Logarithmic tensor category theory, II: Logarithmic formal calculus and properties of logarithmic intertwining operators

Yi-Zhi Huang, James Lepowsky and Lin Zhang

Abstract

This is the second part in a series of papers in which we introduce and develop a natural, general tensor category theory for suitable module categories for a vertex (operator) algebra. In this paper (Part II), we develop logarithmic formal calculus and study logarithmic intertwining operators.

In this paper, Part II of a series of eight papers on logarithmic tensor category theory, we develop logarithmic formal calculus and study logarithmic intertwining operators. The sections, equations, theorems and so on are numbered globally in the series of papers rather than within each paper, so that for example equation (a.b) is the b-th labeled equation in Section a, which is contained in the paper indicated as follows: In Part I [HLZ1], which contains Sections 1 and 2, we give a detailed overview of our theory, state our main results and introduce the basic objects that we shall study in this work. We include a brief discussion of some of the recent applications of this theory, and also a discussion of some recent literature. The present paper, Part II, contains Section 3. In Part III [HLZ2], which contains Section 4, we introduce and study intertwining maps and tensor product bifunctors. In Part IV [HLZ3], which contains Sections 5 and 6, we give constructions of the $P(z)$- and $Q(z)$-tensor product bifunctors using what we call "compatibility conditions" and certain other conditions. In Part V [HLZ4], which contains Sections 7 and 8, we study products and iterates of intertwining maps and of logarithmic intertwining operators and we begin the development of our analytic approach. In Part VI [HLZ5], which contains Sections 9 and 10, we construct the appropriate natural associativity isomorphisms between triple tensor product functors. In Part VII [HLZ6], which contains Section 11, we give sufficient conditions for the existence of the associativity isomorphisms. In Part VIII [HLZ7], which contains Section 12, we construct braided tensor category structure.

Acknowledgments The authors gratefully acknowledge partial support from NSF grants DMS-0070800 and DMS-0401302. Y.-Z. H. is also grateful for partial support from NSF grant PHY-0901237 and for the hospitality of Institut des Hautes Études Scientifiques in the fall of 2007.

3 Logarithmic formal calculus, logarithmic intertwining operators and their basic properties

In this section we study the notion of "logarithmic intertwining operator" introduced in [M]. For this, we will need to discuss spaces of formal series in powers of both x and "$\log x$", a new formal variable, with coefficients in certain vector spaces. In this "logarithmic formal calculus," we establish certain properties, some of them quite subtle, of the formal derivative operator d/dx acting on such spaces. Then, following [M] with a slight variant (see Remark 3.24), we introduce the notion of logarithmic intertwining operator. These are the appropriate replacements of ordinary intertwining operators when $L(0)$-semisimplicity is relaxed. In the strongly graded setting, it will be natural to consider the associated "grading-compatible" logarithmic intertwining operators. We work out some important principles and formulas concerning logarithmic intertwining operators, certain of which turn out to be the same as in the ordinary intertwining operator case. Some of these results require proofs that are quite delicate.

Recall the notation $\mathcal{W}\{x\}$ (2.1) for the space of formal series in a formal variable x with coefficients in a vector space $\mathcal{W}$, with arbitrary complex powers of x.

From now on we will sometimes need and use new independent (commuting) formal variables called $\log x$, $\log y$, $\log x_1$, $\log x_2, \dots$, etc. We will work with formal series in such formal variables together with the "usual" formal variables x, y, x_1, $x_2, \dots$, etc., with coefficients in certain vector spaces, and the powers of the monomials in *all* the variables can be arbitrary complex numbers. (Later we will restrict our attention to only nonnegative integral powers of the "log" variables.)

Given a formal variable x, we will use the notation $\frac{d}{dx}$ to denote the linear map on $\mathcal{W}\{x, \log x\}$, for any vector space $\mathcal{W}$ not involving x, defined (and indeed well defined) by the (expected) formula

$$\begin{aligned}
&\frac{d}{dx}\Big(\sum_{m,n\in\mathbb{C}} w_{n,m}x^n(\log x)^m\Big)\\
&\quad= \sum_{m,n\in\mathbb{C}}((n+1)w_{n+1,m} + (m+1)w_{n+1,m+1})x^n(\log x)^m\\
&\quad\Big(= \sum_{m,n\in\mathbb{C}} nw_{n,m}x^{n-1}(\log x)^m + \sum_{m,n\in\mathbb{C}} mw_{n,m}x^{n-1}(\log x)^{m-1}\Big) \qquad (3.1)
\end{aligned}$$

where $w_{n,m} \in \mathcal{W}$ for all $m, n \in \mathbb{C}$. We will also use the same notation for the restriction of $\frac{d}{dx}$ to any subspace of $\mathcal{W}\{x, \log x\}$ which is closed under $\frac{d}{dx}$, e.g., $\mathcal{W}\{x\}[[\log x]]$ or $\mathbb{C}[x, x^{-1}, \log x]$. Clearly, $\frac{d}{dx}$ acting on $\mathcal{W}\{x\}$ coincides with the usual formal derivative.

Remark 3.1 Let f, g and f_i, i in some index set I, all be formal series of the form

$$\sum_{m,n\in\mathbb{C}} w_{n,m}x^n(\log x)^m \in \mathcal{W}\{x, \log x\}, \quad w_{n,m} \in \mathcal{W}. \qquad (3.2)$$

One checks the following straightforwardly: Suppose that the sum of f_i, $i \in I$, exists (in the obvious sense). Then the sum of the $\frac{d}{dx} f_i$, $i \in I$, also exists and is equal to $\frac{d}{dx} \sum_{i \in I} f_i$. More generally, for any $T = p(x)\frac{d}{dx}$, $p(x) \in \mathbb{C}[x, x^{-1}]$, the sum of Tf_i, $i \in I$, exists and is equal to $T \sum_{i \in I} f_i$. Thus the sum of $e^{yT} f_i$, $i \in I$, exists and is equal to $e^{yT} \sum_{i \in I} f_i$ (e^{yT} being the formal exponential series, as usual). Suppose that $\mathcal{W}$ is an (associative) algebra or that the coefficients of either f or g are complex numbers. If the product of f and g exists, then the product of $\frac{d}{dx} f$ and g and the product of f and $\frac{d}{dx} g$ both exist, and $\frac{d}{dx}(fg) = (\frac{d}{dx} f)g + f(\frac{d}{dx} g)$. Furthermore, for any T as before, the product of Tf and g and the product of f and Tg both exist, and $T(fg) = (Tf)g + f(Tg)$. In addition, the product of $e^{yT} f$ and $e^{yT} g$ exists and is equal to $e^{yT}(fg)$, just as in formulas (8.2.6)–(8.2.10) of [FLM]. The point here, of course, is just the formal derivation property of $\frac{d}{dx}$, except that sums and products of expressions do not exist in general.

Remark 3.2 Note that the "equality" $x = e^{\log x}$ does not hold, since the left-hand side is a formal variable, while the right-hand side is a formal series in another formal variable. In fact, this formula should not be assumed, since, for example, the formal delta function $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$ would not exist in the sense of formal calculus, if x were allowed to be replaced by the formal series $e^{\log x}$. By contrast, note that the equality

$$\log e^x = x \tag{3.3}$$

does indeed hold. This is because the formal series e^x is of the form $1 + X$ where X involves only positive integral powers of x and in (3.3), "log" refers to the usual formal logarithmic series

$$\log(1 + X) = \sum_{i \geq 1} \frac{(-1)^{i-1}}{i} X^i, \tag{3.4}$$

not to the "log" of a formal variable. We will use the symbol "log" in both ways, and the meaning will be clear in context.

We will typically use notations of the form $f(x)$, instead of $f(x, \log x)$, to denote elements of $\mathcal{W}\{x, \log x\}$ for some vector space $\mathcal{W}$ as above. For this reason, we need to interpret carefully the meaning of symbols such as $f(x + y)$, or more generally, symbols obtained by replacing x in $f(x)$ by something other than just a single formal variable (since $\log x$ is a formal variable and not the image of some operator acting on x). For the three main types of cases that will be encountered in this work, we use the following notational conventions; the existence of the expressions will be justified in Remark 3.4:

Notation 3.3 For formal variables x and y, and $f(x)$ of the form (3.2), we define

$$\begin{aligned} f(x+y) &= \sum_{m,n \in \mathbb{C}} w_{n,m} (x+y)^n \Big(\log x + \log\Big(1 + \frac{y}{x}\Big)\Big)^m \\ &= \sum_{m,n \in \mathbb{C}} w_{n,m} (x+y)^n \Big(\log x + \sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \Big(\frac{y}{x}\Big)^i \Big)^m \end{aligned} \tag{3.5}$$

(recall (3.4)); in the right-hand side, $(\log x + \sum_{i\geq 1} \frac{(-1)^{i-1}}{i}(\frac{y}{x})^i)^m$, according to the binomial expansion convention, is to be expanded in nonnegative integral powers of the second summand $\sum_{i\geq 1} \frac{(-1)^{i-1}}{i}(\frac{y}{x})^i$, so the right-hand side of (3.5) is equal to

$$\sum_{m,n\in\mathbb{C}} w_{n,m}(x+y)^n \sum_{j\in\mathbb{N}} \binom{m}{j} (\log x)^{m-j} \Big(\sum_{i\geq 1} \frac{(-1)^{i-1}}{i} \Big(\frac{y}{x}\Big)^i \Big)^j \tag{3.6}$$

when expanded one step further. Also define

$$f(xe^y) = \sum_{m,n\in\mathbb{C}} w_{n,m} x^n e^{ny} (\log x + y)^m \tag{3.7}$$

and

$$f(xy) = \sum_{m,n\in\mathbb{C}} w_{n,m} x^n y^n (\log x + \log y)^m, \tag{3.8}$$

where the binomial expansion convention is again of course being used.

Remark 3.4 The existence of the right-hand side of (3.5), or (3.6), can be seen by writing $(x+y)^n$ as $x^n(1+\frac{y}{x})^n$ and observing that

$$\Big(\sum_{i\geq 1} \frac{(-1)^{i-1}}{i} \Big(\frac{y}{x}\Big)^i \Big)^j \in \Big(\frac{y}{x}\Big)^j \mathbb{C}\Big[\Big[\frac{y}{x}\Big]\Big].$$

The existence of the right-hand sides of (3.7) and of (3.8) is clear. Furthermore, both $f(x+y)$ and $f(xe^y)$ lie in $\mathcal{W}\{x, \log x\}[[y]]$, while $f(xy)$ lies in $\mathcal{W}\{xy, \log x\}[[\log y]]$. One might expect that $f(x+y)$ can be written as $e^{y\frac{d}{dx}} f(x)$, and $f(xe^y)$ as $e^{yx\frac{d}{dx}} f(x)$ (cf. Section 8.3 of [FLM]), but these formulas must be verified (see Theorem 3.6 below).

Remark 3.5 It is clear that when there is no $\log x$ involved in $f(x)$, the expression $f(x+y)$ (respectively, $f(xe^y)$, $f(xy)$) coincides with the usual formal operation of substitution of $x+y$ (respectively, xe^y, xy) for x in $f(x)$. In general, it is straightforward to check that if the sum of $f_i(x)$, $i \in I$, exists and is equal to $f(x)$, then the sum of $f_i(x+y)$, $i \in I$ (respectively, $f_i(xe^y), i \in I$, $f_i(xy), i \in I$), also exists and is equal to $f(x+y)$ (respectively, $f(xe^y)$, $f(xy)$). Also, suppose that $\mathcal{W}$ is an (associative) algebra or that the coefficients of either f or g are complex numbers. If the product of $f(x)$ and $g(x)$ exists, then the product of $f(x+y)$ and $g(x+y)$ (respectively, $f(xe^y)$ and $g(xe^y)$, $f(xy)$ and $g(xy)$) also exists and is equal to $(fg)(x+y)$ (respectively, $(fg)(xe^y)$, $(fg)(xy)$).

Note that by (3.5),

$$\log(x+y) = \log x + \sum_{i\geq 1} \frac{(-1)^{i-1}}{i} \Big(\frac{y}{x}\Big)^i = e^{y\frac{d}{dx}} \log x. \tag{3.9}$$

The next result includes a generalization of this to arbitrary elements of $\mathcal{W}\{x, \log x\}$. Formula (3.10) is a formal "Taylor theorem" for logarithmic formal series. In the case of non-logarithmic formal series, this principle is used extensively in vertex operator algebra theory; recall (2.2) above and see Proposition 8.3.1 of [FLM] for the proof in the generality of formal series with arbitrary complex powers of the formal variable. In the logarithmic case below, a much more elaborate proof is required than in the non-logarithmic case. The other formula in Theorem 3.6, formula (3.11), is easier to prove. It too is important (in the non-logarithmic case) in vertex operator algebra theory; again see Proposition 8.3.1 of [FLM].

Theorem 3.6 *For $f(x)$ as in (3.2), we have*

$$e^{y\frac{d}{dx}}f(x) = f(x+y) \tag{3.10}$$

("Taylor's theorem" for logarithmic formal series) and

$$e^{yx\frac{d}{dx}}f(x) = f(xe^y). \tag{3.11}$$

Proof By Remarks 3.1 and 3.5 (or 3.4), we need only prove these equalities for $f(x) = x^n$ and $f(x) = (\log x)^m$, $m, n \in \mathbb{C}$. The case $f(x) = x^n$ easily follows from the direct expansion of the two sides of (3.10) and of (3.11) (see Proposition 8.3.1 in [FLM]). Now assume that $f(x) = (\log x)^m$, $m \in \mathbb{C}$.

Formula (3.11) is easier, so we prove it first. By (3.1) we have

$$x\frac{d}{dx}(\log x)^m = m(\log x)^{m-1},$$

so that for $k \in \mathbb{N}$,

$$\left(x\frac{d}{dx}\right)^k (\log x)^m = m(m-1)\cdots(m-k+1)(\log x)^{m-k} = k!\binom{m}{k}(\log x)^{m-k}.$$

Thus

$$e^{yx\frac{d}{dx}}(\log x)^m = \sum_{k\in\mathbb{N}} \frac{y^k}{k!}\left(x\frac{d}{dx}\right)^k (\log x)^m = \sum_{k\in\mathbb{N}} \binom{m}{k} y^k (\log x)^{m-k} = (\log x + y)^m,$$

as we want.

For (3.10), we shall give two proofs — an analytic proof and an algebraic proof. First, consider the analytic function $(\log z)^m = e^{m \log\log z}$ over, say, $|z-3| < 1$ in the complex plane. In this proof we take the branch of $\log z$ so that

$$-\pi < \Im(\log z) \leq \pi. \tag{3.12}$$

Then by analyticity, for any z in this domain, when $|z_1|$ is small enough the Taylor series expansion $e^{z_1\frac{d}{dz}}(\log z)^m$ converges absolutely to $(\log(z+z_1))^m$. That is,

$$(\log(z+z_1))^m = e^{z_1\frac{d}{dz}}(\log z)^m = e^{\frac{z_1}{z}z\frac{d}{dz}}(\log z)^m. \tag{3.13}$$

Observe that as a formal series, the right-hand side of (3.13) is in the space $(\log z)^m\mathbb{C}[(\log z)^{-1}][[z_1/z]]$.

On the other hand, by the choice of domain and the branch of log we have

$$\log(z+z_1) = \log z + \log(1+z_1/z)$$

and

$$|\log z| > \log 2 > |\log(1+z_1/z)|$$

when $|z_1|$ is small enough. So when $|z_1|$ is small enough we have

$$\begin{aligned}(\log(z+z_1))^m &= (\log z + \log(1+z_1/z))^m \\ &= \sum_{j\in\mathbb{N}}\binom{m}{j}(\log z)^{m-j}\Big(\sum_{i\geq 1}\frac{(-1)^{i-1}}{i}\Big(\frac{z_1}{z}\Big)^i\Big)^j. \end{aligned} \tag{3.14}$$

Since as formal series, the right-hand sides of (3.13) and (3.14) are both in the space

$$(\log z)^m\mathbb{C}[(\log z)^{-1}][[z_1/z]],$$

and both converge to the same analytic function $(\log(z+z_1))^m$ in the above domain, by setting $z_1 = 0$ in these two functions and their derivatives with respect to z_1 we see that their corresponding coefficients of powers of z_1/z and further, of all monomials in $\log z$ and z_1/z must be the same. Hence we can replace z and z_1 by formal variables x and y, respectively, and obtain (3.10) for $f(x) = (\log x)^m$.

An algebraic proof of (3.10) (for $(\log x)^m$) can be given as follows: Since

$$\frac{d}{dx}(\log x)^m = mx^{-1}(\log x)^{m-1}$$

and higher derivatives involve derivatives of products of powers of x and powers of $\log x$, let us first compute $(d/dx)^k(x^n(\log x)^m)$ directly for all $m, n \in \mathbb{C}$ and $k \in \mathbb{N}$. Define linear maps T_0 and T_1 on $\mathbb{C}\{x, \log x\}$ by setting

$$T_0x^n(\log x)^m = nx^{n-1}(\log x)^m \quad \text{and} \quad T_1x^n(\log x)^m = mx^{n-1}(\log x)^{m-1},$$

respectively, and extending to all of $\mathbb{C}\{x, \log x\}$ by formal linearity. Then the formula

$$\frac{d}{dx}x^n(\log x)^m = nx^{n-1}(\log x)^m + mx^{n-1}(\log x)^{m-1}$$

(extended to $\mathbb{C}\{x, \log x\}$) can be written as

$$\frac{d}{dx} = T_0 + T_1$$

on $\mathbb{C}\{x, \log x\}$. So for $k \geq 1$, on $\mathbb{C}\{x, \log x\}$,

$$\begin{aligned}\Big(\frac{d}{dx}\Big)^k &= \sum_{(i_1,\dots,i_k)\in\{0,1\}^k} T_{i_1}\cdots T_{i_k} \\ &= \sum_{j=0}^{k-1}\sum_{0\leq t_1<t_2<\cdots<t_{k-j}<k} T_1^{k-t_{k-j}-1}T_0T_1^{t_{k-j}-t_{k-j-1}-1}T_0\cdots T_0T_1^{t_2-t_1-1}T_0T_1^{t_1} + T_1^k,\end{aligned}$$

where j gives the number of T_1's in the product $T_{i_1} \cdots T_{i_k}$; there are $k-j$ T_0's, which are in the following positions, reading from the right: $t_1+1,\ t_2+1, \ldots,\ t_{k-j}+1$. That is, for $k \geq 1$,

$$\left(\frac{d}{dx}\right)^k x^n (\log x)^m = \sum_{j=0}^{k} m(m-1)\cdots(m-j+1)\cdot$$
$$\cdot\Big(\sum_{0\leq t_1<t_2<\cdots<t_{k-j}<k} (n-t_1)(n-t_2)\cdots(n-t_{k-j})\Big) x^{n-k} (\log x)^{m-j},$$

where it is understood that if $j=k$, then the latter sum (in parentheses) is 1. In this formula, setting $n=0$, multiplying by $y^k/k!$, and then summing over $k \in \mathbb{N}$, we get (noting that $t_1 = 0$ contributes 0)

$$e^{y\frac{d}{dx}} (\log x)^m = \sum_{k\in\mathbb{N}} \left(\frac{y}{x}\right)^k \sum_{j=0}^{k} \binom{m}{j} (\log x)^{m-j} \frac{j!}{k!}\cdot$$
$$\cdot\Big(\sum_{0<t_1<t_2<\cdots<t_{k-j}<k} (-t_1)(-t_2)\cdots(-t_{k-j})\Big). \tag{3.15}$$

So (3.10) for $f(x) = (\log x)^m$ is equivalent to equating the right-hand side of (3.15) to

$$\Big(\log x + \sum_{i\geq 1} \frac{(-1)^{i-1}}{i}\left(\frac{y}{x}\right)^i\Big)^m =$$
$$= \sum_{j\in\mathbb{N}} \binom{m}{j} (\log x)^{m-j} \Big(\sum_{i\geq 1} \frac{(-1)^{i-1}}{i}\left(\frac{y}{x}\right)^i\Big)^j$$
$$= \sum_{j\in\mathbb{N}} \binom{m}{j} (\log x)^{m-j} \sum_{k\geq j} \Bigg(\sum_{\substack{i_1+\cdots+i_j=k \\ 1\leq i_1,\ldots,i_j\leq k}} \frac{(-1)^{k-j}}{i_1 i_2\cdots i_j}\Bigg)\left(\frac{y}{x}\right)^k$$
$$= \sum_{k\in\mathbb{N}} \left(\frac{y}{x}\right)^k \sum_{j=0}^{k} \binom{m}{j} (\log x)^{m-j} \Bigg(\sum_{\substack{i_1+\cdots+i_j=k \\ 1\leq i_1,\ldots,i_j\leq k}} \frac{(-1)^{k-j}}{i_1 i_2\cdots i_j}\Bigg). \tag{3.16}$$

Comparing the right-hand sides of (3.15) and (3.16) we see that it is equivalent to proving the combinatorial identity

$$\frac{j!}{k!} \sum_{0<t_1<t_2<\cdots<t_{k-j}<k} t_1 t_2 \cdots t_{k-j} = \sum_{\substack{i_1+\cdots+i_j=k \\ 1\leq i_1,\ldots,i_j\leq k}} \frac{1}{i_1 i_2 \cdots i_j} \tag{3.17}$$

for all $k \in \mathbb{N}$ and $j = 0, \ldots, k$. Note that there is no m involved here. But for m a nonnegative integer, (3.10) for $f(x) = (\log x)^m$ follows from

$$e^{y\frac{d}{dx}} (\log x)^m = (e^{y\frac{d}{dx}} \log x)^m$$

(recall Remark 3.1) and

$$e^{y\frac{d}{dx}} \log x = \log x + \sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \left(\frac{y}{x}\right)^i$$

(recall (3.9)). Thus the expressions in (3.15) and (3.16) are equal for any such m. Equating coefficients and choosing $m \geq j$ gives us (3.17). Therefore (3.10) also holds for any $m \in \mathbb{C}$. □

Remark 3.7 Here is an amusing sidelight: When we were writing up the proof above, one of us (L.Z.) happened to pick up the then-current issue of the American Mathematical Monthly and happened to notice the following problem from the Problems and Solutions section, proposed by D. Lubell [Lub]:

> Let N and j be positive integers, and let $S = \{(w_1, \dots, w_j) \in \mathbb{Z}_+^j \,|\, 0 < w_1 + \cdots + w_j \leq N\}$ and $T = \{(w_1, \cdots, w_j) \in \mathbb{Z}_+^j \,|\, w_1, \dots, w_j \text{ are distinct and bounded by } N\}$. Show that
>
> $$\sum_S \frac{1}{w_1 \cdots w_j} = \sum_T \frac{1}{w_1 \cdots w_j}.$$

But this follows immediately from (3.17) (which is in fact a refinement), since the left-hand side of (3.17) is equal to

$$j! \sum_{1 \leq w_1 < w_2 < \cdots < w_{j-1} \leq k-1} \frac{1}{w_1 w_2 \cdots w_{j-1} k} = \sum_{T_k} \frac{1}{w_1 w_2 \cdots w_j}$$

where

$$T_k = \{(w_1, \dots, w_j) \in \{1, 2, \dots, k\}^j \,|\, w_i \text{ distinct, with maximum exactly } k\},$$

the right-hand side is

$$\sum_{S_k} \frac{1}{w_1 w_2 \cdots w_j}$$

where

$$S_k = \{(w_1, \dots, w_j) \in \{1, 2, \dots, k\}^j \,|\, w_1 + \cdots + w_j = k\},$$

and one has $S = \coprod_{k=1}^N S_k$ and $T = \coprod_{k=1}^N T_k$.

When we define the notion of logarithmic intertwining operator below, we will impose a condition requiring certain formal series to lie in spaces of the type $\mathcal{W}[\log x]\{x\}$ (so that for each power of x, possibly complex, we have a *polynomial* in $\log x$), partly because such results as the following (which is expected) will indeed hold in our formal setup when the powers of the formal variables are restricted in this way (cf. Remark 3.9 below).

Lemma 3.8 *Let $a \in \mathbb{C}$ and $m \in \mathbb{Z}_+$. If $f(x) \in \mathcal{W}[\log x]\{x\}$ ($\mathcal{W}$ any vector space not involving x or $\log x$) satisfies the formal differential equation*

$$\left(x\frac{d}{dx} - a\right)^m f(x) = 0, \tag{3.18}$$

then $f(x) \in \mathcal{W}x^a \oplus \mathcal{W}x^a \log x \oplus \cdots \oplus \mathcal{W}x^a(\log x)^{m-1}$, and furthermore, if m is the smallest integer so that (3.18) is satisfied, then the coefficient of $x^a(\log x)^{m-1}$ in $f(x)$ is nonzero.

Proof For any $f(x) = \sum_{n,k} w_{n,k}x^n(\log x)^k \in \mathcal{W}\{x, \log x\}$,

$$\begin{aligned} x\frac{d}{dx}f(x) &= \sum_{n,k} nw_{n,k}x^n(\log x)^k + \sum_{n,k} kw_{n,k}x^n(\log x)^{k-1} \\ &= \sum_{n,k}(nw_{n,k} + (k+1)w_{n,k+1})x^n(\log x)^k. \end{aligned}$$

Thus for any $a \in \mathbb{C}$,

$$\left(x\frac{d}{dx} - a\right)f(x) = \sum_{n,k}((n-a)w_{n,k} + (k+1)w_{n,k+1})x^n(\log x)^k. \tag{3.19}$$

Now suppose that $f(x)$ lies in $\mathcal{W}[\log x]\{x\}$. Let us prove the assertion of the lemma by induction on m.

If $m = 1$, by (3.19) we see that $(x\frac{d}{dx} - a)f(x) = 0$ means that

$$(n-a)w_{n,k} + (k+1)w_{n,k+1} = 0 \quad \text{for any } n \in \mathbb{C},\ k \in \mathbb{Z}. \tag{3.20}$$

Fix n. If $w_{n,k} \neq 0$ for some k, let k_n be the smallest nonnegative integer such that $w_{n,k} = 0$ for any $k > k_n$ (such a k_n exists because $f(x) \in \mathcal{W}[\log x]\{x\}$). Then

$$(n-a)w_{n,k_n} = -(k_n+1)w_{n,k_n+1} = 0.$$

But $w_{n,k_n} \neq 0$ by the choice of k_n, so we must have $n = a$. Now (3.20) becomes $(k+1)w_{a,k+1} = 0$ for any $k \in \mathbb{Z}$, so that $w_{a,k} = 0$ unless $k = 0$. Thus $f(x) = w_{a,0}x^a$. If in addition $m = 1$ is the smallest integer such that (3.18) holds, then $f(x) \neq 0$. So $w_{a,0}$, the coefficient of x^a, is not zero.

Suppose the statement is true for m. Then for the case $m+1$, since

$$0 = \left(x\frac{d}{dx} - a\right)^{m+1} f(x) = \left(x\frac{d}{dx} - a\right)^m \left(x\frac{d}{dx} - a\right)f(x) \tag{3.21}$$

implies that

$$\left(x\frac{d}{dx} - a\right)f(x) = \bar{w}_0x^a + \bar{w}_1x^a\log x + \cdots + \bar{w}_{m-1}x^a(\log x)^{m-1} \tag{3.22}$$

for some $\bar{w}_0, \bar{w}_1, \cdots, \bar{w}_{m-1} \in \mathcal{W}$, by (3.19) we get

$$\begin{aligned}(n-a)w_{n,j} + (j+1)w_{n,j+1} &= 0 \qquad \text{for any } n \neq a \text{ and any } j \in \mathbb{Z}\\ (j+1)w_{a,j+1} &= \bar{w}_j \qquad \text{for any } j \in \{0,1,\ldots,m-1\}\\ (j+1)w_{a,j+1} &= 0 \qquad \text{for any } j \notin \{0,1,\ldots,m-1\}\end{aligned}$$

By the same argument as above we get $w_{n,j} = 0$ for any $n \neq a$ and any j. So

$$f(x) = w_{a,0}x^a + \bar{w}_0 x^a \log x + \frac{\bar{w}_1}{2} x^a (\log x)^2 + \cdots + \frac{\bar{w}_{m-1}}{m} x^a (\log x)^m,$$

as we want. If in addition $m+1$ is the smallest integer so that (3.21) is satisfied, then by the induction assumption, $\bar{w}_{m-1}$ in (3.22) is not zero. So the coefficient in $f(x)$ of $x^a(\log x)^m$, $\bar{w}_{m-1}/m$, is not zero, as we want. $\square$

Remark 3.9 Note that there are solutions of the equation (3.18) outside $\mathcal{W}[\log x]\{x\}$, for example,

$$f(x) = wx^b e^{(a-b)\log x} \in x^b \mathcal{W}[[\log x]]$$

for any complex number $b \neq a$ and any $0 \neq w \in \mathcal{W}$.

Following [M], with a slight generalization (see Remark 3.24), we now introduce the notion of logarithmic intertwining operator, together with the notion of "grading-compatible logarithmic intertwining operator," adapted to the strongly graded case. We will later see that the axioms in these definitions correspond exactly to those in the notion of certain "intertwining maps" (see Definition 4.2 below).

Definition 3.10 Let (W_1, Y_1), (W_2, Y_2) and (W_3, Y_3) be generalized modules for a Möbius (or conformal) vertex algebra V. A *logarithmic intertwining operator of type* $\binom{W_3}{W_1\,W_2}$ is a linear map

$$\mathcal{Y}(\cdot, x)\cdot : W_1 \otimes W_2 \to W_3[\log x]\{x\}, \tag{3.23}$$

or equivalently,

$$w_{(1)} \otimes w_{(2)} \mapsto \mathcal{Y}(w_{(1)}, x)w_{(2)} = \sum_{n\in\mathbb{C}} \sum_{k\in\mathbb{N}} w_{(1)}{}^{\mathcal{Y}}_{n;\,k} w_{(2)} x^{-n-1} (\log x)^k \in W_3[\log x]\{x\} \tag{3.24}$$

for all $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, such that the following conditions are satisfied: the *lower truncation condition*: for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $n \in \mathbb{C}$,

$$w_{(1)}{}^{\mathcal{Y}}_{n+m;\,k} w_{(2)} = 0 \quad \text{for} \;\; m \in \mathbb{N} \;\; \text{sufficiently large, independently of } k; \tag{3.25}$$

the *Jacobi identity*:

$$\begin{aligned}&x_0^{-1}\delta\Big(\frac{x_1-x_2}{x_0}\Big) Y_3(v, x_1)\mathcal{Y}(w_{(1)}, x_2)w_{(2)}\\ &\qquad -x_0^{-1}\delta\Big(\frac{x_2-x_1}{-x_0}\Big)\mathcal{Y}(w_{(1)}, x_2)Y_2(v, x_1)w_{(2)}\\ &= x_2^{-1}\delta\Big(\frac{x_1-x_0}{x_2}\Big)\mathcal{Y}(Y_1(v, x_0)w_{(1)}, x_2)w_{(2)}\end{aligned} \tag{3.26}$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ (note that the first term on the left-hand side is meaningful because of (3.25)); the $L(-1)$-*derivative property:* for any $w_{(1)} \in W_1$,

$$\mathcal{Y}(L(-1)w_{(1)}, x) = \frac{d}{dx}\mathcal{Y}(w_{(1)}, x); \tag{3.27}$$

and the $\mathfrak{sl}(2)$-*bracket relations:* for any $w_{(1)} \in W_1$,

$$[L(j), \mathcal{Y}(w_{(1)}, x)] = \sum_{i=0}^{j+1} \binom{j+1}{i} x^i \mathcal{Y}(L(j-i)w_{(1)}, x) \tag{3.28}$$

for $j = -1, 0$ and 1 (note that if V is in fact a conformal vertex algebra, this follows automatically from the Jacobi identity (3.26) by setting $v = \omega$ and then taking $\mathrm{Res}_{x_0}\mathrm{Res}_{x_1} x_1^{j+1}$).

Remark 3.11 We will sometimes write the Jacobi identity (3.26) as

$$\begin{aligned} x_0^{-1}\delta\Big(\frac{x_1 - x_2}{x_0}\Big) & Y(v, x_1)\mathcal{Y}(w_{(1)}, x_2)w_{(2)} \\ & - x_0^{-1}\delta\Big(\frac{x_2 - x_1}{-x_0}\Big)\mathcal{Y}(w_{(1)}, x_2)Y(v, x_1)w_{(2)} \\ = x_2^{-1}\delta\Big(\frac{x_1 - x_0}{x_2}\Big) & \mathcal{Y}(Y(v, x_0)w_{(1)}, x_2)w_{(2)} \end{aligned} \tag{3.29}$$

(dropping the subscripts on the module actions) for brevity.

Remark 3.12 The ordinary intertwining operators (as in, for example, [HL1]) among triples of modules for a vertex operator algebra are exactly the logarithmic intertwining operators that do not involve the formal variable $\log x$, except for our present relaxation of the lower truncation condition. The lower truncation condition that we use here can be equivalently stated as: For any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $n \in \mathbb{C}$, there is no nonzero term involving x^{n-m} appearing in $\mathcal{Y}(w_{(1)}, x)w_{(2)}$ when $m \in \mathbb{N}$ is large enough. In [HL1], the lower truncation condition in the definition of the notion of intertwining operator states: For any $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$,

$$(w_{(1)})_n w_{(2)} = 0 \text{ for } n \text{ whose real part is sufficiently large.} \tag{3.30}$$

This is slightly stronger than the lower truncation condition that we use here, even if no $\log x$ is involved, when the powers of x in $\mathcal{Y}(w_{(1)}, x)w_{(2)}$ belong to infinitely many different congruence classes modulo $\mathbb{Z}$. But in the case that our (generalized) modules satisfy the lower boundedness condition (2.89), this stronger condition (3.30) will hold.

Remark 3.13 Given a generalized module (W, Y_W) for a Möbius (or conformal) vertex algebra V, the vertex operator map Y_W itself is clearly a logarithmic intertwining operator of type $\binom{W}{VW}$; in fact, it does not involve $\log x$ and its powers of x are all integers. In particular, taking (W, Y_W) to be (V, Y) itself, we have that the vertex operator map Y is a logarithmic intertwining operator of type $\binom{V}{VV}$ not involving $\log x$ and having only integral powers of x.

The logarithmic intertwining operators of a fixed type $\binom{W_3}{W_1\,W_2}$ form a vector space.

Definition 3.14 In the setting of Definition 3.10, suppose in addition that V and W_1, W_2 and W_3 are strongly graded (recall Definitions 2.23 and 2.25). A logarithmic intertwining operator $\mathcal{Y}$ as in Definition 3.10 is a *grading-compatible logarithmic intertwining operator* if for $\beta, \gamma \in \tilde{A}$ (recall Definition 2.25) and $w_{(1)} \in W_1^{(\beta)}$, $w_{(2)} \in W_2^{(\gamma)}$, $n \in \mathbb{C}$ and $k \in \mathbb{N}$, we have

$$w_{(1)}{}^{\mathcal{Y}}_{n;\,k} w_{(2)} \in W_3^{(\beta+\gamma)}. \tag{3.31}$$

Remark 3.15 The term “grading-compatible” in Definition 3.14 refers to the $\tilde{A}$-gradings; *any* logarithmic intertwining operator is compatible with the $\mathbb{C}$-gradings of W_1, W_2 and W_3, in view of Proposition 3.20(b) below.

Remark 3.16 Given a strongly graded generalized module (W, Y_W) for a strongly graded Möbius (or conformal) vertex algebra V, the vertex operator map Y_W is a grading-compatible logarithmic intertwining operator of type $\binom{W}{V\,W}$ not involving $\log x$ and having only integral powers of x. Taking (W, Y_W) in particular to be (V, Y) itself, we have that the vertex operator map Y is a grading-compatible logarithmic intertwining operator of type $\binom{V}{V\,V}$ not involving $\log x$ and having only integral powers of x.

In the strongly graded context (the main context for our tensor product theory), we will use the following notation and terminology, traditionally used in the setting of ordinary intertwining operators, as in [FHL]:

Definition 3.17 In the setting of Definition 3.14, the grading-compatible logarithmic intertwining operators of a fixed type $\binom{W_3}{W_1\,W_2}$ form a vector space, which we denote by $\mathcal{V}^{W_3}_{W_1\,W_2}$. We call the dimension of $\mathcal{V}^{W_3}_{W_1\,W_2}$ the *fusion rule* for W_1, W_2 and W_3 and denote it by $N^{W_3}_{W_1\,W_2}$.

Remark 3.18 In the strongly graded context, suppose that W_1, W_2 and W_3 in Definition 3.10 are expressed as finite direct sums of submodules. Then the space $\mathcal{V}^{W_3}_{W_1\,W_2}$ can be naturally expressed as the corresponding (finite) direct sum of the spaces of (grading-compatible) logarithmic intertwining operators among the direct summands, and the fusion rule $N^{W_3}_{W_1\,W_2}$ is thus the sum of the fusion rules for the direct summands.

Remark 3.19 As we shall point out in Remark 3.23 below, it turns out that the notion of fusion rule in Definition 3.17 agrees with the traditional notion, in the case of a vertex operator algebra and ordinary modules. The justification of this assertion uses Parts (b) and (c), or alternatively, Part (a), of the next proposition. Part (a), whose proof uses Lemma 3.8, shows how logarithmic intertwining operators yield expansions involving only finitely many powers of $\log x$. Part (b) is a generalization of formula (2.49).

Proposition 3.20 *Let W_1, W_2, W_3 be generalized modules for a Möbius (or conformal) vertex algebra V, and let $\mathcal{Y}(\cdot, x)\cdot$ be a logarithmic intertwining operator of type $\binom{W_3}{W_1\,W_2}$. Let*

$w_{(1)}$ and $w_{(2)}$ be homogeneous elements of W_1 and W_2 of generalized weights n_1 and $n_2 \in \mathbb{C}$, respectively, and let k_1 and k_2 be positive integers such that

$$(L(0) - n_1)^{k_1} w_{(1)} = 0 \;\; \textit{and} \;\; (L(0) - n_2)^{k_2} w_{(2)} = 0.$$

Then we have:

(a) ([M]) For any $w'_{(3)} \in W_3^*$, $n_3 \in \mathbb{C}$ and $k_3 \in \mathbb{Z}_+$ such that $(L'(0) - n_3)^{k_3} w'_{(3)} = 0$,

$$\begin{aligned}&\langle w'_{(3)}, \mathcal{Y}(w_{(1)}, x) w_{(2)}\rangle \\ &\qquad \in \mathbb{C}x^{n_3-n_1-n_2} \oplus \mathbb{C}x^{n_3-n_1-n_2} \log x \oplus \cdots \oplus \mathbb{C}x^{n_3-n_1-n_2} (\log x)^{k_1+k_2+k_3-3}.\end{aligned} \tag{3.32}$$

(b) For any $n \in \mathbb{C}$ and $k \in \mathbb{N}$, $w_{(1)}{}^{\mathcal{Y}}_{n;\,k} w_{(2)} \in W_3$ is homogeneous of generalized weight $n_1 + n_2 - n - 1$.

(c) Fix $n \in \mathbb{C}$ and $k \in \mathbb{N}$. For each $i, j \in \mathbb{N}$, let m_{ij} be a nonnegative integer such that

$$(L(0) - n_1 - n_2 + n + 1)^{m_{ij}} (((L(0) - n_1)^i w_{(1)})^{\mathcal{Y}}_{n;\,k} (L(0) - n_2)^j w_{(2)}) = 0.$$

Then for all $t \geq \max\{m_{ij} \,|\, 0 \leq i < k_1,\; 0 \leq j < k_2\} + k_1 + k_2 - 2$,

$$w_{(1)}{}^{\mathcal{Y}}_{n;\,k+t} w_{(2)} = 0.$$

We will need the following lemma in the proof:

Lemma 3.21 *Let W_1, W_2, W_3 be generalized modules for a Möbius (or conformal) vertex algebra V. Let*

$$\begin{aligned}\mathcal{Y}(\cdot, x)\cdot : W_1 \otimes W_2 &\to W_3\{x, \log x\} \\ w_{(1)} \otimes w_{(2)} &\mapsto \mathcal{Y}(w_{(1)}, x) w_{(2)} = \sum_{n,k \in \mathbb{C}} w_{(1)}{}^{\mathcal{Y}}_{n;\,k} w_{(2)} x^{-n-1} (\log x)^k\end{aligned} \tag{3.33}$$

be a linear map that satisfies the L(-1)-derivative property (3.27) and the $L(0)$-bracket relation, that is, (3.28) with $j = 0$. Then for any $a, b, c \in \mathbb{C}$, $t \in \mathbb{N}$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$,

$$\begin{aligned}(L(0) - c)^t \mathcal{Y}(w_{(1)}, x) w_{(2)} = &\sum_{i,j,l \in \mathbb{N},\, i+j+l=t} \frac{t!}{i!j!l!} \cdot \\ &\cdot \left(x\frac{d}{dx} - c + a + b \right)^l \mathcal{Y}((L(0) - a)^i w_{(1)}, x)(L(0) - b)^j w_{(2)}.\end{aligned} \tag{3.34}$$

Also, for any $a, b, n, k \in \mathbb{C}$, $t \in \mathbb{N}$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, we have

$$\begin{aligned}&(L(0) - a - b + n + 1)^t (w_{(1)}{}^{\mathcal{Y}}_{n;\,k} w_{(2)}) \\ &\quad = t! \sum_{i,j,l \geq 0,\, i+j+l=t} \binom{k+l}{l} \Big(\frac{(L(0) - a)^i}{i!} w_{(1)} \Big)^{\mathcal{Y}}_{n;\,k+l} \Big(\frac{(L(0) - b)^j}{j!} w_{(2)} \Big);\end{aligned} \tag{3.35}$$

in generating function form, this gives

$$e^{y(L(0)-a-b+n+1)}(w_{(1)}{}^{\mathcal{Y}}_{n;\,k}w_{(2)})$$
$$= \sum_{l\in\mathbb{N}}\binom{k+l}{l}(e^{y(L(0)-a)}w_{(1)})^{\mathcal{Y}}_{n;\,k+l}(e^{y(L(0)-b)}w_{(2)})y^l. \tag{3.36}$$

Proof From (3.27) and (3.28) with $j=0$ we have

$$L(0)\mathcal{Y}(w_{(1)},x)w_{(2)} = \mathcal{Y}(w_{(1)},x)L(0)w_{(2)}$$
$$+x\frac{d}{dx}\mathcal{Y}(w_{(1)},x)w_{(2)} + \mathcal{Y}(L(0)w_{(1)},x)w_{(2)}. \tag{3.37}$$

Hence

$$(L(0)-c)\mathcal{Y}(w_{(1)},x)w_{(2)} = \mathcal{Y}(w_{(1)},x)(L(0)-b)w_{(2)}$$
$$+\left(x\frac{d}{dx}-c+a+b\right)\mathcal{Y}(w_{(1)},x)w_{(2)} + \mathcal{Y}((L(0)-a)w_{(1)},x)w_{(2)}$$

for any complex numbers a, b and c. In view of the fact that the actions of $L(0)$ and d/dx commute with each other, this implies (3.34) essentially because of the expansion formula for powers of a sum of commuting operators, that is, for any commuting operators $T_1,\dots,T_s$ and $t\in\mathbb{N}$,

$$(T_1+\cdots+T_s)^t = \sum_{i_1,\dots,i_s\in\mathbb{N},\; i_1+\cdots+i_s=t}\frac{t!}{i_1!\cdots i_s!}T_1^{i_1}\cdots T_s^{i_s}. \tag{3.38}$$

On the other hand, by taking coefficient of $x^{-n-1}(\log x)^k$ in (3.37) we get

$$L(0)w_{(1)}{}^{\mathcal{Y}}_{n;\,k}w_{(2)} = w_{(1)}{}^{\mathcal{Y}}_{n;\,k}L(0)w_{(2)} + (-n-1)w_{(1)}{}^{\mathcal{Y}}_{n;\,k}w_{(2)}$$
$$+(k+1)w_{(1)}{}^{\mathcal{Y}}_{n;\,k+1}w_{(2)} + (L(0)w_{(1)})^{\mathcal{Y}}_{n;\,k}w_{(2)}.$$

So for any $a,b,n,k\in\mathbb{C}$,

$$(L(0)-a-b+n+1)(w_{(1)}{}^{\mathcal{Y}}_{n;\,k}w_{(2)}) = ((L(0)-a)w_{(1)})^{\mathcal{Y}}_{n;\,k}w_{(2)}$$
$$+w_{(1)}{}^{\mathcal{Y}}_{n;\,k}(L(0)-b)w_{(2)} + (k+1)w_{(1)}{}^{\mathcal{Y}}_{n;\,k+1}w_{(2)}. \tag{3.39}$$

For $p,q\in\mathbb{N}$ and $n,k\in\mathbb{C}$, let us write

$$T_{p,k,q} = ((L(0)-a)^p w_{(1)})^{\mathcal{Y}}_{n;\,k}((L(0)-b)^q w_{(2)}). \tag{3.40}$$

Then from (3.39) we see that for any $p,q\in\mathbb{N}$ and $a,b,n,k\in\mathbb{C}$,

$$(L(0)-a-b+n+1)T_{p,k,q} = T_{p+1,k,q} + (k+1)T_{p,k+1,q} + T_{p,k,q+1}. \tag{3.41}$$

Hence by (3.38) we have

$$(L(0)-a-b+n+1)^t T_{p,k,q} = t!\sum_{i,j,l\geq 0,\; i+j+l=t}\frac{(k+1)(k+2)\cdots(k+l)}{i!j!l!}T_{p+i,k+l,q+j}$$

for any $a, b, n, k \in \mathbb{C}$ and $p, q \in \mathbb{N}$. In particular, by setting $p = q = 0$ we get (3.35), and (3.36) follows easily from (3.35) by multiplying by $y^t/t!$ and then summing over $t \in \mathbb{N}$. $\square$

Proof of Proposition 3.20 (a): Under the assumptions of the proposition, let us show that

$$\left\langle w'_{(3)}, \left(x\frac{d}{dx} - n_3 + n_1 + n_2\right)^{k_3+k_1+k_2-2} \mathcal{Y}(w_{(1)}, x)w_{(2)}\right\rangle = 0 \tag{3.42}$$

by induction on $k_1 + k_2$.

For $k_1 = k_2 = 1$, from (3.34) with $a = n_1$, $b = n_2$, $c = n_3$ and $t = k_3$ we have

$$\begin{aligned} 0 &= \langle (L'(0) - n_3)^{k_3} w'_{(3)}, \mathcal{Y}(w_{(1)}, x)w_{(2)}\rangle \\ &= \langle w'_{(3)}, (L(0) - n_3)^{k_3} \mathcal{Y}(w_{(1)}, x)w_{(2)}\rangle \\ &= \left\langle w'_{(3)}, \left(x\frac{d}{dx} - n_3 + n_1 + n_2\right)^{k_3} \mathcal{Y}(w_{(1)}, x)w_{(2)}\right\rangle, \end{aligned}$$

which is (3.42) in the case $k_1 = k_2 = 1$.

Suppose that (3.42) is true for all the cases with smaller $k_1 + k_2$. Then from (3.34) with $a = n_1$, $b = n_2$, $c = n_3$ and $t = k_3 + k_1 + k_2 - 2$ we have

$$\begin{aligned} 0 &= \langle (L'(0) - n_3)^{k_3+k_1+k_2-2} w'_{(3)}, \mathcal{Y}(w_{(1)}, x)w_{(2)}\rangle \\ &= \langle w'_{(3)}, (L(0) - n_3)^{k_3+k_1+k_2-2} \mathcal{Y}(w_{(1)}, x)w_{(2)}\rangle \\ &= \Bigg\langle w'_{(3)}, \sum_{i,j,k\in\mathbb{N},\ i+j+k=k_3+k_1+k_2-2} \frac{(k_3+k_1+k_2-2)!}{i!j!k!} \cdot \\ &\qquad \cdot \left(x\frac{d}{dx} - n_3 + n_1 + n_2\right)^{k} \mathcal{Y}((L(0) - n_1)^i w_{(1)}, x)(L(0) - n_2)^j w_{(2)}\Bigg\rangle \\ &= \left\langle w'_{(3)}, \left(x\frac{d}{dx} - n_3 + n_1 + n_2\right)^{k_3+k_1+k_2-2} \mathcal{Y}(w_{(1)}, x)w_{(2)}\right\rangle, \end{aligned}$$

where the last equality uses the induction assumption for the pair of elements $(L(0)-n_1)^i w_{(1)}$ and $(L(0)-n_2)^j w_{(2)}$ for all $(i, j) \neq (0, 0)$. So (3.42) is established, that is, we have the formal differential equation

$$\left(x\frac{d}{dx} - n_3 + n_1 + n_2\right)^{k_3+k_1+k_2-2} \langle w'_{(3)}, \mathcal{Y}(w_{(1)}, x)w_{(2)}\rangle = 0.$$

This implies (a) by Lemma 3.8.

(b): This follows from (3.35) with $a = n_1$, $b = n_2$ and the fact that for any $\bar{w}_{(1)} \in W_1$, $\bar{w}_{(2)} \in W_2$ and $\bar{n} \in \mathbb{C}$, there exists $K \in \mathbb{N}$ so that $(\bar{w}_{(1)})^{\mathcal{Y}}_{\bar{n};\bar{k}} \bar{w}_{(2)} = 0$ for all $\bar{k} > K$, due to (3.23).

(c): Let us prove (c) by induction on k_1+k_2 again. For $k_1=k_2=1$, (3.35) with $a=n_1$ and $b=n_2$ gives

$$(L(0)-n_1-n_2+n+1)^t(w_{(1)}{}^{\mathcal{Y}}_{n;\,k}w_{(2)})=((k+t)!/k!)w_{(1)}{}^{\mathcal{Y}}_{n;\,k+t}w_{(2)},$$

that is,

$$w_{(1)}{}^{\mathcal{Y}}_{n;\,k+t}w_{(2)}=(k!/(k+t)!)(L(0)-n_1-n_2+n+1)^t(w_{(1)}{}^{\mathcal{Y}}_{n;\,k}w_{(2)}).$$

So for $t\geq m_{00}$, $w_{(1)}{}^{\mathcal{Y}}_{n;\,k+t}w_{(2)}=0$, proving the statement in case $k_1=k_2=1$.

Suppose that the statement (c) is true for all smaller k_1+k_2. Then for $(i,j)\neq(0,0)$,

$$\Big(\frac{(L(0)-n_1)^i}{i!}w_{(1)}\Big)^{\mathcal{Y}}_{n;\,k+l}\Big(\frac{(L(0)-n_2)^j}{j!}w_{(2)}\Big)=0$$

when $l\geq\max\{m_{i'j'}\,|\,i\leq i'<k_1,\ j\leq j'<k_2\}+(k_1-i)+(k_2-j)-2$, and in particular, when $l\geq\max\{m_{i'j'}\,|\,0\leq i'<k_1,\ 0\leq j'<k_2\}+k_1+k_2-i-j-2$. But then for all $t\geq\max\{m_{ij}\,|\,0\leq i<k_1,\ 0\leq j<k_2\}+k_1+k_2-2$, (3.35) gives $0=((k+t)!/k!)w_{(1)}{}^{\mathcal{Y}}_{n;\,k+t}w_{(2)}$, proving what we need. $\square$

The following corollary is immediate from Proposition 3.20(b):

Corollary 3.22 *Let V be a Möbius (or conformal) vertex algebra and let W_1, W_2 and W_3 be generalized V-modules whose weights are all congruent modulo $\mathbb{Z}$ to complex numbers h_1, h_2 and h_3, respectively. (For example, W_1, W_2 and W_3 might be indecomposable; recall Remark 2.20.) Let $\mathcal{Y}(\cdot,x)\cdot$ be a logarithmic intertwining operator of type $\binom{W_3}{W_1\,W_2}$. Then all powers of x in $\mathcal{Y}(\cdot,x)\cdot$ are congruent modulo $\mathbb{Z}$ to $h_3-h_1-h_2$.* $\square$

Remark 3.23 Let W_1, W_2 and W_3 be (ordinary) modules for a Möbius (or conformal) vertex algebra V. Then any logarithmic intertwining operator of type $\binom{W_3}{W_1\,W_2}$ is just an ordinary intertwining operator of this type, i.e., it does not involve $\log x$. This clearly follows from Proposition 3.20(b) and (c), where k_1 and k_2 are chosen to be 1, k is chosen to be 0, and m_{00} is chosen to be 1. It also follows, alternatively, from Proposition 3.20(a). As a result, for V a vertex operator algebra (viewed as a conformal vertex algebra strongly graded with respect to the trivial group; recall Remark 2.24) and W_1, W_2 and W_3 V-modules in the sense of Remark 2.27, the notion of fusion rule defined in this work (recall Definition 3.17) coincides with the notion of fusion rule defined in, for example, [HL1] (except for the minor issue of the truncation condition for an intertwining operator, discussed in Remark 3.12).

Remark 3.24 Our definition of logarithmic intertwining operator is identical to that in [M] (in case V is a vertex operator algebra) except that in [M], a logarithmic intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1\,W_2}$ is required to be a linear map $W_1\to\mathrm{Hom}(W_2,W_3)\{x\}[\log x]$, instead of as in (3.23), and the lower truncation condition (3.25) is replaced by: For any $w_{(1)}\in W_1$, $w_{(2)}\in W_2$ and $k\in\mathbb{N}$,

$$w_{(1)}{}^{\mathcal{Y}}_{n;k}w_{(2)}=0\ \text{ for } n \text{ whose real part is sufficiently large.}\tag{3.43}$$

Given generalized V-modules W_1, W_2 and W_3, suppose that for each $i = 1, 2, 3$, there exists some $K_i \in \mathbb{Z}_+$ such that $(L(0) - L(0)_s)^{K_i} W_i = 0$ (this is satisfied by many interesting examples and is assumed in [M] for all generalized modules under consideration). Then for any logarithmic intertwining operator $\mathcal{Y}$, any homogeneous elements $w_{(1)} \in W_{1[n_1]}$, $w_{(2)} \in W_{2[n_2]}$, $n_1, n_2 \in \mathbb{C}$, any $n \in \mathbb{C}$ and any $k \in \mathbb{N}$, all the m_{ij}'s in Proposition 3.20(c) can be chosen to be no greater than K_3, while k_1 and k_2 can be chosen to be no greater than K_1 and K_2, respectively. Proposition 3.20(c) thus implies that the largest power of $\log x$ that is involved in $\mathcal{Y}$ is no greater than $K_1 + K_2 + K_3 - 3$. (This follows alternatively from Proposition 3.20(a) and (2.100).) In particular, $\mathcal{Y}$ maps W_1 to $\mathrm{Hom}(W_2, W_3)\{x\}[\log x]$, and in fact, we even have that $K_1 + K_2 + K_3 - 3$ is a global bound on the powers of $\log x$, independently of $w_{(1)} \in W_1$, so that

$$\mathcal{Y}(\cdot, x)\cdot \in \mathrm{Hom}(W_1 \otimes W_2, W_3)\{x\}[\log x], \tag{3.44}$$

and this global bound on the powers of $\log x$ is even independent of $\mathcal{Y}$.

Remark 3.25 In the setting of Definition 3.14, if W_3 is lower bounded, then any grading-compatible logarithmic intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1\, W_2}$ satisfies (3.43), in view of Proposition 3.20(b).

Remark 3.26 Given a logarithmic intertwining operator $\mathcal{Y}$ as in (3.24), set

$$\mathcal{Y}^{(k)}(w_{(1)}, x)w_{(2)} = \sum_{n \in \mathbb{C}} {w_{(1)}}^{\mathcal{Y}}_{n;\,k} w_{(2)} x^{-n-1}$$

for $k \in \mathbb{N}$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, so that

$$\mathcal{Y}(w_{(1)}, x)w_{(2)} = \sum_{k \in \mathbb{N}} \mathcal{Y}^{(k)}(w_{(1)}, x)w_{(2)}(\log x)^k.$$

By taking the coefficients of the powers of $\log x_2$ and $\log x$ in (3.26) and (3.28), respectively, we see that each $\mathcal{Y}^{(k)}$ satisfies the Jacobi identity and the $\mathfrak{sl}(2)$-bracket relations. On the other hand, taking the coefficients of the powers of $\log x$ in (3.27) gives

$$\mathcal{Y}^{(k)}(L(-1)w_{(1)}, x) = \frac{d}{dx}\mathcal{Y}^{(k)}(w_{(1)}, x) + \frac{k+1}{x}\mathcal{Y}^{(k+1)}(w_{(1)}, x) \tag{3.45}$$

for any $k \in \mathbb{N}$ and $w_{(1)} \in W_1$. So $\mathcal{Y}^{(k)}$ does not in general satisfy the $L(-1)$-derivative property. (If $\mathcal{Y}^{(k+1)} = 0$, then $\mathcal{Y}^{(k)}$ of course does satisfy the $L(-1)$-derivative property and so is an (ordinary) intertwining operator; this certainly happens for $k = 0$ in the context of Remark 3.23 and for $k = K_1 + K_2 + K_3 - 3$ in the context of Remark 3.24.) However, in the following we will see that suitable formal linear combinations of certain modifications of $\mathcal{Y}^{(k)}$ (depending on $t \in \mathbb{N}$; see below) form a sequence of logarithmic intertwining operators.

Remark 3.27 Given a logarithmic intertwining operator $\mathcal{Y}$, let us write

$$\begin{aligned}\mathcal{Y}(w_{(1)},x)w_{(2)} &= \sum_{n\in\mathbb{C}}\sum_{k\in\mathbb{N}} {w_{(1)}}^{\mathcal{Y}}_{n;\,k}w_{(2)}x^{-n-1}(\log x)^k\\ &= \sum_{\mu\in\mathbb{C}/\mathbb{Z}}\sum_{\bar{n}=\mu}\sum_{k\in\mathbb{N}} {w_{(1)}}^{\mathcal{Y}}_{n;\,k}w_{(2)}x^{-n-1}(\log x)^k\end{aligned}$$

for any $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, where $\bar{n}$ denotes the equivalence class of n in $\mathbb{C}/\mathbb{Z}$. By extracting summands corresponding to the same congruence class modulo $\mathbb{Z}$ of the powers of x in (3.26), (3.27) and (3.28) we see that for each $\mu \in \mathbb{C}/\mathbb{Z}$,

$$w_{(1)} \otimes w_{(2)} \mapsto \mathcal{Y}^{\mu}(w_{(1)},x)w_{(2)} = \sum_{\bar{n}=\mu}\sum_{k\in\mathbb{N}} {w_{(1)}}^{\mathcal{Y}}_{n;\,k}w_{(2)}x^{-n-1}(\log x)^k \tag{3.46}$$

still defines a logarithmic intertwining operator. In the strongly graded case, if $\mathcal{Y}$ is grading-compatible, then so is the operator $\mathcal{Y}^{\mu}$ in (3.46). Conversely, suppose that we are given a family of logarithmic intertwining operators $\{\mathcal{Y}^{\mu} | \mu \in \mathbb{C}/\mathbb{Z}\}$ parametrized by $\mu \in \mathbb{C}/\mathbb{Z}$ such that the powers of x in $\mathcal{Y}^{\mu}$ are restricted as in (3.46). Then the formal sum $\sum_{\mu\in\mathbb{C}/\mathbb{Z}}\mathcal{Y}^{\mu}$ is well defined and is a logarithmic intertwining operator. In the strongly graded case, if each $\mathcal{Y}^{\mu}$ is grading-compatible, then so is this sum.

In the setting of Definition 3.10, for any integer p, set

$$\mathcal{Y}(w_{(1)}, e^{2\pi ip}x)w_{(2)} = \mathcal{Y}(w_{(1)},y)w_{(2)}\Big|_{y^n=e^{2\pi ipn}x^n,\ (\log y)^k=(2\pi ip+\log x)^k,\ n\in\mathbb{C},\ k\in\mathbb{N}}. \tag{3.47}$$

This is in fact a well-defined element of $W_3[\log x]\{x\}$. Note that this element certainly depends on p, not just on $e^{2\pi ip}$ $(= 1)$. This substitution, which can be thought of as "$x \mapsto e^{2\pi ip}x$," will be considered in a more general form in (3.76) below.

Remark 3.28 It is clear that in Definition 3.10, for any integer p, all the axioms are formally invariant under the substitution

$$x \mapsto e^{2\pi ip}x$$

given by (3.47). That is, if we apply this substitution to each axiom, the axiom keeps the same form, with the operator $\mathcal{Y}(\cdot, x)$ replaced by

$$\mathcal{Y}(\cdot, e^{2\pi ip}x).$$

For example, for the Jacobi identity (3.26), we perform the substitution $x_2 \mapsto e^{2\pi ip}x_2$; the formal delta-functions remain unchanged because they involve only integral powers of x_2 and no logarithms. It follows that $\mathcal{Y}(\cdot, e^{2\pi ip}x)$ is again a logarithmic intertwining operator.

From Remark 3.28, for any $\mu \in \mathbb{C}/\mathbb{Z}$ and logarithmic intertwining operator $\mathcal{Y}^{\mu}$ as in (3.46), the linear map defined by

$$w_{(1)} \otimes w_{(2)} \mapsto \mathcal{Y}^{\mu}(w_{(1)}, e^{2\pi ip}x)w_{(2)} = \sum_{\bar{n}=\mu}\sum_{k\in\mathbb{N}} {w_{(1)}}^{\mathcal{Y}}_{n;\,k}w_{(2)}e^{2\pi ip(-n-1)}x^{-n-1}(\log x + 2\pi ip)^k$$

is also a logarithmic intertwining operator. In the strongly graded case, if the operator in (3.46) is grading-compatible, then so is this one. The right-hand side above can be written as

$$\begin{aligned} & e^{-2\pi i p\mu}\sum_{\bar{n}=\mu}\sum_{k\in\mathbb{N}} w_{(1)}{}^{\mathcal{Y}}_{n;\,k}w_{(2)}x^{-n-1}\sum_{t\in\mathbb{N}}\binom{k}{t}(\log x)^{k-t}(2\pi i p)^{t} \\ &\quad = e^{-2\pi i p\mu}\sum_{t\in\mathbb{N}}(2\pi i p)^{t}\sum_{k\in\mathbb{N}}\binom{k+t}{t}\sum_{\bar{n}=\mu} w_{(1)}{}^{\mathcal{Y}}_{n;\,k+t}w_{(2)}x^{-n-1}(\log x)^{k} \end{aligned}$$

(the coefficient of each power of x being a finite sum over t and k). We now have:

Proposition 3.29 *Let W_1, W_2, W_3 be generalized modules for a Möbius (or conformal) vertex algebra V, and let $\mathcal{Y}(\cdot, x)\cdot$ be a logarithmic intertwining operator of type $\binom{W_3}{W_1\,W_2}$. For $\mu \in \mathbb{C}/\mathbb{Z}$ and $t \in \mathbb{N}$, define*

$$\mathcal{X}_t^{\mu} : W_1 \otimes W_2 \to W_3[\log x]\{x\}$$

by:

$$\mathcal{X}_t^{\mu} : w_{(1)} \otimes w_{(2)} \mapsto \sum_{k\in\mathbb{N}}\binom{k+t}{t}\sum_{\bar{n}=\mu} w_{(1)}{}^{\mathcal{Y}}_{n;\,k+t}w_{(2)}x^{-n-1}(\log x)^{k}.$$

Then each $\mathcal{X}_t^{\mu}$ is a logarithmic intertwining operator of type $\binom{W_3}{W_1\,W_2}$. Equivalently, the operator $\mathcal{X}_t$ defined by

$$\mathcal{X}_t : w_{(1)} \otimes w_{(2)} \mapsto \sum_{k\in\mathbb{N}}\binom{k+t}{t}\sum_{n\in\mathbb{C}} w_{(1)}{}^{\mathcal{Y}}_{n;\,k+t}w_{(2)}x^{-n-1}(\log x)^{k} \tag{3.48}$$

is a logarithmic intertwining operator of the same type. In the strongly graded case, if $\mathcal{Y}$ is grading-compatible, then so are $\mathcal{X}_t^{\mu}$ and $\mathcal{X}_t$.

Proof The equivalence of the two statements follows from Remark 3.27.

We now prove that for each $s \in \mathbb{N}$, $\mathcal{X}_s^{\mu}$ is a logarithmic intertwining operator. The conditions (3.24) and (3.25) for $\mathcal{X}_s^{\mu}$ follow from these conditions for $\mathcal{Y}$. From the above we see that for any integer p, $e^{-2\pi i p\mu}\sum_{t\in\mathbb{N}}(2\pi i p)^{t}\mathcal{X}_t^{\mu}$, and hence

$$\mathcal{Y}_p^{\mu} = \sum_{t\in\mathbb{N}}(2\pi i p)^{t}\mathcal{X}_t^{\mu}, \tag{3.49}$$

is a logarithmic intertwining operator.

To prove (3.26), (3.27) and (3.28) for $\mathcal{X}_s^{\mu}$, we observe that for fixed $w_{(1)}$, $w_{(2)}$ and n such that $\bar{n} = \mu$, and in fact for any finite set of $w_{(1)}$, $w_{(2)}$ and n such that $\bar{n} = \mu$, there exists $S \in \mathbb{N}$ with $S \geq s$ such that the coefficient

$$\mathcal{X}_t^{\mu}(w_{(1)} \otimes w_{(2)})_{[-n-1]}$$

of x^{-n-1} in $\mathcal{X}_t^\mu(w_{(1)}\otimes w_{(2)})$ is 0 if $t>S$. In particular, for any $p\in\mathbb{Z}$, the expression given by (3.49) for the coefficient

$$\mathcal{Y}_p^\mu(w_{(1)}\otimes w_{(2)})_{[-n-1]}$$

of x^{-n-1} in $\mathcal{Y}_p^\mu(w_{(1)}\otimes w_{(2)})$ becomes

$$\mathcal{Y}_p^\mu(w_1\otimes w_2)_{[-n-1]}=\sum_{t=0}^{S}(2\pi ip)^t\mathcal{X}_t^\mu(w_{(1)}\otimes w_{(2)})_{[-n-1]}.$$

Choosing $p\in\mathbb{N}$ and inverting the (finite) Vandermonde matrix

$$((2\pi ip)^t)_{p,t=0,\ldots,S},$$

we obtain that for $t\leq S$ (including $t=s$),

$$\mathcal{X}_t^\mu(w_{(1)}\otimes w_{(2)})_{[-n-1]}=\sum_{p=0}^{S}a_{tp}^{(S)}\mathcal{Y}_p^\mu(w_{(1)}\otimes w_{(2)})_{[-n-1]}$$

for some $a_{tp}^{(S)}\in\mathbb{C}$; note that $\sum_{p=0}^{S}a_{tp}^{(S)}\mathcal{Y}_p^\mu$ is a logarithmic intertwining operator. (Also note, incidentally, that when S is increased, this formula remains true even though the numbers $a_{tp}^{(S)}$ change; the inverses of the (finite) Vandermonde matrices do not stablize as S grows.) From this formula we see easily that the properties (3.26), (3.27) and (3.28) for each $\mathcal{Y}_p^\mu$ imply these properties for $\mathcal{X}_s^\mu$, and so $\mathcal{X}_s^\mu$ is a logarithmic intertwining operator. (A different proof that each $\mathcal{X}_t$, and hence $\mathcal{X}_t^\mu$, is a logarithmic intertwining operator is given in the course of the next three remarks.)

The last assertion is clear. □

Remark 3.30 Let W_i, W^i, $i=1,2,3$, be generalized modules for a Möbius (or conformal) vertex algebra V. If $\mathcal{Y}(\cdot,x)\cdot$ is a logarithmic intertwining operator of type $\binom{W_3}{W_1\,W_2}$ and

$$\sigma_1:W^1\to W_1,\ \sigma_2:W^2\to W_2\ \text{ and }\ \sigma_3:W_3\to W^3$$

are V-module homomorphisms, then it is easy to see that $\sigma_3\mathcal{Y}(\sigma_1\cdot,x)\sigma_2\cdot$ is a logarithmic intertwining operator of type $\binom{W^3}{W^1\,W^2}$. In the strongly graded case, if $\mathcal{Y}$ is grading-compatible, then so is $\sigma_3\mathcal{Y}(\sigma_1\cdot,x)\sigma_2\cdot$ (recall from Remark 2.28 that each σ_j preserves the $\tilde{A}$-grading). That is, in categorical language, let $\mathcal{C}$ be a full subcategory of either the category of V-modules, or the category of generalized V-modules, or $\mathcal{M}_{sg}$ (the category of strongly graded V-modules; recall Notation 2.36), or $\mathcal{GM}_{sg}$ (the category of strongly graded generalized V-modules; again recall Notation 2.36). Then the correspondence from $\mathcal{C}\times\mathcal{C}\times\mathcal{C}$ to the category **Vect** of vector spaces given by

$$(W_1,W_2,W_3)\mapsto\mathcal{V}_{W_1\,W_2}^{W_3}$$

is functorial in the third slot and cofunctorial in the first two slots.

Remark 3.31 Now recall from Remark 2.21 that $L(0) - L(0)_s$ commutes with the actions of both V and $\mathfrak{sl}(2)$. So $(L(0) - L(0)_s)^i$ is a V-module homomorphism from a generalized module to itself for any $i \in \mathbb{N}$, and this remains true in the strongly graded case. Hence by Remark 3.30, given any logarithmic intertwining operator $\mathcal{Y}(\cdot, x)\cdot$ as in (3.23) and any $i, j, k \in \mathbb{N}$,

$$(L(0) - L(0)_s)^k \mathcal{Y}((L(0) - L(0)_s)^i \cdot, x)(L(0) - L(0)_s)^j \cdot \tag{3.50}$$

is again a logarithmic intertwining operator, and in the strongly graded case, if $\mathcal{Y}$ is grading-compatible, so is this operator. In the next remark we will see that the logarithmic intertwining operators (3.48) are just linear combinations of these.

Remark 3.32 Let W_1, W_2, W_3 and $\mathcal{Y}$ be as above and let $w_{(1)} \in W_{1[n_1]}$ and $w_{(2)} \in W_{2[n_2]}$ for some complex numbers n_1 and n_2. Fixing $n \in \mathbb{C}$ and using the notation $T_{p,k,q}$ in (3.40) with $a = n_1$ and $b = n_2$, we rewrite formula (3.41) in the proof of Lemma 3.21 as

$$(k+1)T_{p,k+1,q} = (L(0) - n_1 - n_2 + n + 1)T_{p,k,q} - T_{p+1,k,q} - T_{p,k,q+1}$$

for any $p, q, k \in \mathbb{N}$. From this, by (3.38) we see that for any $t \in \mathbb{N}$,

$$\binom{k+t}{t} T_{p,k+t,q} = \sum_{i,j,l \in \mathbb{N},\ i+j+l=t} \frac{1}{i!j!l!}(-1)^{i+j}(L(0) - n_1 - n_2 + n + 1)^l T_{p+i,k,q+j}.$$

Setting $p = q = 0$ we get

$$\begin{aligned}\binom{k+t}{t} w_{(1)}{}^{\mathcal{Y}}_{n;\,k+t} w_{(2)} = &\sum_{i,j,l \in \mathbb{N},\ i+j+l=t} \frac{1}{i!j!l!}(-1)^{i+j} \cdot \\ &\cdot (L(0) - n_1 - n_2 + n + 1)^l ((L(0) - n_1)^i w_{(1)})^{\mathcal{Y}}_{n;\,k} (L(0) - n_2)^j w_{(2)})\end{aligned} \tag{3.51}$$

for any $t \in \mathbb{N}$. (Note that this formula gives an alternate proof of Proposition 3.20(c).) Now multiplying by $x^{-n-1}(\log x)^k$ and then summing over $n \in \mathbb{C}$ and over $k \in \mathbb{N}$ we see that for every $t \in \mathbb{N}$, $\mathcal{X}_t$ in (3.48) is a linear combination of logarithmic intertwining operators of the form (3.50).

In preparation for generalizing basic results from [FHL] on intertwining operators to the logarithmic case, we need to generalize more of the basic tools. We now define operators "$x^{\pm L(0)}$" for generalized modules, in the natural way:

Definition 3.33 Let W be a generalized module for a Möbius (or conformal) vertex algebra. We define

$$x^{\pm L(0)} : W \to W\{x\}[\log x] \subset W[\log x]\{x\}$$

as follows: For any $w \in W_{[n]}$ ($n \in \mathbb{C}$), define

$$x^{\pm L(0)} w = x^{\pm n} e^{\pm \log x (L(0) - n)} w \tag{3.52}$$

(note that the local nilpotence of $L(0)-n$ on $W_{[n]}$ insures that the formal exponential series terminates) and then extend linearly to all $w \in W$. (Of course, we could also write

$$x^{\pm L(0)} = x^{\pm L(0)_s} e^{\pm \log x (L(0)-L(0)_s)}, \tag{3.53}$$

using the notation $L(0)_s$.) We also define operators $x^{\pm L'(0)}$ on W^* by the condition that for all $w' \in W^*$ and $w \in W$,

$$\langle x^{\pm L'(0)} w', w\rangle = \langle w', x^{\pm L(0)} w\rangle \ (\in \mathbb{C}\{x\}[\log x]), \tag{3.54}$$

so that

$$x^{\pm L'(0)} : W^* \to W^*\{x\}[[\log x]].$$

Remark 3.34 Note that these definitions are (of course) compatible with the usual definitions if W is just an (ordinary) module. In formula (3.52), $x^{\pm L(0)}$ is defined in a naturally factored form reminiscent of the factorization invoked in Remark 3.9 (providing counterexamples there); the symbol $x^{\pm(L(0)-n)}$ is given meaning by its replacement by $e^{\pm \log x (L(0)-n)}$. Also note that in case both V and W are strongly graded, the definition of $x^{\pm L'(0)}$ given by (3.54), when applied to the subspace W' of W^*, coincides with the definition of $x^{\pm L'(0)}$ given by (3.52) induced from the contragredient module action of V on W'. (Recall Theorem 2.34.)

Remark 3.35 Note that for $w \in W_{[n]}$, by definition we have

$$x^{\pm L(0)} w = x^{\pm n} \sum_{i \in \mathbb{N}} \frac{(L(0)-n)^i w}{i!} (\pm \log x)^i \in x^{\pm n} W_{[n]}[\log x]. \tag{3.55}$$

It is also handy to have that for any $w \in W$,

$$x^{L(0)} x^{-L(0)} w = w = x^{-L(0)} x^{L(0)} w, \tag{3.56}$$

which is clear from definition. Later we will also need the formula

$$\frac{d}{dx} x^{\pm L(0)} w = \pm x^{-1} x^{\pm L(0)} L(0) w \tag{3.57}$$

for any $w \in W$, i.e.,

$$x \frac{d}{dx} x^{\pm L(0)} w = \pm x^{\pm L(0)} L(0) w, \tag{3.58}$$

or equivalently,

$$\left(x \frac{d}{dx} \mp L(0) \right) x^{\pm L(0)} w = 0 \tag{3.59}$$

(cf. Lemma 3.8 and Remark 3.9). This can be proved by directly checking that for w homogeneous of generalized weight n,

$$\frac{d}{dx} e^{\pm \log x (L(0)-n)} w = \pm x^{-1} e^{\pm \log x (L(0)-n)} (L(0)-n) w,$$

and hence for such w,

$$\begin{aligned}\frac{d}{dx}x^{\pm L(0)}w &= \frac{d}{dx}(x^{\pm n}e^{\pm \log x(L(0)-n)}w)\\ &= \pm nx^{\pm n-1}e^{\pm \log x(L(0)-n)}w \pm x^{\pm n-1}e^{\pm \log x(L(0)-n)}(L(0)-n)w\\ &= \pm x^{\pm n-1}e^{\pm \log x(L(0)-n)}L(0)w = \pm x^{-1}x^{\pm L(0)}L(0)w.\end{aligned}$$

In the statement (and proof) of the next result, we shall use expressions of the type

$$\begin{aligned}(1-x)^{L(0)} &= \sum_{k\in\mathbb{N}}\binom{L(0)}{k}(-x)^k\\ &= \sum_{k\in\mathbb{N}}\frac{L(0)(L(0)-1)\cdots(L(0)-k+1)}{k!}(-x)^k,\end{aligned}$$

which also equals

$$e^{L(0)\log(1-x)},$$

as well as expressions involving

$$(x(1-yx)^{-1})^n = \sum_{k\in\mathbb{N}}\binom{-n}{k}x^n(-yx)^k$$

for $n \in \mathbb{C}$ and

$$\begin{aligned}\log(x(1-yx)^{-1}) &= \log x + \log(1-yx)^{-1}\\ &= \log x + \sum_{k\geq 1}\frac{1}{k}(yx)^k.\end{aligned}$$

We can now state and prove generalizations to logarithmic intertwining operators of three standard formulas for (ordinary) intertwining operators, namely, formulas (5.4.21), (5.4.22) and (5.4.23) of [FHL]. The result is (see also [M] for Parts (a) and (b)):

Proposition 3.36 *Let $\mathcal{Y}$ be a logarithmic intertwining operator of type $\binom{W_3}{W_1\,W_2}$ and let $w \in W_1$. Then*

(a)

$$e^{yL(-1)}\mathcal{Y}(w,x)e^{-yL(-1)} = \mathcal{Y}(e^{yL(-1)}w,x) = \mathcal{Y}(w,x+y) \tag{3.60}$$

(recall (3.5))

(b)

$$y^{L(0)}\mathcal{Y}(w,x)y^{-L(0)} = \mathcal{Y}(y^{L(0)}w,xy) \tag{3.61}$$

(recall (3.8))

(c)

$$e^{yL(1)}\mathcal{Y}(w,x)e^{-yL(1)} = \mathcal{Y}(e^{y(1-yx)L(1)}(1-yx)^{-2L(0)}w,x(1-yx)^{-1}). \tag{3.62}$$

Proof From (3.28) with $j = -1$ we see that for any $w \in W_1$,

$$L(-1)\mathcal{Y}(w,x) = \mathcal{Y}(L(-1)w,x) + \mathcal{Y}(w,x)L(-1).$$

This implies

$$\frac{y^n(L(-1))^n}{n!}\mathcal{Y}(w,x) = \sum_{i,j\in\mathbb{N},\, i+j=n} \mathcal{Y}\Big(\frac{y^i(L(-1))^i}{i!}w,x\Big)\frac{y^j(L(-1))^j}{j!}$$

for any $n \in \mathbb{N}$, where y is a new formal variable. Summing over $n \in \mathbb{N}$ we see that for any $w \in W_1$,

$$e^{yL(-1)}\mathcal{Y}(w,x) = \mathcal{Y}(e^{yL(-1)}w,x)e^{yL(-1)},$$

and hence by (3.10),

$$e^{yL(-1)}\mathcal{Y}(w,x)e^{-yL(-1)} = \mathcal{Y}(e^{yL(-1)}w,x) = \mathcal{Y}(w,x+y) \tag{3.63}$$

(and this expression also equals $e^{y\frac{d}{dx}}\mathcal{Y}(w,x)$). Note that all the expressions in (3.63) remain well defined if we replace y by any element of $y\mathbb{C}[x][[y]]$ and that (3.63) still holds if we replace y by any such element. (But note that (3.10) would fail to hold if we replaced y by for example $yx \in y\mathbb{C}[x][[y]]$.)

For (b), note that for homogeneous $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, by (3.36) with the formal variable y replaced by the formal variable $\log y$, we get (recalling Proposition 3.20(b) and (3.52))

$$\begin{aligned} &y^{L(0)}(w_{(1)}{}^{\mathcal{Y}}_{n;\,k}w_{(2)}) = \\ &\quad\sum_{l\in\mathbb{N}} \binom{k+l}{k} (y^{L(0)}w_{(1)})^{\mathcal{Y}}_{n;\,k+l}(y^{L(0)}w_{(2)})y^{-n-1}(\log y)^l \end{aligned}$$

for any $n \in \mathbb{C}$ and $k \in \mathbb{N}$. Multiplying this by $x^{-n-1}(\log x)^k$, summing over $n \in \mathbb{C}$ and $k \in \mathbb{N}$ and using (3.8) we get

$$y^{L(0)}\mathcal{Y}(w_{(1)},x)w_{(2)} = \mathcal{Y}(y^{L(0)}w_{(1)},xy)y^{L(0)}w_{(2)}.$$

Formula (3.61) then follows from (3.56).

Finally we prove (c). From (3.28) with $j = 1$ we see that for any $w \in W_1$,

$$L(1)\mathcal{Y}(w,x) = \mathcal{Y}((L(1) + 2xL(0) + x^2L(-1))w,x) + \mathcal{Y}(w,x)L(1).$$

This implies that

$$e^{yL(1)}\mathcal{Y}(w,x) = \mathcal{Y}(e^{y(L(1)+2xL(0)+x^2L(-1))}w,x)e^{yL(1)},$$

or

$$e^{yL(1)}\mathcal{Y}(w,x)e^{-yL(1)} = \mathcal{Y}(e^{y(L(1)+2xL(0)+x^2L(-1))}w,x).$$

Using the identity

$$e^{y(L(1)+2xL(0)+x^2L(-1))} = e^{yx^2(1-yx)^{-1}L(-1)}e^{y(1-yx)L(1)}(1-yx)^{-2L(0)},$$

whose proof is exactly the same as the proof of formula (5.2.41) of [FHL], we obtain

$$e^{yL(1)}\mathcal{Y}(w,x)e^{-yL(1)} = \mathcal{Y}(e^{yx^2(1-yx)^{-1}L(-1)}e^{y(1-yx)L(1)}(1-yx)^{-2L(0)}w,x). \tag{3.64}$$

But by (3.63) with y replaced by $yx^2(1-yx)^{-1}$, the right-hand side of (3.64) is equal to

$$\mathcal{Y}(e^{y(1-yx)L(1)}(1-yx)^{-2L(0)}w, x+yx^2(1-yx)^{-1}). \tag{3.65}$$

Since

$$(x+yx^2(1-yx)^{-1})^n = (x(1-yx)^{-1})^n$$

for $n \in \mathbb{C}$ and

$$\log(x+yx^2(1-yx)^{-1}) = \log(x(1-yx)^{-1}),$$

(3.65) is equal to the right-hand side of (3.62), proving (3.62). □

Remark 3.37 The following formula, also a generalization of the corresponding formula in the ordinary case (see (2.69)), will be needed: For $j = -1, 0, 1$,

$$x^{L(0)}L(j)x^{-L(0)} = x^{-j}L(j). \tag{3.66}$$

To prove this, we first observe that for any $m \in \mathbb{C}$, $[L(0)-m, L(j)] = -jL(j)$ implies that

$$e^{\log x(L(0)-m)}L(j)e^{-\log x(L(0)-m)} = e^{-j\log x}L(j).$$

Hence, for a generalized module element w homogeneous of generalized weight n,

$$\begin{aligned} x^{L(0)}L(j)w &= x^{n-j}e^{\log x(L(0)-n+j)}L(j)w \\ &= x^{n-j}e^{-j\log x}L(j)e^{\log x(L(0)-n+j)}w \\ &= x^{n-j}L(j)e^{\log x(L(0)-n)}w \\ &= x^{-j}L(j)x^{L(0)}w, \end{aligned}$$

and (3.66) then follows immediately from (3.56). (Another proof: By Remark 2.21, the second factor on the right-hand side of (3.53) commutes with $L(j)$.)

Remark 3.38 From (3.66) we see that

$$x^{L(0)}e^{yL(j)}x^{-L(0)} = e^{yx^{-j}L(j)}. \tag{3.67}$$

For an ordinary module W for a vertex operator algebra and any $a \in \mathbb{C}$, the operator $e^{aL(0)}$ on W is defined by

$$e^{aL(0)}w = e^{ah}w \tag{3.68}$$

for any homogeneous $w \in W_{(h)}$, $h \in \mathbb{C}$ and then by linear extension to any $w \in W$. More generally, for a generalized module W for a Möbius (or conformal) vertex algebra and any $a \in \mathbb{C}$, we define the operator $e^{aL(0)}$ on W by

$$e^{aL(0)}w = e^{ah}e^{a(L(0)-h)}w \tag{3.69}$$

for any homogeneous $w \in W_{[h]}$, $h \in \mathbb{C}$ and then by linear extension to all $w \in W$. (Note that for a formal variable x, we already have $e^{xL(0)}w = e^{hx}e^{x(L(0)-h)}w$.) From the definition,

$$e^{aL(0)}e^{-aL(0)}w = w. \tag{3.70}$$

Recalling Remark 2.21 for the notation $L(0)_s$, we see that

$$e^{aL(0)} = e^{aL(0)_s}e^{a(L(0)-L(0)_s)} \text{ on } W, \tag{3.71}$$

where the exponential series $e^{a(L(0)-L(0)_s)}$ terminates on each element of W.

Remark 3.39 The operators defined in (3.68) and (3.69) can be alternatively defined or viewed as the (analytically) convergent sums of the indicated exponential series of operators; these operators act on the (finite-dimensional) subspaces of W generated by the repeated action of $L(0)$ on homogeneous vectors $w \in W$.

Remark 3.40 The operator $e^{a(L(0)-L(0)_s)}$ on W is a V-homomorphism, in view of Remark 2.21 (cf. Remark 3.31). Let r be an integer. Then $e^{2\pi i r L(0)_s}$ is also a V-homomorphism, by Remark 2.20. Thus for $r \in \mathbb{Z}$, $e^{2\pi i r L(0)}$ is a V-homomorphism. In the strongly graded case, all of these V-homomorphisms are grading-preserving.

Remark 3.41 We now recall some identities about the action of $\mathfrak{sl}(2)$ on any of its modules. For convenience we put them in the following form:

$$e^{xL(-1)}\begin{pmatrix} L(-1) \\ L(0) \\ L(1) \end{pmatrix} e^{-xL(-1)} = \begin{pmatrix} 1 & 0 & 0 \\ -x & 1 & 0 \\ x^2 & -2x & 1 \end{pmatrix} \begin{pmatrix} L(-1) \\ L(0) \\ L(1) \end{pmatrix} \tag{3.72}$$

$$e^{xL(0)}\begin{pmatrix} L(-1) \\ L(0) \\ L(1) \end{pmatrix} e^{-xL(0)} = \begin{pmatrix} e^x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-x} \end{pmatrix} \begin{pmatrix} L(-1) \\ L(0) \\ L(1) \end{pmatrix} \tag{3.73}$$

$$e^{xL(1)}\begin{pmatrix} L(-1) \\ L(0) \\ L(1) \end{pmatrix} e^{-xL(1)} = \begin{pmatrix} 1 & 2x & x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} L(-1) \\ L(0) \\ L(1) \end{pmatrix}. \tag{3.74}$$

Formula (3.73) follows from (5.2.12) and (5.2.13) in [FHL]; Formula (3.74) follows from (5.2.14) in [FHL] and $[L(1), L(-1)] = 2L(0)$; and formula (3.72) follows from (3.74) and the fact that

$$L(-1) \mapsto L(1),\ L(0) \mapsto -L(0),\ L(1) \mapsto L(-1)$$

is a Lie algebra automorphism of $\mathfrak{sl}(2)$.

Remark 3.42 It is convenient to note that the $\mathfrak{sl}(2)$-bracket relations (3.28) are equivalent to

$$\mathcal{Y}(L(j)w_{(1)},x)=\sum_{i=0}^{j+1}\binom{j+1}{i}(-x)^i[L(j-i),\mathcal{Y}(w_{(1)},x)] \tag{3.75}$$

for $j=-1$, 0 and 1. This can be easily checked by writing (3.28) as

$$\left(\begin{array}{c}[L(-1),\mathcal{Y}(w_{(1)},x)]\\ [L(0),\mathcal{Y}(w_{(1)},x)]\\ [L(1),\mathcal{Y}(w_{(1)},x)]\end{array}\right)=\left(\begin{array}{ccc}1&0&0\\ x&1&0\\ x^2&2x&1\end{array}\right)\left(\begin{array}{c}\mathcal{Y}(L(-1)w_{(1)},x)\\ \mathcal{Y}(L(0)w_{(1)},x)\\ \mathcal{Y}(L(1)w_{(1)},x)\end{array}\right)$$

and then multiplying this by the inverse of the invertible matrix on the right-hand side, obtained by replacing x by $-x$. (Of course, in the case where V is conformal, this equivalence is already encoded in the symmetry of the Jacobi identity.)

We have already defined the natural process of multiplying the formal variable in a logarithmic intertwining operator by $e^{2\pi ip}$ for $p\in\mathbb{Z}$ (recall (3.47)), and this process yields another logarithmic intertwining operator (recall Remark 3.28). It is natural to generalize this substitution process to that of multiplying the formal variable in a logarithmic intertwining operator by the exponential e^{ζ} of any complex number ζ. As in the special case $\zeta=2\pi ip$, the process will depend on ζ, not just on e^{ζ}, but we will still find it convenient to use the shorthand symbol e^{ζ} in our notation for the process. In Section 7 of [HL2], we introduced this procedure in the case of ordinary (nonlogarithmic) intertwining operators, and we now carry it out in the general logarithmic case. We are about to use this substitution mostly for

$$\zeta=(2r+1)\pi i,\ r\in\mathbb{Z}.$$

Let (W_1,Y_1), (W_2,Y_2) and (W_3,Y_3) be generalized modules for a Möbius (or conformal) vertex algebra V. Let $\mathcal{Y}$ be a logarithmic intertwining operator of type $\binom{W_3}{W_1\,W_2}$. For any complex number ζ and any $w_{(1)}\in W_1$ and $w_{(2)}\in W_2$, set

$$\mathcal{Y}(w_{(1)},e^{\zeta}x)w_{(2)}=\mathcal{Y}(w_{(1)},y)w_{(2)}\Big|_{y^n=e^{\zeta n}x^n,\ (\log y)^k=(\zeta+\log x)^k,\ n\in\mathbb{C},\ k\in\mathbb{N}}, \tag{3.76}$$

a well-defined element of $W_3[\log x]\{x\}$. Note that this element indeed depends on ζ, not just on e^{ζ}.

Remark 3.43 In Section 4 below we will take the further step of specializing the formal variable x to 1 (or equivalently, the formal variable y to e^{ζ}) in (3.76); that is, we will consider $\mathcal{Y}(w_{(1)},e^{\zeta})w_{(2)}$.

Given any $r\in\mathbb{Z}$, we define

$$\Omega_r(\mathcal{Y}):W_2\otimes W_1\to W_3[\log x]\{x\}$$

by the formula

$$\Omega_r(\mathcal{Y})(w_{(2)},x)w_{(1)}=e^{xL(-1)}\mathcal{Y}(w_{(1)},e^{(2r+1)\pi i}x)w_{(2)} \tag{3.77}$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. This expression is indeed well defined because of the truncation condition (3.25) (recall Remark 3.12). The following result generalizes Proposition 7.1 of [HL2] (for the ordinary intertwining operator case) and has essentially the same proof as that proposition, which in turn generalized Proposition 5.4.7 of [FHL]:

Proposition 3.44 *The operator $\Omega_r(\mathcal{Y})$ is a logarithmic intertwining operator of type $\binom{W_3}{W_2\, W_1}$. Moreover,*

$$\Omega_{-r-1}(\Omega_r(\mathcal{Y})) = \Omega_r(\Omega_{-r-1}(\mathcal{Y})) = \mathcal{Y}. \tag{3.78}$$

In the strongly graded case, if $\mathcal{Y}$ is grading-compatible, then so is $\Omega_r(\mathcal{Y})$, and in particular, the correspondence $\mathcal{Y} \mapsto \Omega_r(\mathcal{Y})$ defines a linear isomorphism from $\mathcal{V}_{W_1\, W_2}^{W_3}$ to $\mathcal{V}_{W_2\, W_1}^{W_3}$, and we have

$$N_{W_1\, W_2}^{W_3} = N_{W_2\, W_1}^{W_3}.$$

Proof The lower truncation condition (3.25) is clear. From the Jacobi identity (3.26) for $\mathcal{Y}$,

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1-y}{x_0}\right) & Y_3(v,x_1)\mathcal{Y}(w_{(1)},y)w_{(2)} \\ & -x_0^{-1}\delta\left(\frac{y-x_1}{-x_0}\right)\mathcal{Y}(w_{(1)},y)Y_2(v,x_1)w_{(2)} \\ & = y^{-1}\delta\left(\frac{x_1-x_0}{y}\right)\mathcal{Y}(Y_1(v,x_0)w_{(1)},y)w_{(2)}, \end{aligned} \tag{3.79}$$

with $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, we obtain

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1-y}{x_0}\right) e^{-yL(-1)}Y_3(v,x_1)\mathcal{Y}(w_{(1)},y)w_{(2)}\Big|_{y^n=e^{(2r+1)\pi i n}x_2^n,\ (\log y)^k=((2r+1)\pi i+\log x_2)^k,\ n\in\mathbb{C},\ k\in\mathbb{N}} \\ & \quad -x_0^{-1}\delta\left(\frac{y-x_1}{-x_0}\right) e^{-yL(-1)}\mathcal{Y}(w_{(1)},y)Y_2(v,x_1)w_{(2)}\Big|_{y^n=e^{(2r+1)\pi i n}x_2^n,\ (\log y)^k=((2r+1)\pi i+\log x_2)^k,\ n\in\mathbb{C},\ k\in\mathbb{N}} \\ & = y^{-1}\delta\left(\frac{x_1-x_0}{y}\right) e^{-yL(-1)}\mathcal{Y}(Y_1(v,x_0)w_{(1)},y)w_{(2)}\Big|_{y^n=e^{(2r+1)\pi i n}x_2^n,\ (\log y)^k=((2r+1)\pi i+\log x_2)^k,\ n\in\mathbb{C},\ k\in\mathbb{N}} \end{aligned} \tag{3.80}$$

The first term of the left-hand side of (3.80) is equal to

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1-y}{x_0}\right) Y_3(v,x_1-y)e^{-yL(-1)}\mathcal{Y}(w_{(1)},y)w_{(2)}\Big|_{y^n=e^{(2r+1)\pi i n}x_2^n,\ (\log y)^k=((2r+1)\pi i+\log x_2)^k,\ n\in\mathbb{C},\ k\in\mathbb{N}} \\ & \qquad = x_0^{-1}\delta\left(\frac{x_1+x_2}{x_0}\right) Y_3(v,x_0)\Omega_r(\mathcal{Y})(w_{(2)},x_2)w_{(1)}, \end{aligned}$$

the second term is equal to

$$-x_0^{-1}\delta\left(\frac{-x_2-x_1}{-x_0}\right)\Omega_r(\mathcal{Y})(Y_2(v,x_1)w_{(2)},x_2)w_{(1)}$$

and the right-hand side of (3.80) is equal to

$$-x_2^{-1}\delta\left(\frac{x_1-x_0}{-x_2}\right)\Omega_r(\mathcal{Y})(w_{(2)},x_2)Y_1(v,x_0)w_{(1)}.$$

Substituting into (3.80) we obtain

$$\begin{aligned}x_0^{-1}\delta\left(\frac{x_1+x_2}{x_0}\right)&Y_3(v,x_0)\Omega_r(\mathcal{Y})(w_{(2)},x_2)w_{(1)}\\ &-x_0^{-1}\delta\left(\frac{x_2+x_1}{x_0}\right)\Omega_r(\mathcal{Y})(Y_2(v,x_1)w_{(2)},x_2)w_{(1)}\\ =-x_2^{-1}\delta&\left(\frac{x_1-x_0}{-x_2}\right)\Omega_r(\mathcal{Y})(w_{(2)},x_2)Y_1(v,x_0)w_{(1)},\end{aligned}\tag{3.81}$$

which is equivalent to

$$\begin{aligned}x_1^{-1}\delta\left(\frac{x_0-x_2}{x_1}\right)&Y_3(v,x_0)\Omega_r(\mathcal{Y})(w_{(2)},x_2)w_{(1)}\\ &-x_1^{-1}\delta\left(\frac{x_2-x_0}{-x_1}\right)\Omega_r(\mathcal{Y})(w_{(2)},x_2)Y_1(v,x_0)w_{(1)}\\ =x_2^{-1}\delta&\left(\frac{x_0-x_1}{x_2}\right)\Omega_r(\mathcal{Y})(Y_2(v,x_1)w_{(2)},x_2)w_{(1)}\end{aligned}\tag{3.82}$$

(recall (2.6)). This in turn is the Jacobi identity for $\Omega_r(\mathcal{Y})$ (with the roles of x_0 and x_1 reversed in (3.26)).

To prove the $L(-1)$-derivative property (3.27) for $\Omega_r(\mathcal{Y})$, first note that from (3.76) and the $L(-1)$-derivative property for $\mathcal{Y}$,

$$\begin{aligned}\frac{d}{dx}\mathcal{Y}(w_{(1)},e^{\zeta}x)w_{(2)} &= e^{\zeta}\left(\frac{d}{dy}\mathcal{Y}(w_{(1)},y)w_{(2)}\right)\Big|_{y^n=e^{\zeta n}x^n,\ (\log y)^k=(\zeta+\log x)^k,\ n\in\mathbb{C},\ k\in\mathbb{N}}\\ &= e^{\zeta}\mathcal{Y}(L(-1)w_{(1)},e^{\zeta}x)w_{(2)},\end{aligned}\tag{3.83}$$

and in particular,

$$\frac{d}{dx}\mathcal{Y}(w_{(1)},e^{(2r+1)\pi i}x)w_{(2)}=-\mathcal{Y}(L(-1)w_{(1)},e^{(2r+1)\pi i}x)w_{(2)}.\tag{3.84}$$

Thus, by using formula (3.28) with $j=-1$ we have

$$\begin{aligned}\frac{d}{dx}\Omega_r(\mathcal{Y})(w_{(2)},x)w_{(1)}&=\frac{d}{dx}e^{xL(-1)}\mathcal{Y}(w_{(1)},e^{(2r+1)\pi i}x)w_{(2)}\\ &=e^{xL(-1)}L(-1)\mathcal{Y}(w_{(1)},e^{(2r+1)\pi i}x)w_{(2)}+e^{xL(-1)}\frac{d}{dx}\mathcal{Y}(w_{(1)},e^{(2r+1)\pi i}x)w_{(2)}\\ &=e^{xL(-1)}L(-1)\mathcal{Y}(w_{(1)},e^{(2r+1)\pi i}x)w_{(2)}-e^{xL(-1)}\mathcal{Y}(L(-1)w_{(1)},e^{(2r+1)\pi i}x)w_{(2)}\\ &=e^{xL(-1)}\mathcal{Y}(w_{(1)},e^{(2r+1)\pi i}x)L(-1)w_{(2)}\\ &=\Omega_r(\mathcal{Y})(L(-1)w_{(2)},x)w_{(1)},\end{aligned}\tag{3.85}$$

as desired.

In the case that V is Möbius, we prove the $\mathfrak{sl}(2)$-bracket relations (3.28) for $\Omega_r(\mathcal{Y})$. By using the $\mathfrak{sl}(2)$-bracket relations for $\mathcal{Y}$ and the relations

$$e^{xL(-1)}L(j)e^{-xL(-1)} = \sum_{i=0}^{j+1}\binom{j+1}{i}(-x)^i L(j-i)$$

for $j = -1, 0$ and -1 (see (3.72)), we have

$$\begin{aligned}
\Omega_r(\mathcal{Y})(L(j)w_{(2)}, x)w_{(1)} &= e^{xL(-1)}\mathcal{Y}(w_{(1)}, e^{(2r+1)\pi i}x)L(j)w_{(2)} \\
&= e^{xL(-1)}L(j)\mathcal{Y}(w_{(1)}, e^{(2r+1)\pi i}x)w_{(2)} \\
&\quad - e^{xL(-1)}\sum_{i=0}^{j+1}\binom{j+1}{i}(-x)^i\mathcal{Y}(L(j-i)w_{(1)}, e^{(2r+1)\pi i}x)w_{(2)} \\
&= \sum_{i=0}^{j+1}\binom{j+1}{i}(-x)^i L(j-i)e^{xL(-1)}\mathcal{Y}(w_{(1)}, e^{(2r+1)\pi i}x)w_{(2)} \\
&\quad - e^{xL(-1)}\sum_{i=0}^{j+1}\binom{j+1}{i}(-x)^i\mathcal{Y}(L(j-i)w_{(1)}, e^{(2r+1)\pi i}x)w_{(2)} \\
&= \sum_{i=0}^{j+1}\binom{j+1}{i}(-x)^i(L(j-i)\Omega_r(\mathcal{Y})(w_{(2)}, x) - \Omega_r(\mathcal{Y})(w_{(2)}, x)L(j-i))w_{(1)},
\end{aligned}$$

which is the alternative form (3.75) of the $\mathfrak{sl}(2)$-bracket relations for $\Omega_r(\mathcal{Y})$.

The identity (3.78) is clear from the definitions of $\Omega_r(\mathcal{Y})$ and $\Omega_{-r-1}(\mathcal{Y})$, and the remaining assertions are clear. $\square$

Remark 3.45 For each triple $s_1, s_2, s_3 \in \mathbb{Z}$, the logarithmic intertwining operator $\mathcal{Y}$ gives rise to a logarithmic intertwining operator $\mathcal{Y}_{[s_1,s_2,s_3]}$ of the same type, defined by

$$\mathcal{Y}_{[s_1,s_2,s_3]}(w_{(1)}, x) = e^{2\pi i s_1 L(0)}\mathcal{Y}(e^{2\pi i s_2 L(0)}w_{(1)}, x)e^{2\pi i s_3 L(0)}$$

for $w_{(1)} \in W_1$, by Remarks 3.30 and 3.40. In the strongly graded case, if $\mathcal{Y}$ is grading-compatible, so is $\mathcal{Y}_{[s_1,s_2,s_3]}$. Clearly,

$$\mathcal{Y}_{[0,0,0]} = \mathcal{Y}$$

and for $r_1, r_2, r_3, s_1, s_2, s_3 \in \mathbb{Z}$,

$$(\mathcal{Y}_{[r_1,r_2,r_3]})_{[s_1,s_2,s_3]} = \mathcal{Y}_{[r_1+s_1,r_2+s_2,r_3+s_3]}.$$

For any $a \in \mathbb{C}$, we have the formula

$$e^{aL(0)}\mathcal{Y}(w_{(1)}, x)e^{-aL(0)} = \mathcal{Y}(e^{aL(0)}w_{(1)}, e^a x) \tag{3.86}$$

(cf. (3.61)). This is proved by imitating the proof of (3.61), replacing $y^{L(0)}$ by $e^{aL(0)}$, y by e^a and $\log y$ by a in that proof, using (3.69) in place of (3.52) and keeping in mind formula (3.76). (When (3.36) is used in this proof, for homogeneous elements $w_{(1)}$ and $w_{(2)}$, the exponential series all terminate, as does the sum over $l \in \mathbb{N}$.) From this, we see that (3.78) generalizes to

$$\Omega_s(\Omega_r(\mathcal{Y})) = \mathcal{Y}_{[r+s+1,-(r+s+1),-(r+s+1)]} = \mathcal{Y}(\cdot, e^{2\pi i(r+s+1)}\cdot)$$

for all $r, s \in \mathbb{Z}$.

In case V, W_1, W_2 and W_3 are strongly graded, which we now assume, we have the concept of "r-contragredient operator" as follows (in the ordinary intertwining operator case this was introduced in [HL2]): Given a grading-compatible logarithmic intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1\,W_2}$ and an integer r, we define the *r-contragredient operator of $\mathcal{Y}$* to be the linear map

$$\begin{array}{rcl} W_1 \otimes W_3' & \to & W_2'\{x\}[[\log x]] \\ w_{(1)} \otimes w'_{(3)} & \mapsto & A_r(\mathcal{Y})(w_{(1)}, x)w'_{(3)} \end{array}$$

given by

$$\begin{aligned} &\langle A_r(\mathcal{Y})(w_{(1)}, x)w'_{(3)}, w_{(2)}\rangle_{W_2} \\ &\quad = \langle w'_{(3)}, \mathcal{Y}(e^{xL(1)}e^{(2r+1)\pi i L(0)}(x^{-L(0)})^2 w_{(1)}, x^{-1})w_{(2)}\rangle_{W_3}, \end{aligned} \tag{3.87}$$

for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W_3'$, where we use the notation

$$f(x^{-1}) = \sum_{m\in\mathbb{N},\, n\in\mathbb{C}} w_{n,m}x^{-n}(-\log x)^m$$

for any

$$f(x) = \sum_{m\in\mathbb{N},\, n\in\mathbb{C}} w_{n,m}x^{n}(\log x)^m \in \mathcal{W}\{x\}[[\log x]],$$

$\mathcal{W}$ any vector space (not involving x). Note that for the case $W_1 = V$ and $W_2 = W_3 = W$, the operator $A_r(\mathcal{Y})$ agrees with the contragredient vertex operator $\mathcal{Y}'$ (recall (2.57) and (2.73)) for any $r \in \mathbb{Z}$.

We have the following result generalizing Proposition 7.3 in [HL2] for ordinary intertwining operators, and having essentially the same proof (and also generalizing Theorem 5.5.1 and Proposition 5.5.2 of [FHL]):

Proposition 3.46 *The r-contragredient operator $A_r(\mathcal{Y})$ of a grading-compatible logarithmic intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1\,W_2}$ is a grading-compatible logarithmic intertwining operator of type $\binom{W_2'}{W_1\,W_3'}$. Moreover,*

$$A_{-r-1}(A_r(\mathcal{Y})) = A_r(A_{-r-1}(\mathcal{Y})) = \mathcal{Y}. \tag{3.88}$$

In particular, the correspondence $\mathcal{Y} \mapsto A_r(\mathcal{Y})$ defines a linear isomorphism from $\mathcal{V}_{W_1 W_2}^{W_3}$ to $\mathcal{V}_{W_1 W_3'}^{W_2'}$, and we have

$$N_{W_1 W_2}^{W_3} = N_{W_1 W_3'}^{W_2'}.$$

Proof First we need to show that for $w_{(1)} \in W_1$ and $w'_{(3)} \in W'_3$,

$$A_r(\mathcal{Y})(w_{(1)}, x)w'_{(3)} \in W'_2[\log x]\{x\}, \tag{3.89}$$

that is, for each power of x there are only finitely many powers of $\log x$. This and the lower truncation condition (3.25) as well as the grading-compatibility condition (3.31) follow from a variant of the argument proving the lower truncation condition for contragredient vertex operators (recall (2.101)):

Fix elements $w_{(1)} \in W_1$ and $w'_{(3)} \in W'_3$ homogeneous with respect to the double gradings of W_1 and W'_3 (it will suffice to prove the desired assertions for such elements), and in fact take

$$w_{(1)} \in W_1^{(\beta)} \;\text{ and }\; w'_{(3)} \in (W'_3)^{(\gamma)},$$

where β and γ are elements of the abelian group $\tilde{A}$, in the notation of Definition 2.25, and fix $n \in \mathbb{C}$. The right-hand side of (3.87) is a (finite) sum of terms of the form

$$\langle w'_{(3)}, \mathcal{Y}(w, x^{-1})w_{(2)}\rangle x^p(\log x)^q \tag{3.90}$$

where $w \in W_1$ is doubly homogeneous and in fact $w \in W_1^{(\beta)}$ (by (2.88)), and where $p \in \mathbb{C}$ and $q \in \mathbb{N}$. (The pairing $\langle\cdot,\cdot\rangle$ is between W'_3 and W_3.) Let $w_{(2)}^{(-\beta-\gamma)}$ be the component of (the arbitrary element) $w_{(2)}$ in $W_2^{(-\beta-\gamma)}$, with respect to the $\tilde{A}$-grading. Then (3.90) equals

$$\langle w'_{(3)}, \mathcal{Y}(w, x^{-1})w_{(2)}^{(-\beta-\gamma)}\rangle x^p(\log x)^q \tag{3.91}$$

because of the grading-compatibility condition (3.31) together with (2.97). (This is why we need our logarithmic intertwining operators to be grading-compatible.) This shows in particular that

$$w_{(1)}{}^{A_r(\mathcal{Y})}_{N;\,K} w'_{(3)} \in (W'_2)^{(\beta+\gamma)} \tag{3.92}$$

for $N \in \mathbb{C}$ and $K \in \mathbb{N}$, so that (3.31) holds for $A_r(\mathcal{Y})$. Let us write (3.91) as

$$\begin{aligned}\sum_{l\in\mathbb{C}}\sum_{k\in\mathbb{N}}&\langle w'_{(3)}, w^{\mathcal{Y}}_{l;\,k} w_{(2)}^{(-\beta-\gamma)}\rangle x^{l+1+p}(-1)^k(\log x)^{k+q}\\ &= \sum_{m\in\mathbb{C}}\sum_{k\in\mathbb{N}}\langle w'_{(3)}, w^{\mathcal{Y}}_{n-p-1-m;\,k} w_{(2)}^{(-\beta-\gamma)}\rangle x^{n-m}(-1)^k(\log x)^{k+q}\end{aligned} \tag{3.93}$$

(recall that we have fixed $n \in \mathbb{C}$). But each term $\langle w'_{(3)}, w^{\mathcal{Y}}_{n-p-1-m;\,k} w_{(2)}^{(-\beta-\gamma)}\rangle$ in (3.93) can be replaced by

$$\langle w'_{(3)}, w^{\mathcal{Y}}_{n-p-1-m;\,k} u_{[m]}\rangle, \tag{3.94}$$

where $u_{[m]} \in W_2^{(-\beta-\gamma)}$ is the component of $w_{(2)}^{(-\beta-\gamma)}$, with respect to the generalized-weight grading, of (generalized) weight

$$\mathrm{wt}\ u_{[m]} = \mathrm{wt}\ w'_{(3)} - \mathrm{wt}\ w + n - p - m,$$

by Proposition 3.20(b). To see that the coefficient of x^n in (3.93) involves only finitely many powers of $\log x$, independently of the element $w_{(2)}$, we take $m = 0$ in (3.93) and we observe that the possible elements $u_{[0]}$ range through the space

$$(W_2)^{(-\beta-\gamma)}_{[\mathrm{wt}\ w'_{(3)} - \mathrm{wt}\ w + n - p]},$$

which is finite dimensional by the grading restriction condition (2.86). This proves (3.89). To prove the lower truncation condition (3.25), what we must show is that for sufficiently large $m \in \mathbb{N}$, the coefficient of x^{n-m} in (3.93) is 0 (independently of $w_{(2)}$). But by the grading restriction condition (2.85),

$$(W_2)^{(-\beta-\gamma)}_{[\mathrm{wt}\ w'_{(3)} - \mathrm{wt}\ w + n - p - m]} = 0 \quad \text{for} \quad m \in \mathbb{N} \text{ sufficiently large.}$$

Hence the coefficient of x^{n-m} in (3.93) is zero for $m \in \mathbb{N}$ sufficiently large, as desired, proving the lower truncation condition.

For the Jacobi identity, we need to show that

$$\begin{aligned}
&\left\langle x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_2'(v, x_1) A_r(\mathcal{Y})(w_{(1)}, x_2) w'_{(3)}, w_{(2)}\right\rangle_{W_2} \\
&\quad - \left\langle x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) A_r(\mathcal{Y})(w_{(1)}, x_2) Y_3'(v, x_1) w'_{(3)}, w_{(2)}\right\rangle_{W_2} \\
&\quad = \left\langle x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) A_r(\mathcal{Y})(Y_1(v, x_0) w_{(1)}, x_2) w'_{(3)}, w_{(2)}\right\rangle_{W_2}. \qquad (3.95)
\end{aligned}$$

By the definitions (2.73) and (3.87) we have

$$\begin{aligned}
&\langle Y_2'(v, x_1) A_r(\mathcal{Y})(w_{(1)}, x_2) w'_{(3)}, w_{(2)}\rangle_{W_2} \\
&\quad = \langle w'_{(3)}, \mathcal{Y}(e^{x_2 L(1)} e^{(2r+1)\pi i L(0)} (x_2^{-L(0)})^2 w_{(1)}, x_2^{-1}) \cdot \\
&\qquad\qquad \cdot Y_2(e^{x_1 L(1)} (-x_1^{-2})^{L(0)} v, x_1^{-1}) w_{(2)}\rangle_{W_3}, \qquad (3.96)
\end{aligned}$$

$$\begin{aligned}
&\langle A_r(\mathcal{Y})(w_{(1)}, x_2) Y_3'(v, x_1) w'_{(3)}, w_{(2)}\rangle_{W_2} \\
&\quad = \langle w'_{(3)}, Y_3(e^{x_1 L(1)} (-x_1^{-2})^{L(0)} v, x_1^{-1}) \cdot \\
&\qquad\qquad \cdot \mathcal{Y}(e^{x_2 L(1)} e^{(2r+1)\pi i L(0)} (x_2^{-L(0)})^2 w_{(1)}, x_2^{-1}) w_{(2)}\rangle_{W_3}, \qquad (3.97)
\end{aligned}$$

$$\begin{aligned}
&\langle A_r(\mathcal{Y})(Y_1(v, x_0) w_{(1)}, x_2) w'_{(3)}, w_{(2)}\rangle_{W_2} \\
&\quad = \langle w'_{(3)}, \mathcal{Y}(e^{x_2 L(1)} e^{(2r+1)\pi i L(0)} (x_2^{-L(0)})^2 Y_1(v, x_0) w_{(1)}, x_2^{-1}) w_{(2)}\rangle_{W_3}. \qquad (3.98)
\end{aligned}$$

From the Jacobi identity for $\mathcal{Y}$ we have

$$\Big\langle w'_{(3)}, \left(\frac{-x_0}{x_1x_2}\right)^{-1}\delta\left(\frac{x_1^{-1}-x_2^{-1}}{-x_0/x_1x_2}\right) Y_3(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v, x_1^{-1})\cdot$$
$$\cdot\mathcal{Y}(e^{x_2L(1)}e^{(2r+1)\pi iL(0)}(x_2^{-L(0)})^2w_{(1)}, x_2^{-1})w_{(2)}\Big\rangle_{W_3}$$
$$-\Big\langle w'_{(3)}, \left(\frac{-x_0}{x_1x_2}\right)^{-1}\delta\left(\frac{x_2^{-1}-x_1^{-1}}{x_0/x_1x_2}\right)\mathcal{Y}(e^{x_2L(1)}e^{(2r+1)\pi iL(0)}(x_2^{-L(0)})^2w_{(1)}, x_2^{-1})\cdot$$
$$\cdot Y_2(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v, x_1^{-1})w_{(2)}\Big\rangle_{W_3}$$
$$= \Big\langle w'_{(3)}, (x_2^{-1})^{-1}\delta\left(\frac{x_1^{-1}+x_0/x_1x_2}{x_2^{-1}}\right)\mathcal{Y}(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v, -x_0/x_1x_2)\cdot$$
$$\cdot e^{x_2L(1)}e^{(2r+1)\pi iL(0)}(x_2^{-L(0)})^2w_{(1)}, x_2^{-1})w_{(2)}\Big\rangle_{W_3}, \tag{3.99}$$

or equivalently,

$$-\Big\langle w'_{(3)}, x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right) Y_3(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v, x_1^{-1})\cdot$$
$$\cdot\mathcal{Y}(e^{x_2L(1)}e^{(2r+1)\pi iL(0)}(x_2^{-L(0)})^2w_{(1)}, x_2^{-1})w_{(2)}\Big\rangle_{W_3}$$
$$+\Big\langle w'_{(3)}, x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\mathcal{Y}(e^{x_2L(1)}e^{(2r+1)\pi iL(0)}(x_2^{-L(0)})^2w_{(1)}, x_2^{-1})\cdot$$
$$\cdot Y_2(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v, x_1^{-1})w_{(2)}\Big\rangle_{W_3}$$
$$= \Big\langle w'_{(3)}, x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\mathcal{Y}(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v, -x_0/x_1x_2)\cdot$$
$$\cdot e^{x_2L(1)}e^{(2r+1)\pi iL(0)}(x_2^{-L(0)})^2w_{(1)}, x_2^{-1})w_{(2)}\Big\rangle_{W_3}. \tag{3.100}$$

Substituting (3.96), (3.97) and (3.98) into (3.95) and then comparing with (3.100), we see that the proof of (3.95) is reduced to the proof of the formula

$$x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\mathcal{Y}(e^{x_2L(1)}e^{(2r+1)\pi iL(0)}(x_2^{-L(0)})^2Y_1(v, x_0)w_{(1)}, x_2^{-1})$$
$$= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\mathcal{Y}(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v, -x_0/x_1x_2)\cdot$$
$$\cdot e^{x_2L(1)}e^{(2r+1)\pi iL(0)}(x_2^{-L(0)})^2w_{(1)}, x_2^{-1}), \tag{3.101}$$

or of

$$\mathcal{Y}(e^{x_2L(1)}e^{(2r+1)\pi iL(0)}(x_2^{-L(0)})^2Y_1(v, x_0)w_{(1)}, x_2^{-1})$$

$$= \mathcal{Y}(Y_1(e^{(x_2+x_0)L(1)}(-(x_2+x_0)^{-2})^{L(0)}v, -x_0/(x_2+x_0)x_2)\cdot \\ \cdot e^{x_2L(1)}e^{(2r+1)\pi iL(0)}(x_2^{-L(0)})^2 w_{(1)}, x_2^{-1}). \tag{3.102}$$

We see that we need only prove

$$e^{x_2L(1)}e^{(2r+1)\pi iL(0)}(x_2^{-L(0)})^2 Y_1(v, x_0) \\ = Y_1(e^{(x_2+x_0)L(1)}(-(x_2+x_0)^{-2})^{L(0)}v, -x_0/(x_2+x_0)x_2)\cdot \\ \cdot e^{x_2L(1)}e^{(2r+1)\pi iL(0)}(x_2^{-L(0)})^2 \tag{3.103}$$

or equivalently, the conjugation formula

$$e^{x_2L(1)}e^{(2r+1)\pi iL(0)}(x_2^{-L(0)})^2 Y_1(v, x_0)(x_2^{L(0)})^2 e^{-(2r+1)\pi iL(0)}e^{-x_2L(1)} \\ = Y_1(e^{(x_2+x_0)L(1)}(-(x_2+x_0)^{-2})^{L(0)}v, -x_0/(x_2+x_0)x_2) \tag{3.104}$$

for $v \in V$, acting on the module W_1. But formula (3.104) follows from (3.61), (3.62) and the formula

$$e^{(2r+1)\pi iL(0)}Y_1(v, x)e^{-(2r+1)\pi iL(0)} = Y_1((-1)^{L(0)}v, -x), \tag{3.105}$$

which is a special case of (3.86). This establishes the Jacobi identity.

The $L(-1)$-derivative property follows from the same argument used in the proof of Theorem 5.5.1 of [FHL]: We have (omitting the subscript W_3 on the pairings after a certain point)

$$\left\langle \frac{d}{dx}A_r(\mathcal{Y})(w_{(1)}, x)w'_{(3)}, w_{(2)}\right\rangle_{W_2} = \frac{d}{dx}\langle A_r(\mathcal{Y})(w_{(1)}, x)w'_{(3)}, w_{(2)}\rangle_{W_2} \\ = \frac{d}{dx}\langle w'_{(3)}, \mathcal{Y}(e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2 w_{(1)}, x^{-1})w_{(2)}\rangle_{W_3} \\ = \langle w'_{(3)}, \frac{d}{dx}\mathcal{Y}(e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2 w_{(1)}, x^{-1})w_{(2)}\rangle \\ = \langle w'_{(3)}, \mathcal{Y}(\frac{d}{dx}(e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2 w_{(1)}), x^{-1})w_{(2)}\rangle \\ +\langle w'_{(3)}, \frac{d}{dx}\mathcal{Y}(w, x^{-1})|_{w=e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2 w_{(1)}} w_{(2)}\rangle \\ = \langle w'_{(3)}, \mathcal{Y}(e^{xL(1)}L(1)e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2 w_{(1)}, x^{-1})w_{(2)}\rangle \\ +\langle w'_{(3)}, \mathcal{Y}(e^{xL(1)}(-2L(0)x^{-1})e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2 w_{(1)}, x^{-1})w_{(2)}\rangle \\ +\langle w'_{(3)}, \frac{d}{dx^{-1}}\mathcal{Y}(w, x^{-1})|_{w=e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2 w_{(1)}} w_{(2)}\rangle(-x^{-2}) \\ = \langle w'_{(3)}, \mathcal{Y}(e^{xL(1)}(2xL(0) - x^2L(1))e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2(-x^{-2})w_{(1)}, x^{-1})w_{(2)}\rangle \\ +\langle w'_{(3)}, \mathcal{Y}(L(-1)e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2 w_{(1)}, x^{-1})w_{(2)}\rangle(-x^{-2}) \\ = \langle w'_{(3)}, \mathcal{Y}(e^{xL(1)}(2xL(0) - x^2L(1))e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2(-x^{-2})w_{(1)}, x^{-1})w_{(2)}\rangle \\ +\langle w'_{(3)}, \mathcal{Y}(L(-1)e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2(-x^{-2})w_{(1)}, x^{-1})w_{(2)}\rangle \tag{3.106}$$

Now by (3.74), with x replaced by $-x$, we have

$$L(-1)e^{xL(1)} = e^{xL(1)}(L(-1) - 2xL(0) + x^2L(1)).$$

Using this together with (3.66) and (3.73) (with x specialized to $-(2r+1)\pi i$, and the convergence of the exponential series invoked; recall Remark 3.39), we see that the right-hand side of (3.106) equals

$$\begin{aligned}
&\langle w'_{(3)}, \mathcal{Y}(e^{xL(1)}L(-1)e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2(-x^{-2})w_{(1)}, x^{-1})w_{(2)}\rangle \\
&\quad = \langle w'_{(3)}, \mathcal{Y}(e^{xL(1)}e^{(2r+1)\pi iL(0)}(-L(-1))(x^{-L(0)})^2(-x^{-2})w_{(1)}, x^{-1})w_{(2)}\rangle \\
&\quad = \langle w'_{(3)}, \mathcal{Y}(e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2L(-1)w_{(1)}, x^{-1})w_{(2)}\rangle \\
&\quad = \langle A_r(\mathcal{Y})(L(-1)w_{(1)}, x)w'_{(3)}, w_{(2)}\rangle_{W_2},
\end{aligned}$$

as desired.

We now show that, in case V is Möbius, the $\mathfrak{sl}(2)$-bracket relations (3.28) hold for $A_r(\mathcal{Y})$. For these, we first see that, for $j = -1, 0, 1$, by using (2.75), (3.87) and the $\mathfrak{sl}(2)$-bracket relations (3.28) for $\mathcal{Y}$ we have

$$\begin{aligned}
&\langle L'(j)A_r(\mathcal{Y})(w_{(1)}, x)w'_{(3)}, w_{(2)}\rangle_{W_2} \\
&\quad = \langle A_r(\mathcal{Y})(w_{(1)}, x)w'_{(3)}, L(-j)w_{(2)}\rangle_{W_2} \\
&\quad = \langle w'_{(3)}, \mathcal{Y}(e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2w_{(1)}, x^{-1})L(-j)w_{(2)}\rangle_{W_3} \\
&\quad = \langle w'_{(3)}, L(-j)\mathcal{Y}(e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2w_{(1)}, x^{-1})w_{(2)}\rangle_{W_3} \\
&\qquad -\Bigg\langle w'_{(3)}, \sum_{i=0}^{-j+1}\binom{-j+1}{i}x^{-i}\cdot \\
&\qquad\qquad \cdot\mathcal{Y}(L(-j-i)e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2w_{(1)}, x^{-1})w_{(2)}\Bigg\rangle_{W_3}. \qquad (3.107)
\end{aligned}$$

Now from (3.74), (3.73) (with x specialized to $-(2r+1)\pi i$) and (3.66), one computes that

$$\begin{aligned}
&(x^{L(0)})^2e^{-(2r+1)\pi iL(0)}e^{-xL(1)}\begin{pmatrix} L(-1) \\ L(0) \\ L(1) \end{pmatrix} e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2 \\
&\qquad = \begin{pmatrix} -x^2 & -2x & -1 \\ 0 & 1 & x^{-1} \\ 0 & 0 & -x^{-2} \end{pmatrix}\begin{pmatrix} L(-1) \\ L(0) \\ L(1) \end{pmatrix}
\end{aligned}$$

on W_1, which implies that

$$\begin{aligned}
&\sum_{i=0}^{-j+1}\binom{-j+1}{i}x^{-i}L(-j-i)e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2 \\
&\qquad = -\sum_{i=0}^{j+1}\binom{j+1}{i}x^{i}e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2L(j-i)
\end{aligned}$$

on W_1. Hence the right-hand side of (3.107) is equal to

$$\begin{aligned}
&\langle L'(j)w'_{(3)}, \mathcal{Y}(e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2w_{(1)}, x^{-1})w_{(2)}\rangle_{W_3}\\
&\quad +\sum_{i=0}^{j+1}\binom{j+1}{i}x^i\langle w'_{(3)}, \mathcal{Y}(e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2L(j-i)w_{(1)}, x^{-1})w_{(2)}\rangle_{W_2}\\
&= \langle A_r(\mathcal{Y})(w_{(1)}, x)L'(j)w'_{(3)}, w_{(2)}\rangle_{W_2}\\
&\quad +\sum_{i=0}^{j+1}\binom{j+1}{i}x^i\langle A_r(\mathcal{Y})(L(j-i)w_{(1)}, x)w'_{(3)}, w_{(2)}\rangle_{W_2},
\end{aligned}$$

and the $\mathfrak{sl}(2)$-bracket relations for $A_r(\mathcal{Y})$ are proved. (Note that this argument essentially generalizes the proof of Lemma 2.22.) We have finished proving that $A_r(\mathcal{Y})$ is a grading-compatible logarithmic intertwining operator.

Finally, for the relation (3.88), we of course identify W_2'' with W_2 and W_3'' with W_3, according to Theorem 2.34. Let us view $\mathcal{Y}$ as a grading-compatible logarithmic intertwining operator of type $\binom{W_2'}{W_1\, W_3'}$, so that $A_r(\mathcal{Y})$ is such an operator of type $\binom{W_3}{W_1\, W_2}$. We have

$$\begin{aligned}
&\langle A_{-r-1}A_r(\mathcal{Y})(w_{(1)}, x)w'_{(3)}, w_{(2)}\rangle_{W_2}\\
&\quad = \langle w'_{(3)}, A_r(\mathcal{Y})(e^{xL(1)}e^{(-2r-1)\pi iL(0)}(x^{-L(0)})^2w_{(1)}, x^{-1})w_{(2)}\rangle_{W_3}\\
&\quad = \langle \mathcal{Y}(e^{x^{-1}L(1)}e^{(2r+1)\pi iL(0)}(x^{L(0)})^2e^{xL(1)}e^{(-2r-1)\pi iL(0)}(x^{-L(0)})^2w_{(1)}, x)w'_{(3)}, w_{(2)}\rangle_{W_2}\\
&\quad = \langle \mathcal{Y}(w_{(1)}, x)w'_{(3)}, w_{(2)}\rangle_{W_2},
\end{aligned}$$

where the last equality is due to the relation

$$e^{(2r+1)\pi iL(0)}(x^{L(0)})^2e^{xL(1)}(x^{-L(0)})^2e^{-(2r+1)\pi iL(0)} = e^{-x^{-1}L(1)} \tag{3.108}$$

on W_1, whose proof is similar to that of formula (5.3.1) of [FHL]. Namely, (3.108) follows from the relation

$$e^{(2r+1)\pi iL(0)}(x^{L(0)})^2xL(1)(x^{-L(0)})^2e^{-(2r+1)\pi iL(0)} = -x^{-1}L(1), \tag{3.109}$$

which realizes the transformation $x \mapsto -\frac{1}{x}$, and (3.109) follows from (3.66) together with (3.73) specialized as above. $\square$

Remark 3.47 The last argument in the proof shows that for any $r, s \in \mathbb{Z}$, formula (3.88) generalizes to:

$$A_s(A_r(\mathcal{Y})) = \mathcal{Y}_{[0,r+s+1,0]}$$

(recall Remark 3.45).

With V, W_1, W_2 and W_3 strongly graded, set

$$N_{W_1W_2W_3} = N_{W_1\, W_2}^{W_3'}. \tag{3.110}$$

Then Proposition 3.44 gives

$$N_{W_1W_2W_3} = N_{W_2W_1W_3}$$

and Proposition 3.46 gives

$$N_{W_1W_2W_3} = N_{W_1W_3W_2}.$$

Thus for any permutation σ of $(1,2,3)$,

$$N_{W_1W_2W_3} = N_{W_{\sigma(1)}W_{\sigma(2)}W_{\sigma(3)}}. \tag{3.111}$$

It is clear from Proposition 3.20(b) that in the nontrivial logarithmic intertwining operator case, taking projections of $\mathcal{Y}(w_{(1)}, x)w_{(2)}$ to (generalized) weight subspaces is not enough to recover its coefficients of $x^n(\log x)^k$ for $n \in \mathbb{C}$ and $k \in \mathbb{N}$, in contrast with the (ordinary) intertwining operator case (cf. [HL1], the paragraph containing formula (4.17)). However, taking projections of certain related intertwining operators does indeed suffice for this purpose:

Proposition 3.48 *Let W_1, W_2, W_3 be generalized modules for a Möbius (or conformal) vertex algebra V and let $\mathcal{Y}$ be a logarithmic intertwining operator of type $\binom{W_3}{W_1\, W_2}$. Let $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ be homogeneous of generalized weights n_1 and n_2, respectively. Then for any $n \in \mathbb{C}$ and any $r \in \mathbb{N}$, ${w_{(1)}}^{\mathcal{Y}}_{n;\,r}w_{(2)}$ can be written as a certain linear combination of products of the component of weight $n_1 + n_2 - n - 1$ of*

$$(L(0) - n_1 - n_2 + n + 1)^l \mathcal{Y}((L(0) - n_1)^i w_{(1)}, x)(L(0) - n_2)^j w_{(2)} \tag{3.112}$$

for certain $i, j, l \in \mathbb{N}$ with monomials of the form $x^{n+1}(\log x)^m$ for certain $m \in \mathbb{N}$.

Proof Multiplying (3.51) by $x^{-n-1}(\log x)^k$ and summing over $k \in \mathbb{N}$ (a finite sum by definition) we have that for any $t \in \mathbb{N}$,

$$\begin{aligned}
&\sum_{k\in\mathbb{N}} \binom{k+t}{t} {w_{(1)}}^{\mathcal{Y}}_{n;\,k+t} w_{(2)} x^{-n-1}(\log x)^k \ \left(= x^{-n-1} \sum_{k\in\mathbb{N}} \binom{k}{t} {w_{(1)}}^{\mathcal{Y}}_{n;\,k} w_{(2)} (\log x)^{k-t} \right)\\
&\quad = \sum_{i,j,l\in\mathbb{N},\, i+j+l=t} \frac{1}{i!j!l!}(-1)^{i+j} \sum_{k\in\mathbb{N}} (L(0) - n_1 - n_2 + n + 1)^l \cdot\\
&\qquad \cdot((L(0) - n_1)^i w_{(1)})^{\mathcal{Y}}_{n;\,k}(L(0) - n_2)^j w_{(2)}) x^{-n-1}(\log x)^k\\
&\quad = \sum_{i,j,l\in\mathbb{N},\, i+j+l=t} \frac{1}{i!j!l!}(-1)^{i+j} \pi_{n_1+n_2-n-1}((L(0) - n_1 - n_2 + n + 1)^l \cdot\\
&\qquad \cdot\mathcal{Y}((L(0) - n_1)^i w_{(1)}, x)(L(0) - n_2)^j w_{(2)}).
\end{aligned} \tag{3.113}$$

Let K be a positive integer such that ${w_{(1)}}^{\mathcal{Y}}_{n;\,k'} w_{(2)} = 0$ for all $k' \geq K$, Denote the right-hand side of (3.113) by $\pi(t, w_{(1)}, w_{(2)}, x, \log x)$. Then by putting the identities (3.113) for $t = 0, 1, \ldots, K-1$ together in matrix form we have

$$x^{-n-1} A \begin{pmatrix} {w_{(1)}}^{\mathcal{Y}}_{n;\,0} w_{(2)} \\ {w_{(1)}}^{\mathcal{Y}}_{n;\,1} w_{(2)} \\ \vdots \\ {w_{(1)}}^{\mathcal{Y}}_{n;\,K-1} w_{(2)} \end{pmatrix} = \begin{pmatrix} \pi(0, w_{(1)}, w_{(2)}, x, \log x) \\ \pi(1, w_{(1)}, w_{(2)}, x, \log x) \\ \vdots \\ \pi(K-1, w_{(1)}, w_{(2)}, x, \log x) \end{pmatrix} \tag{3.114}$$

where A is the $K\times K$ matrix whose (i,j)-entry is equal to $\binom{j-1}{i-1}(\log x)^{j-i}$. Letting P_K be the triangular matrix whose (i,j)-entry is $\binom{j-1}{i-1}$ (an upper triangular "Pascal matrix"), we have

$$A = \mathrm{diag}(1, (\log x)^{-1}, \dots, (\log x)^{-(K-1)}) \cdot P_K \cdot \mathrm{diag}(1, \log x, \dots, (\log x)^{K-1}).$$

Its inverse is

$$A^{-1} = \mathrm{diag}(1, (\log x)^{-1}, \dots, (\log x)^{-(K-1)}) \cdot P_K^{-1} \cdot \mathrm{diag}(1, \log x, \dots, (\log x)^{K-1})$$

and the (i,j)-entry of P_K^{-1} is $(-1)^{i+j}\binom{j-1}{i-1}$. Now multiplying the left-hand side of (3.114) by $x^{n+1}A^{-1}$ we obtain

$$\begin{pmatrix} w_{(1)}{}^{\mathcal{Y}}_{n;\,0}w_{(2)} \\ w_{(1)}{}^{\mathcal{Y}}_{n;\,1}w_{(2)} \\ \vdots \\ w_{(1)}{}^{\mathcal{Y}}_{n;\,K-1}w_{(2)} \end{pmatrix} = x^{n+1}A^{-1}\begin{pmatrix} \pi(0, w_{(1)}, w_{(2)}, x, \log x) \\ \pi(1, w_{(1)}, w_{(2)}, x, \log x) \\ \vdots \\ \pi(K-1, w_{(1)}, w_{(2)}, x, \log x) \end{pmatrix}$$

or explicitly,

$$(w_{(1)})^{\mathcal{Y}}_{n;\,r}w_{(2)} = x^{n+1}\sum_{t=r}^{K-1}(-1)^{r+t}\binom{t}{r}(\log x)^{t-r}\pi(t, w_{(1)}, w_{(2)}, x, \log x) \tag{3.115}$$

for $r = 0, 1, \dots, K-1$. (In particular, all x's and $\log x$'s cancel out in the right-hand side of (3.115).) $\square$

Remark 3.49 Proposition 3.48 says that taking projections of (3.112) to (generalized) weight subspaces is enough to recover the coefficients of $\mathcal{Y}$. Although we will not need to use Proposition 3.48 in the proof of our main results, we will heavily use certain analytic results, in particular, Propositions 7.8 and 7.9 and Corollary 7.10 in Section 7, to determine, using projections, coefficients of products of powers of complex variables and of their logs, in expressions involving logarithmic intertwining operators, in many results that we will need, including in particular Propositions 7.16 and 7.18 and their respective corollaries.

References

[FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, preprint, 1989; *Memoirs Amer. Math. Soc.* **104**, 1993.

[FLM] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Appl. Math., Vol. 134, Academic Press, Boston, 1988.

[HL1] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, I, *Selecta Mathematica (New Series)* **1** (1995), 699–756.

[HL2] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, II, *Selecta Mathematica (New Series)* **1** (1995), 757–786.

[HLZ1] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory for generalized modules for a conformal vertex algebra, I: Introduction and strongly graded algebras and their generalized modules, to appear.

[HLZ2] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, III: Intertwining maps and tensor product bifunctors, to appear.

[HLZ3] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, IV: Constructions of tensor product bifunctors and the compatibility conditions, to appear.

[HLZ4] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, V: Convergence condition for intertwining maps and the corresponding compatibility condition, to appear.

[HLZ5] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, VI: Expansion condition, associativity of logarithmic intertwining operators, and the associativity isomorphisms, to appear.

[HLZ6] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, VII: Convergence and extension properties and applications to expansion for intertwining maps, to appear.

[HLZ7] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, VIII: Braided tensor category structure on categories of generalized modules for a conformal vertex algebra, to appear.

[Lub] D. Lubell, Problem 10992, Problems and Solutions, *American Mathematical Monthly* **110** (2003), 155.

[M] A. Milas, Weak modules and logarithmic intertwining operators for vertex operator algebras, in *Recent Developments in Infinite-Dimensional Lie Algebras and Conformal Field Theory*, ed. S. Berman, P. Fendley, Y.-Z. Huang, K. Misra, and B. Parshall, Contemp. Math., Vol. 297, American Mathematical Society, Providence, RI, 2002, 201–225.

Department of Mathematics, Rutgers University, Piscataway, NJ 08854 (permanent address)
and
Beijing International Center for Mathematical Research, Peking University, Beijing, China
E-mail address: yzhuang@math.rutgers.edu

Department of Mathematics, Rutgers University, Piscataway, NJ 08854
E-mail address: lepowsky@math.rutgers.edu

Department of Mathematics, Rutgers University, Piscataway, NJ 08854
E-mail address: linzhang@math.rutgers.edu