry over to the case of $Q(z)$, for example, the results above on the finitely reductive case, as were presented in [HL1].

Definition 4.36 Let (W_1, Y_1), (W_2, Y_2) and (W_3, Y_3) be generalized V-modules. A $Q(z)$-*intertwining map of type* $\binom{W_3}{W_1\, W_2}$ is a linear map

$$I : W_1 \otimes W_2 \to \overline{W}_3$$

such that the following conditions are satisfied: the *grading compatibility condition*: for $\beta, \gamma \in \tilde{A}$ and $w_{(1)} \in W_1^{(\beta)}$, $w_{(2)} \in W_2^{(\gamma)}$,

$$I(w_{(1)} \otimes w_{(2)}) \in \overline{W_3^{(\beta+\gamma)}}; \tag{4.71}$$

the *lower truncation condition:* for any elements $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, and any $n \in \mathbb{C}$,

$$\pi_{n-m} I(w_{(1)} \otimes w_{(2)}) = 0 \quad \text{for} \;\; m \in \mathbb{N} \;\; \text{sufficiently large} \tag{4.72}$$

(which follows from (4.71), in view of the grading restriction condition (2.85); cf. (4.3)); the *Jacobi identity*:

$$\begin{aligned} &z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) Y_3^o(v, x_0) I(w_{(1)} \otimes w_{(2)}) \\ &\quad = x_0^{-1}\delta\left(\frac{x_1 - z}{x_0}\right) I(Y_1^o(v, x_1)w_{(1)} \otimes w_{(2)}) \\ &\qquad -x_0^{-1}\delta\left(\frac{z - x_1}{-x_0}\right) I(w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \end{aligned} \tag{4.73}$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ (recall (2.57) for the notation Y^o, and note that the left-hand side of (4.73) is meaningful because any infinite linear combination of v_n of the form $\sum_{n<N} a_n v_n$ ($a_n \in \mathbb{C}$) acts on any $I(w_{(1)} \otimes w_{(2)})$, in view of (4.72)); and the $\mathfrak{sl}(2)$-*bracket relations*: for any $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$,

$$\begin{aligned} L(-j)I(w_{(1)} \otimes w_{(2)}) \;\; = \;\; & \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i I((L(-j+i)w_{(1)}) \otimes w_{(2)}) \\ & -\sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i I(w_{(1)} \otimes L(j-i)w_{(2)}) \end{aligned} \tag{4.74}$$

for $j = -1, 0$ and 1 (note that if V is in fact a conformal vertex algebra, this follows automatically from (4.73) by setting $v = \omega$ and taking $\mathrm{Res}_{x_1}\mathrm{Res}_{x_0} x_0^{j+1}$). The vector space of $Q(z)$-intertwining maps of type $\binom{W_3}{W_1\, W_2}$ is denoted by

$$\mathcal{M}[Q(z)]_{W_1 W_2}^{W_3}.$$

Remark 4.37 As was explained in [HL1], the symbol $Q(z)$ represents the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with one negatively oriented puncture at z and two ordered positively oriented punctures at ∞ and 0, with local coordinates $w - z$, $1/w$ and w, respectively, vanishing at these punctures. In fact, this structure is conformally equivalent to the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with one negatively oriented puncture at ∞ and two ordered positively oriented punctures $1/z$ and 0, with local coordinates $z/(zw-1)$, $(zw-1)/z^2w$ and $z^2w/(zw-1)$ vanishing at ∞, $1/z$ and 0, respectively.

Remark 4.38 In the case of $\mathbb{C}$-graded ordinary modules for a vertex operator algebra, where the grading restriction condition (2.90) for a module W is replaced by the (more restrictive) condition

$$W_{(n)} = 0 \quad \text{for } \ n \in \mathbb{C} \ \text{ with sufficiently negative real part} \tag{4.75}$$

as in [HL1] (and where, in our context, the abelian groups A and $\tilde{A}$ are trivial), the notion of $Q(z)$-intertwining map above agrees with the earlier one introduced in [HL1]; in this case, the conditions (4.71) and (4.72) are automatic.

Remark 4.39 (cf. Remark 4.5) If W_3 in Definition 4.36 is lower bounded, then (4.72) can be strengthened to:

$$\pi_n I(w_{(1)} \otimes w_{(2)}) = 0 \quad \text{for } \ \Re(n) \ \text{ sufficiently negative.} \tag{4.76}$$

In view of Remarks 4.3 and 4.37, we can now give a natural correspondence between $P(z)$- and $Q(z)$-intertwining maps. (See the next three results.) Recall that since our generalized V-modules are strongly graded, we have contragredient generalized modules of generalized modules.

Proposition 4.40 *Let $I : W_1 \otimes W_2 \to \overline{W}_3$ and $J : W_3' \otimes W_2 \to \overline{W_1'}$ be linear maps related to each other by:*

$$\langle w_{(1)}, J(w'_{(3)} \otimes w_{(2)})\rangle = \langle w'_{(3)}, I(w_{(1)} \otimes w_{(2)})\rangle \tag{4.77}$$

for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W_3'$. Then I is a $Q(z)$-intertwining map of type $\binom{W_3}{W_1\,W_2}$ if and only if J is a $P(z)$-intertwining map of type $\binom{W_1'}{W_3'\,W_2}$.

Proof Suppose that I is a $Q(z)$-intertwining map of type $\binom{W_3}{W_1\,W_2}$. We shall show that J is a $P(z)$-intertwining map of type $\binom{W_1'}{W_3'\,W_2}$.

Since I satisfies the grading compatibility condition, it is clear that J also satisfies this condition. For the lower truncation condition for J, it suffices to show that for any $w_{(2)} \in W_2^{(\beta)}$ and $w'_{(3)} \in (W_3')^{(\gamma)}$, where $\beta, \gamma \in \tilde{A}$, and any $n \in \mathbb{C}$,

$$\langle \pi_{[n-m]} W_1^{(-\beta-\gamma)}, J(w'_{(3)} \otimes w_{(2)})\rangle = 0$$

for $m \in \mathbb{N}$ sufficiently large, or that

$$\langle w'_{(3)}, I(\pi_{[n-m]} W_1^{(-\beta-\gamma)} \otimes w_{(2)})\rangle = 0 \quad \text{for } \ m \in \mathbb{N} \ \text{ sufficiently large.} \tag{4.78}$$

But (4.78) follows immediately from (2.85).

Now we prove the Jacobi identity for J. The Jacobi identity for I gives

$$\begin{aligned}
&z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)\langle w'_{(3)}, Y_3^o(v,x_0)I(w_{(1)}\otimes w_{(2)})\rangle\\
&\quad = x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right)\langle w'_{(3)}, I(Y_1^o(v,x_1)w_{(1)}\otimes w_{(2)})\rangle\\
&\quad\quad -x_0^{-1}\delta\left(\frac{z-x_1}{-x_0}\right)\langle w'_{(3)}, I(w_{(1)}\otimes Y_2(v,x_1)w_{(2)})\rangle
\end{aligned}\tag{4.79}$$

for any $v\in V$, $w_{(1)}\in W_1$, $w_{(2)}\in W_2$ and $w'_{(3)}\in W'_3$. By (2.73) the left-hand side is equal to

$$z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)\langle Y'_3(v,x_0)w'_{(3)}, I(w_{(1)}\otimes w_{(2)})\rangle$$

So by (4.77), the identity (4.79) can be written as

$$\begin{aligned}
&z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)\langle w_{(1)}, J(Y'_3(v,x_0)w'_{(3)}\otimes w_{(2)})\rangle\\
&\quad = x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right)\langle Y_1^o(v,x_1)w_{(1)}, J(w'_{(3)}\otimes w_{(2)})\rangle\\
&\quad\quad -x_0^{-1}\delta\left(\frac{z-x_1}{-x_0}\right)\langle w_{(1)}, J(w'_{(3)}\otimes Y_2(v,x_1)w_{(2)})\rangle.
\end{aligned}$$

Applying (2.73) to the first term of the right-hand side we see that this can be written as

$$\begin{aligned}
&z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)\langle w_{(1)}, J(Y'_3(v,x_0)w'_{(3)}\otimes w_{(2)})\rangle\\
&\quad = x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right)\langle w_{(1)}, Y'_1(v,x_1)J(w'_{(3)}\otimes w_{(2)})\rangle\\
&\quad\quad -x_0^{-1}\delta\left(\frac{z-x_1}{-x_0}\right)\langle w_{(1)}, J(w'_{(3)}\otimes Y_2(v,x_1)w_{(2)})\rangle
\end{aligned}$$

for any $v\in V$, $w_{(1)}\in W_1$, $w_{(2)}\in W_2$ and $w'_{(3)}\in W'_3$. This is exactly the Jacobi identity for J.

The $\mathfrak{sl}(2)$-bracket relations can be proved similarly, as follows: The $\mathfrak{sl}(2)$-bracket relations for I give

$$\begin{aligned}
\langle w'_{(3)}, L(-j)I(w_{(1)}\otimes w_{(2)})\rangle &= \sum_{i=0}^{j+1}\binom{j+1}{i}(-z)^i\langle w'_{(3)}, I((L(-j+i)w_{(1)})\otimes w_{(2)})\rangle\\
&\quad -\sum_{i=0}^{j+1}\binom{j+1}{i}(-z)^i\langle w'_{(3)}, I(w_{(1)}\otimes L(j-i)w_{(2)})\rangle
\end{aligned}$$

for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w'_{(3)} \in W'_3$ and $j = -1, 0, 1$. Using (2.75) and then applying (4.77) we get

$$\begin{aligned}\langle w_{(1)}, J(L'(j)w'_{(3)} \otimes w_{(2)})\rangle &= \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i \langle L(-j+i)w_{(1)}, J(w'_{(3)} \otimes w_{(2)})\rangle \\ &\quad - \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i \langle w_{(1)}, J(w'_{(3)} \otimes L(j-i)w_{(2)})\rangle,\end{aligned}$$

or

$$\begin{aligned}J(L'(j)w'_{(3)} \otimes w_{(2)}) &= \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i L(j-i) J(w'_{(3)} \otimes w_{(2)}) \\ &\quad - \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i J(w'_{(3)} \otimes L(j-i)w_{(2)}),\end{aligned}$$

for $j = -1, 0, 1$. This is the alternative form (4.8) of the $\mathfrak{sl}(2)$-bracket relations for J. Hence J is a $P(z)$-intertwining map.

The other direction of the proposition is proved by simply reversing the order of the arguments. $\square$

Let W_1, W_2 and W_3 be generalized V-modules, as above. We shall call an element λ of $(W_1 \otimes W_2 \otimes W_3)^*$ *$\tilde{A}$-compatible* if

$$\lambda((W_1)^{(\beta)} \otimes (W_2)^{(\gamma)} \otimes (W_3)^{(\delta)}) = 0$$

for $\beta, \gamma, \delta \in \tilde{A}$ satisfying

$$\beta + \gamma + \delta \neq 0.$$

Recall from Definitions 2.18 and 2.32 that for a generalized V-module W, $\overline{W'}$ can be viewed as a (usually proper) subspace of W^*. We shall call a linear map

$$I : W_1 \otimes W_2 \to W_3^*$$

$\tilde{A}$-compatible if its image lies in $\overline{W'_3}$, that is,

$$I : W_1 \otimes W_2 \to \overline{W'_3}, \tag{4.80}$$

and if I satisfies the usual grading compatibility condition (4.2) or (4.71) for $P(z)$- or $Q(z)$-intertwining maps. Now an element λ of $(W_1 \otimes W_2 \otimes W_3)^*$ amounts exactly to a linear map

$$I_\lambda : W_1 \otimes W_2 \to W_3^*.$$

If λ is $\tilde{A}$-compatible, then for $w_{(1)} \in W_1^{(\beta)}$, $w_{(2)} \in W_2^{(\gamma)}$ and $w_{(3)} \in W_3^{(\delta)}$ such that

$$\delta \neq -(\beta + \gamma),$$

we have

$$\langle w_{(3)}, I_\lambda(w_{(1)} \otimes w_{(2)})\rangle = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) = 0,$$

so that

$$I_\lambda(w_{(1)} \otimes w_{(2)}) \in \overline{(W_3')^{(\beta+\gamma)}}$$

and I_λ is $\tilde{A}$-compatible. Similarly, if I_λ is $\tilde{A}$-compatible, then so is λ. Thus we have the following straightforward result relating $\tilde{A}$-compatibility of λ with that of I_λ:

Lemma 4.41 *The linear functional $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ is $\tilde{A}$-compatible if and only if I_λ is $\tilde{A}$-compatible. The map given by $\lambda \mapsto I_\lambda$ is the unique linear isomorphism from the space of $\tilde{A}$-compatible elements of $(W_1 \otimes W_2 \otimes W_3)^*$ to the space of $\tilde{A}$-compatible linear maps from $W_1 \otimes W_2$ to $\overline{W_3'}$ such that*

$$\langle w_{(3)}, I_\lambda(w_{(1)} \otimes w_{(2)})\rangle = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. Similarly, there are canonical linear isomorphisms from the space of $\tilde{A}$-compatible elements of $(W_1 \otimes W_2 \otimes W_3)^*$ to the space of $\tilde{A}$-compatible linear maps from $W_1 \otimes W_3$ to $\overline{W_2'}$ and to the space of $\tilde{A}$-compatible linear maps from $W_2 \otimes W_3$ to $\overline{W_1'}$ satisfying the corresponding conditions. In particular, there is a canonical linear isomorphism from the space of $\tilde{A}$-compatible linear maps from $W_1 \otimes W_2$ to $\overline{W_3}$ to the space of $\tilde{A}$-compatible linear maps from $W_3' \otimes W_2$ to $\overline{W_1'}$ given by (4.77). $\square$

Using this lemma and Proposition 4.40, we have:

Corollary 4.42 *The formula (4.77) gives a canonical linear isomorphism between the space of $Q(z)$-intertwining maps of type $\binom{W_3}{W_1\,W_2}$ and the space of $P(z)$-intertwining maps of type $\binom{W_1'}{W_3'\,W_2}$.* $\square$

Remark 4.43 If the generalized modules under consideration are lower bounded, then the spaces of intertwining maps satisfy the stronger conditions (4.7) and (4.76).

We can now use Proposition 4.8 together with Proposition 4.40 and Corollary 4.42 to construct a correspondence between the logarithmic intertwining operators of type $\binom{W_1'}{W_3'W_2}$ and the $Q(z)$-intertwining maps of type $\binom{W_3}{W_1W_2}$; this generalizes the corresponding result in the finitely reductive case, with ordinary modules, in [HL1]. Fix an integer p. Let $\mathcal{Y}$ be a logarithmic intertwining operator of type $\binom{W_1'}{W_3'W_2}$, and use (4.15) to define a linear map

$$I_{\mathcal{Y},p} : W_3' \otimes W_2 \to \overline{W_1'};$$

by Proposition 4.8, this is a $P(z)$-intertwining map of the same type. Then use Proposition 4.40 and Corollary 4.42 to define a $Q(z)$-intertwining map

$$I^{Q(z)}_{\mathcal{Y},p} : W_1 \otimes W_2 \to \overline{W}_3$$

of type $\binom{W_3}{W_1 W_2}$ (uniquely) by

$$\begin{aligned}\langle w'_{(3)}, I^{Q(z)}_{\mathcal{Y},p}(w_{(1)}\otimes w_{(2)})\rangle_{W_3} &= \langle w_{(1)}, I_{\mathcal{Y},p}(w'_{(3)}\otimes w_{(2)})\rangle_{W'_1}\\ &= \langle w_{(1)}, \mathcal{Y}(w'_{(3)}, e^{l_p(z)})w_{(2)}\rangle_{W'_1}\end{aligned} \tag{4.81}$$

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w'_{(3)} \in W'_3$. (We are using the symbol $Q(z)$ to distinguish this from the $P(z)$ case above.) Then the correspondence

$$\mathcal{Y} \mapsto I^{Q(z)}_{\mathcal{Y},p}$$

is an isomorphism from $\mathcal{V}^{W'_1}_{W'_3 W_2}$ to $\mathcal{M}[Q(z)]^{W_3}_{W_1 W_2}$. From Proposition 4.8 and (4.18), its inverse is given by sending a $Q(z)$-intertwining map I of type $\binom{W_3}{W_1 W_2}$ to the logarithmic intertwining operator

$$\mathcal{Y}^{Q(z)}_{I,p} : W'_3 \otimes W_2 \to W'_1[\log x]\{x\}$$

defined by

$$\begin{aligned}&\langle w_{(1)}, \mathcal{Y}^{Q(z)}_{I,p}(w'_{(3)}, x)w_{(2)}\rangle_{W'_1}\\ &\quad = \langle y^{-L'(0)}x^{-L'(0)}w'_{(3)}, I(y^{L(0)}x^{L(0)}w_{(1)} \otimes y^{-L(0)}x^{-L(0)}w_{(2)})\rangle_{W_3}\Big|_{y=e^{-l_p(z)}}\end{aligned}$$

for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$. Thus we have:

Proposition 4.44 *For $p \in \mathbb{Z}$, the correspondence*

$$\mathcal{Y} \mapsto I^{Q(z)}_{\mathcal{Y},p}$$

is a linear isomorphism from the space $\mathcal{V}^{W'_1}_{W'_3 W_2}$ of logarithmic intertwining operators of type $\binom{W'_1}{W'_3\, W_2}$ to the space $\mathcal{M}[Q(z)]^{W_3}_{W_1 W_2}$ of $Q(z)$-intertwining maps of type $\binom{W_3}{W_1 W_2}$. Its inverse is given by

$$I \mapsto \mathcal{Y}^{Q(z)}_{I,p}.$$

□

Remark 4.45 If the generalized modules under consideration are lower bounded, then the stronger conditions (3.43) and (4.76) hold.

We now give the definition of $Q(z)$-tensor product.

Definition 4.46 Let $\mathcal{C}_1$ be either $\mathcal{M}_{sg}$ or $\mathcal{GM}_{sg}$. For $W_1, W_2 \in \mathrm{ob}\,\mathcal{C}_1$, a *$Q(z)$-product of W_1 and W_2* is an object (W_3, Y_3) of $\mathcal{C}_1$ together with a $Q(z)$-intertwining map I_3 of type $\binom{W_3}{W_1 W_2}$. We denote it by $(W_3, Y_3; I_3)$ or simply by (W_3, I_3). Let $(W_4, Y_4; I_4)$ be another $Q(z)$-product of W_1 and W_2. A *morphism* from $(W_3, Y_3; I_3)$ to $(W_4, Y_4; I_4)$ is a module map η from W_3 to W_4 such that the diagram

$$\begin{tikzcd}
& W_1 \otimes W_2 \arrow[dl, "I_3"'] \arrow[dr, "I_4"] & \\
\overline{W}_3 \arrow[rr, "\overline{\eta}"] & & \overline{W}_4
\end{tikzcd}$$

commutes, that is,

$$I_4 = \overline{\eta} \circ I_3.$$

where, as before, $\overline{\eta}$ is the natural map from $\overline{W}_3$ to $\overline{W}_4$ extending η.

Definition 4.47 Let $\mathcal{C}$ be a full subcategory of either $\mathcal{M}_{sg}$ or $\mathcal{GM}_{sg}$. For $W_1, W_2 \in \mathrm{ob}\,\mathcal{C}$, a *$Q(z)$-tensor product of W_1 and W_2 in $\mathcal{C}$* is a $Q(z)$-product $(W_0, Y_0; I_0)$ with $W_0 \in \mathrm{ob}\,\mathcal{C}$ such that for any $Q(z)$-product $(W, Y; I)$ with $W \in \mathrm{ob}\,\mathcal{C}$, there is a unique morphism from $(W_0, Y_0; I_0)$ to $(W, Y; I)$. Clearly, a $Q(z)$-tensor product of W_1 and W_2 in $\mathcal{C}$, if it exists, is unique up to unique isomorphism. In this case we will denote it by

$$(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; \boxtimes_{Q(z)})$$

and call the object

$$(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)})$$

the *$Q(z)$-tensor product (generalized) module of W_1 and W_2 in $\mathcal{C}$*. Again we will skip the phrase "in $\mathcal{C}$" if the category $\mathcal{C}$ under consideration is clear in context.

The following immediate consequence of Definition 4.47 and Proposition 4.44 relates module maps from a $Q(z)$-tensor product module with $Q(z)$-intertwining maps and logarithmic intertwining operators:

Proposition 4.48 *Suppose that $W_1 \boxtimes_{Q(z)} W_2$ exists. We have a natural isomorphism*

$$\begin{array}{rcl} \mathrm{Hom}_V(W_1 \boxtimes_{Q(z)} W_2, W_3) & \xrightarrow{\sim} & \mathcal{M}[Q(z)]_{W_1 W_2}^{W_3} \\ \eta & \mapsto & \overline{\eta} \circ \boxtimes_{Q(z)} \end{array}$$

and for $p \in \mathbb{Z}$, a natural isomorphism

$$\begin{array}{rcl} \mathrm{Hom}_V(W_1 \boxtimes_{Q(z)} W_2, W_3) & \xrightarrow{\sim} & \mathcal{V}_{W_3' W_2}^{W_1'} \\ \eta & \mapsto & \mathcal{Y}_{\eta,p}^{Q(z)} \end{array}$$

where $\mathcal{Y}_{\eta,p}^{Q(z)} = \mathcal{Y}_{I,p}^{Q(z)}$ with $I = \overline{\eta} \circ \boxtimes_{Q(z)}$. $\square$

Suppose that the $Q(z)$-tensor product $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; \boxtimes_{Q(z)})$ of W_1 and W_2 exists. We will sometimes denote the action of the canonical $Q(z)$-intertwining map

$$w_{(1)} \otimes w_{(2)} \mapsto \boxtimes_{Q(z)}(w_{(1)} \otimes w_{(2)}) = \boxtimes_{Q(z)}(w_{(1)}, z)w_{(2)} \in \overline{W_1 \boxtimes_{Q(z)} W_2} \tag{4.82}$$

on elements simply by $w_{(1)} \boxtimes_{Q(z)} w_{(2)}$:

$$w_{(1)} \boxtimes_{Q(z)} w_{(2)} = \boxtimes_{Q(z)}(w_{(1)} \otimes w_{(2)}) = \boxtimes_{Q(z)}(w_{(1)}, z)w_{(2)}. \tag{4.83}$$

Using Propositions 3.44 and 3.46, we have the following result, generalizing Proposition 4.9 and Corollary 4.10 in [HL1]:

Proposition 4.49 *For any integer r, there is a natural isomorphism*

$$B_r : \mathcal{V}_{W_1W_2}^{W_3} \to \mathcal{V}_{W_3'W_2}^{W_1'}$$

defined by the condition that for any logarithmic intertwining operator $\mathcal{Y}$ in $\mathcal{V}_{W_1W_2}^{W_3}$ and $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w'_{(3)} \in W'_3$,

$$\begin{aligned}\langle w_{(1)}, B_r(\mathcal{Y})(w'_{(3)}, x)w_{(2)}\rangle_{W'_1} \\ = \langle e^{-x^{-1}L(1)}w'_{(3)}, \mathcal{Y}(e^{xL(1)}w_{(1)}, x^{-1})e^{-xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2w_{(2)}\rangle_{W_3}.\end{aligned} \tag{4.84}$$

Proof From Proposition 3.44, for any integer r_1 we have an isomorphism Ω_{r_1} from $\mathcal{V}_{W_1W_2}^{W_3}$ to $\mathcal{V}_{W_2W_1}^{W_3}$, and from Proposition 3.46, for any integer r_2 we have an isomorphism A_{r_2} from $\mathcal{V}_{W_2W_1}^{W_3}$ to $\mathcal{V}_{W_2W'_3}^{W'_1}$. By Proposition 3.44 again, for any integer r_3 there is an isomorphism, which we again denote Ω_{r_3}, from $\mathcal{V}_{W_2W'_3}^{W'_1}$ to $\mathcal{V}_{W'_3W_2}^{W'_1}$. Thus for any triple (r_1, r_2, r_3) of integers, we have an isomorphism $\Omega_{r_3} \circ A_{r_2} \circ \Omega_{r_1}$ from $\mathcal{V}_{W_1W_2}^{W_3}$ to $\mathcal{V}_{W'_3W_2}^{W'_1}$. Let $\mathcal{Y}$ be a logarithmic intertwining operator in $\mathcal{V}_{W_1W_2}^{W_3}$ and $w_{(1)}$, $w_{(2)}$, $w'_{(3)}$ elements of W_1, W_2, W'_3, respectively. From the definitions of Ω_{r_1}, A_{r_2} and Ω_{r_3}, we have

$$\begin{aligned}&\langle(\Omega_{r_3} \circ A_{r_2} \circ \Omega_{r_1})(\mathcal{Y})(w'_{(3)}, x)w_{(2)}, w_{(1)}\rangle_{W_1} = \\ &\quad = \langle e^{xL(-1)}A_{r_2}(\Omega_{r_1}(\mathcal{Y}))(w_{(2)}, e^{(2r_3+1)\pi i}x)w'_{(3)}, w_{(1)}\rangle_{W_1} \\ &\quad = \langle A_{r_2}(\Omega_{r_1}(\mathcal{Y}))(w_{(2)}, e^{(2r_3+1)\pi i}x)w'_{(3)}, e^{xL(1)}w_{(1)}\rangle_{W_1} \\ &\quad = \langle w'_{(3)}, \Omega_{r_1}(\mathcal{Y})(e^{-xL(1)}e^{(2r_2+1)\pi iL(0)}e^{-2(2r_3+1)\pi iL(0)}(x^{-L(0)})^2w_{(2)}, \\ &\qquad\qquad e^{-(2r_3+1)\pi i}x^{-1})e^{xL(1)}w_{(1)}\rangle_{W_3} \\ &\quad = \langle w'_{(3)}, e^{-x^{-1}L(-1)}\mathcal{Y}(e^{xL(1)}w_{(1)}, e^{(2r_1+1)\pi i}e^{-(2r_3+1)\pi i}x^{-1}) \cdot \\ &\qquad\qquad \cdot e^{-xL(1)}e^{(2r_2+1)\pi iL(0)}e^{-2(2r_3+1)\pi iL(0)}(x^{-L(0)})^2w_{(2)}\rangle_{W_3} \\ &\quad = \langle e^{-x^{-1}L(1)}w'_{(3)}, \mathcal{Y}(e^{xL(1)}w_{(1)}, e^{2(r_1-r_3)\pi i}x^{-1}) \cdot \\ &\qquad\qquad \cdot e^{-xL(1)}e^{(2(r_2-2r_3-1)+1)\pi iL(0)}(x^{-L(0)})^2w_{(2)}\rangle_{W_3}.\end{aligned} \tag{4.85}$$

From (4.85) we see that $\Omega_{r_3} \circ A_{r_2} \circ \Omega_{r_1}$ depends only on $r_2 - 2r_3 - 1$ and $r_1 - r_3$, and the operators $\Omega_{r_3} \circ A_{r_2} \circ \Omega_{r_1}$ with different $r_1 - r_3$ but the same $r_2 - 2r_3 - 1$ differ from each other only by automorphisms of $\mathcal{V}_{W_1W_2}^{W_3}$ (recall Remarks 3.30, 3.40 and 3.45). Thus for our purpose, we need only consider those isomorphisms such that $r_1 - r_3 = 0$. Given any integer r, we choose two integers r_2 and r_3 such that $r = r_2 - 2r_3 - 1$ and we define

$$B_r = \Omega_{r_3} \circ A_{r_2} \circ \Omega_{r_3}. \tag{4.86}$$

From (4.85) we see that B_r is independent of the choices of r_2 and r_3 and that (4.84) holds.
□

Combining the last two results, we obtain:

Corollary 4.50 *For any $W_1, W_2, W_3 \in \mathrm{ob}\,\mathcal{C}$ such that $W_1 \boxtimes_{Q(z)} W_2$ exists and any integers p and r, we have a natural isomorphism*

$$\begin{array}{rcl} \mathrm{Hom}_V(W_1 \boxtimes_{Q(z)} W_2, W_3) & \overset{\sim}{\to} & \mathcal{V}_{W_1 W_2}^{W_3} \\ \eta & \mapsto & B_r^{-1}(\mathcal{Y}_{\eta,p}^{Q(z)}). \qquad \square \end{array} \tag{4.87}$$

4.3 $P(z)$-tensor products and $Q(z^{-1})$-tensor products

Here we prove the following result:

Theorem 4.51 *Let W_1 and W_2 be objects of a full subcategory $\mathcal{C}$ of either $\mathcal{M}_{sg}$ or $\mathcal{GM}_{sg}$. Then the $P(z)$-tensor product of W_1 and W_2 exists if and only if the $Q(z^{-1})$-tensor product of W_1 and W_2 exists.*

Proof Recalling our choice of branch (4.9), let

$$p = -\frac{\log(z^{-1}) + \log z}{2\pi i}.$$

Then p is an integer and we have

$$-(\log(z^{-1}) + 2\pi p i) = \log z,$$

and

$$e^{-n(\log(z^{-1})+2\pi p i)} = e^{n \log z}$$

for $n \in \mathbb{C}$.

From Propositions 4.8, 4.49 and 4.44, we see that for $W_1, W_2, W_3 \in \mathrm{ob}\,\mathcal{C}$, there is a linear isomorphism $\mu_{W_1 W_2}^{W_3} : \mathcal{M}[P(z)]_{W_1 W_2}^{W_3} \to \mathcal{M}[Q(z^{-1})]_{W_1 W_2}^{W_3}$ defined by

$$\mu_{W_1 W_2}^{W_3}(I) = I_{B_{2p}(\mathcal{Y}_{I,0}),p}^{Q(z^{-1})}$$

for $I \in \mathcal{M}[P(z)]_{W_1 W_2}^{W_3}$. By definition, $\mu_{W_1 W_2}^{W_3}(I)$ is determined uniquely by (recalling (4.9)–(4.10))

$$\begin{aligned} &\langle w'_{(3)}, \mu_{W_1 W_2}^{W_3}(I)(w_{(1)} \otimes w_{(2)})\rangle \\ &\quad = \langle w'_{(3)}, I_{B_{2p}(\mathcal{Y}_{I,0}),p}^{Q(z^{-1})}(w_{(1)} \otimes w_{(2)})\rangle \\ &\quad = \langle w_{(1)}, B_{2p}(\mathcal{Y}_{I,0})(w'_{(3)}, e^{l_p(z^{-1})})w_{(2)}\rangle \\ &\quad = \langle w_{(1)}, B_{2p}(\mathcal{Y}_{I,0})(w'_{(3)}, e^{\log(z^{-1})+2\pi p i})w_{(2)}\rangle \\ &\quad = \langle e^{-zL(1)}w'_{(3)}, \mathcal{Y}_{I,0}(e^{z^{-1}L(1)}w_{(1)}, e^{\log z})e^{-z^{-1}L(1)}e^{(2(2p)+1)i\pi L(0)}e^{-2(\log z^{-1}+2\pi p i))L(0)}w_{(2)}\rangle \\ &\quad = \langle e^{-zL(1)}w'_{(3)}, \mathcal{Y}_{I,0}(e^{z^{-1}L(1)}w_{(1)}, z)e^{-z^{-1}L(1)}e^{i\pi L(0)}e^{-2(\log z^{-1})L(0)}w_{(2)}\rangle \\ &\quad = \langle e^{-zL(1)}w'_{(3)}, I((e^{z^{-1}L(1)}w_{(1)}) \otimes (e^{-z^{-1}L(1)}e^{i\pi L(0)}e^{-2(\log z^{-1})L(0)}w_{(2)}))\rangle \end{aligned} \tag{4.88}$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$. From (4.88), we also see that for $J \in \mathcal{M}[Q(z^{-1})]_{W_1 W_2}^{W_3}$, $(\mu_{W_1W_2}^{W_3})^{-1}(J)$ is determined uniquely by

$$\begin{aligned}
&\langle w'_{(3)}, (\mu_{W_1W_2}^{W_3})^{-1}(J)(w_{(1)} \otimes w_{(2)})\rangle \\
&\quad = \langle e^{zL(1)} w'_{(3)}, J((e^{-z^{-1}L(1)} w_{(1)}) \otimes (e^{2(\log z^{-1})L(0)} e^{-i\pi L(0)} e^{z^{-1}L(1)} w_{(2)}))\rangle
\end{aligned} \tag{4.89}$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$.

Assume that the $P(z)$-tensor product $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ exists. Then

$$\boxtimes_{Q(z^{-1})} = \mu_{W_1W_2}^{W_1 \boxtimes_{P(z)} W_2}(\boxtimes_{P(z)}) = I^{Q(z^{-1})}_{B_{2p}(\mathcal{Y}_{\boxtimes_{P(z)},0}),p}$$

is a $Q(z^{-1})$-intertwining map of type $\binom{W_1 \boxtimes_{P(z)} W_2}{W_1 W_2}$. We claim that $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{Q(z^{-1})})$ is the $Q(z^{-1})$-tensor product of W_1 and W_2.

In fact, for any $Q(z^{-1})$-product $(W, Y; I)$ of W_1 and W_2,

$$(\mu_{W_1W_2}^{W})^{-1}(I) = I_{B_{2p}^{-1}(\mathcal{Y}_{I,p}^{Q(z^{-1})}),0}$$

is a $P(z)$-intertwining map of type $\binom{W}{W_1 W_2}$ and thus $(W, Y; (\mu_{W_1W_2}^{W})^{-1}(I))$ is a $P(z)$-product of W_1 and W_2. Since $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ is the $P(z)$-tensor product of W_1 and W_2, there is a unique morphism of $P(z)$-products from $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ to $(W, Y; (\mu_{W_1W_2}^{W})^{-1}(I))$, that is, there exists a unique module map

$$\eta^{P(z)} : W_1 \boxtimes_{P(z)} W_2 \to W$$

such that

$$(\mu_{W_1W_2}^{W})^{-1}(I) = \overline{\eta^{P(z)}} \circ \boxtimes_{P(z)},$$

or equivalently,

$$\begin{aligned}
I &= \mu_{W_1W_2}^{W}(\overline{\eta^{P(z)}} \circ \boxtimes_{P(z)}) \\
&= \mu_{W_1W_2}^{W}(\overline{\eta^{P(z)}} \circ (\mu_{W_1W_2}^{W_1 \boxtimes_{P(z)} W_2})^{-1}(\mu_{W_1W_2}^{W_1 \boxtimes_{P(z)} W_2}(\boxtimes_{P(z)}))) \\
&= \mu_{W_1W_2}^{W}(\overline{\eta^{P(z)}} \circ (\mu_{W_1W_2}^{W_1 \boxtimes_{P(z)} W_2})^{-1}(\boxtimes_{Q(z^{-1})})).
\end{aligned} \tag{4.90}$$

From (4.88) and (4.89), we see that the right-hand side of (4.90) is determined uniquely by

$$\begin{aligned}
&\langle w', (\mu_{W_1W_2}^{W}(\overline{\eta^{P(z)}} \circ (\mu_{W_1W_2}^{W_1 \boxtimes_{P(z)} W_2})^{-1}(\boxtimes_{Q(z^{-1})})))(w_{(1)} \otimes w_{(2)})\rangle \\
&\quad = \langle e^{-zL(1)} w', (\overline{\eta^{P(z)}} \circ (\mu_{W_1W_2}^{W_1 \boxtimes_{P(z)} W_2})^{-1}(\boxtimes_{Q(z^{-1})})) \\
&\qquad\qquad ((e^{z^{-1}L(1)} w_{(1)}) \otimes (e^{-z^{-1}L(1)} e^{i\pi L(0)} e^{-2(\log z^{-1})L(0)} w_{(2)}))\rangle \\
&\quad = \langle e^{-zL(1)} w', \overline{\eta^{P(z)}}((\mu_{W_1W_2}^{W_1 \boxtimes_{P(z)} W_2})^{-1}(\boxtimes_{Q(z^{-1})})
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad ((e^{z^{-1}L(1)}w_{(1)}) \otimes (e^{-z^{-1}L(1)}e^{i\pi L(0)}e^{-2(\log z^{-1})L(0)}w_{(2)})))\rangle\\
&= \langle (\eta^{P(z)})'(e^{-zL(1)}w'), (\mu_{W_1W_2}^{W_1\boxtimes_{P(z)}W_2})^{-1}(\boxtimes_{Q(z^{-1})})\\
&\qquad\qquad ((e^{z^{-1}L(1)}w_{(1)}) \otimes (e^{-z^{-1}L(1)}e^{i\pi L(0)}e^{-2(\log z^{-1})L(0)}w_{(2)})))\rangle\\
&= \langle (\eta^{P(z)})'(w'), \boxtimes_{Q(z^{-1})}(w_{(1)} \otimes w_{(2)})\rangle\\
&= \langle w', (\overline{\eta^{P(z)}} \circ \boxtimes_{Q(z^{-1})})(w_{(1)} \otimes w_{(2)})\rangle
\end{aligned}
\tag{4.91}
$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w' \in W'$. From (4.90) and (4.91), we see that

$$
I = \overline{\eta^{P(z)}} \circ \boxtimes_{Q(z^{-1})}. \tag{4.92}
$$

We also need to show the uniqueness—that any module map $\eta : W_1 \boxtimes_{P(z)} W_2 \to W$ such that $I = \overline{\eta} \circ \boxtimes_{Q(z^{-1})}$ must be equal to $\eta^{P(z)}$. For this, it is sufficient to show that $\eta_1 = 0$, where

$$
\eta_1 = \eta^{P(z)} - \eta,
$$

given that

$$
\overline{\eta_1}(w_{(1)} \boxtimes_{Q(z^{-1})} w_{(2)}) = 0
$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. But for $w' \in (W_1 \boxtimes_{P(z)} W_2)'$

$$
\langle e^{zL(1)}w', \overline{\eta_1}(w_{(1)} \boxtimes_{Q(z^{-1})} w_{(2)})\rangle = 0,
$$

so that

$$
\langle e^{zL(1)}{\eta_1}'(w'), w_{(1)} \boxtimes_{Q(z^{-1})} w_{(2)}\rangle = \langle {\eta_1}'(e^{zL(1)}w'), w_{(1)} \boxtimes_{Q(z^{-1})} w_{(2)}\rangle = 0.
$$

From the definition of $\boxtimes_{Q(z)}$ and (4.88), we have

$$
\begin{aligned}
&\langle e^{zL(1)}{\eta_1}'(w'), w_{(1)} \boxtimes_{Q(z^{-1})} w_{(2)}\rangle\\
&\quad = \langle {\eta_1}'(w'), (e^{z^{-1}L(1)}w_{(1)}) \boxtimes_{P(z)} (e^{-z^{-1}L(1)}e^{i\pi L(0)}e^{-2(\log z^{-1})L(0)}w_{(2)})\rangle,
\end{aligned}
$$

and thus

$$
\langle {\eta_1}'(w'), (e^{z^{-1}L(1)}w_{(1)}) \boxtimes_{P(z)} (e^{-z^{-1}L(1)}e^{i\pi L(0)}e^{-2(\log z^{-1})L(0)}w_{(2)})\rangle = 0. \tag{4.93}
$$

Since $e^{z^{-1}L(1)}$ and $e^{-z^{-1}L(1)}e^{i\pi L(0)}e^{-2(\log z^{-1})L(0)}$ are invertible operators on W_1 and W_2, (4.93) for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ is equivalent to

$$
\langle {\eta_1}'(w'), w_{(1)} \boxtimes_{P(z)} w_{(2)}\rangle = 0
$$

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$. Thus by Proposition 4.23,

$$
{\eta_1}'(w') = 0
$$

for all homogeneous w' and hence for all w', showing that indeed $\eta_1 = 0$ and proving the uniqueness of η. Thus $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{Q(z^{-1})})$ is the $Q(z^{-1})$-tensor product of W_1 and W_2.

Conversely, by essentially reversing these arguments we see that if the $Q(z^{-1})$-tensor product of W_1 and W_2 exists, then so does the $P(z)$-tensor product. $\square$

From Theorem 4.51 and Proposition 4.21, we immediately obtain:

Corollary 4.52 *Let W_1 and W_2 be objects of a full subcategory $\mathcal{C}$ of either $\mathcal{M}_{sg}$ or $\mathcal{GM}_{sg}$. Then the $P(z)$-tensor product of W_1 and W_2 exists if and only if the $Q(z)$-tensor product of W_1 and W_2 exists.* $\square$

Remark 4.53 From the proof we see that as generalized V-modules, $W_1 \boxtimes_{P(z)} W_2$ and $W_1 \boxtimes_{Q(z^{-1})} W_2$ are equivalent, but the main issue is that the intertwining maps $\boxtimes_{P(z)}$ and $\boxtimes_{Q(z^{-1})}$, which encode the geometric information, are very different; as generalized V-modules *only*, $W_1 \boxtimes_{P(z)} W_2$ and $W_1 \boxtimes_{Q(z)} W_2$ are equivalent. Compare this with Remark 4.22.

References

[FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, preprint, 1989; *Memoirs Amer. Math. Soc.* **104**, 1993.

[H] Y.-Z. Huang, A theory of tensor products for module categories for a vertex operator algebra, IV, *J. Pure Appl. Alg.* 100 (1995) 173–216.

[HL1] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, I, *Selecta Mathematica (New Series)* **1** (1995), 699–756.

[HL2] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, II, *Selecta Mathematica (New Series)* **1** (1995), 757–786.

[HL3] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, III, *J. Pure Appl. Alg.* **100** (1995) 141–171.

[HLZ1] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory for generalized modules for a conformal vertex algebra, I: Introduction and strongly graded algebras and their generalized modules, to appear.

[HLZ2] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, II: Logarithmic formal calculus and properties of logarithmic intertwining operators, to appear.

[HLZ3] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, IV: Constructions of tensor product bifunctors and the compatibility conditions, to appear.

[HLZ4] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, V: Convergence condition for intertwining maps and the corresponding compatibility condition, to appear.

[HLZ5] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, VI: Expansion condition, associativity of logarithmic intertwining operators, and the associativity isomorphisms, to appear.

[HLZ6] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, VII: Convergence and extension properties and applications to expansion for intertwining maps, to appear.

[HLZ7] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, VIII: Braided tensor category structure on categories of generalized modules for a conformal vertex algebra, to appear.

Department of Mathematics, Rutgers University, Piscataway, NJ 08854 (permanent address)
and
Beijing International Center for Mathematical Research, Peking University, Beijing, China
E-mail address: yzhuang@math.rutgers.edu

Department of Mathematics, Rutgers University, Piscataway, NJ 08854
E-mail address: lepowsky@math.rutgers.edu

Department of Mathematics, Rutgers University, Piscataway, NJ 08854
E-mail address: linzhang@math.rutgers.edu