General Relativity and Integrability

Robert Penna (IAS)

A remarkable fact about general relativity is that it admits many exact solutions.

The field equations are 10 nonlinear partial differential equations.

Nonlinear PDE's do not usually admit any exact solutions.

What makes general relativity so special?

The "hidden simplicity" of general relativity dates back to the discovery of the Schwarzschild metric in 1916.

Einstein's letter to Schwarzschild:

> *I have read your paper with the utmost interest. I had not expected that one could formulate the exact solution of the problem in such a simple way.*

The Schwarzschild metric is diagonal and it has two commuting Killing vectors, ∂_t and ∂_φ.

It turns out the field equations reduce to a linear differential equation when the metric is diagonal and has two commuting Killing vectors.

So the existence of the Schwarzschild metric is perhaps not too surprising (although it is somewhat surprising that the field equations reduce to a linear differential equation in this case).

A more interesting example is the Kerr metric (1963).

This also has two commuting Killing vectors, ∂_t and ∂_φ, but the metric is not diagonal.

The field equations really are nonlinear in this case.

Why do exact solutions (like Kerr) exist?

Consider vacuum solutions of general relativity with two commuting Killing vectors and zero cosmological constant.

In this case, general relativity reduces to an $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ *nonlinear sigma model on a half-plane, coupled to a pair of scalar fields.*

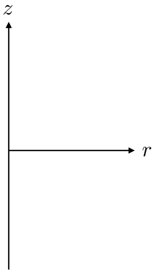
It turns out this as an integrable system (at least classically)!

Vacuum metrics with two commuting Killing vectors (such as Kerr) can be viewed as solutions of this 2d integrable system. This at least partially explains why exact solutions like the Kerr metric exist.

The Half-Plane

If the Killing vectors of the 4d metric are ∂_t and ∂_φ, then the 2d spacetime is a Euclidean half-plane.

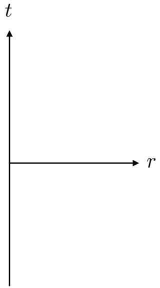
Solutions of the 2d integrable system describe stationary, axisymmetric, 4d black holes, such as the Kerr metric and the N-Kerr metric.



Lorentzian Half-Plane

If the Killing vectors of the 4d metric are ∂_z and ∂_φ, then the 2d spacetime is a Lorentzian half-plane.

In this case, solutions of the 2d integrable system describe 4d cylindrical gravitational waves.

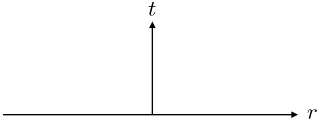


The Cosmological Half-Plane

There is another variant of this story in which the time direction is a half-line.

In this case, the 4d metrics are cosmological solutions.

Solutions of these three versions of the 2d theory are often related by analytic continuations.



2d Fields

A 4d vacuum metric with two commuting Killing vectors (and zero cosmological constant) can be decomposed in terms of 2×2 blocks as

$$g_{\mu\nu} = \begin{pmatrix} \lambda\delta_{ij} & \mathbf{0} \\ \mathbf{0} & \tilde{\sigma}_{ij} \end{pmatrix}$$

We set the off-diagonal blocks to $\mathbf{0}$ using the equations of motion.

We write the 2d metric, $\lambda\delta_{ij}$, as a product of a conformal factor, λ, and a flat metric, δ_{ij}, using a 2d diffeomorphism.

We view the 2×2 matrix in the lower right corner, $\tilde{\sigma}_{ij}$, as a matrix-valued field on 2d spacetime. We rewrite it in terms of a scalar field, $\rho = \det \tilde{\sigma}_{ij}$, and an $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$-valued field, σ_{ij}.

The fields of the 2d theory are ρ, λ, and σ_{ij}.

2d Equations of Motion

The fields of the 2d theory are ρ, λ, and σ_{ij}.

The equation for ρ is trivial: $\Box\rho = 0$. We usually set $\rho = r$.

The equation for λ decouples, so we can ignore it and solve for λ at the end.

The most interesting equation is the equation for σ_{ij}. On the Lorentzian half-plane:

$$-r\partial_t(\sigma^{-1}\partial_t\sigma) + \partial_r(r\sigma^{-1}\partial_r\sigma) = 0$$

This is a nonlinear sigma model with $\sigma \in \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$.

The Geroch Group

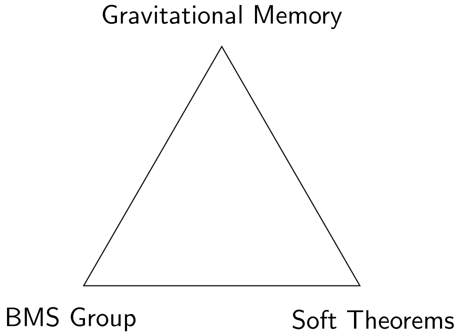
The 2d sigma model is exactly solvable using the inverse scattering method (Belinski-Zakharov 1978).

There is an infinite dimensional hidden symmetry called the **Geroch group** that explains why it is integrable. The Geroch group is an $\mathrm{SL}(2, \mathbb{R})$ loop group. The existence of the Geroch group entails an infinite number of nonlocal conservation laws.

For reviews of the Geroch group, see Breitenlohner and Maison 1987, Nicolai 1991. For rigorous results: Beheshti and Tahvildar-Zadeh 2016.

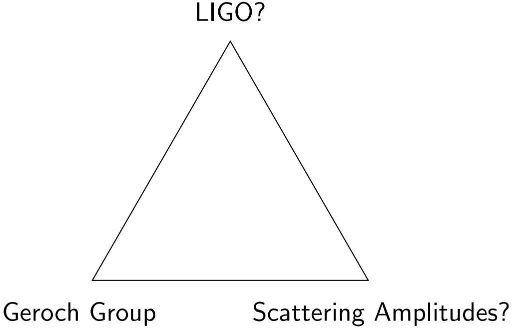
The Infrared Triangle

Recall that the BMS group is an infinite dimensional symmetry of general relativity (unrelated to the Geroch group) and it is part of the "infrared triangle":



Is There an “Integrability Triangle”?

Is there an “integrability triangle” for the Geroch group?



Scattering Amplitudes and Factorization

In ordinary quantum integrable systems, integrability manifests as factorization: multiparticle amplitudes factorize as products of $2 \to 2$ scattering amplitudes.

Is there a limit of graviton scattering amplitudes where the Geroch group manifests as a kind of factorization?

(I don't know.)

Large N?

One could try to get a simpler version of the 2d sigma model by studying the "large N" limit.

The dimensional reduction of general relativity from d spacetime dimensions to two spacetime dimensions is a sigma model with target space $\mathrm{SL}(N, \mathbb{R})/\mathrm{SO}(N)$, where $N = d - 2$.

You could try to study this sigma model using the $1/N$ expansion. This is still hard. One reason is that the target space is negatively curved, so the theory is not asymptotically free.

The Cosmological Constant

Another way to try to get a simpler model might be to turn on a cosmological constant and look for conformal symmetry and/or links to AdS/CFT.

Unfortunately, the Geroch group has only been defined with zero cosmological constant. The dimensional reduction of general relativity with cosmological constant to two spacetime dimensions is not known to be integrable.

Note, however, that there are exact solutions with cosmological constant such as Kerr-(A)dS. So it is tempting to speculate that a generalization of the Geroch group with cosmological constant exists, waiting to be found.

Classical Scattering

Absent much understanding of the action of the Geroch group on quantum gravity scattering amplitudes, we can at least look for the imprint of the Geroch group on classical gravitational wave scattering.

Towards this end, I recently derived some new exact solutions for gravitational pulse wave scattering (Penna 2022). These are cylindrical gravitational waves with a pulse profile in the radial direction. There are two commuting Killing vectors, ∂_φ and ∂_z.

I obtained this solution by a double Wick rotation of the double Kerr metric of Kramer and Neugebauer (1980). The double Kerr metric has a conical singularity at $\rho = 0$ that holds the black holes apart but the double pulse wave has no such conical singularity.

The Double Kerr Metric

The double Kerr metric has the form

$$ds^2 = -f(d\hat{t} - \Omega d\varphi)^2 + f^{-1}\left(e^{2\gamma}(d\rho^2 + d\hat{z}^2) + \rho^2 d\varphi^2\right)$$

There are three functions in the metric: $f(\rho, \hat{z})$, $\Omega(\rho, \hat{z})$, and $\gamma(\rho, \hat{z})$. (ρ is the radial coordinate.)

There are two Killing vectors, $\partial_{\hat{t}}$ and ∂_{φ}.

The Ernst Potential

The metric functions, $f(\rho, \hat{z})$, $\Omega(\rho, \hat{z})$, and $\gamma(\rho, \hat{z})$, are very complicated.

But they can all be derived from a much simpler complex scalar field called the Ernst potential.

To describe the Ernest potential, first replace $\Omega(\rho, \hat{z})$ with a dual potential, $\psi(\rho, \hat{z})$, defined by

$$\partial_\rho \psi = \frac{f^2}{\rho} \partial_{\hat{z}} \Omega \,, \quad \partial_{\hat{z}} \psi = -\frac{f^2}{\rho} \partial_\rho \Omega$$

Then combine $f(\rho, \hat{z})$ and $\psi(\rho, \hat{z})$ into a single complex scalar field, the Ernest potential:

$$\epsilon = f + i\psi$$

The Ernst Potential

The Ernest potential for the double Kerr metric is $\epsilon = (1-\xi)/(1+\xi)$,

$$\xi = \frac{\begin{vmatrix} 1 & 1 & 1 & 1 \\ s_1 & s_2 & s_3 & s_4 \\ k_1 & k_2 & k_3 & k_4 \\ k_1^2 & k_2^2 & k_3^2 & k_4^2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ s_1 & s_2 & s_3 & s_4 \\ k_1 & k_2 & k_3 & k_4 \\ k_1 s_1 & k_2 s_2 & k_3 s_3 & k_4 s_4 \end{vmatrix}}$$

The k_m $(m = 1, \ldots, 4)$ are four real constants.

Free Parameters

The functions in the Ernst potential are

$$s_m = r_m e^{i\omega_m}$$

$(m = 1, \ldots, 4)$ where the ω_m are an additional four real constants and

$$r_m = \left(\rho^2 + (\hat{z} - k_m)^2\right)^{1/2}$$

Thus the double Kerr metric has eight real parameters (k_m and ω_m, $m = 1, \ldots, 4$). They correspond to the masses, spins, NUT parameters, and displacements along $\hat{z}$ of the black holes.

Double Wick Rotation

To get the double pulse wave metric, we make the double analytic continuation (Penna 2022)

$$\hat{t} = iz \,, \quad \hat{z} = it$$

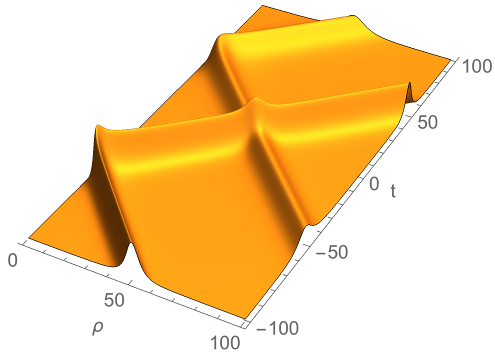
t and z are real coordinates.

We also analytically continue the black hole spins and displacement parameters to imaginary values.

This procedure is based on Piran, Safier, and Katz (1986). They obtained a metric for a single gravitational pulse wave from a double analytic continuation of the (single) Kerr metric.

The Double Pulse Wave

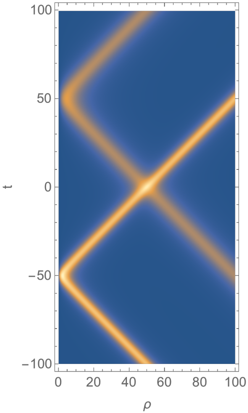
The new metric describes a pair of pulses that enter from infinity, pass through each other, and return to infinity:



Penna 2022

No Time Delay

Ordinary solitons usually experience a time delay after scattering. Not so for gravitational pulse waves:



Penna 2022

Related Works

Boyd, Centrella, and Klasky (1991) studied collisions of cosmological pulse waves. Their pulse waves emerge from a big bang-type singularity in the past.

Tomimatsu (1989) and Tomizawa and Mishimi (2015) studied how the polarizations of ingoing and outgoing pulse waves are related.

LIGO?

The other question in the "integrability triangle" was whether LIGO can "see" Geroch symmetry.

I'm not sure. Geroch conservation laws are non-local, and they are only exact conservation laws when spacetime has two commuting Killing vectors.

However: the Kerr metric is a solution of the 2d integrable model, so there is a sense in which the dynamics of a weakly perturbed Kerr black hole is "close" to an integrable system. It might be worthwhile to look for approximate versions of Geroch conservation laws in black hole perturbation theory.

The Ehlers Group

The dimensional reduction of general relativity to three spacetime dimensions has a hidden $\mathrm{SL}(2,\mathbb{R})$ symmetry called the Ehlers group. The Geroch group is an enhanced version of the Ehlers group.

$d = 3$ Ehlers group

↓

$d = 2$ Geroch group

Is there an even bigger symmetry in $d = 1$?

To get the 2d theory, we decomposed the 4d metric as

$$g_{\mu\nu} = \begin{pmatrix} \lambda\delta_{ij} & \mathbf{0} \\ \mathbf{0} & \tilde{\sigma}_{ij} \end{pmatrix}$$

We used the equations of motion to set the off-diagonal 2×2 blocks to $\mathbf{0}$.

The 2d theory was integrable, so there does not seem to be any room for a further enhancement of the Geroch group.

To get an even bigger symmetry, we will not use the equations of motion to set the off-diagonal 2×2 blocks to $\mathbf{0}$.

The Tetrad

We parametrize the tetrad as (Nicolai 1992)

$$e^a_\mu = \begin{pmatrix} \Delta^{-1/2} & 0 & 0 & 0 \\ 0 & \Delta^{-1/2}\lambda & \Delta^{-1/2}\rho A & \Delta^{1/2} B_- \\ 0 & 0 & \Delta^{-1/2}\rho & \Delta^{1/2} B_2 \\ 0 & 0 & 0 & \Delta^{1/2} \end{pmatrix}$$

The tetrad depends on six scalar fields, $\Delta, B_2, B_-, \rho, A$ and λ. We eliminated four scalar fields using a diffeomorphism.

We assume the six scalar fields are functions of a single spacetime coordinate: $\Delta = \Delta(u)$, $B_2 = B_2(u), \ldots$.

The Metric

The metric is $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$, where η_{ab} is the Minkowski metric,

$$\eta_{ab} = \begin{pmatrix} 0 & -1/2 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The coordinates are (u, v, x^2, x^3). Note that u is a null coordinate.

The reason we choose a null coordinate is that some of the equations of motion are trivially satisfied and the 1d theory is simpler.

Solution Space

The vacuum equations of motion, $R_{\mu\nu} = 0$, are equivalent to three equations (Nicolai 1992, Penna 2021):

$$\lambda^{-1}\rho^{-1}(\lambda'\rho' - \lambda\rho'') = \frac{1}{2}\Delta^{-2}\left(\Delta'^2 + \frac{\Delta^4 B_2'^2}{\rho^2}\right)$$

$$B_-' = AB_2'$$

$$A' = 0$$

Primes are u derivatives.

The 1d theory is a theory of six functions subject to three equations.

The equations are nonlinear, so we can ask: is there a symmetry that maps solutions to solutions, as the Geroch group does in 2d?

$\mathfrak{sl}(3, \mathbb{R})$

Let

$$E_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad F_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$E_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

These six matrices generate $\mathfrak{sl}(3, \mathbb{R})$.

The Matzner-Misner Algebra

Let g be one of the six matrices on the previous slide.

Define an action on e^a_μ by

$$e \to -ge + eh(g)$$

$h(g)$ is a Lorentz transformation. We choose $h(g)$ to restore the upper triangular form of e^a_μ.

This is an $\mathfrak{sl}(3, \mathbb{R})$ symmetry of the 1d theory called the Matzner-Misner algebra.

The Ehlers Algebra

Define a dual potential, $B(u)$, by

$$B' = \frac{\Delta^2}{\rho} B_2'$$

Further define

$$V \equiv \begin{pmatrix} \Delta^{1/2} & B\Delta^{-1/2} \\ 0 & \Delta^{-1/2} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$$

Let $g \in \mathfrak{sl}(2, \mathbb{R})$ act on V by

$$V \to -gV + Vh(g)$$

$h(g)$ is a compensating $\mathfrak{so}(2)$ transformation.

This is a hidden $\mathfrak{sl}(2, \mathbb{R})$ symmetry of the 1d theory called the Ehlers algebra.

Hyperbolic Kac-Moody Symmetry

The Matzner-Misner algebra and the Ehlers algebra do not commute.

Instead, they generate (by repeated commutators) an infinite dimensional algebra which is apparently a hyperbolic Kac-Moody algebra with Dynkin diagram (Nicolai 1992, Penna 2021):

The Geroch algebra is just a tiny (though infinite dimensional) subalgebra of the hyperbolic Kac-Moody algebra.

An Infinite Tower of Dual Potentials

Recall that we defined a dual potential, $B(u)$, according to

$$B' = \frac{\Delta^2}{\rho} B_2'$$

The Ehlers algebra was defined by a local action on B and Δ. Its action on B_2 is nonlocal (it involves an integral over all of u).

Similarly, the action of the Matzner-Misner algebra on B is nonlocal.

To get a local action of the hyperbolic Kac-Moody algebra on solution space, we need to introduce an infinite tower of new dual potentials and new constraint equations.

This makes it hard to prove that the symmetry really is the hyperbolic Kac-Moody algebra on the previous slide. However, there is strong evidence this is the case (Nicolai 1992, Penna 2021).

Root Spaces

One way to organize the hyperbolic Kac-Moody algebra (HKA) is to break it up as a direct sum of root spaces. The Cartan subalgebra is three dimensional, so each root space is labeled by a triple of integers, (n_1, n_2, n_3).

Here are the dimensions of the root spaces with $(n_1, n_2, n_3) = (n, n, 1)$, for $n = 1, 2, 3, \ldots$:

$$1, \quad 2, \quad 3, \quad 5, \quad 7, \quad 11, \quad 15, \quad 22, \quad 30, \quad \ldots$$

These numbers were computed by "brute force," using an algorithm called Peterson's formula.

Amazingly, this sequence is just p_n, the number of partitions of n (Feingold and Frenkel 1983).

Root Spaces

Here are the dimensions of the root spaces with $(n_1, n_2, n_3) = (n, n, 2)$, for $n = 2, 3, 4 \ldots$:

$$1, \quad 3, \quad 7, \quad 15, \quad 30, \quad 56, \quad 101, \quad 176, \quad 297, \quad 490, \quad \color{red}{791}, \quad \ldots$$

This is almost the same thing as p_{2n-3}.

But there are "gaps in the spectrum" starting at $p_{21} = 792$.

Here is the sequence of gaps:

$$1, \quad 2, \quad 5, \quad 10, \quad 21, \quad 38, \quad 70 \quad \ldots$$

Root Spaces

Here are the dimensions of the root spaces with $(n_1, n_2, n_3) = (n, n, 3)$, for $n = 3, 4, 5 \ldots$:

$$1, \quad 5, \quad 15, \quad 42, \quad 101, \quad 231, \quad 490, \quad 1002, \quad 1956, \quad \ldots$$

This is almost the same thing as p_{3n-8}.

But there are gaps starting at $p_{25} = 1958$.

Here is the sequence of gaps:

$$2, \quad 8, \quad 24, \quad 64, \quad 154, \quad 342, \quad 720, \quad 1447, \quad \ldots$$

These are mysterious sequences of integers, possibly related to number theory, that are hidden inside general relativity!

Summary

- The dimensional reduction of general relativity to two spacetime dimensions is an integrable system. It has an infinite dimensional hidden symmetry called the Geroch group that explains why it is integrable.
- We have found new exact solutions for cylindrical gravitational pulse wave scattering. The pulses are solitons of the integrable system.
- It would be interesting to look for signatures of the Geroch group in scattering amplitudes and LIGO observables (the "integrability triangle").
- The dimensional reduction of general relativity to one dimension apparently has a hyperbolic Kac-Moody symmetry.
- Perhaps the rich geometrical structure of general relativity could be used to shed light on this mysterious algebra.