Affine Lie algebras and tensor categories

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Outline

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Affine Lie algebras

- Let $\mathfrak{g}$ be a complex simple Lie algebra of rank r and $(\cdot,\cdot)$ the invariant symmetric bilinear form on $\mathfrak{g}$.
- The affine Lie algebra $\hat{\mathfrak{g}}$ is the vector space $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$ equipped with the bracket operation

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + (a, b)m\delta_{m+n,0}\mathbf{k},$$

$$[a \otimes t^m, \mathbf{k}] = 0,$$

  for $a, b \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$.
- Let $\hat{\mathfrak{g}}_{\pm} = \mathfrak{g} \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}]$. Then

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_{-} \oplus \mathfrak{g} \oplus \mathbb{C}\mathbf{k} \oplus \hat{\mathfrak{g}}_{+}.$$

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Modules

- If $\mathbf{k}$ acts as a complex number ℓ on a $\hat{\mathfrak{g}}$-module, then ℓ is called the level of the module.
- Let M be a $\mathfrak{g}$-module and let $\ell \in \mathbb{C}$. Let $\hat{\mathfrak{g}}_+$ act on M trivially and let $\mathbf{k}$ act as the scalar multiplication by ℓ. Then M becomes a $\mathfrak{g} \oplus \mathbb{C}\mathbf{k} \oplus \hat{\mathfrak{g}}_+$-module.
- We have a $\mathbb{C}$-graded induced $\hat{\mathfrak{g}}$-module

$$\widehat{M}_\ell = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g} \oplus \mathbb{C}\mathbf{k} \oplus \hat{\mathfrak{g}}_+)} M.$$

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- If **k** acts as a complex number ℓ on a $\hat{\mathfrak{g}}$-module, then ℓ is called the level of the module.
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- Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. For $\lambda \in \mathfrak{h}^*$, let $L(\lambda)$ be the irreducible highest weight $\mathfrak{g}$-module with the highest weight λ.
- We use $M(\ell, \lambda)$ to denote the $\hat{\mathfrak{g}}$-module $\widehat{L(\lambda)}_\ell$.
- Let $J(\ell, \lambda)$ be the maximal proper submodule of $M(\ell, \lambda)$ and $L(\ell, \lambda) = M(\ell, \lambda)/J(\ell, \lambda)$.
- $L(\ell, \lambda)$ is the unique irreducible graded $\hat{\mathfrak{g}}$-module such that $\mathbf{k}$ acts as ℓ and the space of all elements annihilated by $\hat{\mathfrak{g}}_+$ is isomorphic to the $\mathfrak{g}$-module $L(\lambda)$.
- $M(\ell, 0)$ and $L(\ell, 0)$ have natural structures of vertex operator algebras. $M(\ell, \lambda)$ and $L(\ell, \lambda)$ are $M(\ell, 0)$-modules and $L(\ell, \lambda)$ is an $L(\ell, 0)$-module for dominant integral λ.

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Module categories

- Let h and $h^\vee$ be the Coxeter number and dual Coxeter number, respectively, of $\mathfrak{g}$.
- For $\ell \in \mathbb{C}$ such that $\ell + h^\vee \notin \mathbb{Q}_{\geq 0}$, let $\mathcal{O}_\ell$ be the category of all the $\hat{\mathfrak{g}}$-modules of level ℓ having a finite composition series all of whose irreducible subquotients are of the form $L(\ell, \lambda)$ for dominant integral $\lambda \in \mathfrak{h}^*$.
- For $\ell \in \mathbb{Z}_+$, let $\widetilde{\mathcal{O}}_\ell$ be the category of $\hat{\mathfrak{g}}$-modules of level ℓ that are isomorphic to direct sums of irreducible $\hat{\mathfrak{g}}$-modules of the form $L(\ell, \lambda)$ for dominant integral $\lambda \in \mathfrak{h}^*$ such that $(\lambda, \theta) \leq \ell$, where θ is the highest root of $\mathfrak{g}$.

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- Admissible modules for affine Lie algebras were studied first by Kac and Wakimoto. The level of these modules are called admissible numbers.
- Let ℓ be an admissible number, that is, $\ell + h^\vee = \frac{p}{q}$ for $p, q \in \mathbb{Z}_+$, $(p, q) = 1$, $p \geq h^\vee$ if $(r^\vee, q) = 1$ and $p \geq h$ if $(r^\vee, q) = r^\vee$, where $r^\vee$ is the "lacety" or lacing number of $\mathfrak{g}$, that is, the maximum number of edges in the Dynkin diagram of $\mathfrak{g}$.
- For such an admissible number, let $\mathcal{O}_{\ell,\mathrm{ord}}$ be the category of $\hat{\mathfrak{g}}$-modules of level ℓ that are isomorphic to direct sums of irreducible modules for the vertex operator algebra $L(\ell, 0)$.

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Conjectures by Moore and Seiberg and their proofs

- The work of Moore-Seiberg on two-dimensional conformal field theory led to a **conjecture**: The category $\widetilde{O}_\ell$ for $\ell \in \mathbb{Z}_+$ has a natural structure of a modular tensor category in the sense of Turaev. This conjecture was proved by me in 2005.
- Moore and Seiberg obtained this conjecture based on two major conjectures on chiral rational conformal field theories: **Operator product expansion of chiral vertex operators** and **modular invariance of chiral vertex operators**. These conjectures were proved by me in 2002 and 2003, respectively.
- Mathematically, chiral vertex operators are called intertwining operators. Conformal field theory can be viewed as the study of intertwining operators.

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- A braided tensor category is a tensor category with a braiding isomorphism such that two hexagon diagrams are commutative.
- A tensor category is rigid if every object has a two-sided dual object.
- A ribbon tensor category is a rigid braided tensor category with a twist satisfying the balancing axioms.
- A modular tensor category is a semisimple ribbon tensor category such that the matrix of the Hopf link invariants is invertible.

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The construction of Kazhdan-Lusztig

Theorem (Kazhdan-Lusztig)

Let $\ell \in \mathbb{C}$ such that $\ell + h^\vee \notin \mathbb{Q}_{\geq 0}$. Then $\mathcal{O}_\ell$ has a natural rigid braided tensor category structure. Moreover, this rigid braided tensor category is equivalent to the rigid braided tensor category of finite-dimensional integrable modules for a quantum group constructed from $\mathfrak{g}$ at $q = e^{\frac{i\pi}{\ell + h^\vee}}$.

- This result was announced in 1991. The detailed constructions were published in 1993 and 1994.
- The construction of the rigid braided tensor category structure, especially the rigidity, depends heavily on the results on the quantum group side.
- This construction cannot be adapted directly to give constructions for the module categories at other levels.

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The vertex-operator-algebraic construction

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The braided tensor category structure

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Direct construction of the equivalence of categories in the case $\ell \in \mathbb{Z}_+$

- Even after the correction in 2013, Finkelberg's proof of the rigidity is not complete. There are a few cases, including the $\mathfrak{g} = \mathfrak{e}_8$ and $\ell = 2$ case, that his method does not work.
- Finkelberg's equivalence between the modular tensor category $\widetilde{\mathcal{O}}_\ell$ and a semisimple subquotient of a rigid braided tensor category of modules for a quantum group is also not complete because of the same few cases, including the $\mathfrak{g} = \mathfrak{e}_8$ and $\ell = 2$ case in which his method does not work.
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Theorem (Creutzig-H.-Yang)

Let ℓ be an admissible number. Then the category $\mathcal{O}_{\ell,\mathrm{ord}}$ has a natural structure of a braided tensor category with a twist.

- This theorem was proved in 2017 using the logarithmic tensor category theory of H.-Lepowsky-Zhang, some results of Kazhdan-Lusztig and the recent results of Arakawa.
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Theorem (Creutzig-H.-Yang)

Let $\mathfrak{g} = \mathfrak{sl}_2$ and $\ell = -2 + \frac{p}{q}$ with p, q coprime positive integers. Then the braided tensor category $\mathcal{O}_{\ell,\mathrm{ord}}$ is a ribbon tensor category and is a modular tensor category if and only if q is odd.

- The idea of the proof of this theorem is to prove that this tensor category is braided equivalent to a full tensor subcategory of the modular tensor category of modules for a minimal Virasoro vertex operator algebra. The modular tensor category structures for minimal Virasoro vertex operator algebras were constructed by me in 2005.

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Two conjectures in the case of admissible case

- Let $\mathfrak{g}$ be a simple Lie algebra and ℓ an admissible number. Then in particular, $\ell = -h^\vee + \frac{p}{q}$ with coprime positive integers p, q.
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  1. $\mathfrak{g} \in \{\mathfrak{sl}_{2n}, \mathfrak{so}_{2n}, \mathfrak{e}_7, \mathfrak{sp}_n\}$ and q even.
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A conjecture on equivalence and an open problem in the case of admissible case

- These conjectures follow from a conjecture on the equivalence of these braided tensor categories with the braided tensor categories coming from module categories for quantum groups constructed from the same finite-dimensional simple Lie algebra $\mathfrak{g}$. The rigidity and modularity of these tensor categories were established in the quantum group side by Sawin in 2003.
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Outline

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