Rigidity and modularity of vertex tensor categories

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Abstract

Let V be a simple vertex operator algebra satisfying the following conditions: (i) $V_{(n)} = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ and V' is isomorphic to V as a V-module. (ii) Every $\mathbb{N}$-gradable weak V-module is completely reducible. (iii) V is C_2-cofinite. (In the presence of Condition (i), Conditions (ii) and (iii) are equivalent to a single condition, namely, that every weak V-module is completely reducible.) Using the results obtained by the author in the formulation and proof of the general version of the Verlinde conjecture and in the proof of the Verlinde formula, we prove that the braided tensor category structure on the category of V-modules is rigid, balanced and nondegenerate. In particular, the category of V-modules has a natural structure of modular tensor category. We also prove that the tensor-categorical dimension of an irreducible V-module is the reciprocal of a suitable matrix element of the fusing isomorphism under a suitable basis.

0 Introduction

In the present paper, we prove the rigidity and modularity of the braided tensor category of modules for a vertex operator algebra satisfying certain natural conditions (see below). Finding proofs of these properties has been an open problem for many years. Our proofs in this paper are based on the results obatined by the author [H7] in the formulation and proof of the general version of the Verlinde conjecture and in the proof of the Verlinde formula.

In 1988, Moore and Seiberg [MS1] [MS2] derived a system of polynomial equations from the axioms for rational conformal field theories. They showed that the Verlinde conjecture [V] is a consequence of these equations. Inspired by an observation of Witten on an analogy with Mac Lane's coherence, Moore and Seiberg [MS2] also demonstrated that the theory of these polynomial equations is actually a conformal-field-theoretic analogue of the theory of the tensor categories. This work of Moore and Seiberg greatly advanced our understanding of the structure of conformal field theories and the name "modular tensor category" was suggested by I. Frenkel for the theory of these Moore-Seiberg equations. Later, the precise notion of modular tensor category was introduced to summarize the properties of these polynomial equations and has played a central role in the development of conformal field theories and three-dimensional topological field theories. See for example [T] and [BK] for the theory of modular tensor categories, their applications and references to many important works of mathematicians and physicists.

Mathematically, Kazhdan and Lusztig [KL1]–[KL5] first constructed a rigid braided tensor category structure on a suitable (nonsemisimple) category of modules of a negative level for an affine Lie algebra. Finkelberg [F1] [F2] transported these braided tensor category structures to the category of integrable highest weight modules of positive integral levels (with a few exceptions) for the same affine Lie algebra. Direct constructions of these braided tensor category structures were also given by Lepowsky and the author [HL5] based on the results in [HL1]–[HL4] [H1] and by Bakalov and Kirillov [BK]. In the general case, for a vertex operator algebra V satisfying suitable conditions (weaker than the conditions in the present paper), the braided tensor category structure on the category of V-modules was constructed by Lepowsky and the author and by the author in a series of papers [HL1]–[HL4] [H1] [H5]. This construction has been generalized to the nonsemisimple (logarithmic) case by Lepowsky, Zhang and the author recently [HLZ1] [HLZ2], and as an application, a different construction of Kazhdan-Lusztig's braided tensor category structure has been given using this logarithmic theory by Zhang [Zha1] [Zha2].

To prove that a semisimple braided tensor category actually carries a modular tensor category structure, we need to prove that it is rigid, balanced and nondegenerate. The balancing isomorphisms or twists in these braided tensor categories are actually trivial to construct and the balancing axioms are easy to prove. On the other hand, the rigidity has been an open problem for many years after the braided tensor category structure on the category of modules for a vertex operator algebra satisfying the conditions in [HL1]–[HL4] [H1] was constructed. The main difficulty is that from the construction, it is not clear why the numbers determining the module maps given by the sequences in the axioms for the rigidity are not 0. The nondegeneracy of the semisimple braided tensor category of modules for a suitable general vertex operator algebra has also been open. Even in the special case of the category of integrable highest weight modules of positive integral levels for affine Lie algebras, as far as the author knows, there have been no proofs of the rigidity or the nondegeneracy property in the literature.

In the present paper, we solve all these problems. Let V be a simple vertex operator algebra satisfying the following conditions: (i) $V_{(n)} = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ and V' is isomorphic to V as a V-module. (ii) Every $\mathbb{N}$-gradable weak V-module is completely reducible. (iii) V is C_2-cofinite. In an early version of the present paper and in the results announced in [H8] and [H9], the proofs of the rigidity and the nondegeneracy property use a condition (Condition (i) in the early version of the present paper) slightly stronger than Condition (i) above. It requires that $V_{(n)} = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ and for any irreducible V-module W, $W_{(0)} = 0$. But actually both proofs still work under Condition (i) in the present version[1]. Also, by results of Li [Li] and Abe-Buhl-Dong [ABD], Conditions (ii) and (iii) are equivalent to a single condition that every weak V-module is completely reducible. In the present paper, using a consequence of a Verlinde formula proved recently by the author in [H7], we show that the braided tensor category structure on the category of V-modules is rigid. Using a formula also obtained easily from this Verlinde formula (see [H7]), we prove that the semisimple rigid balanced braided tensor category structure on the category of V-modules is nondegenerate. In particular, the category of V-modules has a natural structure of modular tensor category.

The results of the present paper have been announced in [H8] and [H9]. See also [Le] for an exposition.

The paper is organized as follows: Section 1 is a review of the tensor product theory developed by Lepowsky and the author in [HL1]–[HL4], [HL6], [H1] and [H5]. Section 2 is a review of the fusing and braiding matrices, the Verlinde conjecture and its consequences studied and proved by the author in [H1]–[H7]. See also [H8] and [H9] for expositions. In Section 3, we prove the rigidity of the braided tensor category of V-modules for a vertex operator algebra satisfying the three conditions mentioned above. We remark that we

[1] I am grateful to Liang Kong for pointing out that the proof of the rigidity in the early version still works with Condition (i) in the present version.

can introduce a notion of rigidity of vertex tensor categories and we actually have proved that the vertex tensor category structure on the category of V-modules is rigid in this sense. At the end of this section, we calculate the tensor-categorical dimension of irreducible V-modules explicitly. In Section 4, we first show that the semi-simple rigid braided tensor category of V-modules with the obvious twists is a ribbon category. Then we show that this ribbon category is nondegenerate. In particular, the category of V-modules is a modular tensor category. We also remark that we can introduce a notion of modular vertex tensor categories and we actually have proved that the category of V-modules is a modular vertex tensor category in this sense.

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1 Review of the tensor product theory

In this section, we review the tensor product theory for modules for a vertex operator algebra developed by Lepowsky and the author in [HL1]–[HL4], [HL6], [H1] and [H5].

Let V be a simple vertex operator algebra and $C_2(V)$ the subspace of V spanned by $u_{-2}v$ for $u, v \in V$. In the present paper, we shall always assume that V satisfies the following conditions:

1. $V_{(n)} = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ and V' is isomorphic to V as a V-module.

2. Every $\mathbb{N}$-gradable weak V-module is completely reducible.

3. V is C_2-cofinite, that is, $\dim V/C_2(V) < \infty$.

These conditions are all natural. Condition 1 says that the vacuum is unique in V and the contragredient of V does not give a new irreducible V-module. As we mentioned in the introduction, Condition 1 is a consequence of the following stronger version of the uniqueness-of-the vacuum condition: $V_{(n)} = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ and for any irreducible V-module W, $W_{(0)} = 0$. Note that finitely generated $\mathbb{N}$-gradable weak V-modules are what naturally appear in the proofs of the theorems on genus-zero and genus-one correlation functions. Thus Condition 2 is natural and necessary because the Verlinde conjecture concerns V-modules, not finitely generated $\mathbb{N}$-gradable weak V-modules. Condition 3 would be a consequence of the finiteness of the dimensions of genus-one conformal blocks, if the conformal field theory had been constructed, and is thus natural and necessary. For vertex operator algebras associated to affine Lie algebras (Wess-Zumino-Novikov-Witten models) and vertex operator algebras associated to the Virasoro algebra (minimal models), Condition 2 can be verified easily by reformulating the corresponding complete reducibility results in terms of the representation theory of affine Lie algebras and the Virasoro algebra. For these vertex operator algebras, Condition 3 can also be easily verified by using results in the representation theory of affine Lie algebras and the Virasoro algebra. In fact, Condition 3 was stated to hold for these algebras in Zhu's paper [Zhu] and was verified by Dong-Li-Mason [DLM] (see also [AN] for the case of minimal models). In addition, as we have mentioned in the abstract and introduction, by results of Li [Li] and Abe-Buhl-Dong [ABD], Conditions (ii) and (iii) are equivalent to a single condition that every weak V-module is completely reducible.

Because of this proposition, we see from Theorem 3.9 in [H5] that V satisfies all the conditions needed in the results proved in [HL1]–[HL4], [H1] and [HL6]. Thus we have:

Theorem 1.1 *Let V be a vertex operator algebra satisfying the conditions above. Then the category of V-modules has a natural structure of braided tensor category.*

The proofs of the rigidity and nondegeneracy in Sections 3 and 4, respectively, depend not only on this theorem, but also on the detailed construction of the braided tensor category structure. For reader's convenience, we now briefly review the structures which are needed in the proof of the theorem above and in the main theorems of the present paper in these later sections.

Let W_1 and W_2 be V-modules. In the present paper, we shall need the $P(z)$-tensor product $W_1 \boxtimes_{P(z)} W_2$ of V-modules W_1 and W_2 for $z \in \mathbb{C}^\times$.

The Jacobi identity for intertwining operators motivates a natural action $\tau_{P(z)}$ of

$$x_0^{-1}\delta\left(\frac{x_1^{-1}-z}{x_0}\right)Y_t(v, x_1)$$

on $(W_1 \otimes W_2)^*$ for $v \in V$ given by

$$\left(\tau_{P(z)}\left(x_0^{-1}\delta\left(\frac{x_1^{-1}-z}{x_0}\right)Y_t(v,x_1)\right)\lambda\right)(w_{(1)}\otimes w_{(2)}) =$$
$$= z^{-1}\delta\left(\frac{x_1^{-1}-x_0}{z}\right)\lambda(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v,x_0)w_{(1)}\otimes w_{(2)})$$
$$+x_0^{-1}\delta\left(\frac{z-x_1^{-1}}{-x_0}\right)\lambda(w_{(1)}\otimes Y_2^*(v,x_1)w_{(2)})$$

where $\lambda \in (W_1 \otimes W_2)^*$ and

$$Y_t(v,x_1) = v\otimes x_1^{-1}\delta\left(\frac{t}{x_1}\right).$$

In particular, we have an action $\tau_{P(z)}(Y_t(v,x_1))$ of $Y_t(v,x_1)$ on $(W_1 \otimes W_2)^*$.

Consider $\lambda \in (W_1 \otimes W_2)^*$ satisfying the following conditions:

The compatibility condition: (a) The *lower truncation condition*: For all $v \in V$, the formal Laurent series $\tau_{P(z)}(Y_t(v,x))\lambda$ involves only finitely many negative powers of x.

(b) The following formula holds:

$$\tau_{P(z)}\left(x_0^{-1}\delta\left(\frac{x_1^{-1}-z}{x_0}\right)Y_t(v,x_1)\right)\lambda =$$
$$= x_0^{-1}\delta\left(\frac{x_1^{-1}-z}{x_0}\right)\tau_{P(z)}(Y_t(v,x_1))\lambda$$

for all $v \in V$.

The local grading-restriction condition: (a) The *grading condition*: λ is a (finite) sum of weight vectors (eigenvectors of $L(0)$) of $(W_1 \otimes W_2)^*$.

(b) Let W_λ be the smallest subspace of $(W_1 \otimes W_2)^*$ containing λ and stable under the component operators $\tau_{P(z)}(v\otimes t^n)$ of the operators $\tau_{P(z)}(Y_t(v,x))$ for $v \in V$, $n \in \mathbb{Z}$. Then the weight spaces $(W_\lambda)_{(n)}$, $n \in \mathbb{C}$, of the (graded) space W_λ have the properties

$$\dim\,(W_\lambda)_{(n)} < \infty \quad \text{for } n \in \mathbb{C},$$
$$(W_\lambda)_{(n)} = 0 \quad \text{for } n \text{ such that } \Re(n) << 0.$$

Let $W_1 \boxtimes_{P(z)} W_2$ be the subspace of $(W_1 \otimes W_2)^*$ consisting of all elements satisfying the two conditions above. Then by Theorem 13.5 in [HL4] and Theorems 3.1 and 3.9 in [H5], $W_1 \boxtimes_{P(z)} W_2$ is a V-module. The $P(z)$-tensor product module $W_1 \boxtimes_{P(z)} W_2$ is defined to be the contragredient module $(W_1 \boxtimes_{P(z)} W_2)'$ of $W_1 \boxtimes_{P(z)} W_2$. We take the $P(1)$-tensor product $\boxtimes_{P(1)}$ to be the tensor product bifunctor and denote it by $\boxtimes$.

For any V-module $W = \coprod_{n\in\mathbb{Q}} W_{(n)}$, we use $\overline{W}$ to denote its algebraic completion $\prod_{n\in\mathbb{Q}} W_{(n)}$. In addition to the $P(z)$-tensor product module, by the definition of $P(z)$-tensor product (Definition 4.1 in [HL1]) and Proposition 12.2 in [HL4], there is also an intertwining operator $\mathcal{Y}$ of type $\binom{W_1\boxtimes_{P(z)}W_2}{W_1W_2}$ such that the following universal property holds: For any intertwining operator $\tilde{\mathcal{Y}}$ of type $\binom{W_3}{W_1W_2}$, there is a module map $f : W_1 \boxtimes_{P(z)} W_2 \to W_3$ such that $\tilde{\mathcal{Y}} = f \circ \mathcal{Y}$. For $w_1 \in W_1$ and $w_2 \in W_2$, the $P(z)$-tensor product of elements w_1 and w_2 to be

$$w_1 \boxtimes w_2 = \mathcal{Y}(w_1, z)w_2 \in \overline{W_1 \boxtimes_{P(z)} W_2},$$

where we use the convention

$$\mathcal{Y}(w_1, z)w_2 = \mathcal{Y}(w_1, x)w_2|_{x^n=e^{n\log z},\ n\in\mathbb{C}}$$

and

$$\log z = \log|z| + i\arg z,\ 0 \le \arg z < 2\pi.$$

(We shall continue to use this convention through this paper.)

The existence of these tensor products of elements is a very important feature of the tensor product theory. They provide a powerful tool for proving theorems in the tensor product theory: We first prove the results on these tensor products of elements. Since the homogeneous components of these tensor products of elements span the tensor product modules, we obtain the results we are interested. (One subtle thing is worth mentioning here. The space spanned by all tensor products of elements has almost no intersection with the tensor product module; in general, only elements of the form $\mathbf{1} \boxtimes_{P(z)} w$ are in $V \boxtimes_{P(z)} W$ where W is an arbitrary V-module.)

For $P(z)$-tensor products with different z, we have parallel transport isomorphisms between them. In this paper, as we have used above, for $z \in \mathbb{C}^\times$, we shall use $\log z$ to denote that value of the logarithm of z satisfying $0 \le \Im(\log z) < 2\pi$. Let W_1 and W_2 be V-modules and $z_1, z_2 \in \mathbb{C}^\times$. Giving a path γ in $\mathbb{C}^\times$ from z_1 to z_2. The parallel isomorphism $\mathcal{T}_\gamma : W_1 \boxtimes_{P(z_1)} W_2 \to$

$W_1 \boxtimes_{P(z_2)} W_2$ (see [HL3] [HL6]) is given as follows: Let $\mathcal{Y}$ be the intertwining operator associated to the $P(z_2)$-tensor product $W_1 \boxtimes_{P(z_2)} W_2$ and $l(z_1)$ the value of the logarithm of z_1 determined uniquely by $\log z_2$ and the path γ. Then $\mathcal{T}_\gamma$ is characterized by

$$\overline{\mathcal{T}}_\gamma(w_1 \boxtimes_{P(z_1)} w_2) = \mathcal{Y}(w_1, x)w_2|_{x^n = e^{nl(z_1)},\ n\in\mathbb{C}}$$

for $w_1 \in W_1$ and $w_2 \in W_2$, where $\overline{\mathcal{T}}_\gamma$ is the natural extension of $\mathcal{T}_\gamma$ to the algebraic completion $\overline{W_1 \boxtimes_{P(z_1)} W_2}$ of $W_1 \boxtimes_{P(z_1)} W_2$. The parallel isomorphism depends only on the homotopy class of γ.

For $z \in \mathbb{C}^\times$, the commutativity isomorphism for the $P(z)$-tensor product

$$\mathcal{C}_{P(z)} : W_1 \boxtimes_{P(z)} W_2 \to W_2 \boxtimes_{P(z)} W_1$$

is characterized as follows: Let γ_z^- be a path from $-z$ to z in the closed upper half plane with 0 deleted and $\mathcal{T}_{\gamma_z^-}$ the corresponding parallel transport isomorphism. Then

$$\overline{\mathcal{C}_{P(z)}}(w_1 \boxtimes_{P(z)} w_2) = e^{zL(-1)}\overline{\mathcal{T}}_{\gamma_z^-}(w_2 \boxtimes_{P(z)} w_1)$$

where $w_1 \in W_1$, $w_2 \in W_2$. When $z = 1$, we obtain a commutativity isomorphism

$$\mathcal{C}_{P(1)} : W_1 \boxtimes W_2 \to W_2 \boxtimes W_1$$

and we shall denote it simply as $\mathcal{C}$.

The tensor product $w_1 \boxtimes w_2$ of $w_1 \in W_1$ and $w_2 \in W_2$ is obtained from two elements w_1 and w_2 of the V-modules W_1 and W_2, respectively. We also need tensor products of more than two elements. Here we describe tensor products of three elements briefly. Let W_1, W_2, W_3 be V-modules and $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$. Let z_1 and z_2 be two nonzero complex numbers. Since in general $w_2 \boxtimes_{P(z_2)} w_3$ does not belong to $W_2 \boxtimes_{P(z_2)} W_3$, we cannot define $w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)$ simply to be the $P(z_2)$-tensor product of w_1 and $w_2 \boxtimes_{P(z_2)} w_3$. But using the convergence of products of intertwining operators, the series

$$\sum_{n\in\mathbb{Z}} w_1 \boxtimes_{P(z_1)} P_n(w_2 \boxtimes_{P(z_2)} w_3)$$

(P_n is the projection map from a V-module to the subspace of weight n) is absolutely convergent in a natural sense when $|z_1| > |z_2|$ and the sum is in

$\overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}$. We define $w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)$ to be this sum. Similarly, when $|z_2| > |z_1 - z_2| > 0$, we have

$$(w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3 \in \overline{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3}.$$

The homogeneous components of these tensor products of three elements also span the corresponding tensor products of three modules.

Let z_1, z_2 be complex numbers satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$ and W_1, W_2 and W_3 V-modules. Then an associativity isomorphism

$$\mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)} : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \to (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$$

was constructed by the author in [H1] and [H5] and is characterized by the property

$$\overline{\mathcal{A}_{P(z_1,P(z_2)}^{P(z_1-z_2),P(z_2)}}(w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)) = (w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3$$

for $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$, where

$$\overline{\mathcal{A}_{P(z_1,P(z_2)}^{P(z_1-z_2),P(z_2)}} : \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \to \overline{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3}$$

is the natural extension of $\mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}$ to the algebraic completion of $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$.

To obtain the associativity isomorphism

$$\mathcal{A} : W_1 \boxtimes (W_2 \boxtimes W_3) \to (W_1 \boxtimes W_2) \boxtimes W_3$$

for the braided tensor category structure, we need certain parallel isomorphisms. Let z_1 and z_2 be real numbers satisfying $z_1 > z_2 > z_1 - z_2 \geq 0$. Let γ_1 and γ_2 be paths in $(0, \infty)$ from 1 to z_1 and z_2, respectively, and γ_3 and γ_4 be paths in $(0, \infty)$ from z_2 and $z_1 - z_2$ to 1, respectively. Then the associativity isomorphism for the braided tensor category structure on the module category for V is given by

$$\mathcal{A} = \mathcal{T}_{\gamma_3} \circ (\mathcal{T}_{\gamma_4} \boxtimes_{P(z_2)} I_{W_3}) \circ \mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)} \circ (I_{W_1} \boxtimes_{P(z_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1},$$

that is, given by the commutative diagram

$$\begin{array}{ccc}
W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) & \xrightarrow{\mathcal{A}_{P(z_2),P(z_2)}^{P(z_2-z_3),P(z_3)}} & (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \\
{\scriptstyle (I_{W_1} \boxtimes_{P(z_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1}} \Big\uparrow & & \Big\downarrow {\scriptstyle \mathcal{T}_{\gamma_3} \circ (\mathcal{T}_{\gamma_4} \boxtimes_{P(z_2)} I_{W_3})} \\
W_1 \boxtimes (W_2 \boxtimes W_3) & \xrightarrow{\mathcal{A}} & (W_1 \boxtimes W_2) \boxtimes W_3
\end{array}$$

The coherence properties are now easy consequences of the constructions and characterizations of the associativity and commutativity isomorphisms. Here we sketch the proof of the commutativity of the pentagon diagram. Let W_1, W_2, W_3 and W_4 be V-modules and let $z_1, z_2, z_3 \in \mathbb{C}$ satisfying

$$|z_1| > |z_2| > |z_3| > |z_1 - z_3| > |z_2 - z_3| > |z_1 - z_2| > 0$$

and

$$|z_2| > |z_1 - z_2| + |z_3|.$$

For example, we can take $z_1 = 7$, $z_2 = 6$ and $z_3 = 4$. We first prove the commutativity of the following diagram:

$$\begin{tikzcd}[column sep=tiny]
 & W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)) \arrow[dl] \arrow[dr] & \\
(W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4) \arrow[d] & & W_1 \boxtimes_{P(z_1)} ((W_2 \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4)) \arrow[d] \\
((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4 & & (W_1 \boxtimes_{P(z_{13})} (W_2 \boxtimes_{P(z_{23})} W_3)) \boxtimes_{P(z_3)} W_4 \arrow[ll]
\end{tikzcd} \tag{1.1}$$

where $z_{12} = z_1 - z_2$ and $z_{23} = z_2 - z_3$. For $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$ and $w_4 \in W_4$, we consider

$$w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} (w_3 \boxtimes_{P(z_3)} w_4)) \in \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4))}.$$

By the characterizations of the associativity isomorphisms, we see that the compositions of the natural extensions of the module maps in the two routes in (1.1) applying to this element both give

$$((w_1 \boxtimes_{P(z_{12})} w_2) \boxtimes_{P(z_{23})} w_3) \boxtimes_{P(z_3)} w_4 \in \overline{((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4}.$$

Since the homogeneous components of

$$w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} (w_3 \boxtimes_{P(z_3)} w_4))$$

for $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$ and $w_4 \in W_4$ span

$$W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)),$$

the diagram (1.1) above is commutative.

On the other hand, by the definition of $\mathcal{A}$, the diagrams

$$\begin{CD}
W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)) @>>> (W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)\\
@VVV @VVV\\
W_1 \boxtimes (W_2 \boxtimes (W_3 \boxtimes W_4)) @>>> (W_1 \boxtimes W_2) \boxtimes (W_3 \boxtimes W_4)
\end{CD} \tag{1.2}$$

$$\begin{CD}
(W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4) @>>> ((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4\\
@VVV @VVV\\
(W_1 \boxtimes W_2) \boxtimes (W_3 \boxtimes W_4) @>>> ((W_1 \boxtimes W_2) \boxtimes W_3) \boxtimes W_4
\end{CD} \tag{1.3}$$

$$\begin{CD}
W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)) @>>> W_1 \boxtimes_{P(z_1)} ((W_2 \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4))\\
@VVV @VVV\\
W_1 \boxtimes (W_2 \boxtimes (W_3 \boxtimes W_4)) @>>> W_1 \boxtimes ((W_2 \boxtimes W_3) \boxtimes W_4)
\end{CD} \tag{1.4}$$

$$\begin{CD}
W_1 \boxtimes_{P(z_1)} ((W_2 \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4)) @>>> (W_1 \boxtimes_{P(z_{13})} (W_2 \boxtimes_{P(z_{23})} W_3)) \boxtimes_{P(z_3)} W_4\\
@VVV @VVV\\
W_1 \boxtimes ((W_2 \boxtimes W_3) \boxtimes W_4) @>>> (W_1 \boxtimes (W_2 \boxtimes W_3)) \boxtimes W_4
\end{CD} \tag{1.5}$$

$$\begin{CD}
(W_1 \boxtimes_{P(z_{13})} (W_2 \boxtimes_{P(z_{23})} W_3)) \boxtimes_{P(z_3)} W_4 @>>> ((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4\\
@VVV @VVV\\
(W_1 \boxtimes (W_2 \boxtimes W_3)) \boxtimes W_4 @>>> ((W_1 \boxtimes W_2) \boxtimes W_3) \boxtimes W_4
\end{CD} \tag{1.6}$$

are all commutative. Combining all the diagrams (1.1)–(1.6) above, we see that the pentagon diagram

$$\begin{tikzcd}[column sep=-2em]
& W_1 \boxtimes (W_2 \boxtimes (W_3 \boxtimes W_4)) \arrow[dl] \arrow[dr] & \\
(W_1 \boxtimes W_2) \boxtimes (W_3 \boxtimes W_4) \arrow[d] & & W_1 \boxtimes ((W_2 \boxtimes W_3) \boxtimes W_4) \arrow[d] \\
((W_1 \boxtimes W_2) \boxtimes W_3) \boxtimes W_4 & & (W_1 \boxtimes (W_2 \boxtimes W_3)) \boxtimes W_4 \arrow[ll]
\end{tikzcd}$$

is also commutative.

The proof of the commutativity of the hexagon diagrams is similar.

The unit object is V. For any $z \in \mathbb{C}^\times$ and any V-module W, the left $P(z)$-unit isomorphism $l_{W;z} : V \boxtimes_{P(z)} W \to W$ is characterized by

$$l_{W;z}(\mathbf{1} \boxtimes_{P(z)} w) = w$$

for $w \in W$ and the right $P(z)$-unit isomorphism $r_{W;z} : W \boxtimes_{P(z)} V \to W$ is characterized by

$$\overline{r_{W;z}}(w \boxtimes_{P(z)} \mathbf{1}) = e^{zL(-1)} w$$

for $w \in W$. In particular, we have the left unit isomorphism $l_W = l_{W;1} : V \boxtimes W \to W$ and the right unit isomorphism $r_W = r_{W;1} : W \boxtimes V \to W$. The proof of the commutativity of the diagrams for unit isomorphisms is similar to the above proof of the commutativity of the pentagon diagram.

2 The fusing and braiding matrices, modular transformations and the Verlinde conjecture

In the proofs in the next two sections of the rigidity and nondegeneracy of the semisimple braided tensor category of V-modules for a vertex operator algebra V satisfying the conditions in the preceding section, we need fusing and braiding matrices and certain consequences of the Verlinde conjecture, which was recently proved by the author [H7] for a vertex operator algebra satisfying conditions slightly weaker than those assumed in the present paper. Here we give a brief review. For details, see [H4]–[H7].

Using the theory of associative algebras and Zhu's algebra for a vertex operator algebra, it is easy to see that for a vertex operator algebra V such that every $\mathbb{N}$-gradable V-module is completely reducible, there are only finitely many inequivalent irreducible V-modules (see Theorem 3.2 in [DLM]). Let $\mathcal{A}$ be the set of equivalence classes of irreducible V-modules. We denote the equivalence class containing V by e. By the main result of [AM] and Theorem 11.3 in [DLM], we know that V-modules are all graded by $\mathbb{R}$. For each $a \in \mathcal{A}$, we choose a representative W^a of a such that $W^e = V$. For $a \in \mathcal{A}$, let h_a be the lowest weight of W^a, that is, $h_a \in \mathbb{R}$ such that $W^a = \coprod_{n \in h_a + \mathbb{N}} W^a_{(n)}$. By Propositions 5.3.1 and 5.3.2 in [FHL], the contragredient module of an irreducible module is also irreducible and the contragredient module of the contragredient module of a V-module is naturally equivalent to the V-module itself. So we have a bijective map

$$\begin{array}{rcl} ' : \mathcal{A} & \to & \mathcal{A} \\ a & \mapsto & a'. \end{array}$$

Let $\mathcal{V}_{a_1 a_2}^{a_3}$ for $a_1, a_2, a_3 \in \mathcal{A}$ be the space of intertwining operators of type $\binom{W^{a_3}}{W^{a_1} W^{a_2}}$ and $N_{a_1 a_2}^{a_3}$ for $a_1, a_2, a_3 \in \mathcal{A}$ the fusion rule, that is, the dimension of the space of intertwining operators of type $\binom{W^{a_3}}{W^{a_1} W^{a_2}}$. The fusion rules $N_{a_1 a_2}^{a_3}$ for $a_1, a_2, a_3 \in \mathcal{A}$ are all finite [GN] [Li] [AN] [H5].

We now discuss matrix elements of fusing and braiding isomorphisms. We need to use different bases of one space of intertwining operators. We shall use $p = 1, 2, 3, 4, 5, 6, \ldots$ to label different bases. For $a_1, a_2, a_3 \in \mathcal{A}$ and $p = 1, 2, 3, 4, 5, 6, \ldots$, let $\{\mathcal{Y}_{a_1 a_2; i}^{a_3; (p)} \mid i = 1, \ldots, N_{a_1 a_2}^{a_3}\}$, be bases of $\mathcal{V}_{a_1 a_2}^{a_3}$. The associativity of intertwining operators proved and studied in [H1], [H4] and [H5] says that there exist

$$F(\mathcal{Y}_{a_1 a_5; i}^{a_4; (1)} \otimes \mathcal{Y}_{a_2 a_3; j}^{a_5; (2)}; \mathcal{Y}_{a_6 a_3; l}^{a_4; (3)} \otimes \mathcal{Y}_{a_1 a_2; k}^{a_6; (4)}) \in \mathbb{C}$$

for $a_1, \ldots, a_6 \in \mathcal{A}$, $i = 1, \ldots, N_{a_1 a_5}^{a_4}$, $j = 1, \ldots, N_{a_2 a_3}^{a_5}$, $k = 1, \ldots, N_{a_6 a_3}^{a_4}$, $l = 1, \ldots, N_{a_1 a_2}^{a_6}$ such that

$$\begin{aligned} &\langle w'_{a_4}, \mathcal{Y}_{a_1 a_5; i}^{a_4; (1)}(w_{a_1}, z_1) \mathcal{Y}_{a_2 a_3; j}^{a_5; (2)}(w_{a_2}, z_2) w_{a_3} \rangle \\ &\quad = \sum_{a_6 \in \mathcal{A}} \sum_{k=1}^{N_{a_6 a_3}^{a_4}} \sum_{l=1}^{N_{a_1 a_2}^{a_6}} F(\mathcal{Y}_{a_1 a_5; i}^{a_4; (1)} \otimes \mathcal{Y}_{a_2 a_3; j}^{a_5; (2)}; \mathcal{Y}_{a_6 a_3; l}^{a_4; (3)} \otimes \mathcal{Y}_{a_1 a_2; k}^{a_6; (4)}) \cdot \\ &\qquad\qquad \cdot \langle w'_{a_4}, \mathcal{Y}_{a_6 a_3; k}^{a_4; (3)}(\mathcal{Y}_{a_1 a_2; l}^{a_6; (4)}(w_{a_1}, z_1 - z_2) w_{a_2}, z_2) w_{a_3} \rangle \end{aligned}$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$, for $a_1, \ldots, a_5 \in \mathcal{A}$, $w_{a_1} \in W^{a_1}$, $w_{a_2} \in W^{a_2}$, $w_{a_3} \in W^{a_3}$, $w'_{a_4} \in (W^{a_4})'$, $i = 1, \ldots, N_{a_1 a_5}^{a_4}$ and $j = 1, \ldots, N_{a_2 a_3}^{a_5}$. The numbers

$$F(\mathcal{Y}_{a_1a_5;i}^{a_4;(1)} \otimes \mathcal{Y}_{a_2a_3;j}^{a_5;(2)}; \mathcal{Y}_{a_6a_3;k}^{a_4;(3)} \otimes \mathcal{Y}_{a_1a_2;l}^{a_6;(4)})$$

together give a matrix which represents a linear isomorphism

$$\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_5}^{a_4} \otimes \mathcal{V}_{a_2a_3}^{a_5} \to \coprod_{a_1,a_2,a_3,a_4,a_6\in\mathcal{A}} \mathcal{V}_{a_6a_3}^{a_4} \otimes \mathcal{V}_{a_1a_2}^{a_6},$$

called the *fusing isomorphism*, such that these numbers are the matrix elements.

By the commutativity of intertwining operators proved and studied in [H2], [H4] and [H5], $r \in \mathbb{Z}$, there exist

$$B^{(r)}(\mathcal{Y}_{a_1a_5;i}^{a_4;(1)} \otimes \mathcal{Y}_{a_2a_3;j}^{a_5;(2)}; \mathcal{Y}_{a_2a_6;l}^{a_4;(3)} \otimes \mathcal{Y}_{a_1a_3;k}^{a_6;(4)}) \in \mathbb{C}$$

for $a_1, \dots, a_6 \in \mathcal{A}$, $i = 1, \dots, N_{a_1a_5}^{a_4}$, $j = 1, \dots, N_{a_2a_3}^{a_5}$, $k = 1, \dots, N_{a_2a_6}^{a_4}$, $l = 1, \dots, N_{a_1a_3}^{a_6}$, such that the analytic extension of the single-valued analytic function

$$\langle w'_{a_4}, \mathcal{Y}_{a_1a_5;i}^{a_4;(1)}(w_{a_1}, z_1)\mathcal{Y}_{a_2a_3;j}^{a_5;(2)}(w_{a_2}, z_2)w_{a_3}\rangle$$

on the region $|z_1| > |z_2| > 0$, $0 \leq \arg z_1, \arg z_2 < 2\pi$ along the path

$$t \mapsto \left(\frac{3}{2} - \frac{e^{(2r+1)\pi i t}}{2}, \frac{3}{2} + \frac{e^{(2r+1)\pi i t}}{2}\right)$$

to the region $|z_2| > |z_1| > 0$, $0 \leq \arg z_1, \arg z_2 < 2\pi$ is

$$\sum_{a_6\in\mathcal{A}} \sum_{k=1}^{N_{a_2a_6}^{a_4}} \sum_{l=1}^{N_{a_1a_3}^{a_6}} B^{(r)}(\mathcal{Y}_{a_1a_5;i}^{a_4;(1)} \otimes \mathcal{Y}_{a_2a_3;j}^{a_5;(2)}; \mathcal{Y}_{a_2a_6;k}^{a_4;(3)} \otimes \mathcal{Y}_{a_1a_3;l}^{a_6;(4)}) \cdot$$
$$\cdot \langle w'_{a_4}, \mathcal{Y}_{a_2a_6;k}^{a_4;(3)}(w_{a_2}, z_1)\mathcal{Y}_{a_1a_3;l}^{a_6;(4)}(w_{a_1}, z_2)w_{a_3}\rangle.$$

The numbers

$$B^{(r)}(\mathcal{Y}_{a_1a_5;i}^{a_4;(1)} \otimes \mathcal{Y}_{a_2a_3;j}^{a_5;(2)}; \mathcal{Y}_{a_2a_6;k}^{a_4;(3)} \otimes \mathcal{Y}_{a_1a_3;l}^{a_6;(4)})$$

together give a linear isomorphism

$$B^{(r)} : \coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_5}^{a_4} \otimes \mathcal{V}_{a_2a_3}^{a_5} \to \coprod_{a_1,a_2,a_3,a_4,a_6\in\mathcal{A}} \mathcal{V}_{a_2a_6}^{a_4} \otimes \mathcal{V}_{a_1a_3}^{a_6},$$

called the *braiding isomorphism*, such that these numbers are the matrix elements.

In this paper, we are mainly interested in the square $(B^{(r)})^2$ of $B^{(r)}$. We shall also use similar notations to denote the matrix elements of the square $(B^{(r)})^2$ of $B^{(r)}$ under the bases above as

$$(B^{(r)})^2(\mathcal{Y}^{a_4;(1)}_{a_1a_5;i}\otimes\mathcal{Y}^{a_5;(2)}_{a_2a_3;j};\mathcal{Y}^{a_4;(3)}_{a_1a_6;i}\otimes\mathcal{Y}^{a_6;(4)}_{a_2a_3;j}).$$

Let

$$\begin{aligned}&(B^{(r)})^2(\langle w_{a_4'},\mathcal{Y}^{a_4;(1)}_{a_1a_5;i}(w_{a_1},z_1)\mathcal{Y}^{a_5;(2)}_{a_2a_3;j}(w_{a_2},z_2)w_{a_3}\rangle)\\&\quad=\sum_{a_6\in\mathcal{A}}\sum_{k=1}^{N^{a_4}_{a_2a_6}}\sum_{l=1}^{N^{a_6}_{a_1a_3}}(B^{(r)})^2(\mathcal{Y}^{a_4;(1)}_{a_1a_5;i}\otimes\mathcal{Y}^{a_5;(2)}_{a_2a_3;j};\mathcal{Y}^{a_4;(3)}_{a_1a_6;i}\otimes\mathcal{Y}^{a_6;(4)}_{a_2a_3;j})\cdot\\&\qquad\qquad\cdot\langle w_{a_4'},\mathcal{Y}^{a_4;(1)}_{a_1a_6;i}(w_{a_1},z_1)\mathcal{Y}^{a_6;(2)}_{a_2a_3;j}(w_{a_2},z_2)w_{a_3}\rangle.\end{aligned}$$

Then by definition, it is in fact the monodromy of the multi-valued analytic extension of

$$\langle w_{a_4'},\mathcal{Y}^{a_4;(1)}_{a_1a_5;i}(w_{a_1},z_1)\mathcal{Y}^{a_5;(2)}_{a_2a_3;j}(w_{a_2},z_2)w_{a_3}\rangle$$

from (z_1,z_2) in the region $|z_1|>|z_2|>0$ to itself along the product of the path in the definition of $B^{(r)}$ above with itself (see Section 1 in [H7] for more details).

We need an action of S_3 on the space

$$\mathcal{V}=\coprod_{a_1,a_2,a_3\in\mathcal{A}}\mathcal{V}^{a_3}_{a_1a_2}.$$

For $r\in\mathbb{Z}$, $a_1,a_2,a_3\in\mathcal{A}$, we have skew-symmetry isomorphisms $\Omega_{-r}:\mathcal{V}^{a_3}_{a_1a_2}\to\mathcal{V}^{a_3}_{a_2a_1}$ and contragredient isomorphisms $A_{-r}:\mathcal{V}^{a_3}_{a_1a_2}\to\mathcal{V}^{a_2'}_{a_1a_3'}$ (see [HL2]). For $a_1,a_2,a_3\in\mathcal{A}$, $\mathcal{Y}\in\mathcal{V}^{a_3}_{a_1a_2}$, we define

$$\begin{aligned}\sigma_{12}(\mathcal{Y})&=e^{\pi i\Delta(\mathcal{Y})}\Omega_{-1}(\mathcal{Y})\\&=e^{-\pi i\Delta(\mathcal{Y})}\Omega_0(\mathcal{Y}),\\\sigma_{23}(\mathcal{Y})&=e^{\pi ih_{a_1}}A_{-1}(\mathcal{Y})\\&=e^{-\pi ih_{a_1}}A_0(\mathcal{Y}),\end{aligned}$$

where $\Delta(\mathcal{Y})=h_{a_3}-h_{a_1}-h_{a_2}$. By Proposition 1.1 in [H7], they generate an action of S_3 on $\mathcal{V}$.

We now want to choose a special basis $\mathcal{Y}^{a_3}_{a_1a_2;i}$, $i=1,\dots,N^{a_3}_{a_1a_2}$, of $\mathcal{V}^{a_3}_{a_1a_2}$ for the triples (a_1,a_2,a_3) of the forms (e,a,a), (a,e,a) and (a,a',e) where

$a \in \mathcal{A}$. For $a \in \mathcal{A}$, we choose $\mathcal{Y}_{ea;1}^{a}$ to be the vertex operator Y_{W^a} defining the module structure on W^a and we choose $\mathcal{Y}_{ae;1}^{a}$ to be $\sigma_{12}(\mathcal{Y}_{ea;1}^{a})$. Since V' as a V-module is equivalent to V, we have $e' = e$ and we know from Remark 5.3.3 in [FHL] that there is a nondegenerate symmetric invariant bilinear form $(\cdot, \cdot)$ on V such that $(\mathbf{1}, \mathbf{1}) = 1$. We identify V and V' using this form. We choose $\mathcal{Y}_{aa';1}^{e} = \mathcal{Y}_{aa';1}^{e'}$ to be the intertwining operator defined using the action of σ_{23} by

$$\mathcal{Y}_{aa';1}^{e'} = \sigma_{23}(\mathcal{Y}_{ae;1}^{a}),$$

that is, $\mathcal{Y}_{aa';1}^{e}$ is defined by

$$(u, \mathcal{Y}_{aa';1}^{e}(w_a, x)w_a') = \langle \mathcal{Y}_{ae;1}^{a}(e^{xL(1)}x^{-2L(0)}w_a, x^{-1})u, w_a'\rangle$$

for $u \in V$, $w_a \in W^a$ and $w_a' \in (W^a)'$.

We now discuss modular transformations. Let $q_\tau = e^{2\pi i\tau}$ for $\tau \in \mathbb{H}$ ($\mathbb{H}$ is the upper-half plane). We consider the q_τ-traces of the vertex operators Y_{W^a} for $a \in \mathcal{A}$ on the irreducible V-modules W^a of the following form:

$$\mathrm{Tr}_{W^a} Y_{W^a}(e^{2\pi izL(0)}u, e^{2\pi iz})q_\tau^{L(0)-\frac{c}{24}} \tag{2.1}$$

for $u \in V$. In [Zhu], under some conditions slightly different from (mostly stronger than) those we assume in this paper, Zhu proved that these q-traces are independent of z, are absolutely convergent when $0 < |q_\tau| < 1$ and can be analytically extended to analytic functions of τ in the upper-half plane. We shall denote the analytic extension of (2.1) by

$$E(\mathrm{Tr}_{W^a} Y_{W^a}(e^{2\pi izL(0)}u, e^{2\pi iz})q_\tau^{L(0)-\frac{c}{24}}).$$

In [Zhu], under his conditions alluded to above, Zhu also proved the following modular invariance property: For

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),$$

let $\tau' = \frac{a\tau+b}{c\tau+d}$. Then there exist unique $A_{a_1}^{a_2} \in \mathbb{C}$ for $a_1, a_2 \in \mathcal{A}$ such that

$$\begin{aligned} E\left(\mathrm{Tr}_{W^{a_1}} Y_{W^{a_1}}\left(e^{\frac{2\pi iz}{c\tau+d}L(0)}\left(\frac{1}{c\tau+d}\right)^{L(0)} u, e^{\frac{2\pi iz}{c\tau+d}}\right) q_{\tau'}^{L(0)-\frac{c}{24}}\right) \\ = \sum_{a_2\in\mathcal{A}} A_{a_1}^{a_2} E(\mathrm{Tr}_{W^{a_2}} Y_{W^{a_2}}(e^{2\pi izL(0)}u, e^{2\pi iz})q_\tau^{L(0)-\frac{c}{24}}) \end{aligned}$$

for $u \in V$. In [DLM], Dong, Li and Mason, among many other things, improved Zhu's results above by showing that the results of Zhu above also hold for vertex operator algebras satisfying the conditions (slightly weaker than what) we assume in this paper. In particular, for

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2, \mathbb{Z}),$$

there exist unique $S_{a_1}^{a_2} \in \mathbb{C}$ for $a_1 \in \mathcal{A}$ such that

$$E\left(\mathrm{Tr}_{W^{a_1}} Y_{W^{a_1}}\left(e^{-\frac{2\pi i z}{\tau}L(0)}\left(-\frac{1}{\tau}\right)^{L(0)} u, e^{-\frac{2\pi i z}{\tau}}\right) q_{-\frac{1}{\tau}}^{L(0)-\frac{c}{24}}\right)$$
$$= \sum_{a_2 \in \mathcal{A}} S_{a_1}^{a_2} E(\mathrm{Tr}_{W^{a_2}} Y_{W^{a_2}}(e^{2\pi i z L(0)} u, e^{2\pi i z}) q_\tau^{L(0)-\frac{c}{24}})$$

for $u \in V$. When $u = \mathbf{1}$, we see that the matrix $S = (S_{a_1}^{a_2})$ actually acts on the space of spanned by the vacuum characters $\mathrm{Tr}_{W^a} q_\tau^{L(0)-\frac{c}{24}}$ for $a \in \mathcal{A}$.

For a vertex operator algebra V satisfying the conditions above, the author proved in [H7] the Verlinde conjecture which states that the action of the modular transformation $\tau \mapsto -1/\tau$ on the space of characters of irreducible V-modules diagonalizes the matrices formed by fusion rules. (Note that in the proof of the Verlinde conjecture, the modular invariance results obtained in [Zhu] and [DLM] are not enough. One needs the results obtained in [H5] and [H6] on intertwining operators and on the modular invariance of the space of q-traces of products of intertwining operators, respectively.) In this paper, we need the following two useful consequences of the Verlinde conjecture: For $a \in \mathcal{A}$,

$$F(\mathcal{Y}_{ae;1}^{a_2} \otimes \mathcal{Y}_{a'a;1}^{e}; \mathcal{Y}_{ea;1}^{a} \otimes \mathcal{Y}_{aa';1}^{e}) \neq 0$$

and

$$S_{a_1}^{a_2} = \frac{S_e^e (B^{(-1)})^2_{a_2,a_1}}{F_{a_1} F_{a_2}} \tag{2.2}$$

where $(S_{a_1}^{a_2})$ is the matrix representing the action of the modular transformation $\tau \mapsto -1/\tau$ on the space of characters of irreducible V-modules,

$$(B^{(-1)})^2_{a_2,a_1} = (B^{(-1)})^2(\mathcal{Y}_{a_2e;1}^{a_2} \otimes \mathcal{Y}_{a_1'a_1;1}^{e}; \mathcal{Y}_{a_2e;1}^{a_2} \otimes \mathcal{Y}_{a_1'a_1;1}^{e})$$

for $a_1, a_2 \in \mathcal{A}$, and

$$F_a = F(\mathcal{Y}_{ae;1}^{a} \otimes \mathcal{Y}_{a'a;1}^{e}; \mathcal{Y}_{ea;1}^{a} \otimes \mathcal{Y}_{aa';1}^{e})$$

for $a \in \mathcal{A}$.

3 The proof of the rigidity and the dimensions of the irreducible modules

In this section, we prove that the braided tensor category structure on the category of V-module is in fact rigid.

First we recall the definition of rigidity (see for example [T] and [BK]). A tensor category with tensor product bifunctor $\boxtimes$ and unit object V is rigid if for every object W in the category, there are right and left dual objects W^* and ${}^*W$ together with morphisms $e_W : W^* \boxtimes W \to V$, $i_W : V \to W \boxtimes W^*$, $e'_W : W \boxtimes {}^*W \to V$ and $i'_W : V \to {}^*W \boxtimes W$ such that the compositions of the morphisms in the sequences

$$\begin{aligned} W &\xrightarrow{l_W^{-1}} V \boxtimes W \xrightarrow{i_W \boxtimes I_W} (W \boxtimes W^*) \boxtimes W \longrightarrow \\ &\xrightarrow{\mathcal{A}^{-1}} W \boxtimes (W^* \boxtimes W) \xrightarrow{I_W \boxtimes e_W} W \boxtimes V \xrightarrow{r_W} W, \end{aligned}$$

$$\begin{aligned} W &\xrightarrow{r_W^{-1}} W \boxtimes V \xrightarrow{I_W \boxtimes i'_W} W \boxtimes ({}^*W \boxtimes W) \longrightarrow \\ &\xrightarrow{\mathcal{A}} (W \boxtimes {}^*W) \boxtimes W \xrightarrow{e'_W \boxtimes I_W} V \boxtimes W \xrightarrow{l_W} W, \end{aligned}$$

$$\begin{aligned} {}^*W &\xrightarrow{l_{{}^*W}^{-1}} V \boxtimes {}^*W \xrightarrow{i'_W \boxtimes I_{{}^*W}} ({}^*W \boxtimes W) \boxtimes {}^*W \longrightarrow \\ &\xrightarrow{\mathcal{A}^{-1}} {}^*W \boxtimes (W \boxtimes {}^*W) \xrightarrow{I_{{}^*W} \boxtimes e'_W} {}^*W \boxtimes V \xrightarrow{r_{{}^*W}} {}^*W, \end{aligned}$$

$$\begin{aligned} W^* &\xrightarrow{r_{W^*}^{-1}} W^* \boxtimes V \xrightarrow{I_{W^*} \boxtimes i_W} W^* \boxtimes (W \boxtimes W^*) \longrightarrow \\ &\xrightarrow{\mathcal{A}} (W^* \boxtimes W) \boxtimes W^* \xrightarrow{e_W \boxtimes I_{W^*}} V \boxtimes W^* \xrightarrow{l_{W^*}} W_*, \end{aligned}$$

are equal to the identity isomorphisms I_W, I_W, $I_{{}^*W}$, I_{W^*}, respectively. Rigidity is a standard notion in the theory of tensor categories. See, for example, [T] and [BK] for details.

In this section, we shall always use the bases $\{\mathcal{Y}^a_{ea;1}\}$, $\{\mathcal{Y}^a_{ae;1}\}$ and $\{\mathcal{Y}^e_{aa';1}\}$ of $\mathcal{V}^a_{ea}$, $\mathcal{V}^a_{ae}$ and $\mathcal{V}^e_{aa'}$ chosen in the preceding section. We take both the left and right duals of a V-module W to be the contragredient module W' of W. Since our tensor category is semisimple, to prove the rigidity, we need only discuss irreducible modules. For $a \in \mathcal{A}$ and $z \in \mathbb{C}^\times$, using the universal property for the tensor product module $(W^a)' \boxtimes_{P(z)} W^a$, we know that there exists a unique module map $\hat{e}_{a;z} : (W^a)' \boxtimes_{P(z)} W^a \to V$ such that

$$\overline{\hat{e}_{a;z}}(w'_a \boxtimes_{P(z)} w_a) = \mathcal{Y}^e_{a'a;1}(w'_a, z)w_a$$

for $w_a \in W^a$ and $w'_a \in (W^a)'$, where $\overline{\hat{e}_{a;z}} : \overline{(W^a)' \boxtimes_{P(z)} W^a} \to \overline{V}$ is the natural extension of $\hat{e}_{a;z}$. When $z = 1$, we shall denote $\hat{e}_{a;1}$ simply by $\hat{e}_a$.

For $a \in \mathcal{A}$ and $z \in \mathbb{C}^\times$, by the universal property for tensor product modules, for any fixed bases $\{\mathcal{Y}^b_{aa';i}\}$ of $\mathcal{V}^b_{aa'}$ for $b \in \mathcal{A}$, there exists an isomorphism

$$f_{a;z} : W^a \boxtimes_{P(z)} (W^a)' \to \coprod_{b\in\mathcal{A}} N^b_{aa'} W^b$$

such that

$$\overline{f_{a;z}}(w_a \boxtimes_{P(z)} w'_a) = \mathcal{Y}(w_a, z) w'_a$$

where

$$\mathcal{Y} = \sum_{b\in\mathcal{A}} \sum_{i=1}^{N^b_{aa'}} \mathcal{Y}^b_{aa';i},$$

an intertwining operator of type $\binom{\coprod_{b\in\mathcal{A}} N^b_{aa'} W^b}{W^a (W^a)'}$. Since $N^e_{aa'} = 1$, there is a unique injective module map

$$g_{a;z} : V \to \coprod_{b\in\mathcal{A}} N^b_{aa'} W^b$$

such that $g_{a;z}(\mathbf{1}) = \mathbf{1} \in V = W^e \subset \coprod_{b\in\mathcal{A}} N^b_{aa'} W^b$. Let

$$i_{a;z} = f^{-1}_{a;z} \circ g_{a;z} : V \to W^a \boxtimes_{P(z)} (W^a)'.$$

When $z = 1$, we shall denote $i_{a;1}$ simply by i_a.

Lemma 3.1 *The map $i_{a;z}$ is independent of the choice of the bases $\mathcal{Y}^b_{aa';i}$ of $\mathcal{V}^b_{aa'}$ for $b \in \mathcal{A}$, $b \neq e$.*

Proof. We choose bases $\widetilde{\mathcal{Y}}^b_{aa';i}$ of $\mathcal{V}^b_{aa'}$ for $b \in \mathcal{A}$ such that $\widetilde{\mathcal{Y}}^e_{aa';1} = \mathcal{Y}^e_{aa';1}$. Then we also obtain an isomorphism

$$\tilde{f}_{a;z} : W^a \boxtimes_{P(z)} (W^a)' \to \coprod_{b\in\mathcal{A}} N^b_{aa'} W^b.$$

Since $\widetilde{\mathcal{Y}}^e_{aa';1} = \mathcal{Y}^e_{aa';1}$, $\pi_a \circ \tilde{f}_{a;z} = \pi_a \circ f_{a;z}$ where $\pi_a : \coprod_{b\in\mathcal{A}} N^b_{aa'} W^b \to W^e = V$ is the projection. By the definition of $g_{a;z}$, we see that $f^{-1}_{a;z} \circ g_{a;z} = \tilde{f}^{-1}_{a;z} \circ g_{a;z}$. So $i_{a;z}$ is independent of the choices of $\mathcal{Y}^b_{aa';i}$ of $\mathcal{V}^b_{aa'}$ for $b \in \mathcal{A}$, $b \neq e$. ∎

For $a \in \mathcal{A}$ and $z \in \mathbb{C}^\times$, using the universal property for the tensor product module $W^a \boxtimes_{P(z)} (W^a)'$, we know that there exists a unique module map $\hat{e}'_{a;z} : W^a \boxtimes_{P(z)} (W^a)' \to V$ such that

$$\overline{\hat{e}'_{a;z}}(w_a \boxtimes_{P(z)} w'_a) = \mathcal{Y}^e_{aa';1}(w_a, z)w'_a$$

for $w_a \in W^a$ and $w'_a \in (W^a)'$, where $\overline{\hat{e}'_{a;z}} : \overline{W^a \boxtimes_{P(z)} (W^a)'} \to \overline{V}$ is the natural extension of $\hat{e}'_{a;z}$. When $z = 1$, we shall denote $\hat{e}'_{a;1}$ simply by $\hat{e}'_a$.

For any fixed $a \in \mathcal{A}$ and $z \in \mathbb{C}^\times$, by the universal property for tensor product modules, for any fixed basis $\mathcal{Y}^b_{a'a;i}$ of $\mathcal{V}^b_{a'a}$ as we choose in Section 1 for $b \in \mathcal{A}$, $b \neq e$, there exists an isomorphism

$$f'_{a;z} : (W^a)' \boxtimes_{P(z)} W^a \to \coprod_{b\in\mathcal{A}} N^b_{a'a} W^b$$

such that

$$\overline{f'_{a;z}}(w'_a \boxtimes_{P(z)} w_a) = \mathcal{Y}(w'_a, z)w_a$$

where

$$\mathcal{Y} = \sum_{b\in\mathcal{A}} \sum_{i=1}^{N^b_{a'a}} \mathcal{Y}^b_{a'a;i},$$

an intertwining operator of type $\binom{\coprod_{b\in\mathcal{A}} N^b_{a'a}W^b}{(W^a)'W^a}$. Since $N^e_{a'a} = 1$, there is a unique injective module map

$$g'_{a;z} : V \to \coprod_{b\in\mathcal{A}} N^b_{a'a} W^b$$

such that $g'_{a;z}(\mathbf{1}) = \mathbf{1} \in V = W^e \subset \coprod_{b\in\mathcal{A}} N^b_{a'a}W^b$. Let

$$i'_{a;z} = (f'_{a;z})^{-1} \circ g'_{a;z} : V \to (W^a)' \boxtimes_{P(z)} W^a.$$

When $z = 1$, we shall denote $i'_{a;1}$ simply by i'_a.

Lemma 3.2 *The map $i'_{a;z}$ is independent of the choice of the bases $\mathcal{Y}^b_{a'a;i}$ of $\mathcal{V}^b_{a'a}$ for $b \in \mathcal{A}$, $b \neq e$.*

Proof. The proof is the same as the one for Lemma 3.1. ∎

The composition

$$r_a \circ (I_{W^a} \boxtimes \hat{e}_a) \circ \mathcal{A}^{-1} \circ (i_a \boxtimes I_{W^a}) \circ l_a^{-1}$$

of the module maps in the sequence

$$\begin{array}{llllll} W^a & \xrightarrow{l_a^{-1}} & V \boxtimes W^a & \xrightarrow{i_a \boxtimes I_{W^a}} & (W^a \boxtimes (W^a)') \boxtimes W^a & \longrightarrow \\ & \xrightarrow{\mathcal{A}^{-1}} & W^a \boxtimes ((W^a)' \boxtimes W^a) & \xrightarrow{I_{W^a} \boxtimes \hat{e}_a} & W^a \boxtimes V & \xrightarrow{r_a} W_a \end{array} \tag{3.1}$$

is a module map from W^a to itself. Since W^a is irreducible, there must exist $\lambda_a \in \mathbb{C}$ such that this module map is equal to λ_a times the identity map on W^a, that is,

$$r_a \circ (I_{W^a} \boxtimes \hat{e}_a) \circ \mathcal{A}^{-1} \circ (i_a \boxtimes I_{W^a}) \circ l_a^{-1} = \lambda_a I_{W^a}. \tag{3.2}$$

We need to calculate λ_a.

Let z_1, z_2 be any nonzero complex numbers. We first calculate the composition of the module maps given by the sequence

$$\begin{array}{llll} W^a & \xrightarrow{l_{a;z_2}^{-1}} & V \boxtimes_{P(z_2)} W^a & \\ & \xrightarrow{i_{a;z_1-z_2} \boxtimes_{P(z_2)} I_{W^a}} & (W^a \boxtimes_{P(z_1-z_2)} (W^a)') \boxtimes_{P(z_2)} W^a & \\ & \xrightarrow{\left(\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}\right)^{-1}} & W^a \boxtimes_{P(z_1)} ((W^a)' \boxtimes_{P(z_2)} W^a) & \\ & \xrightarrow{I_{W^a} \boxtimes_{P(z_1)} \hat{e}_{a;z_2}} & W^a \boxtimes_{P(z_1)} V & \xrightarrow{r_{a;z_1}} W_a, \end{array} \tag{3.3}$$

that is, we first calculate

$$r_{a;z_1} \circ (I_{W^a} \boxtimes_{P(z_1)} \hat{e}_{a;z_2}) \circ \left(\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}\right)^{-1} \circ (i_{a;z_1-z_2} \boxtimes_{P(z_2)} I_{W^a}) \circ l_{a;z_2}^{-1}.$$

Since this is also a module map from an irreducible module to itself, there exists $\lambda_{a,z_1,z_2} \in \mathbb{C}$ such that it is equal to $\lambda_{a;z_1,z_2} I_{W^a}$.

Proposition 3.3 *For $a \in \mathcal{A}$ and $z_1, z_2 \in \mathbb{C}^\times$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$, we have*

$$\begin{aligned} \lambda_{a;z_1,z_2} &= F_a \\ &= F(\mathcal{Y}^a_{ae;1} \otimes \mathcal{Y}^e_{a'a;1}; \mathcal{Y}^a_{ea;1} \otimes \mathcal{Y}^e_{aa';1}). \end{aligned} \tag{3.4}$$

Proof. Let $w_1, w_2 \in W^a$ and $w_1' \in (W^a)'$. Then

$$\mathcal{Y}^a_{ea;1}(\mathcal{Y}^e_{aa';1}(w_1, z_1 - z_2)w_1', z_2)w_2 \in \overline{W^a}.$$

By the definition of $l_{a;z_2}$, we have

$$\overline{l^{-1}_{a;z_2}}(\mathcal{Y}^a_{ea;1}(\mathcal{Y}^e_{aa';1}(w_1, z_1 - z_2)w_1', z_2)w_2) = (\mathcal{Y}^e_{aa';1}(w_1, z_1 - z_2)w_1') \boxtimes_{P(z_2)} w_2. \tag{3.5}$$

By definitions and the associativity of intertwining operators, we have

$$\begin{aligned}
&\overline{r_{a;z_1}}\Big(\overline{\big(I_{W^a} \boxtimes_{P(z_1)} \hat{e}_{a;z_2}\big)}\Big(\overline{\Big(\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}\Big)^{-1}}((w_1 \boxtimes_{P(z_1-z_2)} w_1') \boxtimes_{P(z_2)} w_2)\Big)\Big)\\
&\quad= \overline{r_{a;z_1}}\Big(\overline{\big(I_{W^a} \boxtimes_{P(z_1)} \hat{e}_{a;z_2}\big)}(w_1 \boxtimes_{P(z_1)} (w_1' \boxtimes_{P(z_2)} w_2))\Big)\\
&\quad= \overline{r_{a;z_1}}(w_1 \boxtimes_{P(z_1)} (\mathcal{Y}^e_{a'a;1}(w_1', z_2)w_2))\\
&\quad= \mathcal{Y}^a_{ae;1}(w_1, z_1)\mathcal{Y}^e_{a'a;1}(w_1', z_2)w_2\\
&\quad= \sum_{b\in\mathcal{A}}\sum_{k=1}^{N^a_{ba}}\sum_{l=1}^{N^b_{aa'}} F(\mathcal{Y}^a_{ae;1} \otimes \mathcal{Y}^e_{a'a;1}; \mathcal{Y}^a_{ba;k} \otimes \mathcal{Y}^b_{aa';l}) \cdot\\
&\qquad\qquad\qquad \cdot\mathcal{Y}^a_{ba;k}(\mathcal{Y}^b_{aa';l}(w_1, z_1 - z_2)w_1', z_2)w_2,
\end{aligned} \tag{3.6}$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$. Since both sides of (3.6) are well-defined in the region $|z_2| > |z_1 - z_2| > 0$, the left- and right-hand sides of (3.6) are equal in this larger region as series in (rational) powers of z_2 and $z_1 - z_2$.

Let $\eta^a_{ba;k} : W^b \boxtimes_{P(z_2)} W^a \to W^a$ and $\eta^b_{aa';l} : W^a \boxtimes_{P(z_1-z_2)} (W^a)' \to W^b$ be module maps determined by

$$\eta^a_{ba;k}(w_3 \boxtimes_{P(z_2)} w_2) = \mathcal{Y}^a_{ba;k}(w_3, z_2)w_2$$

and

$$\eta^b_{aa';l}(w_1 \boxtimes_{P(z_2)} w_1') = \mathcal{Y}^a_{aa';k}(w_1, z_1 - z_2)w_1',$$

respectively, for $w_1, w_2 \in W^a$, $w_3 \in W^b$ and $w_1' \in (W^a)'$. Then (3.6) gives

$$\begin{aligned}
&\overline{r_{a;z_1}}\Big(\overline{\big(I_{W^a} \boxtimes_{P(z_1)} \hat{e}_{a;z_2}\big)}\Big(\overline{\Big(\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}\Big)^{-1}}((w_1 \boxtimes_{P(z_1-z_2)} w_1') \boxtimes_{P(z_2)} w_2)\Big)\Big)\\
&\quad= \sum_{b\in\mathcal{A}}\sum_{k=1}^{N^a_{ba}}\sum_{l=1}^{N^b_{aa'}} F(\mathcal{Y}^a_{ae;1} \otimes \mathcal{Y}^e_{a'a;1}; \mathcal{Y}^a_{ba;k} \otimes \mathcal{Y}^b_{aa';l}) \cdot\\
&\qquad\qquad \cdot(\overline{\eta^a_{ba;k}} \circ (\overline{\eta^b_{aa';l}} \boxtimes_{P(z_1-z_2)} I_{(W^a)'}))((w_1 \boxtimes_{P(z_1-z_2)} w_1') \boxtimes_{P(z_2)} w_2)
\end{aligned} \tag{3.7}$$

for $w_1, w_2 \in W^a$ and $w_1' \in (W^a)'$, when $|z_2| > |z_1 - z_2| > 0$. Since the components of elements of the form $(w_1 \boxtimes_{P(z_1-z_2)} w_1') \boxtimes_{P(z_2)} w_2$ for $w_1, w_2 \in W^a$

and $w_1' \in (W^a)'$ span $(W^a \boxtimes_{P(z_1-z_2)} (W^a)') \boxtimes_{P(z_2)} W^a$, (3.7) gives

$$\begin{aligned}
&r_{a;z_1} \circ \left(I_{W^a} \boxtimes_{P(z_1)} \hat{e}_{a;z_2}\right) \circ \left(\mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}\right)^{-1}\\
&\quad = \sum_{b\in\mathcal{A}} \sum_{k=1}^{N_{ba}^a} \sum_{l=1}^{N_{aa'}^b} F(\mathcal{Y}_{ae;1}^a \otimes \mathcal{Y}_{a'a;1}^e; \mathcal{Y}_{ba;k}^a \otimes \mathcal{Y}_{aa';l}^b) \cdot\\
&\qquad\qquad \cdot(\eta_{ba;k}^a \circ (\eta_{aa';l}^b \boxtimes_{P(z_1-z_2)} I_{(W^a)'}))
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&r_{a;z_1} \circ \left(I_{W^a} \boxtimes_{P(z_1)} \hat{e}_{a;z_2}\right) \circ \left(\mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}\right)^{-1} \circ (i_{a;z_1-z_2} \boxtimes_{P(z_2)} I_{(W^a)'}) \circ l_{a;z_2}^{-1}\\
&= \sum_{b\in\mathcal{A}} \sum_{k=1}^{N_{ba}^a} \sum_{l=1}^{N_{aa'}^b} F(\mathcal{Y}_{ae;1}^a \otimes \mathcal{Y}_{a'a;1}^e; \mathcal{Y}_{ba;k}^a \otimes \mathcal{Y}_{aa';l}^b) \cdot\\
&\qquad \cdot(\eta_{ba;k}^a \circ (\eta_{aa';l}^b \boxtimes_{P(z_1-z_2)} I_{(W^a)'}) \circ (i_{a;z_1-z_2} \boxtimes_{P(z_2)} I_{(W^a)'}) \circ l_{a;z_2}^{-1})\\
&= \sum_{b\in\mathcal{A}} \sum_{k=1}^{N_{ba}^a} \sum_{l=1}^{N_{aa'}^b} F(\mathcal{Y}_{ae;1}^a \otimes \mathcal{Y}_{a'a;1}^e; \mathcal{Y}_{ba;k}^a \otimes \mathcal{Y}_{aa';l}^b) \cdot\\
&\qquad \cdot(\eta_{ba;k}^a \circ ((\eta_{aa';l}^b \circ i_{a;z_1-z_2}) \boxtimes_{P(z_1-z_2)} I_{(W^a)'}) \circ l_{a;z_2}^{-1}). \qquad (3.8)
\end{aligned}$$

Let $\pi_{b;l} : \coprod_{d\in\mathcal{A}} N_{aa'}^d W^d \to W^b$ for $b \in \mathcal{A}$ and $l = 1, \ldots, N_{aa'}^b$ be the projection to the l-th copy of W^b. Then

$$\begin{aligned}
\eta_{aa';l}^b(w_1 \boxtimes_{P(z_1-z_2)} w_1') &= \mathcal{Y}_{aa';l}^b(w_1, z_1 - z_2)w_1'\\
&= (\pi_{b;l} \circ f_{a;z_1-z_2})(w_1 \boxtimes_{P(z_1-z_2)} w_1')
\end{aligned}$$

for $w_1 \in W^a$ and $w_1' \in (W^a)'$. So we have $\eta_{aa';l}^b = \pi_{b;l} \circ f_{a;z_1-z_2}$ and

$$\begin{aligned}
\eta_{aa';l}^b \circ i_{a;z_1-z_2} &= \pi_{b;l} \circ f_{a;z_1-z_2} \circ f_{a;z_1-z_2}^{-1} \circ g_{a;z_1-z_2}\\
&= \pi_{b;l} \circ g_{a;z_1-z_2}\\
&= \delta_{eb}\delta_{1l} I_V. \qquad (3.9)
\end{aligned}$$

Also,

$$\begin{aligned}
(\eta_{ea;1}^a \circ l_{a;z_2}^{-1})(w_1) &= \eta_{ea;1}^a(l_{a;z_2}^{-1}(w_1))\\
&= \eta_{ea;1}^a(\mathbf{1} \boxtimes_{P(z_2)} w_1)\\
&= Y_{W^a}(\mathbf{1}, z_2)w_1\\
&= w_1
\end{aligned}$$

for $w_1 \in W^a$. So we have

$$\eta^a_{ea;1} \circ l^{-1}_{a;z_2} = I_{W^a}. \tag{3.10}$$

Using (3.9) and (3.10), the right-hand side of (3.8) becomes

$$\begin{aligned}
&\sum_{b\in\mathcal{A}}\sum_{k=1}^{N^a_{ba}}\sum_{l=1}^{N^b_{aa'}} F(\mathcal{Y}^a_{ae;1}\otimes\mathcal{Y}^e_{a'a;1};\mathcal{Y}^a_{ba;k}\otimes\mathcal{Y}^b_{aa';l})\cdot\\
&\qquad\qquad \cdot\delta_{eb}\delta_{1l}(\eta^a_{ba;k}\circ l^{-1}_{a;z_2})\\
&\quad = F(\mathcal{Y}^a_{ae;1}\otimes\mathcal{Y}^e_{a'a;1};\mathcal{Y}^a_{ea;1}\otimes\mathcal{Y}^e_{aa';1})\cdot\\
&\qquad\qquad \cdot(\eta^a_{ea;1}\circ l^{-1}_{a;z_2})\\
&\quad = F(\mathcal{Y}^a_{ae;1}\otimes\mathcal{Y}^e_{a'a;1};\mathcal{Y}^a_{ea;1}\otimes\mathcal{Y}^e_{aa';1})I_{W^a}.
\end{aligned} \tag{3.11}$$

From (3.8), (3.11) and the definition of $\lambda_{a;z_1,z_2}$, we obtain (3.4). ∎

Proposition 3.4 *For $a \in \mathcal{A}$, we have*

$$\begin{aligned}
\lambda_a &= F_a\\
&= F(\mathcal{Y}^a_{ae;1}\otimes\mathcal{Y}^e_{a'a;1};\mathcal{Y}^a_{ea;1}\otimes\mathcal{Y}^e_{aa';1}).
\end{aligned} \tag{3.12}$$

Proof. Choose $z_1^0, z_2^0 \in (0,\infty)$ satisfying $z_1^0 > z_2^0 > z_1^0 - z_2^0 > 0$ and choose paths $\gamma_0, \gamma_1, \gamma_2$ in $(0,\infty)$ from $z_1^0 - z_2^0, z_1^0, z_2^0$ to $1, 1, 1$, respectively. Then by the definitions of all the module maps involved, the following diagrams are commutative:

$$\begin{array}{ccc}
W^a & \xrightarrow{l^{-1}_{a;z_2^0}} & V\boxtimes_{P(z_2^0)} W^a\\
{\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{T}_{\gamma_2}}\\
W^a & \xrightarrow{l^{-1}_a} & V\boxtimes W^a
\end{array}$$

$$\begin{array}{ccc}
V\boxtimes_{P(z_2)} W^a & \xrightarrow{i_{a;z_1^0-z_2^0}\boxtimes_{P(z_2^0)} I_{W^a}} & (W^a\boxtimes_{P(z_1^0-z_2^0)}(W^a)')\boxtimes_{P(z_2^0)} W^a\\
{\scriptstyle \mathcal{T}_{\gamma_2}}\big\downarrow & & \big\downarrow{\scriptstyle (\mathcal{T}_{\gamma_0}\boxtimes I_{W^a})\circ\mathcal{T}_{\gamma_2}}\\
V\boxtimes W^a & \xrightarrow{i_a\boxtimes I_{W^a}} & (W^a\boxtimes(W^a)')\boxtimes W^a
\end{array}$$

$$\begin{tikzcd}
(W^a \boxtimes_{P(z_1^0-z_2^0)} (W^a)') \boxtimes_{P(z_2^0} W^a \arrow[r, "\left(\mathcal{A}^{P(z_1^0-z_2^0),P(z_2^0)}_{P(z_1^0),P(z_2^0)}\right)^{-1}"] \arrow[d, "(\mathcal{T}_{\gamma_0} \boxtimes I_{W^a}) \circ \mathcal{T}_{\gamma_2}"'] & W^a \boxtimes_{P(z_1^0)} ((W^a)' \boxtimes_{P(z_2^0)} W^a) \arrow[d, "(I_{W^a} \boxtimes \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1}"] \\
(W^a \boxtimes (W^a)') \boxtimes W^a \arrow[r, "\mathcal{A}^{-1}"] & W^a \boxtimes ((W^a)' \boxtimes W^a)
\end{tikzcd}$$

$$\begin{tikzcd}
W^a \boxtimes_{P(z_1^0)} ((W^a)' \boxtimes_{P(z_2^0)} W^a) \arrow[r, "I_{W^a} \boxtimes_{P(z_1^0)} \hat{e}_{a;z_2^0}"] \arrow[d, "(I_{W^a} \boxtimes \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1}"'] & W^a \boxtimes_{P(z_1^0)} V \arrow[d, "\mathcal{T}_{\gamma_1}"] \\
W^a \boxtimes ((W^a)' \boxtimes W^a) \arrow[r, "I_{W^a} \boxtimes \hat{e}_a"] & W^a \boxtimes V
\end{tikzcd}$$

$$\begin{tikzcd}
W^a \boxtimes_{P(z_1^0)} V \arrow[r, "r_{a;z_1^0}"] \arrow[d, "\mathcal{T}_{\gamma_1}"] & W^a \arrow[d, "="] \\
W^a \boxtimes V \arrow[r, "r_a"] & W^a
\end{tikzcd}$$

Combining these diagrams, we obtain

$$\begin{aligned}
& r_a \circ (I_{W^a} \boxtimes \hat{e}_a) \circ \mathcal{A}^{-1} \circ (i_a \boxtimes I_{W^a}) \circ l_a^{-1} \\
& \quad = r_{a;z_1} \circ (I_{W^a} \boxtimes_{P(z_1)} \hat{e}_{a;z_2}) \ \circ \\
& \qquad\qquad \circ \left(\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}\right)^{-1} \circ (i_{a;z_1-z_2} \boxtimes_{P(z_2)} I_{W^a}) \circ l_{a;z_2}^{-1}.
\end{aligned}$$

From this equality and the definitions of λ_a and $\lambda_{a;z_1^0,z_2^0}$, we obtain

$$\lambda_a = \lambda_{a;z_1^0,z_2^0}.$$

By Proposition 3.3, we obtain (3.12). ∎

The composition

$$l_a \circ (\hat{e}'_a \boxtimes I_{W^a}) \circ \mathcal{A} \circ (I_{W^a} \boxtimes i'_a) \circ r_a^{-1}$$

of the module maps in the sequence

$$\begin{array}{cccccccc}
W^a & \xrightarrow{r_a^{-1}} & W^a \boxtimes V & \xrightarrow{I_{W^a} \boxtimes i'_a} & W^a \boxtimes ((W^a)' \boxtimes W^a) & \longrightarrow & & \\
 & \xrightarrow{\mathcal{A}} & (W^a \boxtimes (W^a)') \boxtimes W^a & \xrightarrow{\hat{e}'_a \boxtimes I_{W^a}} & V \boxtimes W^a & \xrightarrow{l_a} & W^a &
\end{array} \tag{3.13}$$

is a module map from W^a to itself. Again, since W^a is irreducible, there must exist $\mu_a \in \mathbb{C}$ such that this module map is equal to μ_a times the identity map on W^a, that is,

$$l_a \circ (\hat{e}'_a \boxtimes I_{W^a}) \circ \mathcal{A} \circ (I_{W^a} \boxtimes i'_a) \circ r_a^{-1} = \mu_a I_{W^a}. \tag{3.14}$$

We need to calculate μ_a.

Our method to calculate μ_a is similar to the one used to calculate λ_a above. Let z_1, z_2 be any nonzero complex numbers. We first calculate the composition of the module maps given by the sequence

$$\begin{array}{lccc} W^a & \xrightarrow{r_{a;z_2}^{-1}} & W^a \boxtimes_{P(z_1)} V & \\ & \xrightarrow{I_{W^a} \boxtimes_{P(z_1)} i'_{a;z_2}} & (W^a \boxtimes_{P(z_1)} ((W^a)' \boxtimes_{P(z_2)} W^a) & \\ & \xrightarrow{\mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}} & (W^a \boxtimes_{P(z_1-z_2)} (W^a)') \boxtimes_{P(z_2)} W^a & \\ & \xrightarrow{\hat{e}'_{a;z_1-z_2} \boxtimes_{P(z_1)} I_{W^a}} & V \boxtimes_{P(z_2)} W^a & \xrightarrow{l_{a;z_2}} W^a, \end{array} \tag{3.15}$$

that is, we first calculate

$$l_{a;z_2} \circ (\hat{e}'_{a;z_1-z_2} \boxtimes_{P(z_1)} I_{W^a}) \circ \mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)} \circ (I_{W^a} \boxtimes_{P(z_1)} i'_{a;z_2}) \circ r_{a;z_2}^{-1}.$$

Since this is also a module map from an irreducible module to itself, there exists $\mu_{a,z_1,z_2} \in \mathbb{C}$ such that it is equal to $\mu_{a;z_1,z_2} I_{W^a}$.

Proposition 3.5 *For $a \in \mathcal{A}$ and $z_1, z_2 \in \mathbb{C}^\times$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$, we have*

$$\begin{aligned} \mu_{a;z_1,z_2} &= F_a \\ &= F(\mathcal{Y}_{ae;1}^a \otimes \mathcal{Y}_{a'a;1}^e; \mathcal{Y}_{ea;1}^a \otimes \mathcal{Y}_{aa';1}^e). \end{aligned} \tag{3.16}$$

Proof. Let $w_1, w_2 \in W^a$ and $w'_1 \in (W^a)'$. Then

$$\mathcal{Y}_{ae;1}^a(w_1, z_1)\mathcal{Y}_{a'a;1}^e(w'_1, z_2)w_2 \in \overline{W^a}.$$

By the definition of $r_{a;z_2}$, we have

$$\overline{r_{a;z_2}^{-1}}(\mathcal{Y}_{ae;1}^a(w_1, z_1)\mathcal{Y}_{a'a;1}^e(w'_1, z_2)w_2) = w_1 \boxtimes_{P(z_1)} \mathcal{Y}_{a'a;1}^e(w'_1, z_2)w_2). \tag{3.17}$$

By definitions and the associativity of intertwining operators, we have

$$\begin{aligned}
&\overline{l_{a;z_2}}\Big(\overline{\big(\hat{e}'_{a;z_1-z_2}\boxtimes_{P(z_1)} I_{W^a}\big)}\Big(\overline{\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}}(w_1\boxtimes_{P(z_1)}(w'_1\boxtimes_{P(z_2)} w_2))\Big)\Big)\\
&\quad=\overline{l_{a;z_2}}\Big(\overline{\big(\hat{e}'_{a;z_1-z_2}\boxtimes_{P(z_1)} I_{W^a}\big)}((w_1\boxtimes_{P(z_1-z_2)} w'_1)\boxtimes_{P(z_2)} w_2)\Big)\\
&\quad=\overline{l_{a;z_2}}((\mathcal{Y}^e_{aa';1}(w_1,z_1-z_2)w'_1)\boxtimes_{P(z_2)} w_2)\\
&\quad=\mathcal{Y}^a_{ea;1}(\mathcal{Y}^e_{aa';1}(w_1,z_1-z_2)w'_1,z_2)w_2\\
&\quad=\sum_{b\in\mathcal{A}}\sum_{k=1}^{N^a_{ab}}\sum_{l=1}^{N^b_{a'a}}F^{-1}(\mathcal{Y}^a_{ea;1}\otimes\mathcal{Y}^e_{aa';1};\mathcal{Y}^a_{ab;k}\otimes\mathcal{Y}^b_{a'a;l})\cdot\\
&\qquad\qquad\qquad\cdot\mathcal{Y}^a_{ab;k}(w_1,z_1)\mathcal{Y}^b_{a'a;l}(w'_1,z_2)w_2, \qquad (3.18)
\end{aligned}$$

when $|z_1|>|z_2|>|z_1-z_2|>0$. Since both sides of (3.18) are well-defined in the region $|z_1|>|z_2|>0$, the left- and right-hand sides of (3.18) are equal in this larger region as series in (rational) powers of z_1 and z_2.

Let $\eta^a_{ab;k}: W^a\boxtimes_{P(z_1)} W^b\to W^a$ and $\eta^b_{a'a;l}:(W^a)'\boxtimes_{P(z_2)} W^a\to W^b$ be module maps determined by

$$\eta^a_{ab;k}(w_1\boxtimes_{P(z_1)} w_3)=\mathcal{Y}^a_{ab;k}(w_1,z_2)w_3$$

and

$$\eta^b_{a'a;l}(w'_1\boxtimes_{P(z_2)} w_2)=\mathcal{Y}^b_{a'a;k}(w'_1,z_2)w_2,$$

respectively, for $w_1,w_2\in W^a$, $w_3\in W^b$ and $w'_1\in(W^a)'$. Then (3.18) gives

$$\begin{aligned}
&\overline{l_{a;z_2}}\Big(\overline{\big(\hat{e}'_{a;z_1-z_2}\boxtimes_{P(z_1)} I_{W^a}\big)}\Big(\overline{\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}}(w_1\boxtimes_{P(z_1)}(w'_1\boxtimes_{P(z_2)} w_2))\Big)\Big)\\
&\quad=\sum_{b\in\mathcal{A}}\sum_{k=1}^{N^a_{ab}}\sum_{l=1}^{N^b_{a'a}}F^{-1}(\mathcal{Y}^a_{ea;1}\otimes\mathcal{Y}^e_{aa';1};\mathcal{Y}^a_{ab;k}\otimes\mathcal{Y}^b_{a'a;l})\cdot\\
&\qquad\cdot(\overline{\eta^a_{ab;k}}\circ(\overline{I_{W^a}\boxtimes_{P(z_1)}\eta^b_{a'a;l}}))(w_1\boxtimes_{P(z_1)}(w'_1\boxtimes_{P(z_2)} w_2)) \qquad (3.19)
\end{aligned}$$

for $w_1,w_2\in W^a$ and $w'_1\in(W^a)'$, when $|z_1|>|z_2|>0$. Since the components of elements of the form $w_1\boxtimes_{P(z_1)}(w'_1\boxtimes_{P(z_2)} w_2)$ for $w_1,w_2\in W^a$ and $w'_1\in(W^a)'$ span $W^a\boxtimes_{P(z_1)}((W^a)'\boxtimes_{P(z_2)} W^a)$, (3.19) gives

$$\begin{aligned}
&l_{a;z_2}\circ\big(\hat{e}'_{a;z_1-z_2}\boxtimes_{P(z_1)} I_{W^a}\big)\circ\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}\\
&\quad=\sum_{b\in\mathcal{A}}\sum_{k=1}^{N^a_{ab}}\sum_{l=1}^{N^b_{a'a}}F^{-1}(\mathcal{Y}^a_{ea;1}\otimes\mathcal{Y}^e_{aa';1};\mathcal{Y}^a_{ab;k}\otimes\mathcal{Y}^b_{a'a;l})\cdot\\
&\qquad\cdot(\eta^a_{ab;k}\circ(I_{W^a}\boxtimes_{P(z_1)}\eta^b_{a'a;l}))
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&l_{a;z_2} \circ \left(\hat{e}'_{a;z_1-z_2} \boxtimes_{P(z_1)} I_{W^a}\right) \circ \mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)} \circ (I_{W^a} \boxtimes_{P(z_1)} i'_{a;z_2}) \circ r^{-1}_{a;z_2}\\
&\quad = \sum_{b\in\mathcal{A}} \sum_{k=1}^{N_{ab}^a} \sum_{l=1}^{N_{a'a}^b} F^{-1}(\mathcal{Y}^a_{ea;1} \otimes \mathcal{Y}^e_{aa';1}; \mathcal{Y}^a_{ab;k} \otimes \mathcal{Y}^b_{a'a;l}) \cdot\\
&\qquad\qquad \cdot(\eta^a_{ab;k} \circ (I_{W^a} \boxtimes_{P(z_1)} \eta^b_{a'a;l}) \circ (I_{W^a} \boxtimes_{P(z_1)} i'_{a;z_2}) \circ r^{-1}_{a;z_2})\\
&\quad = \sum_{b\in\mathcal{A}} \sum_{k=1}^{N_{ab}^a} \sum_{l=1}^{N_{a'a}^b} F^{-1}(\mathcal{Y}^a_{ea;1} \otimes \mathcal{Y}^e_{aa';1}; \mathcal{Y}^a_{ab;k} \otimes \mathcal{Y}^b_{a'a;l}) \cdot\\
&\qquad\qquad \cdot(\eta^a_{ab;k} \circ (I_{W^a} \boxtimes_{P(z_1)} (\eta^b_{a'a;l} \circ i'_{a;z_2})) \circ r^{-1}_{a;z_2}).
\end{aligned} \tag{3.20}$$

Let $\pi'_{b;l} : \coprod_{d\in\mathcal{A}} N^d_{a'a} W^d \to W^b$ for $b \in \mathcal{A}$ and $l = 1, \ldots, N^b_{a'a}$ be the projection to the l-th copy of W^b. Then from

$$\begin{aligned}
\eta^b_{a'a;l}(w'_1 \boxtimes_{P(z_2)} w_1) &= \mathcal{Y}^b_{aa';l}(w'_1, z_2)w_1\\
&= (\pi_{b;l} \circ f'_{a;z_2})(w'_1 \boxtimes_{P(z_1-z_2)} w_1)
\end{aligned}$$

for $w_1 \in W^a$ and $w'_1 \in (W^a)'$, we obtain $\eta^b_{a'a;l} = \pi_{b;l} \circ f'_{a;z_2}$ and

$$\begin{aligned}
\eta^b_{a'a;l} \circ i'_{a;z_2} &= \pi_{b;l} \circ f'_{a;z_2} \circ (f'_{a;z_2})^{-1} \circ g'_{a;z_2}\\
&= \pi_{b;l} \circ g'_{a;z_2}\\
&= \delta_{eb}\delta_{1l} I_V.
\end{aligned} \tag{3.21}$$

We also have

$$\begin{aligned}
(\overline{\eta^a_{ae;1} \circ r^{-1}_{a;z_2}})(e^{z_2L(-1)}w_1) &= \overline{\eta^a_{ae;1}}(\overline{r^{-1}_{a;z_2}}(e^{z_2L(-1)}w_1))\\
&= \overline{\eta^a_{ae;1}}(w_1 \boxtimes_{P(z_2)} \mathbf{1})\\
&= \mathcal{Y}^a_{ae;1}(w_1, z_2)\mathbf{1}\\
&= e^{z_2L(-1)} Y_{W^a}(\mathbf{1}, -z_2)w_1\\
&= e^{z_2L(-1)}w_1
\end{aligned} \tag{3.22}$$

for $w_1 \in W^a$. Since the homogeneous components of elements of the form $e^{z_2L(-1)}w_1$ span W^a, we see that (3.22) gives

$$\eta^a_{ae;1} \circ r^{-1}_{a;z_2} = I_{W^a}. \tag{3.23}$$

From (3.21) and (3.23), the right-hand side of (3.20) becomes

$$
\begin{aligned}
&\sum_{b\in\mathcal{A}}\sum_{k=1}^{N_{ab}^{a}}\sum_{l=1}^{N_{a'a}^{b}} F^{-1}(\mathcal{Y}_{ea;1}^{a}\otimes\mathcal{Y}_{aa';1}^{e};\mathcal{Y}_{ab;k}^{a}\otimes\mathcal{Y}_{a'a;l}^{b})\cdot\\
&\qquad\qquad\cdot\delta_{eb}\delta_{1l}(\eta_{ab;k}^{a}\circ r_{a;z_2}^{-1})\\
&\quad = F^{-1}(\mathcal{Y}_{ea;1}^{a}\otimes\mathcal{Y}_{aa';1}^{e};\mathcal{Y}_{ab;k}^{a}\otimes\mathcal{Y}_{a'a;l}^{b})(\eta_{ae;1}^{a}\circ r_{a;z_2}^{-1})\\
&\quad = F^{-1}(\mathcal{Y}_{ea;1}^{a}\otimes\mathcal{Y}_{aa';1}^{e};\mathcal{Y}_{ab;k}^{a}\otimes\mathcal{Y}_{a'a;l}^{b})I_{W^a}. \qquad (3.24)
\end{aligned}
$$

From (3.20), (3.24) and the definition of $\mu_{a;z_1,z_2}$, we obtain

$$\mu_{a;z_1,z_2} = F^{-1}(\mathcal{Y}_{ae;1}^{a}\otimes\mathcal{Y}_{a'a;1}^{e};\mathcal{Y}_{ea;1}^{a}\otimes\mathcal{Y}_{aa';1}^{e}). \qquad (3.25)$$

By Proposition 3.2 in [H7] and the properties of the bases $\mathcal{Y}_{ae;1}^{a}$, $\mathcal{Y}_{a'a;1}^{e}$, $\mathcal{Y}_{ea;1}^{a}$ and $\mathcal{Y}_{aa';1}^{e}$ under the action of the S_3 symmetry for $a\in\mathcal{A}$, we have

$$F^{-1}(\mathcal{Y}_{ae;1}^{a}\otimes\mathcal{Y}_{a'a;1}^{e};\mathcal{Y}_{ea;1}^{a}\otimes\mathcal{Y}_{aa';1}^{e}) = F(\mathcal{Y}_{ae;1}^{a}\otimes\mathcal{Y}_{a'a;1}^{e};\mathcal{Y}_{ea;1}^{a}\otimes\mathcal{Y}_{aa';1}^{e}). \qquad (3.26)$$

From (3.25) and (3.26), we obtain (3.16). ∎

Proposition 3.6 *For $a\in\mathcal{A}$, we have*

$$
\begin{aligned}
\mu_a &= F_a\\
&= F(\mathcal{Y}_{ae;1}^{a}\otimes\mathcal{Y}_{a'a;1}^{e};\mathcal{Y}_{ea;1}^{a}\otimes\mathcal{Y}_{aa';1}^{e}). \qquad (3.27)
\end{aligned}
$$

Proof. Choose $z_1^0, z_2^0\in(0,\infty)$ satisfying $z_1^0 > z_2^0 > z_1^0 - z_2^0 > 0$ and choose paths $\gamma_0,\gamma_1,\gamma_2$ in $(0,\infty)$ from $z_1^0 - z_2^0, z_1^0, z_2^0$ to $1,1,1$, respectively. Then by the definitions of all the module maps involved for general z_1, z_2, the following diagrams are commutative:

$$
\begin{tikzcd}
W^a \arrow[r, "r^{-1}_{a;z_2^0}"] \arrow[d, "="'] & W^a\boxtimes_{P(z_2^0)} V \arrow[d, "\mathcal{T}_{\gamma_2}"]\\
W^a \arrow[r, "r_a^{-1}"] & W^a\boxtimes V
\end{tikzcd}
$$

$$
\begin{tikzcd}
W^a\boxtimes_{P(z_2^0)} V \arrow[r, "I_{W^a}\boxtimes_{P(z_1^0)} i'_{a;z_2}"] \arrow[d, "\mathcal{T}_{\gamma_2}"'] & W^a\boxtimes_{P(z_1^0)}((W^a)'\boxtimes_{P(z_2^0)} W^a) \arrow[d, "(I_{W^a}\boxtimes\mathcal{T}_{\gamma_2})\circ\mathcal{T}_{\gamma_1}"]\\
W^a\boxtimes V \arrow[r, "I_{W^a}\boxtimes i'_a"] & W^a\boxtimes((W^a)'\boxtimes W^a)
\end{tikzcd}
$$

$$\begin{array}{ccc}
W^a \boxtimes_{P(z_1^0)} ((W^a)' \boxtimes_{P(z_2^0)} W^a) & \xrightarrow{\mathcal{A}^{P(z_1^0-z_2^0),P(z_2^0)}_{P(z_1^0),P(z_2^0)}} & (W^a \boxtimes_{P(z_1^0-z_2^0)} (W^a)') \boxtimes_{P(z_2^0)} W^a \\
{\scriptstyle (I_{W^a}\boxtimes \mathcal{T}_{\gamma_2})\circ \mathcal{T}_{\gamma_1}}\downarrow & & \downarrow{\scriptstyle (\mathcal{T}_{\gamma_0}\boxtimes I_{W^a})\circ \mathcal{T}_{\gamma_2}} \\
W^a \boxtimes ((W^a)' \boxtimes W^a) & \xrightarrow{\mathcal{A}} & (W^a \boxtimes (W^a)') \boxtimes W^a
\end{array}$$

$$\begin{array}{ccc}
(W^a \boxtimes_{P(z_1^0-z_2^0)} (W^a)') \boxtimes_{P(z_2^0)} W^a & \xrightarrow{\hat{e}'_{a;z_1-z_2}\boxtimes_{P(z_1)} I_{W^a}} & V \boxtimes_{P(z_2^0)} W^a \\
{\scriptstyle (\mathcal{T}_{\gamma_0}\boxtimes I_{W^a})\circ \mathcal{T}_{\gamma_2}}\downarrow & & \downarrow{\scriptstyle \mathcal{T}_{\gamma_2}} \\
(W^a \boxtimes (W^a)') \boxtimes W^a & \xrightarrow{\hat{e}'_a\boxtimes I_{W^a}} & V \boxtimes W^a
\end{array}$$

$$\begin{array}{ccc}
V \boxtimes_{P(z_2^0)} W^a & \xrightarrow{l_{a;z_2^0}} & W^a \\
{\scriptstyle \mathcal{T}_{\gamma_2}}\downarrow & & \downarrow{\scriptstyle =} \\
V \boxtimes W^a & \xrightarrow{l_a} & W^a
\end{array}$$

Combining these diagrams, we obtain

$$\begin{aligned}
&l_a \circ (\hat{e}'_a \boxtimes I_{W^a}) \circ \mathcal{A} \circ (I_{W^a} \boxtimes i'_a) \circ r_a^{-1} \\
&\quad = l_{a;z_2} \circ (\hat{e}'_{a;z_1-z_2} \boxtimes_{P(z_1)} I_{W^a}) \circ \mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)} \circ (I_{W^a} \boxtimes_{P(z_1)} i'_{a;z_2}) \circ r^{-1}_{a;z_2}.
\end{aligned}$$

From this equality and the definitions of μ_a and $\mu_{a;z_1^0,z_2^0}$, we obtain

$$\mu_a = \mu_{a;z_1^0,z_2^0}.$$

By Proposition 3.5, we obtain (3.27). ∎

For the compositions

$$\begin{aligned}
\lambda'_a &= r_{a'} \circ (I_{(W^a)'} \boxtimes \hat{e}'_a) \circ \mathcal{A}^{-1} \circ (i'_a \boxtimes I_{W^a}) \circ l_{a'}^{-1}, \\
\mu'_a &= l_{a'} \circ (\hat{e}_a \boxtimes I_{(W^a)'}) \circ \mathcal{A} \circ (I_{W^a} \boxtimes i_a) \circ r_{a'}^{-1}
\end{aligned}$$

of the module maps in the sequences

$$\begin{aligned}
(W^a)' &\xrightarrow{l_{a'}^{-1}} V \boxtimes (W^a)' \longrightarrow \\
&\xrightarrow{i'_a\boxtimes I_{W^a}} ((W^a)' \boxtimes W^a) \boxtimes (W^a)' \longrightarrow \\
&\xrightarrow{\mathcal{A}^{-1}} (W^a)' \boxtimes (W^a \boxtimes (W^a)') \xrightarrow{I_{(W^a)'}\boxtimes \hat{e}'_a} (W^a)' \boxtimes V \xrightarrow{r_{a'}} (W^a)',
\end{aligned} \tag{3.28}$$

$$
\begin{array}{llllll}
(W^a)' & \xrightarrow{r_{a'}^{-1}} & (W^a)' \boxtimes V & \longrightarrow & & \\
& \xrightarrow{I_{W^a} \boxtimes i_a} & (W^a)' \boxtimes (W^a \boxtimes (W^a)') & \longrightarrow & & \\
& \xrightarrow{\mathcal{A}} & ((W^a)' \boxtimes W^a) \boxtimes (W^a)' & \xrightarrow{\hat{e}_a \boxtimes I_{(W^a)'}} & V \boxtimes (W^a)' & \xrightarrow{l_{a'}} (W^a)',
\end{array}
\tag{3.29}
$$

respectively, we have the following:

Proposition 3.7 *For $a \in \mathcal{A}$, we have*

$$
\begin{aligned}
\lambda'_a &= \mu'_a \\
&= F_a \\
&= F(\mathcal{Y}^a_{ae;1} \otimes \mathcal{Y}^e_{a'a;1}; \mathcal{Y}^a_{ea;1} \otimes \mathcal{Y}^e_{aa';1}).
\end{aligned}
\tag{3.30}
$$

The proof of Proposition 3.7 is completely analogous to the proofs of Propositions 3.4 and 3.6 and is omitted here.

Since $F_a \neq 0$, we let

$$
\begin{aligned}
e_a &= \frac{1}{F_a} \hat{e}_a, \\
e'_a &= \frac{1}{F_a} \hat{e}'_a
\end{aligned}
$$

for $a \in \mathcal{A}$.

Theorem 3.8 *Let V be a simple vertex operator algebra satisfying the conditions in Section 1. Then with the left duals, the right duals and the module maps i_a, e_a, i'_a and e'_a above, the braided tensor category structure on the category of V-modules constructed in [HL1]–[HL4], [H1] and [H5] is rigid.*

Proof. Since our tensor category is semisimple, to prove the rigidity, we need only discuss irreducible modules. By Propositions 3.4, 3.6 and 3.7 and the definitions of the maps e_a and e'_a, the composition of the maps in the sequences

$$
\begin{array}{llllll}
W^a & \xrightarrow{l_a^{-1}} & V \boxtimes W^a & \xrightarrow{i_a \boxtimes I_{W^a}} & (W^a \boxtimes (W^a)') \boxtimes W^a & \longrightarrow \\
& \xrightarrow{\mathcal{A}^{-1}} & W^a \boxtimes ((W^a)' \boxtimes W^a) & \xrightarrow{I_{W^a} \boxtimes e_a} & W^a \boxtimes V & \xrightarrow{r_a} W_a,
\end{array}
$$

$$
\begin{array}{llllll}
W^a & \xrightarrow{r_a^{-1}} & W^a \boxtimes V & \xrightarrow{I_{W^a} \boxtimes i'_a} & W^a \boxtimes ((W^a)' \boxtimes W^a) & \longrightarrow \\
& \xrightarrow{\mathcal{A}} & (W^a \boxtimes (W^a)') \boxtimes W^a & \xrightarrow{e'_a \boxtimes I_{W^a}} & V \boxtimes W^a & \xrightarrow{l_a} W^a,
\end{array}
$$

$$\begin{aligned}
(W^a)' &\xrightarrow{l_{a'}^{-1}} V \boxtimes (W^a)' \longrightarrow \\
&\xrightarrow{i'_a \boxtimes I_{W^a}} ((W^a)' \boxtimes W^a) \boxtimes (W^a)' \longrightarrow \\
&\xrightarrow{\mathcal{A}^{-1}} (W^a)' \boxtimes (W^a \boxtimes (W^a)') \xrightarrow{I_{(W^a)'} \boxtimes e'_a} (W^a)' \boxtimes V \xrightarrow{r_{a'}} (W^a)',
\end{aligned}$$

$$\begin{aligned}
(W^a)' &\xrightarrow{r_{a'}^{-1}} (W^a)' \boxtimes V \longrightarrow \\
&\xrightarrow{I_{W^a} \boxtimes i_a} (W^a)' \boxtimes (W^a \boxtimes (W^a)') \longrightarrow \\
&\xrightarrow{\mathcal{A}} ((W^a)' \boxtimes W^a) \boxtimes (W^a)' \xrightarrow{e_a \boxtimes I_{(W^a)'}} V \boxtimes (W^a)' \xrightarrow{l_{a'}} (W^a)'
\end{aligned}$$

are the identity maps. So the tensor category is rigid. ∎

The calculations above also allow us to calculate the tensor-categorical dimensions of V-modules:

Theorem 3.9 *For $a \in \mathcal{A}$, the tensor-categorical dimension of W^a is $\frac{1}{F_a}$.*

Proof. For any $a \in \mathcal{A}$, choose $w_a \in w^a$ and $w'_a \in (W^a)'$ to be lowest weight vectors such that $\langle w'_a, w_a \rangle = 1$. Then by the definitions of i_a and $\hat{e}'_a$,

$$\begin{aligned}
i_a(\mathbf{1}) &= P_0(w_a \boxtimes w'_a), \\
\overline{\hat{e}'_a}(w_a \boxtimes w'_a) &= \mathcal{Y}^e_{aa';1}(w_a, 1)w'_a.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\hat{e}'_a(i_a(\mathbf{1})) &= \hat{e}'_a(P_0(w_a \boxtimes w'_a)) \\
&= P_0(\overline{\hat{e}'_a}(w_a \boxtimes w'_a)) \\
&= P_0(\mathcal{Y}^e_{aa';1}(w_a, 1)w'_a) \\
&= \mathrm{Res}_x x^{2h_a - 1} \mathcal{Y}^e_{aa';1}(w_a, x)w'_a.
\end{aligned}$$

Since V is irreducible and $V_{(0)} = \mathbb{C}\mathbf{1}$, there exists $\nu \in \mathbb{C}$ such that

$$\mathrm{Res}_x x^{2h_a - 1} \mathcal{Y}^e_{aa';1}(w_a, x)w'_a = \nu \mathbf{1}.$$

We now calculate ν. By the definition of the intertwining operator $\mathcal{Y}^e_{aa';1}$ and the assumption that w_a and w'_a are lowest weight vectors and $\langle w'_a, w_a \rangle = 1$, we have

$$\mathrm{Res}_x x^{2h_a - 1} (\mathcal{Y}^e_{aa';1}(w_a, x)w'_a, \mathbf{1})$$

$$
\begin{aligned}
&= e^{\pi i h_a}\mathrm{Res}_x x^{2h_a-1}\langle w_a', \mathcal{Y}_{ae;1}^a(e^{xL(1)}e^{-\pi i L(0)}x^{-2L(0)}w_a, x^{-1})\mathbf{1}\rangle\\
&= \mathrm{Res}_x x^{-1}\langle w_a', \mathcal{Y}_{ae;1}^a(w_a, x^{-1})\mathbf{1}\rangle\\
&= \mathrm{Res}_x x^{-1}\langle w_a', e^{x^{-1}L(-1)}w_a\rangle\\
&= \mathrm{Res}_x x^{-1}\langle e^{x^{-1}L(1)}w_a', w_a\rangle\\
&= \langle w_a', w_a\rangle\\
&= 1.
\end{aligned}
$$

Since $(\mathbf{1}, \mathbf{1}) = 1$, we obtain $\nu = 1$. Since V is irreducible, the calculation above shows that

$$\hat{e}_a' \circ i_a = I_V.$$

Thus

$$e_a' \circ i_a = \frac{1}{F_a} I_V.$$

By the definition of the tensor-categorical dimension (see [T]), we see that the dimension of W^a is $\frac{1}{F_a}$. ∎

Remark 3.10 In [HL3], the notion of vertex tensor category was introduced and it was proved in [HL3] and [H5] that for a vertex operator algebra V satisfying the conditions (weaker than) in this paper, the category of V-modules has a natural structure of vertex tensor category. In the proof of the rigidity above, we have calculated the compositions of module maps in the sequences (3.3), (3.15) and the corresponding sequences (which we have omitted) for the proof of (3.7). If we replace $\hat{e}_{a;z_1,z_2}$ and $\hat{e}'_{a;z_1,z_2}$ by

$$
\begin{aligned}
e_{a;z_1,z_2} &= \frac{1}{F_a}\hat{e}_{a;z_1,z_2},\\
e'_{a;z_1,z_2} &= \frac{1}{F_a}\hat{e}'_{a;z_1,z_2},
\end{aligned}
$$

these compositions are equal to identity maps. Actually for any vertex tensor category whose objects have left and right duals and also have the associated morphisms, these sequences still make sense. So we can introduce a notion of rigidity for vertex tensor categories by requiring the compositions of the morphisms in these sequences to be equal to the identity on the object for every object in the category. Then our result above shows that the vertex tensor category of V-modules is rigid. We know that a vertex tensor category gives a braided tensor category [HL3]. Then the proofs of Propositions 3.4 and 3.6 and the corresponding parts in the proof (which we have omitted) of Proposition 3.7 shows that if a vertex tensor category is rigid, then the corresponding braided tensor category is also rigid.

4 Balancing axioms, the nondegeneracy property and the modular tensor category

In this section we prove that the category of V-modules for a vertex operator algebra V satisfying the conditions in Section 1 has a natural structure of modular tensor category. The main work is a proof of the nondegeneracy property.

We first recall the notion of modular tensor category (see [T] and [BK] for details). A *ribbon category* is a rigid braided tensor category (with tensor product bifunctor $\boxtimes$, the braiding isomorphism $\mathcal{C}$, the unit object V and the right dual functor $*$) together with an isomorphism $\theta_W \in \mathrm{Hom}(W, W)$ for each object W satisfying the following *balancing axioms*: (i) $\theta_{W_1\boxtimes W_2} = \mathcal{C}^2 \circ (\theta_{W_1} \boxtimes \theta_{W_2})$. (ii) $\theta_V = I_V$. (iii) $\theta_{W^*} = (\theta_W)^*$. A semisimple ribbon category with finitely many inequivalent irreducible objects is a *modular tensor category* if it has the following *nondegeneracy property*: Let $\{W_1, \dots, W_m\}$ be a complete set of representatives of equivalence classes of irreducible objects. Then the $m \times m$ matrix formed by the traces of the morphism $\mathcal{C}^2 \in \mathrm{Hom}(W_i \boxtimes W_j, W_i \boxtimes W_j)$ in the ribbon category is invertible. See, for example, [T] and [BK] for details of the theory of modular tensor categories.

Now we consider a vertex operator algebra V satisfying the conditions in Section 1. For a V-module W, let

$$\theta_W = e^{2\pi i L(0)}.$$

Theorem 4.1 *The rigid braided tensor category of V-modules together with the balancing isomorphism or the twist θ_W for each V-module W is a ribbon category.*

Proof. Let W_1 and W_2 be V-modules and let $\mathcal{Y}$ be the intertwining operator of type $\binom{W_1\boxtimes W_2}{W_1 W_2}$ such that $w_1 \boxtimes w_2 = \mathcal{Y}(w_1, 1)w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$. From the definition of the braiding isomorphism $\mathcal{C}$, we have

$$\begin{aligned} \overline{\mathcal{C}^2}(w_1 \boxtimes w_2) &= \overline{\mathcal{C}^2}(\mathcal{Y}(w_1, 1)w_2) \\ &= \mathcal{Y}(w_1, x)w_2|_{x^n=e^{2\pi ni},\ n\in\mathbb{R}} \end{aligned}$$

for $w_1 \in W_1$ and $w_2 \in W_2$. Then for $w_1 \in W_1$ and $w_2 \in W_2$,

$$\begin{aligned}
\overline{\theta_{W_1 \boxtimes W_2}}(w_1 \boxtimes w_2) &= e^{2\pi i L(0)}(w_1 \boxtimes w_2) \\
&= e^{2\pi i L(0)}\mathcal{Y}((w_1, 1)w_2) \\
&= \mathcal{Y}(e^{2\pi i L(0)}w_1, x)e^{2\pi i L(0)}w_2|_{x^n = e^{2\pi n i},\ n\in\mathbb{R}} \\
&= \overline{\mathcal{C}^2}((e^{2\pi i L(0)}w_1) \boxtimes (e^{2\pi i L(0)}w_2)) \\
&= \overline{\mathcal{C}^2} \circ \overline{\theta_{W_1} \boxtimes \theta_{W_2}}(w_1 \boxtimes w_2) \\
&= \overline{\mathcal{C}^2 \circ \theta_{W_1} \boxtimes \theta_{W_2}}(w_1 \boxtimes w_2).
\end{aligned}$$

Since the homogeneous components of $w_1 \boxtimes w_2$ for all $w_1 \in W_1$ and $w_2 \in W_2$ span the V-module $W_1 \boxtimes W_2$, we obtain

$$\theta_{W_1 \boxtimes W_2} = \mathcal{C}^2 \circ (\theta_{W_1} \boxtimes \theta_{W_2}).$$

Since V is $\mathbb{Z}$-graded, we have $\theta_V = I_V$. For any V-module W,

$$\begin{aligned}
(\theta_W)^* &= (e^{2\pi i L(0)})^* \\
&= e^{2\pi i L'(0)} \\
&= \theta_{W'}.
\end{aligned}$$

Thus all the balancing axioms are satisfied. So the category of V-module is a ribbon category. ∎

To prove the nondegeneracy property, we calculate the compositions of the maps in the sequence

$$
\begin{tikzcd}
V \arrow[d, "i'_{a_2}"'] \\
(W^{a_2})' \boxtimes W^{a_2} \arrow[d, "I_{(W^{a_2})'} \boxtimes r_{a_2}^{-1}"'] \\
(W^{a_2})' \boxtimes (W^{a_2} \boxtimes V) \arrow[d, "I_{(W^{a_2})'} \boxtimes (I_{W^{a_2}} \boxtimes i'_{a_1;})"'] \\
(W^{a_2})' \boxtimes (W^{a_2} \boxtimes ((W^{a_1})' \boxtimes W^{a_1})) \arrow[d, "I_{(W^{a_2})'} \boxtimes \mathcal{A}"'] \\
(W^{a_2})' \boxtimes ((W^{a_2} \boxtimes (W^{a_1})') \boxtimes W^{a_1}) \arrow[d, "I_{(W^{a_2})'} \boxtimes (\mathcal{C}^2 \boxtimes I_{W^{a_2}})"'] \\
(W^{a_2})' \boxtimes ((W^{a_2} \boxtimes (W^{a_1})') \boxtimes W^{a_1}) \arrow[d, "I_{(W^{a_2})'} \boxtimes \mathcal{A}"'] \\
(W^{a_2})' \boxtimes (W^{a_2} \boxtimes ((W^{a_1})' \boxtimes W^{a_1})) \arrow[d, "I_{(W^{a_2})'} \boxtimes (I_{W^{a_2}} \boxtimes \hat{e}_{a_1})"'] \\
(W^{a_2})' \boxtimes (W^{a_2} \boxtimes V) \arrow[d, "I_{(W^{a_2})'} \boxtimes r_{a_2}"'] \\
(W^{a_2})' \boxtimes W^{a_2} \arrow[d, "\hat{e}_{a_2}"'] \\
V
\end{tikzcd}
\tag{4.1}
$$

This composition will give us the trace of $\mathcal{C}^2 \in \mathrm{Hom}((W^{a_1})', W^{a_2})$. Similarly to the calculations of the maps in the proof of the rigidity in the preceding section, we first calculate the composition of certain other module maps.

For $z_1, z_2, z_3 \in \mathbb{C}^\times$ satisfying $|z_1| > |z_2| > |z_3| > |z_2 - z_3| > 0$, consider the module map given by the sequence

$$\begin{array}{c}
V \\
\Big\downarrow {\scriptstyle i'_{a_2;z_1}} \\
(W^{a_2})' \boxtimes_{P(z_1)} W^{a_2} \\
\Big\downarrow {\scriptstyle I_{(W^{a_2})'} \boxtimes_{P(z_1)} r^{-1}_{a_2,z_2}} \\
(W^{a_2})' \boxtimes_{P(z_1)} (W^{a_2} \boxtimes_{P(z_2)} V) \\
\Big\downarrow {\scriptstyle I_{(W^{a_2})'} \boxtimes_{P(z_1)} (I_{W^{a_2}} \boxtimes_{P(z_2)} i'_{a_1;z_3})} \\
(W^{a_2})' \boxtimes_{P(z_1)} (W^{a_2} \boxtimes_{P(z_2)} ((W^{a_1})' \boxtimes_{P(z_3)} W^{a_1})) \\
\Big\downarrow {\scriptstyle I_{(W^{a_2})'} \boxtimes_{P(z_1)} \mathcal{A}^{P(z_2-z_3),P(z_3)}_{P(z_2),P(z_2)}} \\
(W^{a_2})' \boxtimes_{P(z_1)} ((W^{a_2} \boxtimes_{P(z_2-z_3)} (W^{a_1})') \boxtimes_{P(z_3)} W^{a_1}) \\
\Big\downarrow {\scriptstyle I_{(W^{a_2})'} \boxtimes_{P(z_1)} ((\mathcal{C}_{P(z_2-z_3)})^2 \boxtimes_{P(z_3)} I_{W^{a_2}})} \\
(W^{a_2})' \boxtimes_{P(z_1)} ((W^{a_2} \boxtimes_{P(z_2-z_3)} (W^{a_1})') \boxtimes_{P(z_3)} W^{a_1}) \\
\Big\downarrow {\scriptstyle I_{(W^{a_2})'} \boxtimes_{P(z_1)} (\mathcal{A}^{P(z_2-z_3),P(z_3)}_{P(z_2),P(z_2)})^{-1}} \\
(W^{a_2})' \boxtimes_{P(z_1)} (W^{a_2} \boxtimes_{P(z_2)} ((W^{a_1})' \boxtimes_{P(z_3)} W^{a_1})) \\
\Big\downarrow {\scriptstyle I_{(W^{a_2})'} \boxtimes_{P(z_1)} (I_{W^{a_2}} \boxtimes_{P(z_2)} \hat{e}_{a_1;z_3})} \\
(W^{a_2})' \boxtimes_{P(z_1)} (W^{a_2} \boxtimes_{P(z_2)} V) \\
\Big\downarrow {\scriptstyle I_{(W^{a_2})'} \boxtimes_{P(z_1)} r_{a_2;z_2}} \\
(W^{a_2})' \boxtimes_{P(z_1)} W^{a_2} \\
\Big\downarrow {\scriptstyle \hat{e}_{a_2;z_1}} \\
V
\end{array} \tag{4.2}$$

Proposition 4.2 *The composition of the module maps in the sequence (4.2) is equal to* $(B^{(-1)})^2_{a_2,a_1} I_V$.

Proof. Since V is an irreducible V-module, we know that the composition must be proportional to I_V. So we need only show that the proportional constant is $(B^{(-1)})^2_{a_2,a_1}$. To show this, we need only calculate the composition applied to any nonzero element of V or the natural extension of the composition applied to any nonzero element of $\overline{V}$.

By definition, we have

$$w' \boxtimes_{P(z)} w = \overline{(f'_{a;z})^{-1}} \left(\sum_{b\in\mathcal{A}} \sum_{k=1}^{N^b_{a'a}} \mathcal{Y}^b_{a'a;k}(w', z)w \right)$$

for $a \in \mathcal{A}$, $z \in \mathbb{C}$, $w \in W^a$ and $w' \in (W^a)'$. Then for any module maps $\alpha : (W^a)' \to (W^a)'$ and $\beta : W^a \to W^a$,

$$\begin{aligned}
&(\overline{\alpha \boxtimes_{P(z)} \beta}) \left(\overline{(f'_{a;z})^{-1}} \left(\sum_{b\in\mathcal{A}} \sum_{k=1}^{N^b_{a'a}} \mathcal{Y}^b_{a'a;k}(w', z)w \right) \right)\\
&\quad = (\overline{\alpha \boxtimes_{P(z)} \beta})(w' \boxtimes_{P(z)} w)\\
&\quad = \alpha(w') \boxtimes_{P(z)} \beta(w)\\
&\quad = \overline{(f'_{a;z})^{-1}} \left(\sum_{b\in\mathcal{A}} \sum_{k=1}^{N^b_{a'a}} \mathcal{Y}^b_{a'a;k}(\alpha(w'), z)\beta(w) \right). \qquad (4.3)
\end{aligned}$$

Since $f'_{a;z}$ is an equivalence of V-modules and the homogeneous components of elements of the form $w' \boxtimes_{P(z)} w$ for $w \in W^a$ and $w' \in (W^a)'$ span $(W^a)' \boxtimes_{P(z)} W^a$, the homogeneous components of elements of the form

$$\sum_{b\in\mathcal{A}} \sum_{k=1}^{N^b_{a'a}} \mathcal{Y}^b_{a'a;k}(w', z)w \qquad (4.4)$$

for $w \in W^a$ and $w' \in (W^a)'$ span $\coprod_{b\in\mathcal{A}} N^b_{a'a} W^b$. In particular, by the definition of $g'_{a;z}$, for $w_a \in W^a$ and $w'_a \in (W^a)'$, $\overline{g'_{a;z}}(\mathcal{Y}^e_{a'a;1}(w'_a, z)w_a)$ can be obtained as a sum of homogeneous components of elements of the form (4.4) and $\overline{g'_{a;z}}(\mathcal{Y}^e_{a'a;1}(\alpha(w'_a), z)\beta(w_a))$ is the sum of the same homogeneous components of the same elements of the form (4.4) with w' and w replaced by $\alpha(w')$ and $\beta(w)$, respectively. Thus from (4.3) we obtain

$$\begin{aligned}
&(\overline{\alpha \boxtimes_{P(z)} \beta})(\overline{(f'_{a;z})^{-1}}(\overline{g'_{a;z}}(\mathcal{Y}^e_{a'a;1}(w'_a, z)w_a)))\\
&\quad = \overline{(f'_{a;z})^{-1}}(\overline{g'_{a;z}}(\mathcal{Y}^e_{a'a;1}(\alpha(w'_a), z)\beta(w_a))). \qquad (4.5)
\end{aligned}$$

From the definition of $i'_{a;z}$ and (4.5), we obtain

$$\overline{(\alpha \boxtimes_{P(z)} \beta) \circ i'_{a;z}}(\mathcal{Y}^{e}_{a'a;1}(w'_a, z)w_a) = \overline{i'_{a;z}}(\mathcal{Y}^{e}_{a'a;1}(\alpha(w'_a), z)\beta(w_a)) \tag{4.6}$$

for module maps $\alpha : (W^a)' \to (W^a)'$ and $\beta : W^a \to W^a$, $a \in \mathcal{A}$, $z \in \mathbb{C}$, $w_a \in W^a$ and $w'_a \in (W^a)'$. Using (4.6), we obtain

$$\begin{aligned}
&(\overline{I_{(W^{a_2})'} \boxtimes_{P(z_1)} (I_{W^{a_2}} \boxtimes_{P(z_2)} i'_{a_1;z_3})})\Big((\overline{I_{(W^{a_2})'} \boxtimes_{P(z_1)} r^{-1}_{a_2,z_2}})\\
&\quad\Big(\overline{i'_{a_2;z_1}}\Big(\mathcal{Y}^{e}_{a'_2a_2;1}(w'_{a_2}, z_1)\mathcal{Y}^{a_2}_{a_2e;1}(w_{a_2}, z_2)\mathcal{Y}^{e}_{a'_1a_1;1}(w_{a'_1}, z_3)w_{a_1}\Big)\Big)\Big)\\
&\quad= \overline{i'_{a_2;z_1}}\Big(\mathcal{Y}^{e}_{a'_2a_2;1}(w'_{a_2}, z_1)\\
&\qquad\qquad\Big(w_{a_2} \boxtimes_{P(z_2)} \Big(\overline{i'_{a_1;z_3}}\Big(\mathcal{Y}^{e}_{a'_1a_1;1}(w_{a'_1}, z_3)w_{a_1}\Big)\Big)\Big)\Big)
\end{aligned} \tag{4.7}$$

for $w_{a_1} \in W^{a_1}$, $w_{a_2} \in W^{a_2}$, $w'_{a_1} \in (W^{a_1})'$ and $w'_{a_2} \in (W^{a_2})'$.

We also have the following commutative diagram

$$\begin{array}{ccc}
W^{a_2} \boxtimes_{P(z_2)} ((W^{a_1})' \boxtimes_{P(z_3)} W^{a_1}) & \xrightarrow{r_{a_2;z_2}\circ(I_{W^{a_2}}\boxtimes_{P(z_2)}\hat{e}_{a'_1;z_3})} & W^{a_2}\\
\mathcal{M}\Big\downarrow & & \Big\downarrow m_{a_2,a_1}\\
W^{a_2} \boxtimes_{P(z_2)} ((W^{a_1})' \boxtimes_{P(z_3)} W^{a_1}) & \xrightarrow{r_{a_2;z_2}\circ(I_{W^{a_2}}\boxtimes_{P(z_2)}\hat{e}_{a'_1;P(z_3)})} & W^{a_2}
\end{array} \tag{4.8}$$

where

$$\mathcal{M} = \left(\mathcal{A}^{P(z_2-z_3),P(z_3)}_{P(z_2),P(z_2)}\right)^{-1} \circ ((\mathcal{C}_{P(z_2-z_3)})^2 \boxtimes_{P(z_3)} I_{W^{a_2}}) \circ \mathcal{A}^{P(z_2-z_3),P(z_3)}_{P(z_2),P(z_2)}$$

and the module map m_{a_2,a_1} is defined as follows: Consider the element

$$\mathcal{Y}^{a_2}_{a_2e;1}(w_{a_2}, z_2)\mathcal{Y}^{e}_{a'_1a_1;1}(w'_{a_1}, z_3)w_{a_1} \tag{4.9}$$

of $\overline{W^{a_2}}$ and the path γ in $M^2 = \{(z_2, z_3) \in \mathbb{C}^2 \mid z_2, z_3 \neq 0, z_2 \neq z_3\}$ given by

$$\gamma(t) = (z_3 + e^{-2\pi it}(z_2 - z_3), z_3).$$

Starting with the element (4.9) above, the analytic extension along the path γ gives a value at the endpoint of γ. This is again an element of $\overline{W^{a_2}}$ which we denote

$$\overline{m_{a_2,a_1}}(\mathcal{Y}^{a_2}_{a_2e;1}(w_{a_2}, z_2)\mathcal{Y}^{e}_{a'_1a_1;1}(w'_{a_1}, z_3)w_{a_1}).$$

The correspondence

$$\begin{aligned}&\mathcal{Y}^{a_2}_{a_2e;1}(w_{a_2}, z_2)\mathcal{Y}^{e}_{a_1'a_1;1}(w'_{a_1}, z_3)w_{a_1}\\ &\qquad \mapsto \overline{m_{a_2,a_1}}(\mathcal{Y}^{a_2}_{a_2e;1}(w_{a_2}, z_2)\mathcal{Y}^{e}_{a_1'a_1;1}(w'_{a_1}, z_3)w_{a_1})\end{aligned}$$

determines a unique module map m_{a_2,a_1} from W^{a_2} to itself.

Note that by definition, the module map $\mathcal{M}$ can also be directly obtained in the same way as that for m_{a_2,a_1}: Starting with the element

$$w_{a_2} \boxtimes_{P(z_2)} (w'_{a_1} \boxtimes_{P(z_3)} w_{a_1}),$$

the analytic extension along the path γ gives a value at the endpoint of γ. This value is

$$\overline{\mathcal{M}}(w_{a_2} \boxtimes_{P(z_2)} (w'_{a_1} \boxtimes_{P(z_3)} w_{a_1}))$$

and the correspondence

$$w_{a_2} \boxtimes_{P(z_2)} (w'_{a_1} \boxtimes_{P(z_3)} w_{a_1}) \mapsto \overline{\mathcal{M}}(w_{a_2} \boxtimes_{P(z_2)} (w'_{a_1} \boxtimes_{P(z_3)} w_{a_1}))$$

determines the map $\mathcal{M}$. From this construction of $\mathcal{M}$, we see that the diagram (4.8) is commutative since analytic extensions certainly commute with the module map

$$r_{a_2;z_2} \circ (I_{W^{a_2}} \boxtimes_{P(z_2)} e_{a_1';z_3}).$$

Now this commutativity of (4.8) gives

$$\begin{aligned}r_{a_2;z_2} \circ (I_{W^{a_2}} \boxtimes_{P(z_2)} e_{a_1';z_3}) &\circ \left(\mathcal{A}^{P(z_2-z_3),P(z_3)}_{P(z_2,P(z_2)}\right)^{-1}\\ &\circ((\mathcal{C}_{P(z_2-z_3)})^2 \boxtimes_{P(z_3)} I_{W^{a_2}}) \circ \mathcal{A}^{P(z_2-z_3),P(z_3)}_{P(z_2,P(z_2)}\\ &= m_{a_2,a_1} \circ (r_{a_2;z_2} \circ (I_{W^{a_2}} \boxtimes_{P(z_2)} e_{a_1';z_3})).\end{aligned} \tag{4.10}$$

On the other hand,

$$\begin{aligned}\overline{m_{a_2,a_1} \circ (r_{a_2;z_2} \circ (I_{W^{a_2}} \boxtimes_{P(z_2)} e_{a_1';z_3}))}&(w_{a_2} \boxtimes_{P(z_2)} (w'_{a_1} \boxtimes_{P(z_3)} w_{a_1}))\\ &= \overline{m_{a_2,a_1}}(\mathcal{Y}^{a_2}_{a_2e;1}(w_{a_2}, z_2)\mathcal{Y}^{e}_{a_1'a_1;1}(w'_{a_1}, z_3)w_{a_1}).\end{aligned} \tag{4.11}$$

But by definition, for any $w'_{a_2} \in (W^{a_2})'$, we have

$$\begin{aligned}&\langle w'_{a_2}, \overline{m_{a_2,a_1}}(\mathcal{Y}^{a_2}_{a_2e;1}(w_{a_2}, z_2)\mathcal{Y}^{e}_{a_1'a_1;1}(w'_{a_1}, z_3)w_{a_1})\rangle\\ &\qquad = (B^{(-1)})^2(\langle w'_{a_2}, \mathcal{Y}^{a_2}_{a_2e;1}(w_{a_2}, z_2)\mathcal{Y}^{e}_{a_1'a_1;1}(w'_{a_1}, z_3)w_{a_1}\rangle).\end{aligned}$$

Thus by the commutativity of intertwining operators, we have

$$\begin{aligned}
&\overline{m_{a_2,a_1}}(\mathcal{Y}^{a_2}_{a_2e;1}(w_{a_2}, z_2)\mathcal{Y}^{e}_{a_1'a_1;1}(w'_{a_1}, z_3)w_{a_1})\\
&\quad= (B^{(-1)})^2(\mathcal{Y}^{a_2}_{a_2e;1} \otimes \mathcal{Y}^{e}_{a_1'a_1;1}; \mathcal{Y}^{a_2}_{a_2e;1} \otimes \mathcal{Y}^{e}_{a_1'a_1;1}) \cdot\\
&\qquad\qquad \cdot\mathcal{Y}^{a_2}_{a_2e;1}(w_{a_2}, z_2)\mathcal{Y}^{e}_{a_1'a_1;1}(w'_{a_1}, z_3)w_{a_1}.
\end{aligned}\tag{4.12}$$

Combining (4.10)–(4.12), we obtain

$$\begin{aligned}
&\overline{(r_{a_2;z_2}}\Big(\overline{(I_{W^{a_2}} \boxtimes_{P(z_2)} \hat{e}_{a_1';z_3}))}\\
&\qquad\Big(\overline{\Big(\mathcal{A}^{P(z_2-z_3),P(z_3)}_{P(z_2),P(z_2)}\Big)^{-1}}\Big(\overline{((\mathcal{C}_{P(z_2-z_3)})^2 \boxtimes_{P(z_3)} I_{W^{a_2}})}\\
&\qquad\qquad\Big(\overline{\mathcal{A}^{P(z_2-z_3),P(z_3)}_{P(z_2),P(z_2)}}(w_{a_2} \boxtimes_{P(z_2)} (w'_{a_1} \boxtimes_{P(z_3)} w_{a_1}))\Big)\Big)\Big)\Big)\\
&\quad= (B^{(-1)})^2(\mathcal{Y}^{a_2}_{a_2e;1} \otimes \mathcal{Y}^{e}_{a_1'a_1;1}; \mathcal{Y}^{a_2}_{a_2e;1} \otimes \mathcal{Y}^{e}_{a_1'a_1;1}) \cdot\\
&\qquad\qquad \cdot\mathcal{Y}^{a_2}_{a_2e;1}(w_{a_2}, z_2)\mathcal{Y}^{e}_{a_1'a_1;1}(w'_{a_1}, z_3)w_{a_1}.
\end{aligned}\tag{4.13}$$

Since $\overline{i'_{a_1;z_3}}(\mathcal{Y}^{e}_{a_1'a_1;1}(w_{a_1'}, z_3)w_{a_1})$ is in $\overline{(W^a)' \boxtimes_{P(z_3)} W^a}$, it can be written as a sum of of homogeneous components of elements of the form $w' \boxtimes_{P(z_3)} w$ for $w' \in (W^a)'$ and $w \in W^a$. Since $f'_{a;z_3}$ is an equivalence of V-modules, $g'_{a;z_3}(\mathcal{Y}^{e}_{a_1'a_1;1}(w_{a_1'}, z_3)w_{a_1})$ is the same sum of the same homogeneous components of the elements of the form

$$\sum_{b\in\mathcal{A}}\sum_{k=1}^{N^b_{a_1'a_1}} \mathcal{Y}^{b}_{a_1'a_1;k}(w_{a_1'}, z_3)w_{a_1}.$$

Then we see that the same sum of the same homogeneous components of the elements of the form $Y^{e}_{a_1'a_1;1}(w_{a_1'}, z_3)w_{a_1}$ must be equal to itself. Using these facts and taking the coefficients of z_3 to all powers in both sides of (4.13) and then taking the sum above, we obtain

$$\begin{aligned}
&\overline{r_{a_2;z_2}}\Big(\overline{(I_{W^{a_2}} \boxtimes_{P(z_2)} \hat{e}_{a_1';z_3})}\\
&\qquad\Big(\overline{\Big(\mathcal{A}^{P(z_2-z_3),P(z_3)}_{P(z_2),P(z_2)}\Big)^{-1}}\Big(\overline{((\mathcal{C}_{P(z_2-z_3)})^2 \boxtimes_{P(z_3)} I_{W^{a_2}})}\\
&\qquad\qquad\Big(\overline{\mathcal{A}^{P(z_2-z_3),P(z_3)}_{P(z_2),P(z_2)}}(w_{a_2} \boxtimes_{P(z_2)} \overline{i'_{a_1;z_3}}(\mathcal{Y}^{e}_{a_1'a_1;1}(w_{a_1'}, z_3)w_{a_1}))\Big)\Big)\Big)\Big)\\
&\quad= (B^{(-1)})^2(\mathcal{Y}^{a_2}_{a_2e;1} \otimes \mathcal{Y}^{e}_{a_1'a_1;1}; \mathcal{Y}^{a_2}_{a_2e;1} \otimes \mathcal{Y}^{e}_{a_1'a_1;1}) \cdot\\
&\qquad\qquad \cdot\mathcal{Y}^{a_2}_{a_2e;1}(w_{a_2}, z_2)\mathcal{Y}^{e}_{a_1'a_1;1}(w'_{a_1}, z_3)w_{a_1}.
\end{aligned}\tag{4.14}$$

From (4.14), we obtain

$$\Big(\overline{\hat{e}_{a_2';z_1}} \circ (\overline{I_{(W^{a_2})'} \boxtimes_{P(z_1)} r_{a_2;z_2}}) \circ (\overline{I_{(W^{a_2})'} \boxtimes_{P(z_1)} (I_{W^{a_2}} \boxtimes_{P(z_2)} \hat{e}_{a_1';z_3})}$$
$$\circ \left(\overline{I_{(W^{a_2})'} \boxtimes_{P(z_1)} \left(\mathcal{A}^{P(z_2-z_3),P(z_3)}_{P(z_2),P(z_2)}\right)^{-1}}\right)$$
$$\circ (\overline{I_{(W^{a_2})'} \boxtimes_{P(z_1)} ((\mathcal{C}_{P(z_2-z_3)})^2 \boxtimes_{P(z_3)} I_{W^{a_2}})}) \circ \left(\overline{I_{(W^{a_2})'} \boxtimes_{P(z_1)} \mathcal{A}^{P(z_2-z_3),P(z_3)}_{P(z_2),P(z_2)}}\right)$$
$$\left(w'_{a_2} \boxtimes_{P(z_1)} \left(w_{a_2} \boxtimes_{P(z_2)} \overline{i'_{a_1;z_3}}\left(\mathcal{Y}^{e}_{a_1'a_1;1}(w'_{a_1}, z_3)w_{a_1}\right)\right)\right)$$
$$= (B^{(-1)})^2(\mathcal{Y}^{a_2}_{a_2e;1} \otimes \mathcal{Y}^{e}_{a_1'a_1;1}; \mathcal{Y}^{a_2}_{a_2e;1} \otimes \mathcal{Y}^{e}_{a_1'a_1;1}) \cdot$$
$$\cdot\Big(\mathcal{Y}^{e}_{a_2'a_2;1}(w'_{a_2}, z_1)\mathcal{Y}^{a_2}_{a_2e;1}(w_{a_2}, z_2)\mathcal{Y}^{e}_{a_1'a_1;1}(w'_{a_1}, z_3)w_{a_1}\Big) \tag{4.15}$$

Noticing that (4.15) holds for all $w_{a_1} \in W^{a_1}$, $w_{a_2} \in W^{a_2}$, $w'_{a_1} \in (W^{a_1})'$ and $w'_{a_2} \in (W^{a_2})'$, and using the same arguments as we have used to obtain (4.14) from (4.13), from (4.15), we obtain

$$\Big(\overline{\hat{e}_{a_2';z_1}} \circ (\overline{I_{(W^{a_2})'} \boxtimes_{P(z_1)} r_{a_2;z_2}}) \circ (\overline{I_{(W^{a_2})'} \boxtimes_{P(z_1)} (I_{W^{a_2}} \boxtimes_{P(z_2)} \hat{e}_{a_1';z_3})}$$
$$\circ \left(\overline{I_{(W^{a_2})'} \boxtimes_{P(z_1)} \left(\mathcal{A}^{P(z_2-z_3),P(z_3)}_{P(z_2),P(z_2)}\right)^{-1}}\right)$$
$$\circ (\overline{I_{(W^{a_2})'} \boxtimes_{P(z_1)} ((\mathcal{C}_{P(z_2-z_3)})^2 \boxtimes_{P(z_3)} I_{W^{a_2}})}) \circ \left(\overline{I_{(W^{a_2})'} \boxtimes_{P(z_1)} \mathcal{A}^{P(z_2-z_3),P(z_3)}_{P(z_2),P(z_2)}}\right)$$
$$\left(\overline{i_{a_2';z_1}}\left(\mathcal{Y}^{e}_{a_2'a_2;1}(w'_{a_2}, z_1)\left(w_{a_2} \boxtimes_{P(z_2)} \overline{i'_{a_1;z_3}}\left(\mathcal{Y}^{e}_{a_1'a_1;1}(w'_{a_1}, z_3)w_{a_1}\right)\right)\right)\right)$$
$$= (B^{(-1)})^2(\mathcal{Y}^{a_2}_{a_2e;1} \otimes \mathcal{Y}^{e}_{a_1'a_1;1}; \mathcal{Y}^{a_2}_{a_2e;1} \otimes \mathcal{Y}^{e}_{a_1'a_1;1}) \cdot$$
$$\cdot\Big(\mathcal{Y}^{e}_{a_2'a_2;1}(w'_{a_2}, z_1)\mathcal{Y}^{a_2}_{a_2e;1}(w_{a_2}, z_2)\mathcal{Y}^{e}_{a_1'a_1;1}(w'_{a_1}, z_3)w_{a_1}\Big). \tag{4.16}$$

Combining (4.7) and (4.16), we obtain

$$\Big(\overline{\hat{e}_{a_2';z_1}} \circ (\overline{I_{(W^{a_2})'} \boxtimes_{P(z_1)} r_{a_2;z_2}}) \circ (\overline{I_{(W^{a_2})'} \boxtimes_{P(z_1)} (I_{W^{a_2}} \boxtimes_{P(z_2)} \hat{e}_{a_1';z_3})}$$
$$\circ \left(\overline{I_{(W^{a_2})'} \boxtimes_{P(z_1)} \left(\mathcal{A}^{P(z_2-z_3),P(z_3)}_{P(z_2),P(z_2)}\right)^{-1}}\right)$$
$$\circ (\overline{I_{(W^{a_2})'} \boxtimes_{P(z_1)} ((\mathcal{C}_{P(z_2-z_3)})^2 \boxtimes_{P(z_3)} I_{W^{a_2}})}) \circ \left(\overline{I_{(W^{a_2})'} \boxtimes_{P(z_1)} \mathcal{A}^{P(z_2-z_3),P(z_3)}_{P(z_2),P(z_2)}}\right)$$
$$\circ (\overline{I_{(W^{a_2})'} \boxtimes_{P(z_1)} (I_{W^{a_2}} \boxtimes_{P(z_2)} i_{a_1';z_3})}) \circ (\overline{I_{(W^{a_2})'} \boxtimes_{P(z_1)} r^{-1}_{a_2,z_2}}) \circ \overline{i_{a_2';z_1}}\Big)$$

$$\Big(\mathcal{Y}^{e}_{a_2'a_2;1}(w'_{a_2},z_1)\mathcal{Y}^{a_2}_{a_2e;1}(w_{a_2},z_2)\Big(\mathcal{Y}^{e}_{a_1'a_1;1}(w'_{a_1},z_3)w_{a_1}\Big)\Big)$$
$$= (B^{(-1)})^2(\mathcal{Y}^{a_2}_{a_2e;1}\otimes\mathcal{Y}^{e}_{a_1'a_1;1};\mathcal{Y}^{a_2}_{a_2e;1}\otimes\mathcal{Y}^{e}_{a_1'a_1;1})\cdot$$
$$\cdot\Big(\mathcal{Y}^{e}_{a_2'a_2;1}(w'_{a_2},z_1)\mathcal{Y}^{a_2}_{a_2e;1}(w_{a_2},z_2)\mathcal{Y}^{e}_{a_1'a_1;1}(w'_{a_1},z_3)w_{a_1}\Big) \tag{4.17}$$

Since we can always choose $w_{a_1}\in W^{a_1}$, $w_{a_2}\in W^{a_2}$, $w'_{a_1}\in (W^{a_1})'$ and $w'_{a_2}\in (W^{a_2})'$ such that

$$\mathcal{Y}^{e}_{a_2'a_2;1}(w'_{a_2},z_1)\mathcal{Y}^{a_2}_{a_2e;1}(w_{a_2},z_2)\mathcal{Y}^{e}_{a_1'a_1;1}(w'_{a_1},z_3)w_{a_1}\neq 0,$$

(4.17) gives

$$\Big(\hat{e}_{a_2';z_1}\circ(I_{(W^{a_2})'}\boxtimes_{P(z_1)} r_{a_2;z_2})\circ(I_{(W^{a_2})'}\boxtimes_{P(z_1)}(I_{W^{a_2}}\boxtimes_{P(z_2)}\hat{e}_{a_1';z_3}))$$
$$\circ\left(I_{(W^{a_2})'}\boxtimes_{P(z_1)}\left(\mathcal{A}^{P(z_2-z_3),P(z_3)}_{P(z_2),P(z_2)}\right)^{-1}\right)$$
$$\circ(I_{(W^{a_2})'}\boxtimes_{P(z_1)}((\mathcal{C}_{P(z_2-z_3)})^2\boxtimes_{P(z_3)} I_{W^{a_2}}))\circ\Big(I_{(W^{a_2})'}\boxtimes_{P(z_1)}\mathcal{A}^{P(z_2-z_3),P(z_3)}_{P(z_2),P(z_2)}\Big)$$
$$\circ(I_{(W^{a_2})'}\boxtimes_{P(z_1)}(I_{W^{a_2}}\boxtimes_{P(z_2)} i_{a_1';z_3}))\circ(I_{(W^{a_2})'}\boxtimes_{P(z_1)} r^{-1}_{a_2,z_2})\circ i_{a_2';z_1}\Big)$$
$$= (B^{(-1)})^2(\mathcal{Y}^{a_2}_{a_2e;1}\otimes\mathcal{Y}^{e}_{a_1'a_1;1};\mathcal{Y}^{a_2}_{a_2e;1}\otimes\mathcal{Y}^{e}_{a_1'a_1;1})I_V$$
$$= (B^{(-1)})^2_{a_2,a_1}I_V. \tag{4.18}$$

∎

Proposition 4.3 *The composition of the module maps in the sequence (4.1) is equal to* $(B^{(-1)})^2_{a_2,a_1}I_V$.

Proof. Choose $z_1^0,z_2^0,z_3^0\in(0,\infty$ satisfying $z_1^0>z_2^0>z_3^0>z_2^0-z_3^0>0$. Let $\gamma_1,\gamma_2,\gamma_3,\gamma_{23}$ be paths in $(0,\infty)$ from $z_1^0,z_2^0,z_3^0,z_2^0-z_3^0$, respectively, to 1. From the definitions of the module maps involved, we have the following equalities:

$$i'_{a_2}=\mathcal{T}_{\gamma_1}\circ i'_{a_2;z_1^0},$$
$$(I_{(W^{a_2})'}\boxtimes\ r^{-1}_{a_2})\circ\mathcal{T}_{\gamma_1}=((I_{(W^{a_2})'}\boxtimes\mathcal{T}_{\gamma_2})\circ\mathcal{T}_{\gamma_1})\circ(I_{(W^{a_2})'}\boxtimes_{P(z_1^0)} r^{-1}_{a_2,z_2^0})$$

$$(I_{(W^{a_2})'}\boxtimes(I_{W^{a_2}}\boxtimes i'_{a_1}))\circ((I_{(W^{a_2})'}\boxtimes\mathcal{T}_{\gamma_2})\circ\mathcal{T}_{\gamma_1})$$
$$=((I_{(W^{a_2})'}\boxtimes(I_{W^{a_2}}\boxtimes\mathcal{T}_{\gamma_3}))\circ(I_{(W^{a_2})'}\boxtimes\mathcal{T}_{\gamma_2})\circ\mathcal{T}_{\gamma_1})\circ$$
$$\circ(I_{(W^{a_2})'}\boxtimes_{P(z_1^0)}(I_{W^{a_2}}\boxtimes_{P(z_2^0)} i'_{a_1;z_3^0})),$$

$$
\begin{aligned}
&(I_{(W^{a_2})'} \boxtimes \mathcal{A}) \circ ((I_{(W^{a_2})'} \boxtimes (I_{W^{a_2}} \boxtimes \mathcal{T}_{\gamma_3})) \circ (I_{(W^{a_2})'} \boxtimes \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1})\\
&\quad = ((I_{(W^{a_2})'} \boxtimes (\mathcal{T}_{\gamma_{23}} \boxtimes I_{W^{a_2}})) \circ (I_{(W^{a_2})'} \boxtimes \mathcal{T}_{\gamma_3}) \circ \mathcal{T}_{\gamma_1}) \circ\\
&\qquad\qquad \circ \left(I_{(W^{a_2})'} \boxtimes_{P(z_1^0)} \mathcal{A}^{P(z_2^0 - z_3^0), P(z_3^0)}_{P(z_2^0), P(z_2^0)} \right),
\end{aligned}
$$

$$
\begin{aligned}
&(I_{(W^{a_2})'} \boxtimes (\mathcal{C}^2 \boxtimes I_{W^{a_2}})) \circ (((I_{(W^{a_2})'} \boxtimes (\mathcal{T}_{\gamma_{23}} \boxtimes I_{W^{a_2}})) \circ (I_{(W^{a_2})'} \boxtimes \mathcal{T}_{\gamma_3}) \circ \mathcal{T}_{\gamma_1})\\
&\quad = (((I_{(W^{a_2})'} \boxtimes (\mathcal{T}_{\gamma_{23}} \boxtimes I_{W^{a_2}})) \circ (I_{(W^{a_2})'} \boxtimes \mathcal{T}_{\gamma_3}) \circ \mathcal{T}_{\gamma_1}) \circ\\
&\qquad\qquad \circ \left(I_{(W^{a_2})'} \boxtimes_{P(z_1^0)} \left(\left(\mathcal{C}_{P(z_2^0 - z_3^0)} \right)^2 \boxtimes_{P(z_3^0)} I_{W^{a_2}} \right) \right),
\end{aligned}
$$

$$
\begin{aligned}
&(I_{(W^{a_2})'} \boxtimes \mathcal{A}^{-1}) \circ (((I_{(W^{a_2})'} \boxtimes (\mathcal{T}_{\gamma_{23}} \boxtimes I_{W^{a_2}})) \circ (I_{(W^{a_2})'} \boxtimes \mathcal{T}_{\gamma_3}) \circ \mathcal{T}_{\gamma_1})\\
&\quad = ((I_{(W^{a_2})'} \boxtimes (I_{W^{a_2}} \boxtimes \mathcal{T}_{\gamma_3})) \circ (I_{(W^{a_2})'} \boxtimes \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1}) \circ\\
&\qquad\qquad \circ \left(I_{(W^{a_2})'} \boxtimes_{P(z_1^0)} \left(\mathcal{A}^{P(z_2^0 - z_3^0), P(z_3^0)}_{P(z_2^0), P(z_2^0)} \right)^{-1} \right),
\end{aligned}
$$

$$
\begin{aligned}
&(I_{(W^{a_2})'} \boxtimes (I_{W^{a_2}} \boxtimes \hat{e}_{a_1})) \circ ((I_{(W^{a_2})'} \boxtimes (I_{W^{a_2}} \boxtimes \mathcal{T}_{\gamma_3})) \circ (I_{(W^{a_2})'} \boxtimes \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1})\\
&\quad = ((I_{(W^{a_2})'} \boxtimes \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1}) \circ (I_{(W^{a_2})'} \boxtimes_{P(z_1^0)} (I_{W^{a_2}} \boxtimes_{P(z_2^0)} \hat{e}_{a_1; z_3^0})),
\end{aligned}
$$

$$
(I_{(W^{a_2})'} \boxtimes r_{a_2}) \circ ((I_{(W^{a_2})'} \boxtimes \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1}) = \mathcal{T}_{\gamma_1} \circ (I_{(W^{a_2})'} \boxtimes_{P(z_1^0)} r_{a_2; z_2^0}),
$$

$$
\hat{e}_{a_2; z_1^0} \circ \mathcal{T}_{\gamma_1} = \hat{e}_{a_2}.
$$

Each of these equalities is equivalent to a commutative diagram. For example, the first and the second equalities above are equivalent to the commutative diagrams

$$
\begin{array}{ccc}
V & \xrightarrow{i'_{a_2; z_1^0}} & (W^{a_2})' \boxtimes_{P(z_1^0)} W^{a_2}\\
{\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{T}_{\gamma_1}}\\
V & \xrightarrow{i'_{a_2}} & (W^{a_2})' \boxtimes W^{a_2}
\end{array}
$$

and

$$
\begin{array}{ccc}
(W^{a_2})' \boxtimes_{P(z_1^0)} W^{a_2} & \xrightarrow{I_{(W^{a_2})'} \boxtimes_{P(z_1^0)} r^{-1}_{a_2, z_2^0}} & (W^{a_2})' \boxtimes_{P(z_1^0)} (W^{a_2} \boxtimes_{P(z_2^0)} V)\\
{\scriptstyle \mathcal{T}_{\gamma_1}}\big\downarrow & & \big\downarrow{\scriptstyle (I_{(W^{a_2})'} \boxtimes \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1}}\\
(W^{a_2})' \boxtimes W^{a_2} & \xrightarrow{I_{(W^{a_2})'} \boxtimes r^{-1}_{a_2}} & (W^{a_2})' \boxtimes (W^{a_2} \boxtimes V)
\end{array}
$$

(Since the size of some of the diagram is too big, we leave the other commutative diagrams to the reader.)

Combining all the diagrams above or equivalently, all the equalities above, we see that the composition of the module maps in the sequence (4.1) and the composition of the module maps in the sequence (4.2) are equal. Then by Proposition 4.2, we obtain the conclusion of the proposition. ∎

Corollary 4.4 *The composition of the module maps in the sequence obtained from (4.1) by replacing $\hat{e}_{a_1}$ and $\hat{e}_{a_1}$ by e_{a_1} and e_{a_1}, respectively, is equal to*

$$\frac{(B^{(-1)})^2_{a_2,a_1}}{F_{a_1}F_{a_2}} I_V,$$

that is,

$$\begin{aligned} & e_{a_2'} \circ (I_{(W^{a_2})'} \boxtimes r_{a_2}) \circ (I_{(W^{a_2})'} \boxtimes (I_{W^{a_2}} \boxtimes e_{a_1'})) \\ & \quad \circ (I_{(W^{a_2})'} \boxtimes \mathcal{A}^{-1}) \circ (I_{(W^{a_2})'} \boxtimes (\mathcal{C}^2 \boxtimes I_{W^{a_2}})) \circ (I_{(W^{a_2})'} \boxtimes \mathcal{A}) \\ & \quad \circ (I_{(W^{a_2})'} \boxtimes (I_{W^{a_2}} \boxtimes i_{a_1'})) \circ (I_{(W^{a_2})'} \boxtimes r_{a_2}^{-1}) \circ i_{a_2'} \\ & \quad = \frac{(B^{(-1)})^2_{a_2,a_1}}{F_{a_1}F_{a_2}} I_V. \end{aligned} \tag{4.19}$$

Proof. From (4.18) and the definitions of e_{a_1} and e_{a_2}, we obtain (4.19). ∎

Theorem 4.5 *Let V be a simple vertex operator algebra satisfying the conditions in Section 1. Then the ribbon category structure on the category of V-modules constructed in [HL1]–[HL4], [H1] and [H5] is nondegenerate.*

Proof. By definition, the traces of $\mathcal{C}^2 \in \mathrm{Hom}((W^{a_1})', W^{a_2})$ for $a_1, a_2 \in \mathcal{A}$ can be calculated as follows: For any $a_1, a_2 \in \mathcal{A}$, consider the composition of the module maps in the sequence obtained from (4.1) by replacing $\hat{e}_{a_1}$ and $\hat{e}_{a_1}$ by e_{a_1} and e_{a_1}, respectively, that is, consider the module map given by the left-hand side of (4.19). This is a module map from V to V. Since V is an irreducible V-module, the module map we are considering is equal to the identity operator on V multiplied by a number. This number is the trace of $\mathcal{C}^2 \in \mathrm{Hom}((W^{a_1})', W^{a_2})$.

Thus by Proposition 4.3, the traces of $\mathcal{C}^2 \in \mathrm{Hom}((W^{a_1})', W^{a_2})$ for $a_1, a_2 \in \mathcal{A}$ are

$$\frac{(B^{(-1)})^2_{a_2,a_1}}{F_{a_1}F_{a_2}}.$$

By (2.2), they are actually equal to

$$\frac{S^{a_2}_{a_1}}{S^e_e},\ a_1, a_2 \in \mathcal{A},$$

which form an invertible matrix. So the tensor category is nondegenerate. ∎

From Theorem 4.5 and the definition of modular tensor category (see for example [T] and [BK]), we immediately obtain the main result of the present paper:

Theorem 4.6 *Let V be a simple vertex operator algebra satisfying the conditions in Section 1. Then the category of V-modules has a natural structure of modular tensor category.*

Remark 4.7 As in the discussion in Remark 3.10, we can also introduce a notion of nondegeneracy for semisimple rigid vertex tensor categories and a notion of modular vertex tensor category. In fact, for any semisimple rigid vertex tensor category, the sequence (4.2) still makes sense. So we can introduce the notion of nondegeneracy for vertex tensor categories by requiring the compositions of the morphisms in this sequence to be equal to the identity on the unit object. Twists and balancing axioms for vertex tensor categories can also be introduced in an obvious way. A modular vertex tensor category is a semisimple vertex tensor category which is also rigid, balanced and nondegenerate. Then our result above shows that the vertex tensor category of V-modules is nondegenerate and thus the category has a natural structure of modular vertex tensor categories. Moreover, the proof of Proposition 4.3 actually shows that if a semisimple rigid vertex tensor category is nondegenerate, then the corresponding semisimple rigid braided tensor category is also nondegenerate. In particular, a modular vertex tensor category gives a modular tensor category.

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