Lecture notes on the representation theory of vertex operator algebras and conformal field theory

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Contents

1 Definitions of conformal field theory, modular functor and weakly conformal field theory

1.1 An algebraic structure on the moduli space of Riemann surfaces with boundaries

Consider the following category $\mathcal{B}$ constructed geometrically: Objects of $\mathcal{B}$ are finite sets (including the empty set) of copies of the unit circle S^1. Given two objects, morphisms from one object to another are conformal equivalence classes of Riemann surfaces (including degenerate ones, e.g., circles, and possibly disconnected) with oriented, ordered and parametrized boundary components such that the copies of S^1 in the domain and codomain parametrize the negatively oriented and positively oriented boundary components, respectively. For simplicity, we shall call a Riemann surface with the additional data in the description of morphisms above a rigged Riemann surface. For an object containing n copies of the unit circle, the identity on it is the degenerate surface given by the n unit circles with the trivial riggings of the boundary components. Given two composable morphisms, that is, two rigged Riemann surfaces Σ_1 and Σ_2 such that the codomain of Σ_1 is the same as the domain of Σ_2, we can compose them by identifying the positively oriented boundary components of Σ_1 with the negatively oriented boundary components of Σ_2. Then the composition or sewing operation satisfies the associativity and any morphism composed with an identity is equal to itself. Thus we indeed have a category.

This category has a symmetric strict monoidal category structure where the monoid or tensor product bifunctor is defined by disjoint unions of objects and morphisms. Such a category is an example of PROPs.

We shall use $[\Sigma]$ to denote the morphism in the category $\mathcal{B}$ containing a rigged Riemann surface Σ.

1.2 Definition of conformal field theory

We consider the category $\mathcal{H}$ of Hilbert spaces over $\mathbb{C}$ with trace-class maps as morphisms. Here trace class maps are continuous maps for which we can take traces. There is a tensor product bifunctor $\otimes$ such that $\mathcal{H}$ becomes a symmetric tensor category. More generally we also have a category $\mathcal{N}_{form}$ of complete nuclear spaces over $\mathbb{C}$ with nondegenerate bilinear forms with trace-class maps as morphisms. There is also a tensor product bifunctor $\otimes$ such that $\mathcal{N}_{form}$ becomes a symmetric tensor category. The category $\mathcal{H}$ (or $\mathcal{N}_{form}$) induces a projective category $\mathcal{P}(\mathcal{H})$ (or $\mathcal{P}(\mathcal{N}_{form})$) whose objects are projective spaces of Hilbert spaces and whose morphisms are those maps induces from trace class maps between Hilbert spaces. Then $\mathcal{P}(\mathcal{H})$ and $\mathcal{P}(\mathcal{N}_{form})$ are also symmetric tensor categories.

Let Σ be a rigged Riemann surface such that $[\Sigma]$ is a morphism in $\mathcal{B}$ from m copies of S^1 to n copies of S^1. Then by identifying the i-th incoming boundary component of Σ parametrized S^1 with the j-th outgoing boundary component of Σ parametrized by S^1 in the codomain of $[\Sigma]$, we obtain a rigged Riemann surface $[\Sigma_{\widehat{i,j}}]$. (Note that the two copies of S^1 identified might or might not be on a same connected component of Σ.) See Figure 1 in the case $i = 4$ and $j = 3$.

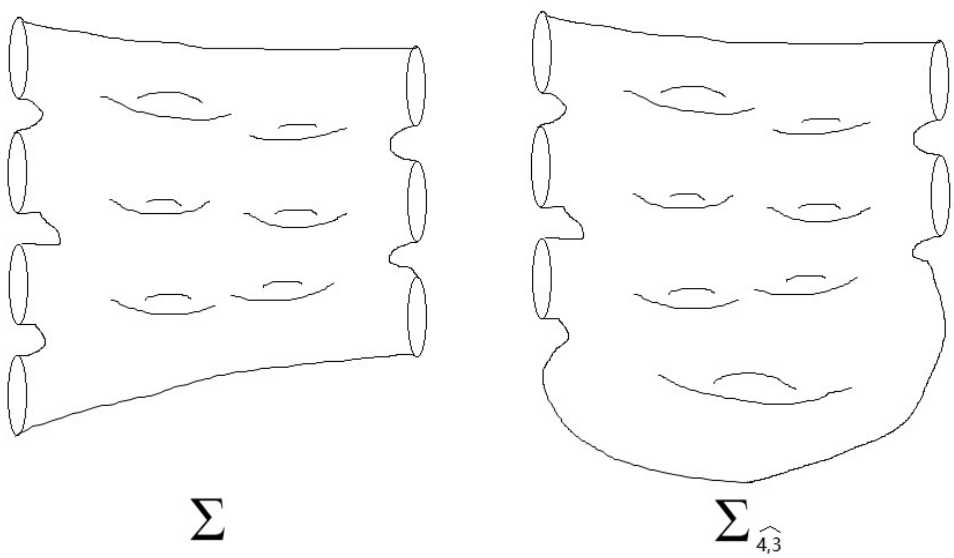

Figure 1: Sewing the 4-th incoming boundary component of a riggeed Riemann surface Σ with the 3-rd outgoing boundary component of Σ to obtain $\Sigma_{\Sigma_{\widehat{4,3}}}$

Let Σ be as above. By changing the i-th incoming boundary component of the Σ to the $n+1$-st outgoing boundary component the same Riemann surface, we obtain a rigged Riemann surface $\Sigma_{i\to n+1}$ with $m-1$ ordered incoming boundary conponenents and $n+1$ ordered outgoing boundary componenents.

We now give a definition of two-dimensional conformal field theory due to Kontsevich and Segal (see [S1] and [S2]).

Definition 1.1. A *two-dimensional conformal field theory* (or simply *conformal field theory* when it is clear that it is two dimensional) is a Hilbert space H and a functor ϕ from $\mathcal{B}$ to $\mathcal{P}(\mathcal{H})$ (or more generally, $\mathcal{P}(\mathcal{N}_{form})$ satisfying the following conditions for morphisms fo the form $[\Sigma]$ from m ordered copies of S^1 to n ordered copies of S^1:

1. The trace between the i-th tensor factor of the domain and the j-th tensor factor of the codomain of $\phi([\Sigma])$ exists and is equal to $\phi([\Sigma_{\widehat{i,j}}])$.

2. The maps $\phi([\Sigma])$ and $\phi([\Sigma_{i\to n+1}])$ are related by the map from $\mathrm{Hom}(P(H)^{\otimes m}), P(H)^{\otimes n}))$ to $\mathrm{Hom}(P(H)^{\otimes m-1}), P(H)^{\otimes (n+1)})$ obtained using the map $H \to H^*$ corresponding to the bilinear form $(\cdot, \cdot)$.

We shall denote a conformal field theory defined in Definition 1.1 by (H, ϕ) or simply by H.

Let Σ be as above and $\overline{\Sigma}$ the rigged Riemann surface obtained by taking the complex conjugate complex structure of the one on $[\Sigma]$ and takeing the incoming and outgoing boundary components of Σ to be the outgoing and incoming blundary components of $\overline{\Sigma}$ with the same orders (note that the orientations of the boundary components are reversed). The correspondence $\Sigma \to \overline{\Sigma}$ is a functor from the category of rigged Riemann surfaces to it self.

Definition 1.2. A *two-dimensional real conformal field theory* (or a *real conformal field theory* is a (two-dimensional) conformal field theory (H, ϕ) together with an anti-linear involution θ from H to itself such that $\phi([\overline{\Sigma}]) = P(\theta)^{\otimes m} \circ \phi^*([\Sigma]) \circ (P(\theta)^{-1})^{\otimes n}$ where $\phi^*([\Sigma])$ is the adjoint of $\Phi([\Sigma])$ and $P(\theta)$ is the map from $P(H)$ to itself induced from θ.

More precise definitions can be given using determinant line and canonical isomorphisms between them. We omit these here.

1.3 Definition of modular functor and the corresponding colored PROP

The definition of conformal field theory in the preceding subsection above do not reveal many important ingredients in concrete models. In particular, they do not give the detailed structure of chiral and anti-chiral parts of conformal field theories, that is, parts of conformal field theories depending on the moduli space parameters analytically and anti-analytically. It is known that meromorphic fields in a conformal field theory form a vertex operator algebra. The representations of this vertex operator algebra form the chiral parts of the theory. Therefore to construct conformal field theories from vertex operator algebras, it is necessary to study first chiral and anti-chiral parts of conformal field theories. Axiomatically, chiral and anti-chiral parts of conformal field theories are weakly conformal field theories introduced by G. Segal in [S1] and [S2] and are generalizations of conformal field theories defined above.

To describe weakly conformal field theories, we first need to describe modular functors, which was also introduced by Segal in [S1] and [S2]. We need to consider rigged Riemann surfaces with additional labels on their boundary components by a set. Let $\mathcal{A}$ be a set. An *$\mathcal{A}$-labeled and rigged Riemann surfaces* (or simply a *labeled and rigged Riemann surfaces* when the set $\mathcal{A}$ of "labels" is clear) is a rigged Riemann surfaces with a the boundary components labeled by elements of $\mathcal{A}$. We consider a category whose objects are conformal equivalence classes of $\mathcal{A}$-labeled and rigged Riemann surfaces and whose morphisms are given by the sewing operation, that is, if one such equivalent class can be obtained from another using the sewing operation, then the procedure of obtaining the second surface from the first one is a morphism. We also use $[\Sigma]$ to denote the conformal equivalence class of a surface Σ.

Definition 1.3. Let $\mathcal{A}$ be a set. A *modular functor with the labeling set* $\mathcal{A}$ (or simply a *modular functor* when the set $\mathcal{A}$ is clear) is a functor E from the category of $\mathcal{A}$-labeled and rigged Riemann surfaces to the category of finite-dimensional vector spaces over $\mathbb{C}$ satisfying the following conditions:

1. $E([\Sigma])$ is independent of the orientation of the boundary components of $[\Sigma]$.

2. $E([\Sigma_1 \sqcup \Sigma_2])$ is naturally isomorphic to $E([\Sigma_1]) \otimes E([\Sigma_2])$.

3. If Σ is obtained from another surface Σ_a by sewing two boundary components with opposite orientations (meaning one incoming and one outgoing) but with the same label $a \in \mathcal{A}$ of Σ_a, then $E([\Sigma])$ is naturally isomorphic to $\oplus_{b\in\mathcal{A}} E([\Sigma_b])$ where for $b \neq a$, Σ_b is the surface obtained from Σ_a by changing the label a to b on the boundary components to be sewn.

4. $E([S_a])$ is canonically isomorphic to $\mathbb{C}$, where S_a is the degenerate $\mathcal{A}$-labeled and rigged Riemann surface given by the unit circle with one incoming boundary component and one outgoing boundary component (both are the same as the degenerate surface itself, the obvious parametrizations and the labels by $a \in \mathcal{A}$.

5. $E([\Sigma])$ depends on Σ holomorphically.

The simplest nontrivial example of a modular functor is given by the determinant line bundles.

From a modular functor E, we can construct a symmetric strict monoidal category $\mathcal{B}_E$ extending the category $\mathcal{B}$ as follows: Objects of $\mathcal{B}_E$ are are finite sets (including the empty set) of pairs of the form of a copy of the unit circle S^1 and an element of the set $\mathcal{A}$. Morphisms of $\mathcal{B}_E$ are pairs of the form of an equivalence class $[\Sigma]$ of $\mathcal{A}$-labeled and rigged Riemann surface (including the degenerate ones) and an element λ of the vector space $E([\Sigma])$, such that each boundary component of Σ and the element of $\mathcal{A}$ labeling the boundary component of Σ match with a pair of a copy of S^1 and an element of $\mathcal{A}$ in the domain or codomain depending on whether the boundary component is incoming or outgoing, respectively. The identity on an object is given by the obvious degenerate $\mathcal{A}$-labeled and rigged Riemann surface S (given by the same number of copies of S^1 as the object) with the obvious labeling by elements of $\mathcal{A}$ determined by the object and the element in $E(S)$ corresponding to $1 \in \mathbb{C}$ (since $E(S)$ is canonically isomorphic to $\mathbb{C}$) . Let $([\Sigma_1], \lambda_1)$ and $([\Sigma_2], \lambda_2)$ be two composable morphisms. Let Σ be the $\mathcal{A}$-labeled and rigged Riemann surface obtained by sewing Σ_1 and Σ_2. The sewing procedure to obtain Σ from Σ_1 and Σ_2 is a morphism of the category of $\mathcal{A}$-labeled and rigged Riemann surfaces above. Since E is a functor, applying E to this morphism, we have a linear map from $E([\Sigma_1 \sqcup \Sigma_2])$ to $E([\Sigma])$. Since $E([\Sigma_1 \sqcup \Sigma_2])$ is naturally isomorphic to $E([\Sigma_1]) \otimes E([\Sigma_2])$, we also have a linear map from $E([\Sigma_1]) \otimes E([\Sigma_2])$ to $E([\Sigma])$. We define the composition of $([\Sigma_1], \lambda_1)$ and $([\Sigma_2], \lambda_2)$ is $([\Sigma], \lambda)$, where Σ, as discussed above, is the $\mathcal{A}$-labeled and rigged Riemann surface obtained by sewing Σ_1 and Σ_2 and λ is the image of $\lambda_1 \otimes \lambda_2 \in E([\Sigma_1]) \otimes E([\Sigma_2])$ under the linear map from $E([\Sigma_1 \sqcup \Sigma_2])$ to $E([\Sigma])$. Then the composition clearly satisfies the associativity and any morphism composed with an identity is equal to itself. In particular, we obtain a category $\det^c_{\mathcal{B}} \otimes \overline{\det}^{\bar{c}}_{\mathcal{B}}$.

The category $\mathcal{B}_E$ is also a symmetric strict monoidal category: The monoid or tensor product bifunctor is defined by disjoint unions of objects, the disjoint union of the rigged surface part of the morphisms and the tensor products of the elements in the corresponding finite-dimensional vector spaces.

The category $\mathcal{B}_E$ has a structure of $\mathcal{A}$-colored PROP.

1.4 Definition of weakly conformal field theory

We consider the category $\mathcal{N}$ of complete nuclear spaces over $\mathbb{C}$ with trace-class maps as morphisms. Just as the category $\mathcal{N}_{form}$, $\mathcal{T}$ is also a symmetric monoidal category.

We also have operations on $\mathcal{A}$-labeled and rigged Riemann surfaces similar to those on rigged Riemann surface in Subsection 1.2. We shall use the same notations to describe them.

Let Σ be an $\mathcal{A}$-labeled and rigged Riemann surface such that $[\Sigma]$ is a morphism in $\mathcal{B}$ from m copies of S^1 to n copies of S^1. Assuming that the labels at the i-th incoming boundary component and the j-th outgoing boundary component of Σ are the same. Then by identifying the i-th incoming boundary component of Σ with the j-th outgoing boundary component of Σ in the codomain of $[\Sigma]$ using the parametrizations of these boundary components, we obtain a $\mathcal{A}$-labeled and rigged Riemann surface $[\Sigma_{\widehat{i,j}}]$.

Since from Σ to $\Sigma_{\widehat{i,j}}$ is an sewing procedure, it is a morphism in the category of $\mathcal{A}$-labeled and rigged Riemann surfaces. By the definition of modular functor, we have a linear map $\rho^E_{[\Sigma];i,j} : E([\Sigma]) \to E([\Sigma_{\widehat{i,j}}])$.

Let Σ be as above. By changing the i-th incoming boundary component of the Σ to the $n+1$-st outgoing boundary component the same Riemann surface, we obtain a $\mathcal{A}$-labeled and rigged Riemann surface $\Sigma_{i\to n+1}$ with $m-1$ ordered incoming boundary conponenents and $n+1$ ordered outgoing boundary componenents. Since $E([\Sigma])$ is independent of the orientation of the boundary components of $[\Sigma]$, $E([\Sigma])$ is in fact the same as $E([\Sigma_{i\to n+1}])$. We shall denote the identity map from $E([\Sigma])$ to $E([\Sigma_{i\to n+1}])$ by $\rho_{[\Sigma];i\to n+1}$.

Definition 1.4. Let E be a modular functor labeled by $\mathcal{A}$. A *weakly conformal field theory over E* is a set $\{H^a \mid a \in \mathcal{A}\}$ of Hilbert spaces and a functor ϕ from $E_{\mathcal{B}}$ to $\mathcal{H}$ (or to $\mathcal{N}$) satisfying the following conditions for morphisms in $E_{\mathcal{B}}$ of the form $([\Sigma], \lambda)$, where $[\Sigma]$ is a morphism in $\mathcal{B}$ from m ordered copies of S^1 to n ordered copies of S^1 and $\lambda \in E([\Sigma])$:

1. If the i-th tensor factor of the domain and the j-th tensor factor of the codomain of $\phi([\Sigma], \lambda)$ are labeled by the same element of A, then the trace between the i-th tensor factor of the domain and the j-th tensor factor of the codomain of $\phi([\Sigma], \lambda)$ exists and is equal to $\phi([\Sigma_{\widehat{i,j}}], \rho^E_{[\Sigma];i,j}(\lambda))$.

2. The maps $\Phi([\Sigma], \lambda)$ and $\Phi([\Sigma_{i\to n+1}], \rho_{[\Sigma];i\to n+1}(\lambda))$ are related by the map from

$$\mathrm{Hom}(H^{a_1} \otimes \cdots \otimes H^{a_m}, H^{b_1} \otimes \cdots \otimes H^{b_n})$$

to

$$\mathrm{Hom}(H^{a_1} \otimes \otimes \widehat{H^{a_i}} \otimes \cdots \otimes H^{a_m}, H^{b_1} \otimes \cdots \otimes H^{b_n} \otimes H^{a^i})$$

obtained using the map $H \to H^*$ corresponding to the bilinear form $(\cdot, \cdot)$, where we use $\widehat{H^{a_i}}$ to denote the tensor factor H^{a_i} is missing.

2 The precise formulation and proof of the associativity of intertwining operators

2.1 The formulation of the theorem

Let V be a vertex operator algebra. We need to introduce some conditions on V and lower-bounded generalized V-modules.

We say that V is of *positive energy* (or *CFT type*) if $V_{(n)} = 0$ for $n \in -\mathbb{Z}_+$ and $V_{(0)} = \mathbb{C}\mathbf{1}$. It is clear that the affine vertex operator algebras $V(\ell, 0)$ and $L(\ell, 0)$ are of positive energy.

A grading-restricted (or lower-bounded) generalized V-module W is said to be *irreducible* if the only grading-restrticted (or lower-bounded) generalized V-submodule of W are 0 and W. It is easy to show that an irreducible grading-restrticted generalized V-module must be an (ordinary) V-module. Note that if a lower-bounded generalized V-mdoule is a direct sum of finitely many irreduicible (ordinary) V-modules, then it is an (ordinary) V-module.

Let W be a lower-bounded generalized V-module. For $n \in \mathbb{Z}_+$, let $C_n(W)$ be the subspace of W spanned by elements of the form $\mathrm{Res}_x x^{-n} Y_W(v, x)w$ for $v \in V_+ = \coprod_{n \in \mathbb{Z}_+} V_{(n)}$ and $w \in W$. We say that W is C_n*-cofinite* if $W/C_n(W)$ is finite dimensional. From the definition, it is not difficult to show that W is C_n-cofinite implies that W is C_{n-1}-cofinite. We are interested in C_1- and C_2-cofinite lower-bounded generalized V-modules.

In the case that $\ell \in \mathbb{Z}_+$ and $\mathfrak{g}$ a finite-dimensional simple Lie algebra, every lower-bounded generalized $L(\ell, 0)$-module is a direct sum of irrecucible $L(\ell, 0)$-modules. Also in this case, there are only finitely many inequivalent irreducible $L(\ell, 0)$-modules. See [FZ] and [LL] for details.

The vertex operator algebra V is always C_1-cofinite. In fact, by the creation property for vertex operator algebra, for $v \in V$, $v = \mathrm{Res}_x x^{-1} Y_V(v, x)\mathbf{1}$. So elements of V_+ are all in $C_1(V)$. Thus $V/C_1(V)$ is linear isomorphic to $\coprod_{n \in -\mathbb{N}} V_{(n)}$. But $\coprod_{n \in -\mathbb{N}} V_{(n)}$ is finite dimensional since V is grading restricted. Thus $V/C_1(V)$ is finite dimensional. On the other hand, even ordinary modules are in general not C_1-cofinite. But for affine Lie algebra vertex operator algebras, we have the following result:

Proposition 2.1. *Let W be an (ordinary) $V(\ell, 0)$-module generated by a finite-dimensional $\mathfrak{g}$-module constructed in Subsection 11.4 in [H8]. Then W is C_1-cofinite.*

Proof. Let M be the finite-dimensional $\mathfrak{g}$-module generating W. Then by Proposition 11.17 in [H8], we know that W is spanned by elements of the form

$$a_1(-n_1) \cdots a_k(-n_k)w,$$

where $a_1, \ldots, a_k \in \mathfrak{g}$, $n_1, \ldots, n_k \in \mathbb{Z}_+$ and $w \in M$. By the definition of the vertex operator map Y_W, we have

$$a_1(-n_1) = \mathrm{Res}_x x^{-n_1} Y_W(a_1(-1)\mathbf{1}, x).$$

Using the $L(-1)$-derivative property for vertex operators, we have

$$\frac{d^{n_1-1}}{dx^{n_1-1}} Y_W(a_1(-1)\mathbf{1}, x) = Y_W(L_{V(\ell,0)}(-1)^{n_1-1}a_1(-1)\mathbf{1}, x).$$

Then we have

$$\begin{aligned}(n_1-1)!a_1(-n_1) &= \mathrm{Res}_x x^{-1} \frac{d^{n_1-1}}{dx^{n_1-1}} Y_W(a_1(-1)\mathbf{1}, x)\\ &= \mathrm{Res}_x x^{-1} Y_W(L_{V(\ell,0)}(-1)^{n_1-1}a_1(-1)\mathbf{1}, x).\end{aligned}$$

Since $L_{V(\ell,0)}(-1)^{n_1-1}a_1(-1)\mathbf{1} \in V_{(n_1)} \subset V_+$, in the case $k \neq 0$, we have

$$a_1(-n_1)\cdots a_k(-n_k)w = \mathrm{Res}_x x^{-1} Y_W(L_{V(\ell,0)}(-1)^{n_1-1}a_1(-1)\mathbf{1}, x)w \in C_1(W).$$

Then $M + C_1(W) = W$. Since M is finite dimensional, we see that $W/C_1(W)$ is finite dimensional. ∎

Next we consider the case that the level ℓ is a positive integer. A (ordinary) $V(\ell, 0)$-module W generated by a finite-dimensional irreducible $\mathfrak{g}$-module as in Proposition 2.1 is also a $\hat{\mathfrak{g}}$-module. Such a $\hat{\mathfrak{g}}$-module has a maximal proper $\hat{\mathfrak{g}}$-submodule so that the quotient of W by this maximal proper $\hat{\mathfrak{g}}$-submodule is an irreducible $\hat{\mathfrak{g}}$-module. It can be shown that this quotient is in fact an irreducible $L(\ell, 0)$-module. In fact, every irreducible $L(\ell, 0)$-module can be constructed in this way. See [FZ] and [LL] for details.

Proposition 2.2. *Let ℓ be a positive integer and W a (ordinatry) $L(\ell, 0)$-module. Then W is C_1-cofinite.*

Proof. Using the construction of the irreducible $L(\ell, 0)$-modules discussed above and the same proof as the one for Proposition 2.1, we see that irreduible $L(\ell, 0)$-modules are C_1-cofinite. In this case, as is mentioned above, every lower-bounded generalized $L(\ell, 0)$-module is a direct sum of irreduible $L(\ell, 0)$-modules. Since there are only finitely many inequivalent irreducible $L(\ell, 0)$-modules (see also the discussion above, a (ordinary) $L(\ell, 0)$-module must be a direct sum of finitely many irreduible $L(\ell, 0)$-modules. In particular, W is a a direct sum of finitely many irreduible $L(\ell, 0)$-modules. Since direct sums of finitely many C_1-cofinite V-modules for any vertex operator algebra V are also C_1-cofinite (exercise), W is C_1-cofinite. ∎

In fact we have the following stronger result:

Proposition 2.3. *Let ℓ be a positive integer and W a (ordinary) $L(\ell, 0)$-module. Then W is C_2-cofinite.*

The proof of this result is more complicated and is omitted here. It is obtained from the C_2-cofiniteness of $L(\ell, 0)$ (see for example Proposition 12.6 in [DLM]) and Theorem 5.2 in [ABD] stating in particular that if a vertex operator algebra V is a C_2-cofinite, then every irreducible V-module is C_2-cofinite.

Irreducible modules for the Heisenberg vertex operator algebras are C_1-cofinite but Heisenberg vertex operator algebras are not C_2-cofinite.

Irreducible modules for the Virasoro vertex operator algebra $V(c, 0)$ are C_1-cofinite if and only if it is not a Verma module for the Virasoro algebra. Here a Verma module for the Virasoro algebra is a module obtained from a one dimensional space with the positive part of the Virasoro algebra acts as 0, $L(0)$ acts as a number h and the module is obtained using the induced module construction. See Corollary 2.2.7 in [CJHRY].

For $z \in \mathbb{C}^\times$, we use $\log z$ to denote $\log |z| + i \arg z$, where $0 \leq \arg z < 2\pi$. For a formal series $f(x_1, x_2)$ in (nonintegral) powers of x_1 and x_2 and nonnegative powers of $\log x_1$ and $\log x_2$, we use $f(z_1, z_2)$ to denote the series obtained by sunstituting $e^{n \log z_1}$, $\log z_1$, $e^{n \log z_2}$, $\log z_2$ for x_1^n, $\log x_1$, x_2^n, $\log x_2$, respectively.

We are ready to formulate the theorem on the associativity of intertwining operators.

Theorem 2.4 ([H7]). *Let V be a C_2-cofinite vertex operator algebra of positive energy. Let W_1, W_2, W_3, W_4, W_5 be grading-restricted generalized V-modules and $\mathcal{Y}_1$ and $\mathcal{Y}_2$ intertwining operators of types $\binom{W_4}{W_1\, W_5}$ and $\binom{W_5}{W_2\, W_3}$, respectively. Then we have the following:*

1. *For $w_1 \in W_1, w_2 \in W_2$, $w_3 \in W_3$ and $w_4' \in W_4'$, the series*

$$\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle$$

is absolutely convergent in the region $|z_1| > |z_2| > 0$ and its sum can be analytically continued to a multivalued analytic function

$$F(\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle)$$

on the region

$$M^2 = \{(z_1, z_2) \mid z_1, z_2 \neq 0, z_1 - z_2 \neq 0\} \subset \mathbb{C}^n$$

and the only possible singular points $z_1 = 0$, $z_2 = 0$ and $z_1 = z_2$ are regular singular points.

2. *There exist a grading-restricted generalized V-module W_6 and intertwining operators $\mathcal{Y}_3$ and $\mathcal{Y}_4$ of the types $\binom{W_4}{W_6\, W_3}$ and $\binom{W_6}{W_1\, W_2}$, respectively, such that for $w_1 \in W_1, w_2 \in W_2$, $w_3 \in W_3$ and $w_4' \in W_4'$,*

$$\langle w_4', \mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(w_2, x_2)w_3\rangle = \langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, x_0)w_2, x_2)w_3\rangle$$

in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$.

A regular singular point of a multivalued analytic function is a point on which the function is not defined but in the neighborhood of the point, the function can be expanded as a series in powers of the variables and a polynomial in the logarithms of the variables.

This theorem has generalizations to the case that every irreducible V-modules are C_1-cofinite but some other technical conditions are needed. For details, see also [H7]. When we prove this theorem below, we mostly assume only that grading-restricted generalized V-modules involved are C_1-cofinite.

2.2 Finitely generated modules over the ring of rational functions

We first give some identities for the product of two intertwining operators.

Proposition 2.5. *Let W_1, W_2, W_3, W_4, W_5 be generalized V-modules and $\mathcal{Y}_1$ and $\mathcal{Y}_2$ intertwining operators of types $\binom{W_4}{W_1\, W_5}$ and $\binom{W_5}{W_2\, W_3}$, respectively. Then we have*

$$\begin{aligned}
&\langle w_4', \mathcal{Y}_1(u_{-1}w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle\\
&\quad = \sum_{k\in\mathbb{N}} z_1^k \langle u^*_{-1-k}w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle\\
&\qquad + \sum_{k\in\mathbb{N}} (z_1-z_2)^{-1-k}\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(u_kw_2, z_2)w_3\rangle\\
&\qquad + \sum_{k\in\mathbb{N}} z_1^{-1-k}\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)u_kw_3\rangle, \qquad (2.1)\\
&\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(u_{-1}w_2, z_2)w_3\rangle\\
&\quad = \sum_{k\in\mathbb{N}} z_2^k \langle u^*_{-1-k}w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle\\
&\qquad - \sum_{k\in\mathbb{N}} (-1)^k(z_1-z_2)^{-1-k}\langle w_4', \mathcal{Y}_1(u_kw_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle\\
&\qquad + \sum_{k\in\mathbb{N}} z_2^{-1-k}\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)u_kw_3\rangle, \qquad (2.2)\\
&\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)u_{-1}w_3\rangle\\
&\quad = \langle u^*_{-1}w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle\\
&\qquad - \sum_{k\in\mathbb{N}} (-1)^k z_1^{-1-k}\langle w_4', \mathcal{Y}_1(u_kw_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle\\
&\qquad - \sum_{k\in\mathbb{N}} (-1)^k z_2^{-1-k}\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(u_kw_2, z_2)w_3\rangle, \qquad (2.3)\\
&\langle u_{-1}w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle\\
&\quad = \sum_{k\in\mathbb{N}} (-1)^k z_1^{1+k}\langle w_4', \mathcal{Y}_1(e^{z_1^{-1}L(1)}(-z_1^2)^{L(0)}u_k(-z_1^{-2})^{L(0)}e^{-z_1^{-1}L(1)}w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle\\
&\qquad + \sum_{k\in\mathbb{N}} (-1)^k z_2^{1+k}\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(e^{z_2^{-1}L(1)}(-z_2^2)^{L(0)}u_k(-z_2^{-2})^{L(0)}e^{-z_2^{-1}L(1)}w_2, z_2)w_3\rangle
\end{aligned}$$

$$+ \langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)u^*_{-1}w_3\rangle. \tag{2.4}$$

Proof. We prove only (2.1). We have the associator formula

$$\begin{aligned}
&\mathcal{Y}_1(Y_{W_1}(u, x_0)w_1, x_1)\\
&\quad = Y_{W_4}(u, x_0 + x_1)\mathcal{Y}_1(w_1, x_1) - \mathrm{Res}_{y_1} x_0^{-1}\delta\left(\frac{-x_1 + y_1}{x_0}\right)\mathcal{Y}_1(w_1, x_1)Y_{W_5}(u, y)
\end{aligned} \tag{2.5}$$

(see (12.56) in [H8]). Take $\mathrm{Res}_{x_0}x_0^{-1}$ on both sides of (2.5), we obtain

$$\begin{aligned}
&\mathcal{Y}_1(u_{-1}w_1, x_1)\\
&\quad = \mathrm{Res}_{x_0}x_0^{-1}Y_{W_4}(u, x_0 + x_1)\mathcal{Y}_1(w_1, x_1) - \mathrm{Res}_{y_1}(-x_1 + y_1)^{-1}\mathcal{Y}_1(w_1, x_1)Y_{W_5}(u, y_1)\\
&\quad = \sum_{n\in\mathbb{Z}} \mathrm{Res}_{x_0}x_0^{-1}u_n(x_0 + x_1)^{-n-1}\mathcal{Y}_1(w_1, x_1) - \mathrm{Res}_{y_1}(-x_1 + y_1)^{-1}\mathcal{Y}_1(w_1, x_1)Y_{W_5}(u, y_1)\\
&\quad = \sum_{k\in\mathbb{N}} x_1^k u_{-1-k}\mathcal{Y}_1(w_1, x_1) - \mathrm{Res}_{y_1}(-x_1 + y_1)^{-1}\mathcal{Y}_1(w_1, x_1)Y_{W_5}(u, y_1).
\end{aligned} \tag{2.6}$$

Then we have

$$\begin{aligned}
&\mathcal{Y}_1(u_{-1}w_1, x_1)\mathcal{Y}_2(w_2, x_2)\\
&\quad = \sum_{k\in\mathbb{N}} x_1^k u_{-1-k}\mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(w_2, x_2) - \mathrm{Res}_{y_1}(-x_1 + y_1)^{-1}\mathcal{Y}_1(w_1, x_1)Y_{W_5}(u, y_1)\mathcal{Y}_2(w_2, x_2).
\end{aligned} \tag{2.7}$$

Using the commutator formula

$$Y_{W_5}(u, y_1)\mathcal{Y}_2(w_2, x_2) = \mathcal{Y}_2(w_2, x_2)Y_{W_3}(u, y_1) + \mathrm{Res}_{y_2}y_1^{-1}\delta\left(\frac{x_2 + y_2}{y_1}\right)\mathcal{Y}_2(Y_{W_2}(u, y_2)w_2, x_2)$$

(see (12.55) in [H8]), we see that the second term in the right-hand side of (2.7) is equal to

$$\begin{aligned}
&- \mathrm{Res}_{y_1}(-x_1 + y_1)^{-1}\mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(w_2, x_2)Y_{W_3}(u, y_1)\\
&\qquad - \mathrm{Res}_{y_1}(-x_1 + y_1)^{-1}\mathcal{Y}_1(w_1, x_1)\mathrm{Res}_{y_2}y_1^{-1}\delta\left(\frac{x_2 + y_2}{y_1}\right)\mathcal{Y}_2(Y_{W_2}(u, y_2)w_2, x_2)\\
&\quad = -\sum_{k\in\mathbb{N}}\sum_{n\in\mathbb{Z}}\binom{-1}{k}\mathrm{Res}_{y_1}(-x_1)^{-1-k}y_1^k\mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(w_2, x_2)u_n y_1^{-n-1}\\
&\qquad - \mathrm{Res}_{y_1}\mathrm{Res}_{y_2}(-x_1 + x_2 + y_2)^{-1}\mathcal{Y}_1(w_1, x_1)y_1^{-1}\delta\left(\frac{x_2 + y_2}{y_1}\right)\mathcal{Y}_2(Y_{W_2}(u, y_2)w_2, x_2)\\
&\quad = -\sum_{k\in\mathbb{N}}\binom{-1}{k}(-x_1)^{-1-k}\mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(w_2, x_2)u_k\\
&\qquad - \sum_{k\in\mathbb{N}}\sum_{n\in\mathbb{N}}\binom{-1}{k}\mathrm{Res}_{y_2}(-x_1 + x_2)^{-1-k}y_2^k\mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(u_n y_2^{-n-1}w_2, x_2)
\end{aligned}$$

$$= \sum_{k\in\mathbb{N}} x_1^{-1-k}\mathcal{Y}_1(w_1,x_1)\mathcal{Y}_2(w_2,x_2)u_k$$
$$+ \sum_{k\in\mathbb{N}} (x_1-x_2)^{-1-k}\mathcal{Y}_1(w_1,x_1)\mathcal{Y}_2(u_kw_2,x_2). \tag{2.8}$$

From (2.7) and (2.8), we obtain

$$\begin{aligned}
&\mathcal{Y}_1(u_{-1}w_1,x_1)\mathcal{Y}_2(w_2,x_2)\\
&\quad= \sum_{k\in\mathbb{N}} x_1^k u_{-1-k}\mathcal{Y}_1(w_1,x_1)\mathcal{Y}_2(w_2,x_2) + \sum_{k\in\mathbb{N}} x_1^{-1-k}\mathcal{Y}_1(w_1,x_1)\mathcal{Y}_2(w_2,x_2)u_k\\
&\qquad+ \sum_{k\in\mathbb{N}} (x_1-x_2)^{-1-k}\mathcal{Y}_1(w_1,x_1)\mathcal{Y}_2(u_kw_2,x_2).
\end{aligned} \tag{2.9}$$

Applying both sides of (2.9) to w_3, pairing with w_4' and then substituting $e^{n\log z_1}$, $\log z_1$, $e^{n\log z_2}$, $\log z_2$ for x_1^n, $\log x_1$, x_2^n, $\log x_2$, respectively, we obtain (2.1). ∎

Let $R = \mathbb{C}[z_1^{\pm1}, z_2^{\pm1}, (z_1-z_2)^{-1}]$ and $T = R\otimes W_4'\otimes W_1\otimes W_2\otimes W_3$. Then T is a (free) R-module. For simplicity, we shall write $f(z_1,z_2)\otimes w_4'\otimes w_1\otimes w_2\otimes w_3$ as $f(z_1,z_2)w_4'\otimes w_1\otimes w_2\otimes w_3$ in T.

Let J be the R-submodule of T generated by elements of the form

$$\begin{aligned}
&\mathcal{A}(u,w_4',w_1,w_2,w_3)\\
&\quad= \sum_{k\geq0} z_1^k u^*_{-1-k}w_4'\otimes w_1\otimes w_2\otimes w_3 - w_4'\otimes u_{-1}w_1\otimes w_2\otimes w_3\\
&\qquad+ \sum_{k\geq0}(z_1-z_2)^{-1-k}w_4'\otimes w_1\otimes u_kw_2\otimes w_3\\
&\qquad+ \sum_{k\geq0} z_1^{-1-k}w_4'\otimes w_1\otimes w_2\otimes u_kw_3,\\
&\mathcal{B}(u,w_4',w_1,w_2,w_3)\\
&\quad= \sum_{k\geq0} z_2^k u^*_{-1-k}w_4'\otimes w_1\otimes w_2\otimes w_3\\
&\qquad+ \sum_{k\geq0}(-1)^k(z_1-z_2)^{-1-k}w_4'\otimes u_kw_1\otimes w_2\otimes w_3 - w_4'\otimes w_1\otimes u_{-1}w_2\otimes w_3\\
&\qquad+ \sum_{k\geq0} z_2^{-1-k}w_4'\otimes w_1\otimes w_2\otimes u_kw_3,\\
&\mathcal{C}(u,w_4',w_1,w_2,w_3)\\
&\quad= u^*_{-1}w_4'\otimes w_1\otimes w_2\otimes w_3 - \sum_{k\geq0}(-1)^k z_1^{-1-k}w_4'\otimes u_kw_1\otimes w_2\otimes w_3\\
&\qquad- \sum_{k\geq0}(-1)^k z_2^{-1-k}w_4'\otimes w_1\otimes u_kw_2\otimes w_3 - w_4'\otimes w_1\otimes w_2\otimes u_{-1}w_3,
\end{aligned}$$

$$
\begin{aligned}
&\mathcal{D}(u, w_4', w_1, w_2, w_3)\\
&\quad = u_{-1}w_4' \otimes w_1 \otimes w_2 \otimes w_3\\
&\qquad - \sum_{k\geq 0}(-1)^k z_1^{1+k} w_4' \otimes e^{z_1^{-1}L(1)}(-z_1^2)^{L(0)} u_k (-z_1^{-2})^{L(0)} e^{-z_1^{-1}L(1)} w_1 \otimes w_2 \otimes w_3\\
&\qquad - \sum_{k\geq 0}(-1)^k z_2^{1+k} w_4' \otimes w_1 \otimes e^{z_2^{-1}L(1)}(-z_2^2)^{L(0)} u_k (-z_2^{-2})^{L(0)} e^{-z_2^{-1}L(1)} w_2 \otimes w_3\\
&\qquad - w_4' \otimes w_1 \otimes w_2 \otimes u^*_{-1} w_3
\end{aligned}
$$

for $u \in V_+$, $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$ and $w_4' \in W_4'$.

The gradings on W_1, W_2, W_3, W_4' induce a grading (also called weight) on $W_4' \otimes W_1 \otimes W_2 \otimes W_3$ and then by defining the weights of elements of R to be 0, we also obtain a grading on T (still called weight). Let $T_{(r)}$ be the homogeneous subspace of weight r for $r \in \mathbb{C}$. Then $T = \coprod_{r\in\mathbb{R}} T_{(r)}$.

We say that a generalized V-module $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ is quasi-finite dimensional if for any real number r, $\coprod_{\Re(n)\leq r} W_{[n]}$ is finite dimensional. In it clear that a quasi-finite dimensional generalized V-module is grading restricted.

We now assume that W_1, W_2, W_3, W_4' are lower-bounded and C_1-cofinite. Then it can be shown that they must be quasi-finite dimensional. (In the case that V is C_2-cofinite and of positive energy, a grading-restricted generalized V-module must be C_2-cofinite and thus is also C_1-cofinite.) In this case, $T_{(s)}$ for $s \in \mathbb{C}$ are finitely-generated R-modules and $T_{(s)} = 0$ when $\Re(s)$ is sufficiently negative. Let $F_r(T) = \coprod_{s\leq r} T_{(s)}$ for $r \in \mathbb{R}$. Then $F_r(T)$, $r \in \mathbb{R}$, are finitely-generated R-modules, $F_{r_1}(T) \subset F_{r_2}(T)$ for $r_1 \leq r_2$ and $\cup_{r\in\mathbb{R}} F_r(T) = T$. Let $F_r(J) = J \cap F_r(T)$ for $r \in \mathbb{R}$. Then $F_r(J)$ for $r \in \mathbb{R}$ are finitely-generated R-modules, $F_{r_1}(J) \subset F_{r_2}(J)$ for $r_1 \leq r_2$ and $\cup_{r\in\mathbb{R}} F_r(J) = J$.

Proposition 2.6. *There exists $M \in \mathbb{Z}$ such that for any $r \in \mathbb{R}$, $F_r(T) \subset F_r(J) + F_M(T)$. In particular, $T = J + F_M(T)$.*

Proof. Since W_1, W_2, W_3, W_4' are all C_1-cofinite, there exists $M \in \mathbb{Z}$ such that

$$
\begin{aligned}
\coprod_{\Re(s)>M} T_{(s)} \subset\ & R(C_1(W_4') \otimes W_1 \otimes W_2 \otimes W_3) + R(W_4' \otimes C_1(W_1) \otimes W_2 \otimes W_3)\\
& + R(W_4' \otimes W_1 \otimes C_1(W_2) \otimes W_3) + R(W_4' \otimes W_1 \otimes W_2 \otimes C_1(W_3)). \qquad (2.10)
\end{aligned}
$$

If $r < M$, then certainly we have $F_r \subset F_M(T) = F_r(J) + F_M(T)$. For $s > M$, we assume that $F_r(T) \subset F_r(J) + F_M(T)$ for $r < s$. We want to show that any homogeneous element of $T_{(s)}$ can be written as a sum of an element of $F_s(J)$ and an element of $F_M(T)$. Since $s > M$, by (2.10), any element of $T_{(s)}$ is an element of the right-hand side of (2.10). We shall discuss only the case that this element is in $R(W_4' \otimes C_1(W_1) \otimes W_2 \otimes W_3)$; the other cases are completely similar.

We need only discuss elements of the form $w_4' \otimes u_{-1}w_1 \otimes w_2 \otimes w_3$ where $u \in V_+$, $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$ and $w_4' \in W_4'$. By assumption, the weight of $w_4' \otimes u_{-1}w_1 \otimes w_2 \otimes w_3$ is s and the weights of $u^*_{-1-k}w_4' \otimes w_1 \otimes w_2 \otimes w_3$, $w_4' \otimes w_1 \otimes w_2 \otimes u_k w_3$ and $w_4' \otimes w_1 \otimes u_k w_2 \otimes w_3$ for

$k \geq 0$, are all less than the weight of $w_4' \otimes u_{-1}w_1 \otimes w_2 \otimes w_3$. So $\mathcal{A}(u, w_4', w_1, w_2, w_3) \in F_s(J)$. Thus we see that $w_4' \otimes u_{-1}w_1 \otimes w_2 \otimes w_3$ can be written as a sum of an element of $F_s(J)$ and elements of T of weights less than s. Since elements of T of weights $r < s$ is in $F_r(J) + F_M(T) \subset F_s(J) + F_M(T)$, $w_4' \otimes u_{-1}w_1 \otimes w_2 \otimes w_3$ can be written as the sum of an element of $F_s(J)$ and an element of $F_M(T)$.

Now we have

$$\begin{aligned} T &= \cup_{r\in\mathbb{R}} F_r(T) \\ &\subset \cup_{r\in\mathbb{R}} (F_r(J) + F_M(T)) \\ &= J + F_M(T). \end{aligned}$$

But we know that $J + F_M(T) \subset T$. So we obtain $T = J + F_M(T)$. ∎

Corollary 2.7. *The quotient R-module T/J is finitely generated.*

Proof. Since $T = J + F_M(T)$ and $F_M(T)$ is finitely-generated, T/J is finitely-generated. ∎

For an element $\mathcal{W} \in T$, we shall use $[\mathcal{W}]$ to denote the coset $\mathcal{W} + J$ in T/J.

Corollary 2.8. *For $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$ and $w_4' \in W_4'$, let M_1 and M_2 be the R-submodules of T/J generated by $[w_4' \otimes L_{W_1}(-1)^j w_1 \otimes w_2 \otimes w_3]$, $j \geq 0$, and by $[w_4' \otimes w_1 \otimes L_{W_2}(-1)^j w_2 \otimes w_3]$, $j \geq 0$, respectively. Then M_1 and M_2 are finitely generated. In particular, for $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$ and $w_4' \in W_4'$, there exist $a_k(z_1, z_2), b_l(z_1, z_2) \in R$ for $k = 1, \dots, m$ and $l = 1, \dots, n$ such that*

$$\begin{aligned} &[w_4' \otimes L(-1)^m w_1 \otimes w_2 \otimes w_3] + a_1(z_1, z_2)[w_4' \otimes L(-1)^{m-1} w_1 \otimes w_2 \otimes w_3] \\ &\quad + \cdots + a_m(z_1, z_2)[w_4' \otimes w_1 \otimes w_2 \otimes w_3] = 0, \end{aligned} \tag{2.11}$$

$$\begin{aligned} &[w_4' \otimes w_1 \otimes L(-1)^n w_2 \otimes w_3] + b_1(z_1, z_2)[w_4' \otimes w_1 \otimes L(-1)^{n-1} w_2 \otimes w_3] \\ &\quad + \cdots + b_n(z_1, z_2)[w_4' \otimes w_1 \otimes w_2 \otimes w_3] = 0. \end{aligned} \tag{2.12}$$

Proof. Since R is a Noetherian ring, any R-submodule of the finitely-generated R-module T/J is also finitely generated. In particular, M_1 and M_2 are finitely generated. The second conclusion follows immediately. ∎

2.3 Differential equations of regular singular points and the convergence of the products of intertwining operators

Now we show that the series obtained from the products and iterates of intertwining operators satisfy systems of differential equations.

Theorem 2.9 ([H3])**.** *Let W_1, W_2, W_3, W_4' be C_1-cofinite grading-restricted generalized V-modules. Then for $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$ and $w_4' \in W_4'$, there exist*

$$a_k(z_1, z_2), b_l(z_1, z_2) \in R = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, (z_1 - z_2)^{-1}]$$

for $k = 1, \dots, m$ and $l = 1, \dots, n$ such that for grading-restricted V-modules W_5 and W_6, intertwining operators $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Y}_3$, $\mathcal{Y}_4$ of types $\binom{W_4}{W_1W_5}$, $\binom{W_5}{W_2W_3}$, $\binom{W_4}{W_6W_3}$, $\binom{W_6}{W_1W_2}$, respectively, both series

$$\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle, \tag{2.13}$$
$$\langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3\rangle \tag{2.14}$$

satisfy the expansions of the system of differential equations

$$\frac{\partial^m \varphi}{\partial z_1^m} + a_1(z_1, z_2)\frac{\partial^{m-1}\varphi}{\partial z_1^{m-1}} + \cdots + a_m(z_1, z_2)\varphi = 0, \tag{2.15}$$
$$\frac{\partial^n \varphi}{\partial z_2^n} + b_1(z_1, z_2)\frac{\partial^{n-1}\varphi}{\partial z_2^{n-1}} + \cdots + b_n(z_1, z_2)\varphi = 0 \tag{2.16}$$

in the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively.

Proof. We consider the maps

$$\begin{aligned}
&\phi_{\mathcal{Y}_1,\mathcal{Y}_2} : T \to \mathbb{C}\{z_1, z_2\}[\log z_1, \log z_2],\\
&\psi_{\mathcal{Y}_3,\mathcal{Y}_4} : T \to \mathbb{C}\{z_2, z_1 - z_2\}[\log z_2, \log(z_1 - z_2)],
\end{aligned}$$

defined by

$$\begin{aligned}
&\phi_{\mathcal{Y}_1,\mathcal{Y}_2}(f(z_1, z_2)w_4' \otimes w_1 \otimes w_2 \otimes w_3)\\
&\quad = \iota_{|z_1|>|z_2|>0}(f(z_1, z_2))\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle,\\
&\psi(f(z_1, z_2)w_0 \otimes w_1 \otimes w_2 \otimes w_3)\\
&\quad = \iota_{|z_2|>|z_1-z_2|>0}(f(z_1, z_2))\langle w_4', \mathcal{Y}_4(\mathcal{Y}_3(w_1, z_1 - z_2)w_2, z_2)w_3\rangle,
\end{aligned}$$

respectively, where

$$\begin{aligned}
\iota_{|z_1|>|z_2|>0} &: R \to \mathbb{C}[[z_1^{\pm 1}, z_2^{\pm 1}]],\\
\iota_{|z_2|>|z_1-z_2|>0} &: R \to \mathbb{C}[[z_2^{\pm 1}, (z_1 - z_2)^{\pm 1}]],
\end{aligned}$$

are the maps expanding elements of R as series in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively.

By Proposition 2.5, we have $\phi_{\mathcal{Y}_1,\mathcal{Y}_2}(J) = \psi_{\mathcal{Y}_3,\mathcal{Y}_4}(J) = 0$. Thus we have induced maps

$$\begin{aligned}
&\bar{\phi}_{\mathcal{Y}_1,\mathcal{Y}_2} : T/J \to \mathbb{C}\{z_1, z_2\}[\log z_1, \log z_2],\\
&\bar{\psi}_{\mathcal{Y}_3,\mathcal{Y}_4} : T/J \to \mathbb{C}\{z_2, z_1 - z_2\}[\log z_2, \log(z_1 - z_2)].
\end{aligned}$$

Applying $\bar{\phi}_{\mathcal{Y}_1,\mathcal{Y}_2}$ and $\bar{\psi}_{\mathcal{Y}_3,\mathcal{Y}_4}$ and to (2.11) and (2.12) and then use the $L(-1)$-derivative property for intertwining operators, we see that (2.13) and (2.14) indeed satisfy the expansions of the system the equations (2.15) and (2.16) in the regions $|z_1 > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively. ∎

Remark 2.10. Note that in the theorems above, $a_k(z_1, z_2)$ for $k = 1, \ldots, m-1$ and $b_l(z_1, z_2)$ for $l = 1, \ldots, n-1$, and consequently the corresponding system, are independent of $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Y}_3$ and $\mathcal{Y}_4$.

The following result can be proved using the same method and the proof is omitted:

Theorem 2.11. *Let W_i for $i = 0, \ldots, p+1$ be C_1-cofinite grading-restricted generalized V-modules. Then for any $w_i \in W_i$ for $i = 0, \ldots, p+1$, there exist*

$$a_{k_l,\, l}(z_1, \ldots, z_p) \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_p^{\pm 1}, (z_1 - z_2)^{-1}, (z_1 - z_3)^{-1}, \ldots, (z_{p-1} - z_p)^{-1}], \tag{2.17}$$

for $k_l = 1, \ldots, m_l$ and $l = 1, \ldots, p$, such that for anygrading-restricted V-modules $\widetilde{W}_q$ for $q = 1, \ldots, p-1$, any intertwining operators $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_{p-1}, \mathcal{Y}_p$, of types $\binom{W_0}{W_1 \widetilde{W}_1}$, $\binom{\widetilde{W}_1}{W_2 \widetilde{W}_2}, \ldots,$ $\binom{\widetilde{W}_{p-2}}{W_{p-1} \widetilde{W}_{p-1}}$, $\binom{\widetilde{W}_{p-1}}{W_p W_{p+1}}$, respectively, the series

$$\langle w_0', \mathcal{Y}_1(w_1, z_1) \cdots \mathcal{Y}_p(w_p, z_p) w_{p+1} \rangle \tag{2.18}$$

satisfy the expansions of the system of differential equations

$$\frac{\partial^{m_l} \varphi}{\partial z_l^{m_l}} + \sum_{k_l=1}^{m_l} a_{k_l,\, l}(z_1, \ldots, z_p) \frac{\partial^{m_l - k_l} \varphi}{\partial z_l^{m_l - k_l}} = 0, \quad l = 1, \ldots, p$$

in the region $|z_1| > \cdots |z_p| > 0$.

We also need to prove that the singular points of the differential equations are regular. A singular point of a system of differential equations are said to be *regular* if every solution can be expanded in the neighborhood of the singular point as a series in powers of the variables and a polynomial in the logarithms of the variables.

For a differential equation of one independent variable of the form

$$\frac{d^n}{dz^n}\phi(z) + a_{n-1}(z)\frac{d^{n-1}}{dz^{n-1}}\phi(z) + \cdots + a_0(z)\phi(z) = 0,$$

if $z = z_0$ is an isolated singular point of $a_{n-1}(z), \ldots, a_0(z)$ such that $z = z_0$ is a removable singular point of $(z - z_0)^i a_{n-i}(z)$ for $i = 1, \ldots, n$, then $z = z_0$ must be a regular singular point of this equation.

We need to prove the system of equations satisfied by products and iterates of intertwining operators are of regular singular points. We need only prove each equation as an ordinary differential equation is of regular singular points. This is because we can prove the convergence of the products of intertwining operators using induction.

We prove that the singular point $z_1 = z_2$ of (2.15) is regular. We need certain filtrations associated to the singular point $z_1 = z_2$ on R and on the R-module T.

For $n \in \mathbb{Z}_+$, let $F_n^{(z_1 = z_2)}(R)$ be the vector subspace of R spanned by elements of the form $f(z_1, z_2)(z_1 - z_2)^{-n}$ for $f(z_1, z_2) \in \mathbb{C}[z_1^{\pm}, z_2^{\pm}]$. Then with respect to this filtration, R is a filtered algebra, that is, $F_m^{(z_1=z_2)}(R) \subset F_n^{(z_1=z_2)}(R)$ for $m \le n$, $R = \cup_{n\in\mathbb{Z}} F_n^{(z_1=z_2)}(R)$ and $F_m^{(z_1=z_2)}(R)F_n^{(z_1=z_2)}(R) \subset F_{m+n}^{(z_1=z_2)}(R)$ for any $m, n \in \mathbb{Z}_+$.

For convenience, we shall use σ to denote $\mathrm{wt}\, w_4' + \mathrm{wt}\, w_1 + \mathrm{wt}\, w_2 + \mathrm{wt}\, w_3$, when the dependence on w_4', w_1, w_2 and w_3 are clear. Let $F_r^{(z_1=z_2)}(T)$ for $r \in \mathbb{R}$ be the subspace of T spanned by elements of the form $f(z_1, z_2)(z_1 - z_2)^{-n} w_4' \otimes w_1 \otimes w_2 \otimes w_3$ where $f(z_1, z_2) \in \mathbb{C}[z_1^{\pm}, z_2^{\pm}]$, $n \in \mathbb{Z}_+$ and $w_4' \in W_4'$, $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$ satisfying $n + \sigma \le r$. These subspaces give a filtration of T in the following sense: $F_r^{(z_1=z_2)}(T) \subset F_s^{(z_1=z_2)}(T)$ for $r \le s$; $T = \cup_{r\in\mathbb{R}} F_r^{(z_1=z_2)}(T)$; $F_n^{(z_1=z_2)}(R)F_r^{(z_1=z_2)}(T) \subset F_{r+n}^{(z_1=z_2)}(T)$.

Let $F_r^{(z_1=z_2)}(J) = F_r^{(z_1=z_2)}(T) \cap J$ for $r \in \mathbb{R}$. We need the following refinement of Proposition 2.6:

Proposition 2.12. *For any $r \in \mathbb{R}$, $F_r^{(z_1=z_2)}(T) \subset F_r^{(z_1=z_2)}(J) + F_M(T)$.*

Proof. The proof is a refinement of the proof of Proposition 2.6. The only additional property we need is that the elements $\mathcal{A}(u, w_4', w_1, w_2, w_3)$, $\mathcal{B}(u, w_4', w_1, w_2, w_3)$, $\mathcal{C}(u, w_4', w_1, w_2, w_3)$ and $\mathcal{D}(u, w_4', w_1, w_2, w_3)$ are all in $F_{\mathrm{wt}\, u+\sigma}^{(z_1=z_2)}(J)$. This is clear. ∎

We also consider the ring $\mathbb{C}[z_1^{\pm}, z_2^{\pm}]$ and the $\mathbb{C}[z_1^{\pm}, z_2^{\pm}]$-module

$$T^{(z_1=z_2)} = \mathbb{C}[z_1^{\pm}, z_2^{\pm}] \otimes W_4' \otimes W_1 \otimes W_2 \otimes W_3.$$

Let $T_{(r)}^{(z_1=z_2)}$ for $r \in \mathbb{R}$ be the space of elements of $T^{(z_1=z_2)}$ of weight r. Then $T^{(z_1=z_2)} = \coprod_{r\in\mathbb{R}} T_{(r)}^{(z_1=z_2)}$.

Let $w_4' \in W_4'$, $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$. Then by Proposition 2.12,

$$w_4' \otimes w_1 \otimes w_2 \otimes w_3 = \mathcal{W}_1 + \mathcal{W}_2$$

where $\mathcal{W}_1 \in F_\sigma^{(z_1=z_2)}(J)$ and $\mathcal{W}_2 \in F_M(T)$.

Lemma 2.13. *For any $s \in [0, 1)$, there exist $S \in \mathbb{R}$ such that $s + S \in \mathbb{Z}_+$ and for any $w_4' \in W_4'$, $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$, $\sigma \in s + \mathbb{Z}$, $(z_1 - z_2)^{\sigma+S}\mathcal{W}_2 \in T^{(z_1=z_2)}$.*

Proof. Let S be a real number such that $s + S \in \mathbb{Z}_+$ and such that for any $r \in \mathbb{R}$ satisfying $r \le -S$, $T_{(r)} = 0$. By definition, elements of $F_r^{(z_1=z_2)}(T)$ for any $r \in \mathbb{R}$ are sums of elements of the form $f(z_1, z_2)(z_1 - z_2)^{-n}\tilde{w}_4' \otimes \tilde{w}_1 \otimes \tilde{w}_2 \otimes \tilde{w}_3$ where $f(z_1, z_2) \in \mathbb{C}[z_1^{\pm}, z_2^{\pm}]$, $n \in \mathbb{Z}_+$ and $\tilde{w}_4' \in W_4'$, $\tilde{w}_1 \in W_1$, $\tilde{w}_2 \in W_2$ and $\tilde{w}_3 \in W_3$ satisfying $n + \mathrm{wt}\, \tilde{w}_4' + \mathrm{wt}\, \tilde{w}_1 + \mathrm{wt}\, \tilde{w}_2 + \mathrm{wt}\, \tilde{w}_3 \le r$. Since $\mathrm{wt}\, \tilde{w}_4' + \mathrm{wt}\, \tilde{w}_1 + \mathrm{wt}\, \tilde{w}_2 + \mathrm{wt}\, \tilde{w}_3 > -S$, we obtain $r - n > -S$ or $r + S - n > 0$. Thus $(z_1 - z_2)^{r+S} F_r^{(z_1=z_2)}(T) \in T^{(z_1=z_2)}$ if $r + S \in \mathbb{Z}$.

By definition,

$$\mathcal{W}_2 = w_4' \otimes w_1 \otimes w_2 \otimes w_3 - \mathcal{W}_1,$$

where

$$\mathcal{W}_1 \in F_\sigma^{(z_1=z_2)}(J) \subset F_\sigma^{(z_1=z_2)}(T).$$

By the discussion above, $(z_1-z_2)^{\sigma+S}\mathcal{W}_1 \in T^{(z_1=z_2)}$ and by definition,

$$w'_4 \otimes w_1 \otimes w_2 \otimes w_3 \in T^{(z_1=z_2)}.$$

Thus $(z_1-z_2)^{\sigma+S}\mathcal{W}_2 \in T^{(z_1=z_2)}$. ∎

Theorem 2.14. *Let W_1, W_2, W_3, W'_4 and $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$ and $w'_4 \in W'_4$ be the same as in Theorem 2.9. For any possible singular point of the form $z_1 = 0$ or $z_2 = 0$ or $z_1 = \infty$ or $z_2 = \infty$ or $z_1 = z_2$ or $z_1^{-1}(z_1-z_2) = 0$ or $z_2^{-1}(z_1-z_2) = 0$, there exist*

$$a_k(z_1, z_2) \in \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, (z_1-z_2)^{-1}]$$

for $k = 1, \dots, m$, such that this singular point of the equation (2.15) satisfied by (2.13) and (2.14) is regular.

Proof. We shall only prove the theorem for the singular point $z_1 = z_2$. By Proposition 2.12,

$$w'_4 \otimes L(-1)^k w_1 \otimes w_2 \otimes w_3 = \mathcal{W}_1^{(k)} + \mathcal{W}_2^{(k)}$$

for $k \geq 0$, where $\mathcal{W}_1^{(k)} \in F_{\sigma+k}^{(z_1=z_2)}(J)$ and $\mathcal{W}_2^{(k)} \in F_M(T)$.

By Lemma 2.13, there exists $S \in \mathbb{R}$ such that $\sigma + S \in \mathbb{Z}_+$ and

$$(z_1-z_2)^{\sigma+k+S}\mathcal{W}_2^{(k)} \in T^{(z_1=z_2)}$$

and thus

$$(z_1-z_2)^{\sigma+k+S}\mathcal{W}_2^{(k)} \in \coprod_{r\leq M} T_{(r)}^{(z_1=z_2)}$$

for $k \geq 0$. Since $\mathbb{C}[z_1^{\pm}, z_2^{\pm}]$ is a Noetherian ring and $\coprod_{r\leq M} T_{(r)}^{(z_1=z_2)}$ is a finitely-generated $\mathbb{C}[z_1^{\pm}, z_2^{\pm}]$-module, the submodule of $\coprod_{r\leq M} T_{(r)}^{(z_1=z_2)}$ generated by $(z_1-z_2)^{\sigma+k+S}\mathcal{W}_2^{(k)}$ for $k \geq 0$ is also finitely generated. Let $(z_1-z_2)^{\sigma+k+S}\mathcal{W}_2^{(k)}$ for $k = 0, \dots, m-1$ be a set of generators of this submodule. Then there exist $c_k(z_1, z_2) \in \mathbb{C}[z_1^{\pm}, z_2^{\pm}]$ for $k = 0, \dots, m-1$ such that

$$(z_1-z_2)^{\sigma+m+S}\mathcal{W}_2^{(m)} = -\sum_{k=0}^{m-1} c_k(z_1, z_2)(z_1-z_2)^{\sigma+k+S}\mathcal{W}_2^{(k)}$$

or equivalently

$$\mathcal{W}_2^{(m)} + \sum_{k=0}^{m-1} c_k(z_1, z_2)(z_1-z_2)^{k-m}\mathcal{W}_2^{(k)} = 0.$$

Thus

$$\begin{aligned} &w'_4 \otimes L(-1)^m w_1 \otimes w_2 \otimes w_3 \\ &\quad + \sum_{k=0}^{m-1} c_k(z_1, z_2)(z_1-z_2)^{k-m} w'_4 \otimes L(-1)^k w_1 \otimes w_2 \otimes w_3 \end{aligned}$$

$$= \mathcal{W}_1^{(m)} + \sum_{k=0}^{m-1} c_k(z_1, z_2)(z_1 - z_2)^{k-m} \mathcal{W}_1^{(k)}. \tag{2.19}$$

Since $\mathcal{W}_1^{(k)} \in F_{\sigma+k}^{(z_1=z_2)}(J) \subset J$, the right-hand side of (2.19) is in J. Thus we obtain

$$\begin{aligned}[] [w_4' \otimes L(-1)^m w_1 \otimes w_2 \otimes w_3] \\ + \sum_{k=0}^{m-1} c_k(z_1, z_2)(z_1 - z_2)^{k-m} [w_4' \otimes L(-1)^k w_1 \otimes w_2 \otimes w_3] = 0. \end{aligned} \tag{2.20}$$

Now it is clear that the singular point $z_1 = z_2$ is regular. ∎

Theorem 2.15. *Let W_1, W_2, W_3, W_4' and $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$ and $w_4' \in W_4'$ be the same as in Theorem 2.9. Then (2.13) and (2.14) are absolutely congvergent in the region $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, and can be analyticlally extended to multivalued functions on the region*

$$M^2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1, z_2 \neq 0,\ z_1 \neq z_2\}.$$

Conclusion 1 of Theorem 2.4 follows immediately from Theorem 2.15.

2.4 Tensor product modules and a characterization of intertwining operators

We have proved the convergence of products of intertwining operators. But this is only the first step in the proof of the associativity or operator product expansion of intertwining operators. We still need to show that the analytic extension of products of intertwining operators can be written as iterates of other two intertwining operators. To do this, we first need a characterization of intertwining operators. This characterization can be described using a suitable subspace of the full dual space of the tensor product of the underlying vector spaces of two lower-bounded generalized modules. But this characterization also gives a construction of tensor product of two such modules. Since we do need to discuss the tensor category structure on a suitable module category for a vertex operator agebra, we use the tensor product to discuss the characterization of intertwining operators.

In [H8], for $z \in \mathbb{C}^\times$ and lower-bounded generalized V-modules W_1 and W_2, we have introduced the notion of $P(z)$-tensor product $W_1 \boxtimes_{P(z)} W_2$ of W_1 and W_2. Here we recall the definition in the category of grading-restricted generalized V-modules.

Definition 2.16. Let $z \in \mathbb{C}^\times$ and W_1 and W_2 grading-restricted generalized V-modules. A *$P(z)$-product of W_1 and W_2 in the category of grading-restricted generalized V-modules* is a pair (W_3, I) consisiting of a grading-restricted generalized V-module W_3 and the value $I = \mathcal{Y}(\cdot, z)\cdot : W_1 \otimes W_2 \to \overline{W}_3$ of an intertwining operator $\mathcal{Y}(\cdot, x)\cdot : W_1 \otimes W_2 \to W_3\{x\}[\log x]$ at z (with the choice of the value $\log z = \log |z| + i \arg z$ where $0 \leq \arg z < 2\pi$). A *$P(z)$-tensor product of W_1 and W_2 in the category of grading-restricted generalized V-modules* is a

$P(z)$-product $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$ such that the following universal property holds: Given any $P(z)$-product (W_3, I) of W_1 and W_2 in the category of grading-restricted generalized V-modules, there exists a unique module map $f : W_1 \boxtimes_{P(z)} W_2 \to W_3$ such that $I = \bar{f} \circ boxtimes_{P(z)}$, where $\bar{f} : \overline{W_1 \boxtimes_{P(z)} W_2} \to \overline{W}_3$ is the unique extenstion of f to $\overline{W_1 \boxtimes_{P(z)} W_2}$ (note that f as a module map must preserve wegihts).

We now give a construction of the $P(z)$-tensor product $W_1 \boxtimes_{P(z)} W_2$ in the category of grading-restricted generalized V-modules. For a grading-restricted generalized V-module W_3, $w_3' \in W_3'$ and an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$, we have an element $\lambda^z_{\mathcal{Y}}(w_3') \in (W_1 \otimes W_2)^*$ given by

$$\lambda_{\mathcal{Y}; w_3'}(w_1 \otimes w_2) = \langle w_3', \mathcal{Y}(w_1, z) w_2 \rangle.$$

Then we obtain a linear map $\lambda^z_{\mathcal{Y}} : W_3' \to (W_1 \otimes W_2)^*$. Let $W_1 \boxbslash_{P(z)} W_2$ be the subspace of $(W_1 \otimes W_2)^*$ spanned by all elements of the form $\lambda^z_{\mathcal{Y}}(w_3')$ for a grading-restricted generalized V-module W_3, $w_3' \in W_3'$ and an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$. Then $\lambda^z_{\mathcal{Y}}$ is in fact a linear map from W_3' to $W_1 \boxbslash_{P(z)} W_2$.

We define a vertex operator map

$$\begin{aligned} Y_{W_1 \boxbslash_{P(z)} W_2} : V \otimes (W_1 \boxbslash_{P(z)} W_2) &\to (W_1 \boxbslash_{P(z)} W_2)[[x, x^{-1}]] \\ v \otimes \lambda &\mapsto Y_{W_1 \boxbslash_{P(z)} W_2}(v, x) \lambda \end{aligned}$$

by

$$(Y_{W_1 \boxbslash_{P(z)} W_2}(v, x) \lambda^z_{\mathcal{Y}}(w_3'))(w_1 \otimes w_2) = \langle Y_{W_3}(v, x) w_3', \mathcal{Y}(w_1, z) w_2 \rangle$$

for $w_1 \in W_1$, $w_2 \in W_2$, $w_3' \in W_3'$ and $v \in V$.

Proposition 2.17. *The space $W_1 \boxbslash_{P(z)} W_2$ equipped with $Y_{W_1 \boxbslash_{P(z)} W_2}$ is a generalized V-module.*

Let W_3 be a grading-restricted generalized V-module and $J : W_3' \to W_1 \boxbslash_{P(z)} W_2$ a V-module map. Let $\mathcal{Y}_J(\cdot, z)\cdot : W_1 \otimes W_2 \to \overline{W}_3$ be defined by

$$\langle w_3', \mathcal{Y}_J(w_1, z) w_2 \rangle = (J(w_3'))(w_1 \otimes w_2).$$

Then we define

$$\mathcal{Y}_J(w_1, x) w_2 = x^{L_{W_3}(0)} z^{-L_{W_3}(0)} \mathcal{Y}_J(x^{-L_{W_1}(0)} z^{L_{W_1}(0)} w_1, z) x^{-L_{W_2}(0)} z^{L_{W_2}(0)} w_2.$$

In this way, we obtain a linear map

$$\mathcal{Y}_J : W_1 \otimes W_2 \to W_3\{x\}[\log x].$$

Proposition 2.18. *The linear map $\mathcal{Y}_J$ is an intertwining operator of type $\binom{W_3}{W_1 W_2}$ such that $\mathcal{Y}_{\lambda^z_{\mathcal{Y}}} = \mathcal{Y}$ and $\lambda^z_{\mathcal{Y}_J} = J$.*

This proposition says that an intertwining operator of type $\binom{W_3}{W_1 W_2}$ is equivalent to a V-module map from W_3' to $W_1 \boxbslash_{P(z)} W_2$.

Assume that $W_1 \boxbslash_{P(z)} W_2$ is a grading-restricted generalized V-module. Then its graded dual $(W_1 \boxbslash_{P(z)} W_2)'$ is also a grading-restricted generalized V-module. Consider the identity operator $1_{W_1 \boxbslash_{P(z)} W_2}$ on $W_1 \boxbslash_{P(z)} W_2$. This is certainly a V-module map from the graded dual $W_1 \boxbslash_{P(z)} W_2$ of $(W_1 \boxbslash_{P(z)} W_2)'$ to $W_1 \boxbslash_{P(z)} W_2$. Then by Proposition 2.18, we have an intertwining operator $\mathcal{Y}_{1_{W_1 \boxbslash_{P(z)} W_2}}$ of type $\binom{(W_1 \boxbslash_{P(z)} W_2)'}{W_1 W_2}$. We denote the evaluation $\mathcal{Y}_{1_{W_1 \boxbslash_{P(z)} W_2}}(\cdot, z)$ of $\mathcal{Y}_{1_{W_1 \boxbslash_{P(z)} W_2}}$ at z by $\boxtimes_{P(z)}$.

Theorem 2.19. *The pair $((W_1 \boxbslash_{P(z)} W_2)', \boxtimes_{P(z)})$ is a $P(z)$-tensor product of W_1 and W_2.*

By the universal property of the $P(z)$-tensor product, we know that all $P(z)$-tensor products are naturally isomorphic. Therefore we shall denote $(W_1 \boxbslash_{P(z)} W_2)'$ by $W_1 \boxtimes_{P(z)} W_2$.

The construction above depends on the assumption that $W_1 \boxbslash_{P(z)} W_2$ is a grading-restricted generalized V-module. We have the the following result:

Theorem 2.20 ([HL2])**.** *Assume that V satisfies the following condition:*

1. *There are only finitely many irreducible V-modules (up to equivalence).*
2. *Every (ordinary) V-module is completely reducible (and is in particular a finite direct sum of irreducible modules).*
3. *All the fusion rules for V are finite (for triples of irreducible modules and hence arbitrary modules).*

Then $W_1 \boxbslash_{P(z)} W_2$ is a (ordinary) V-module.

Theorem 2.21 ([H7])**.** *Assume that V is of positive energy ($V_{(n)} = 0$ for $n < 0$ and $V_{(0)} = \mathcal{V}\mathbf{1}$) and C_2-cofinite. Then $W_1 \boxbslash_{P(z)} W_2$ is a grading-restricted generalized V-module.*

Let

$$Y_t(v, x) = \sum_{n \in \mathbb{Z}} (v \otimes t^n) x^{-n-1} \in (V \otimes \mathbb{C}[t, t^{-1}])[[x, x^{-1}]]$$

for $v \in V$.

Note that the coefficients of the formal series

$$x_0^{-1} \delta\left(\frac{x_1^{-1} - z}{x_0}\right) Y_t(v, x_1)$$

in x_0 and x_1 span

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}].$$

Definition 2.22. Define the linear action $\tau_{P(z)}$ of

$$V \otimes \iota_+\mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$$

on $(W_1 \otimes W_2)^*$ by

$$\begin{aligned}
&\left(\tau_{P(z)}\left(x_0^{-1}\delta\left(\frac{x_1^{-1} - z}{x_0}\right)Y_t(v, x_1)\right)\lambda\right)(w_{(1)} \otimes w_{(2)})\\
&\quad = z^{-1}\delta\left(\frac{x_1^{-1} - x_0}{z}\right)\lambda(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)})\\
&\qquad + x_0^{-1}\delta\left(\frac{z - x_1^{-1}}{-x_0}\right)\lambda(w_{(1)} \otimes Y_2^o(v, x_1)w_{(2)})
\end{aligned}\tag{2.21}$$

for $\xi \in V \otimes \iota_+\mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$.

Let $L_{W_1 \boxtimes_{P(z)} W_2}(n)$ for $n \in \mathbb{Z}$ be the operators on $W_1 \boxtimes_{P(z)} W_2$ defined by

$$\tau_{P(z)}(Y_t(\omega, x)) = \sum_{n\in\mathbb{Z}} L_{W_1 \boxtimes_{P(z)} W_2}(n)x^{-n-2}.$$

From the definition of $\tau_{P(z)}$ and $Y_{W_1 \boxtimes_{P(z)} W_2}$, we have

$$Y_{W_1 \boxtimes_{P(z)} W_2}(v, x) = \tau_{P(z)}(Y_t(v, x))\Big|_{W_1 \boxtimes_{P(z)} W_2}\tag{2.22}$$

for $v \in V$.

The element $\lambda_{\mathcal{Y}}^z(w_3') \in W_1 \boxtimes_{P(z)} W_2 \subset (W_1 \otimes W_2)^*$ satisfies the following properties for $\lambda \in (W_1 \otimes W_2)^*$:

$P(z)$-compatibility condition

(a) The Laurent series $Y_{W_1 \boxtimes_{P(z)} W_2}(v, x)\lambda$ involves only finitely many negative powers of x.

(b) The following formula holds:

$$\tau_{P(z)}\left(x_0^{-1}\delta\left(\frac{x_1^{-1} - z}{x_0}\right)Y_t(v, x_1)\right)\lambda = x_0^{-1}\delta\left(\frac{x_1^{-1} - z}{x_0}\right)Y_{W_1 \boxtimes_{P(z)} W_2}(v, x_1)\lambda\tag{2.23}$$

for all $v \in V$.

The $P(z)$-local grading restriction condition

(a) The $P(z)$-*grading condition*: λ is a (finite) sum of generalized eigenvectors for the operator $L_{W_1 \boxtimes_{P(z)} W_2}(0)$ on $(W_1 \otimes W_2)^*$.

(b) Let W_λ be the smallest graded subspace of $(W_1 \otimes W_2)^*$ (with respect to $L_{W_1 \boxtimes_{P(z)} W_2}(0)$) containing λ and stable under the component operators of $Y_{W_1 \boxtimes_{P(z)} W_2}(v, x)$ for $v \in V$. Then W_λ has the properties

$$\dim (W_\lambda)_{[n]} < \infty, \tag{2.24}$$

$$(W_\lambda)_{[n+k]} = 0, \tag{2.25}$$

for $k \in \mathbb{Z}$ sufficiently negative, $n \in \mathbb{C}$.

Theorem 2.23. *Assume that $W_1 \boxtimes_{P(z)} W_2$ is a grading-restricted generalized V-module. Then an element $\lambda \in (W_1 \otimes W_2)^*$ satisfying the $P(z)$-compatibility condition and the $P(z)$-local grading-restriction condition is in $W_1 \boxtimes_{P(z)} W_2$, that is, it is of the form $\lambda^z_{\mathcal{Y}}(w_3')$, where W_3 is a grading-restricted generalized V-module, $w_3' \in W_3'$ and $\mathcal{Y}$ an intertwining operator of type $\binom{W_3}{W_1 W_2}$.*

The proof of this theorem is rather technical. We will not give the proof here. The reader can find a proof in [HL4] using the results in [HL2] and [HL3] and a direct proof in Section 6 in [HLZ1].

2.5 The proof of the associativity of intertwining operators

We now outline the proof of Conclusion 2 of Theorem 2.4.

Let W_1, W_2, W_3, W_4, W_5 be grading-restricted generalized V-modules and $\mathcal{Y}_1$ and $\mathcal{Y}_2$ intertwining operators of types $\binom{W_4}{W_1\, W_5}$ and $\binom{W_5}{W_2\, W_3}$, respectively. $w_1 \in W_1, w_2 \in W_2$, $w_3 \in W_3$ and $w_4' \in W_4'$,

$$\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle$$

is absolutely convergent in the region $|z_1| > |z_2| > 0$ and can be analytically extended to a multivalued function

$$F(\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle)$$

on the region

$$M_2 = \{(z_1, z_2) \mid z_1, z_2 \neq 0, z_1 - z_2 \neq 0\}.$$

For fixed $z_1, z_2 \in \mathbb{C}^\times$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$, let

$$\mu_{w_4', w_3}(w_1, w_2) = \langle w_4', \mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(w_2, x_2)w_3\rangle.$$

Then for fixed w_4', w_3, we obtain an element $\mu_{w_4', w_3} \in (W_1 \otimes W_2)^*$.

Proposition 2.24. *The element μ_{w_4', w_3} satisfies the $P(z_1 - z_2)$-compatibility condition.*

But μ_{w_4', w_3} does not satisfy the $P(z_1 - z_2)$-local grading restriction condition. Since our differential equations are of regular singular points, the multivalued analytic function

$$F(\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle)$$

can be expanded in the region $|z_2| > |z_1 - z_2| > 0$ as a series of the from

$$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{n_1,n_2\in\mathbb{N}}\sum_{k_1,k_2=0}^{K} a_{i,,jn_1,n_2,k_1,k_2}(w_4', w_1, w_2, w_3)z_2^{r_i+n_1}(\log z_2)^{k_1}(z_1-z_2)^{s_j+n_2}(\log(z_1-z_2))^{k_2},$$

where $r_i, s_j \in \mathbb{C}$. Let

$$\mu_{w_4',w_3;s_j+n_2}(w_1, w_2) = \sum_{i=1}^{N}\sum_{n_1\in\mathbb{N}}\sum_{k_2,k_{12}=0}^{K} a_{i,j,n_1,n_2,k_1,k_2}(w_4', w_1, w_2, w_3)z_2^{r_i+n_1}(\log z_2)^{k_1}(z_1-z_2)^{s_i+n_2}(\log(z_1-z_2))^{k_2}$$

for $j = 1, \ldots, J$ and $n_2 \in \mathbb{N}$. Then for fixed w_4', w_3 and $j = 1, \ldots, J$, $n_2 \in \mathbb{N}$, we obtain an element $\mu_{w_4',w_3;s_j+n_2} \in (W_1 \otimes W_2)^*$. Moreover, we have

Proposition 2.25. *The element $\mu_{w_4',w_3;s_j+n_2}$ satisfies the $P(z_1 - z_2)$-compatibility condition and $P(z_1 - z_2)$-local grading restriction condition.*

Under the conditions in Theorem 2.4, we know that $W_1 \boxtimes_{P(z)} W_2$ is a grading-restricted generalized V-module. Then by Theorem 2.23,

$$\mu_{w_4',w_3;s_j+n_2} \in W_1 \boxtimes_{P(z_1-z_2)} W_2.$$

Moreover, it is easy to calculate to see that $\mu_{w_4',w_3;s_j+n_2}$ is homogeneous. By definition, we have

$$\mu_{w_4',w_3} = \sum_{j=1}^{J}\sum_{n_2\in\mathbb{N}} \mu_{w_4',w_3;s_j+n_2}.$$

Thus we see that $\mu_{w_4',w_3} \in \overline{W_1 \boxtimes_{P(z)} W_2}$.

As we have discussed above, the identity operator $1_{W_1 \boxtimes_{P(z_1-z_2)} W_2}$ on $W_1 \boxtimes_{P(z_1-z_2)} W_2$ gives an intertwining operator $\mathcal{Y}_4$ of type $\binom{W_1\boxtimes_{P(z_1-z_2)}W_2}{W_1W_2}$ such that for $w_1 \in W_1$, $w_2 \in W_2$, $\lambda \in W_1 \boxtimes_{P(z_1-z_2)} W_2$,

$$\langle \lambda, \mathcal{Y}_4(w_1, z)w_2\rangle = \lambda(w_1 \otimes w_2).$$

Then we have

$$\langle \mu_{w_4',w_3;s_j+n_2}, \mathcal{Y}_4(w_1, z)w_2\rangle = \mu_{w_4',w_3;s_j+n_2}(w_1, w_2).$$

Taking sum over j and n_2, we obtain

$$\langle \mu_{w_4',w_3}, \mathcal{Y}_4(w_1, z)w_2\rangle = \mu_{w_4',w_3}(w_1, w_2) = \langle w_4', \mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(w_2, x_2)w_3\rangle.$$

On the other hand, for any element $w \in W_1 \boxtimes_{P(z_1-z_2)} W_2$, we have an element $\nu_w \in (W_4' \otimes W_3)^*$ defined by

$$\nu_w(w_4' \otimes w_3) = \langle \mu_{w_4',w_3}, w\rangle$$

for $w_4' \in W_4'$ and $w_3 \in W_3$. Then we have

Proposition 2.26. *The element ν_w satisfies the $Q(z_2)$-compatibility condition and $Q(z_2)$-local grading-restricted condition.*

Here the $Q(z_2)$-compatibility condition and $Q(z)$-local grading-restriction condition are analogous to the $P(z_2)$-compatibility condition and $Q(z)$-local grading-restriction condition, respectively. But $Q(z_2)$ is the conformal equivalence class of the sphere $\mathbb{C} \cup \{\infty\}$ with the negatively oriented puncture at z and the positively oriented punctures at 0 and ∞ and with the standard local coordinates. We also have a theorem completely analogous to Theorem 2.23 stating that an element of $(W_4' \otimes W_3)^*$ satisfying the $Q(z_2)$-compatibility condition and $Q(z)$-local grading-restriction condition must be an element obtained using an intertwining operator $\mathcal{Y}$ of the type $\binom{W_4}{WW_3}$ for some grading-restricted generalized V-module W by

$$w_4' \otimes w_3 \mapsto \langle w_4', \mathcal{Y}(w, z_2)w_3\rangle.$$

Here we omit the details. Using this result, we see that there exists an intertwining operator $\mathcal{Y}_3$ of the type $\binom{W_4}{(W_1 \boxtimes_{P(z_1-z_2)} W_2)\ W_3}$ such that

$$\nu_w(w_4' \otimes w_3) = \langle w_4', \mathcal{Y}_3(w, z_2)w_3\rangle.$$

Thus we have

$$\begin{aligned} \langle w_4', \mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(w_2, x_2)w_3\rangle &= \langle \mu_{w_4', w_3}, \mathcal{Y}_4(w_1, z)w_2\rangle \\ &= \nu_{\mathcal{Y}_4(w_1, z)w_2}(w_4' \otimes w_3) \\ &= \langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z)w_2, z_2)w_3\rangle. \end{aligned}$$

This is the associativity of intertwining operators at the point (z_1, z_2).

3 Tensor categories

We review the basic concepts and properties in the theory of tensor categories in this section. The main references for this section are [J], [M], [T] and [EGNO].

3.1 Basic concepts in category theory

Definition 3.1. A *category* consists of the following data:

1. A collection of *objects*.

2. For two objects A and B, a set $\mathrm{Hom}(A, B)$ of *morphisms from A to B*.

3. For an object A, an *identity* $1_A \in \mathrm{Hom}(A, A)$.

4. For three objects A, B, C, a map

$$\begin{aligned} \circ : \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) &\to \mathrm{Hom}(A, C) \\ (f, g) &\mapsto f \circ g \end{aligned}$$

called *composition* or *multiplication*.

These data must satisfy the following axioms:

1. The composition is associative, that is, for objects A, B, C, D and $f \in \mathrm{Hom}(C, D)$, $g \in \mathrm{Hom}(B, C)$, $h \in \mathrm{Hom}(A, B)$, we have $f \circ (g \circ h) = (f \circ g) \circ h$.

2. For an object A, the identity 1_A is the identity for the composition of morphisms when the morphisms involving A, that is, for an object B, $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, A)$, we have $1_A \circ g = g$ and $f \circ 1_A = f$.

We shall use $\mathcal{C}$, $\mathcal{D}$ and so on to denote categories. For a category $\mathcal{C}$, we use $\mathrm{Ob}\,\mathcal{C}$ to denote the collection of objects of $\mathcal{C}$.

Definition 3.2. Let $\mathcal{C}$ be a category. For any $A, B \in \mathrm{Ob}\,\mathcal{C}$, an element $f \in \mathrm{Hom}(A, B)$ is called an isomorphism if there exists $f^{-1} \in \mathrm{Hom}(B, A)$ such that $f \circ f^{-1} = 1_B$ and $f^{-1} \circ f = 1_A$.

Definition 3.3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant functor* (or a *contravariant functor*) from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data:

1. A map $\mathcal{F}$ from the collection $\mathrm{Ob}\,\mathcal{C}$ of objects of $\mathcal{C}$ to the collection $\mathrm{Ob}\,\mathcal{D}$ of objects of $\mathcal{D}$.

2. Given objects A and B of $\mathcal{C}$, a map, still denoted by $\mathcal{F}$, from $\mathrm{Hom}(A, B)$ to $\mathrm{Hom}(\mathcal{F}(A), \mathcal{F}(B))$ (or from $\mathrm{Hom}(A, B)$ to $\mathrm{Hom}(\mathcal{F}(B), \mathcal{F}(A))$ for a contravariant functor).

These data must satisfy the following axioms:

1. For objects A, B, C of $\mathcal{C}$ and morphisms $f \in \mathrm{Hom}(B, c)$, $g \in \mathrm{Hom}(A, B)$, we have
$$\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$$
(or
$$\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f)$$
for a contravariant functor).

2. For an object A of $\mathcal{C}$, $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$.

We shall denote the functor defined above by $\mathcal{F}$.

Definition 3.4. Let $\mathcal{F}$ and $\mathcal{G}$ be functors from $\mathcal{C}$ to $\mathcal{D}$. A *natural transformation η from $\mathcal{F}$ to $\mathcal{G}$* consists of an element $\eta_A \in \mathrm{Hom}(\mathcal{F}(A), \mathcal{G}(A))$ for each object $A \in \mathrm{Ob}\,\mathcal{C}$ such that the following diagram is commutative for $A, B \in \mathrm{Ob}\,\mathcal{C}$ and $f, g \in \mathrm{Hom}(A, B)$:

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\eta_A"] \arrow[d, "\mathcal{F}(f)"'] & \mathcal{G}(A) \arrow[d, "\mathcal{F}(g)"] \\
\mathcal{F}(B) \arrow[r, "\eta_B"'] & \mathcal{G}(B).
\end{tikzcd}$$

A *natural isomorphism* from $\mathcal{C}$ to $\mathcal{D}$ is a natural transformation η from $\mathcal{C}$ to $\mathcal{D}$ such that $\eta_A \in \mathrm{Hom}(\mathcal{F}(A), \mathcal{G}(A))$ for each object $A \in \mathrm{Ob}\,\mathcal{C}$ is an isomorphism.

Definition 3.5. Let $\mathcal{F}$ be a functor from a category $\mathcal{C}$ to a category $\mathcal{D}$ and $\mathcal{G}$ a functor from a category $\mathcal{D}$ to a category $\mathcal{E}$. The *composition* $\mathcal{G} \circ \mathcal{F}$ of $\mathcal{G}$ and $\mathcal{F}$ is a functor from $\mathcal{C}$ to $\mathcal{E}$ given by $(\mathcal{G} \circ \mathcal{F})(A) = \mathcal{G}(\mathcal{F}(A))$ for $A \in \mathrm{Ob}\,\mathcal{C}$ and $(\mathcal{G} \circ \mathcal{F})(f) = \mathcal{G}(\mathcal{F}(f))$ for $f \in \mathrm{Hom}(A, B)$ and $A, B \in \mathrm{Ob}\,\mathcal{C}$. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We say that $\mathcal{C}$ *is isomorphic to* $\mathcal{D}$ if there is a functor $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$ and a functor $\mathcal{F}^{-1}$ such that $\mathcal{F} \circ \mathcal{F}^{-1} = 1_{\mathcal{D}}$ and $\mathcal{F}^{-1} \circ \mathcal{F} = 1_{\mathcal{C}}$. We say that $\mathcal{C}$ *is equivalent to* $\mathcal{D}$ if there is a functor $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$ and a functor $\mathcal{G}$ such that $\mathcal{F} \circ \mathcal{G}$ is naturally isomorphic to $1_{\mathcal{D}}$ and $\mathcal{G} \circ \mathcal{F}$ is naturally isomorphic to $1_{\mathcal{C}}$.

Definition 3.6. Let A_j for $j \in I$ be objects of a category $\mathcal{C}$. A *product* of A_j for $j \in I$ is an object $\prod_{j \in I} A_j$ together with morphisms $p_j : \prod_{j \in I} A_j \to A_j$ satisfying the following universal property: For any object A of $\mathcal{C}$ and any morphism $f_j : A \to A_i$, there exists a unique morphism $f : A \to \prod_{j \in I} A_j$ such that such that $f_j = p_j \circ f$ for $i \in I$. A *coproduct* of A_j for $j \in J$ is an object $\coprod_{j \in I} A_j$ together with morphisms $i_j : A_j \to \prod_{j \in I} A_j$ satisfying the following universal property: For any object A of $\mathcal{C}$ and any morphism $f_j : A_j \to A$, there exists a unique morphism $f : \prod_{j \in I} A_i \to A$ such that $f_j = f \circ i_j$ for $i \in I$.

Exercise 3.7. Prove that products and coproducts of objects A_j for $j \in I$ in a category $\mathcal{C}$ are unique up to isomorphisms.

Definition 3.8. An *initial object* in a category $\mathcal{C}$ is an object I in $\mathcal{C}$ such that for any object X in $\mathcal{C}$, $\mathrm{Hom}(I, X)$ has one and only one element. An *terminal object* in a category $\mathcal{C}$ is an object T in $\mathcal{C}$ such that for any object X in $\mathcal{C}$, $\mathrm{Hom}(X, T)$ has one and only one element. A *zero object* in a category $\mathcal{C}$ is both an initial object and a terminal object.

Definition 3.9. Let $\mathcal{C}$ be a category containing a zero object 0. Let A and B be objects of $\mathcal{C}$ and let $f \in \mathrm{Hom}(A, B)$. A *kernel* of f is an object K and a morphism $k \in \mathrm{Hom}(K, A)$ satisfying $f \circ k = 0$ and the following universal property: For any object K' and morphism $k' \in \mathrm{Hom}(K', A)$ satisfying $f \circ k' = 0$, there exists a unique $g \in \mathrm{Hom}(K', K)$ such that $k' = k \circ g$. A *cokernel* of f is an object Q and a morphism $q \in \mathrm{Hom}(B, Q)$ satisfying $q \circ f = 0$ and the following universal property: For any object Q' and morphism $q' \in \mathrm{Hom}(B, Q')$ satisfying $q' \circ f = 0$, there exists a unique $u \in \mathrm{Hom}(Q, Q')$ such that $q' = u \circ q$.

Exercise 3.10. Prove that kernels and cokernels of of a morphism are unique up to isomorphisms.

Definition 3.11. Let $\mathcal{C}$ be a category containing a zero object 0. Let $A_1, \ldots, A_n$ be objects of $\mathcal{C}$. A *biproduct* of $A_1, \ldots, A_n$ is an object $A_1 \oplus \cdots \oplus A_n$ of $\mathcal{C}$ and $p_k : A_1 \oplus \cdots \oplus A_n \to A_k$ and $i_k : A_k \to A_1 \oplus \cdots \oplus A_n$ for $k = 1, \ldots, n$ such that $p_k \circ i_k = 1_{A_k}$ for $k = 1, \ldots, n$, $p_l \circ i_k = 0$ for $l \neq k$, $A_1 \oplus \cdots \oplus A_n$ equipped with p_k for $k = 1, \ldots, n$ is a product of $A_1, \ldots, A_n$ and $A_1 \oplus \cdots \oplus A_n$ equipped with i_k for $k = 1, \ldots, n$ is a coproduct of $A_1, \ldots, A_n$.

Definition 3.12. Let $\mathcal{C}$ be a category. Let A and B be objects of $\mathcal{C}$. A morphism $f \in \mathrm{Hom}(A, B)$ is said to be a *monomorphism* if for any object C and any $g_1, g_2 \in \mathrm{Hom}(C, A)$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$. A morphism $f \in \mathrm{Hom}(A, B)$ is said to be aN *epimorphism* if for any object C and any $g_1, g_2 \in \mathrm{Hom}(B, C)$, $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$.

Definition 3.13. An *abelian category* is a category $\mathcal{C}$ satisfying the following conditions:

1. For any objects A and B, $\mathrm{Hom}(A, B)$ is an abelian group and for any objects A, B and C, the map from $\mathrm{Hom}(B, A) \times \mathrm{Hom}(C, B)$ to $\mathrm{Hom}(C, A)$ given by the composition of morphisms is bilinear.

2. Every finite set of objects has a biproduct.

3. Every morphism has a kernel and cokernel.

4. Every monomorphism is a kernel of some morphism and every epimorphism is a cokernel of some morphism.

3.2 Monoidal categories and tensor categories

Definition 3.14. An *monoidal category* consists of the following data:

1. A category $\mathcal{C}$.

2. A bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *tensor product bifunctor*.

3. A natural isomorphism $\mathcal{A}$ from $\otimes \circ (1_{\mathcal{C}} \times \otimes)$ to $\otimes \circ (\otimes \times 1_{\mathcal{C}})$ called the *associativity isomorphism*.

4. An object $\mathbf{1}$ called the *unit object*.

5. A natural isomorphism l from $\mathbf{1} \otimes \cdot$ to $1_{\mathcal{C}}$ called the *left unit isomorphism* and a natural isomorphism r from $\cdot \otimes \mathbf{1}$ to $1_{\mathcal{C}}$ called the *right unit isomorphism*.

These data satisfy the following axioms:

1. The following *pentagon diagram* is commutative for objects A_1, A_2, A_3, A_4:

$$\begin{tikzcd}
 & A_1 \otimes (A_2 \otimes (A_3 \otimes A_4)) \arrow[dl] \arrow[dr] & \\
(A_1 \otimes A_2) \otimes (A_3 \otimes A_4) \arrow[d] & & A_1 \otimes ((A_2 \otimes A_3) \otimes A_4) \arrow[d] \\
((A_1 \otimes A_2) \otimes A_3) \otimes A_4 & & (A_1 \otimes (A_2 \otimes A_3)) \otimes A_4 \arrow[ll]
\end{tikzcd}$$

2. The following *triangle diagram* is commutative for objects A_1, A_2:

$$\begin{tikzcd}
(A_1 \otimes \mathbf{1}) \otimes A_2 \arrow[r] \arrow[d] & A_1 \otimes (\mathbf{1} \otimes A_2) \arrow[d] \\
A_1 \otimes A_2 \arrow[r, "="'] & A_1 \otimes A_2.
\end{tikzcd}$$

Definition 3.15. A *tensor category* is an abelian category equipped with a monoidal category structure such that the abelain category structure and the monoidal category structure are compatible in the sense that for objects A, B, C and D, the map $\otimes : \mathrm{Hom}(A, B) \times \mathrm{Hom}(C, D) \to \mathrm{Hom}(A \times C, B \otimes D)$ is bilinear.

Definition 3.16. Let $\mathcal{C}$ be a monoidal category. A *graph diagram in* $\mathcal{C}$ is a graph whose vertices are functors obtained from the tensor product bifunctor and the unit objects and the edges are natural isomorphisms obtained from the associativity isomorphisms, the left and the right unit isomorphisms. A graph diagram is *commutative* if the compositions of the isorphisms in any two paths with the same starting and ending vertices must be equal.

Theorem 3.17 (Mac Lane)**.** *Let $\mathcal{C}$ be a monoidal category. Any graph diagram in $\mathcal{C}$ is commutative*

We omit the proof here; see [M] and [EGNO].

Definition 3.18. A *monoidal functor* from a monoidal categoory $\mathcal{C}$ to a monoidal category $\mathcal{D}$ is a triple $(\mathcal{F}, J, \varphi)$ where $\mathcal{F}$ is a functor from $\mathcal{C}$ to $\mathcal{D}$, J a natural transformation from the functor $\mathcal{F}(\cdot) \otimes_{\mathcal{D}} \mathcal{F}(\cdot)$ to the functor $\mathcal{F}(\cdot \otimes_{\mathcal{C}} \cdot)$ and φ an isomorphism from $\mathbf{1}_{\mathcal{D}}$ to $\mathcal{F}(\mathbf{1}_{\mathcal{C}})$ such that the diagram

$$\begin{array}{ccc}
(\mathcal{F}(A_1) \otimes_{\mathcal{D}} \mathcal{F}(A_2)) \otimes_{\mathcal{D}} \mathcal{F}(A_3) & \longrightarrow & \mathcal{F}(A_1) \otimes_{\mathcal{D}} (\mathcal{F}(A_2) \otimes_{\mathcal{D}} \mathcal{F}(A_3)) \\
\downarrow & & \downarrow \\
(\mathcal{F}(A_1 \otimes_{\mathcal{C}} A_2) \otimes_{\mathcal{D}} \mathcal{F}(A_3) & & \mathcal{F}(A_1) \otimes_{\mathcal{D}} \mathcal{F}(A_2 \otimes_{\mathcal{C}} A_3) \\
\downarrow & & \downarrow \\
(\mathcal{F}((A_1 \otimes_{\mathcal{C}} A_2) \otimes_{\mathcal{C}} A_3) & \longrightarrow & \mathcal{F}(A_1 \otimes_{\mathcal{C}} (A_2 \otimes_{\mathcal{C}} A_3))
\end{array}$$

for objects A_1, A_2 and A_3 in $\mathcal{C}$ and the diagram

$$\begin{array}{ccc}
\mathbf{1}_{\mathcal{D}} \otimes_{\mathcal{D}} \mathcal{F}(A) & \longrightarrow & \mathcal{F}(A) \\
\downarrow & & \uparrow \\
\mathcal{F}(\mathbf{1}_{\mathcal{C}} \otimes_{\mathcal{D}} \mathcal{F}(A) & \longrightarrow & \mathcal{F}(\mathbf{1}_{\mathcal{C}} \otimes_{\mathcal{C}} A)
\end{array}$$

for an object A in $\mathcal{C}$ are commutative. A *monoidal eqivalence* from a monoidal categoory $\mathcal{C}$ to a monoidal category $\mathcal{D}$ is a monoidal functor $(\mathcal{F}, J, \varphi)$ from $\mathcal{C}$ to $\mathcal{D}$ such that $\mathcal{F}$ is an equivalence of categories and J is a natural isomorphism.

Definition 3.19. A monoidal category is *strict* if

$$\begin{aligned}
\otimes \circ (1_{\mathcal{C}} \times \otimes) &= \otimes \circ (\otimes \times 1_{\mathcal{C}}), \\
\mathbf{1} \otimes \cdot &= 1_{\mathcal{C}} \\
\cdot \otimes \mathbf{1} &= 1_{\mathcal{C}}
\end{aligned}$$

and the associativity, the left and the right unit isomorphisms are identities.

Theorem 3.20 (Mac Lane)**.** *Any monoidal category is monoidal equivalent to a strict monoidal category.*

Exercise 3.21. Consider the category of bimodules for an associative algebra and the tensor product bifunctor we defined in the section on associative algebras. Show that there exists an associaitivity isomorphism such that the pentagon diagram is commutative.

3.3 Symmetries and braidings

Definition 3.22. Let $\mathcal{C}$ be a monoidal category. A *symmetry* of $\mathcal{C}$ is a natural isomorphism C from $\otimes$ to $\otimes \circ \sigma_{12}$ (σ_{12} being the functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C} \times \mathcal{C}$ induced from the nontrivial element of S_2) such that for objects A_1, A_2, the morphism

$$C_{A_2,A_1} \circ C_{A_1,A_2} : A_1 \otimes A_2 \to A_2 \otimes A_1 \to A_1 \otimes A_2$$

is equal to the identity $1_{A_1 \otimes A_2}$ and for objects A_1, A_2 and A_2, the *hexagon diagram*

$$\begin{tikzcd}
& (A_1 \otimes A_2) \otimes A_3 \arrow[dl] \arrow[dr] & \\
(A_2 \otimes A_1) \otimes A_3 \arrow[d] & & A_1 \otimes (A_2 \otimes A_3) \arrow[d] \\
A_2 \otimes (A_1 \otimes A_3) \arrow[dr] & & (A_2 \otimes A_3) \otimes A_1 \arrow[dl] \\
& A_2 \otimes (A_3 \otimes A_1) &
\end{tikzcd}$$

is commutative. A *symmetric monoidal category* is a monoidal category with a symmetry. A *symmetric tensor category* is a tensor category with a symmetry.

Definition 3.23. Let $\mathcal{C}$ be a monoidal category. A *braiding* of $\mathcal{C}$ is a natural isomorphism R from $\otimes$ to $\otimes \circ \sigma_{12}$ such that for objects A_1, A_2 and A_2, the *hexagon diagrams*

$$
\begin{tikzcd}
& (A_1 \boxtimes A_2) \boxtimes A_3 \arrow[dl, "R^{\pm 1} \boxtimes 1_{A_3}"'] \arrow[dr] & \\
(A_2 \boxtimes A_1) \boxtimes A_3 \arrow[d] & & A_1 \boxtimes (A_2 \boxtimes A_3) \arrow[d, "R^{\pm 1}"] \\
A_2 \boxtimes (A_1 \boxtimes A_3) \arrow[dr, "1_{A_2} \boxtimes R^{\pm 1}"'] & & (A_2 \boxtimes A_3) \boxtimes A_1 \arrow[dl] \\
& A_2 \boxtimes (A_3 \boxtimes A_1) &
\end{tikzcd}
$$

is commutative. A *braided monoidal category* is a monoidal category with a braiding. A *braided tensor category* is a tensor category with a braiding.

3.4 Rigidity

Definition 3.24. Let $\mathcal{C}$ be a monoidal category. For an object A, a *right dual* of A is an object A^* and morphisms $\mathrm{ev}_A : A^* \otimes A \to \mathbf{1}$ and $\mathrm{coev}_A : \mathbf{1} \to A \otimes A^*$ such that the morphism obtained by composing the morphisms in

$$A \to \mathbf{1} \otimes A \to (A \otimes A^*) \otimes A \to A \otimes (A^* \otimes A) \to A \otimes \mathbf{1} \to A$$

is equal to the identity 1_A and the morphism obtained by composing the morphisms in

$$A^* \to A^* \otimes \mathbf{1} \to A^* \otimes (A \otimes A^*) \to (A^* \otimes A) \otimes A^* \to \mathbf{1} \otimes A^* \to A^*$$

is equal to the identity 1_{A^*}. A *left dual* of A is an object ${}^*A$ and morphisms $\mathrm{ev}'_A : A \otimes^* A \to \mathbf{1}$ and $\mathrm{coev}'_A : \mathbf{1} \to^* A \otimes A$ such that the morphism obtained by composing the morphisms in

$$A \to A \otimes \mathbf{1} \to A \otimes ({}^*A \otimes A) \to (A \otimes^* A) \otimes A \to \mathbf{1} \otimes A \to A$$

is equal to the identity 1_A and the morphism obtained by composing the morphisms in

$${}^*A \to \mathbf{1} \otimes^* A \to ({}^*A \otimes A) \otimes^* A \to^* A \otimes (A \otimes^* A) \to^* A \otimes \mathbf{1} \to^* A$$

is equal to the identity $1_{{}^*A}$.

Definition 3.25. A monoidal category $\mathcal{C}$ is said to be *rigid* if there are contravariant functors $^*\cdot : \mathcal{C} \to \mathcal{C}$ and $\cdot^* : \mathcal{C} \to \mathcal{C}$ such that for an object A, $^*A$ and A^* are left and right duals of A.

Exercise 3.26. Show that the category of finite-dimensional representations for a finite group and the category of finite-dimensional modules for a finite-dimensional Lie algebra are rigid symmetric tensor categories.

3.5 Ribbon categories and modular tensor categories

Definition 3.27. Let $\mathcal{C}$ be a braided onoidal category. A *twist* of $\mathcal{C}$ is a natural isomorphism $\theta : 1_{\mathcal{C}} \to 1_{\mathcal{C}}$ such that for objects A_1 and A_2,

$$\theta_{A_1 \otimes A_2} = R_{A_2,A_1} \circ R_{A_1,A_2} \circ (\theta_{A_1} \otimes \theta_{A_2}).$$

Definition 3.28. A *ribbon category* is a rigid braided monoidal category equipped with a twist.

Lemma 3.29. *In a ribbon category, the left dual and right dual can be taken to be the same.*

We omit the proof of this lemma.

Let $\mathcal{C}$ be a ribbon category and let $K = \mathrm{Hom}(\mathbf{1}, \mathbf{1})$. Then K is a monoid (a set with an associative product and an identity).

Lemma 3.30. *K is in fact commutative.*

In a ribbon category, we can define the "trace" of a morphism and the "dimension" of an object as follows:

Definition 3.31. Let $f \in \mathrm{Hom}(A, A)$ be a morphism in a ribbon category. The *trace* of f is defined to be

$$\mathrm{Tr}\, f = \mathrm{ev}_A \circ R_{A,A^*} \circ ((\theta_A \circ f) \otimes 1_{A^*}) \circ \mathrm{coev}_A \in K.$$

The dimension $\dim A$ of an object A is defined to be $\mathrm{Tr}\ 1_A$.

The trace of a morphism satisfies the properties that a trace should have.

Proposition 3.32. *Let $\mathcal{C}$ be a ribbon category. Then we have:*

1. *For $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, A)$, $\mathrm{Tr}\ fg = \mathrm{Tr}\ gf$.*

2. *For $f \in \mathrm{Hom}(A_1, A_2)$ and $g \in \mathrm{Hom}(A_3, A_4)$, $\mathrm{Tr}\ (f \otimes g) = (\mathrm{Tr}\ f)(\mathrm{Tr}\ g)$.*

3. *For $k \in K$, $\mathrm{Tr}\ k = k$.*

Example 3.33. The category of finite-dimensional representations of a finite group and the category of finite-dimensional modules for a finite-dimensional Lie algebra are ribbon categories whose braidings and twists are trivial.

Example 3.34. Let G be an mulplicative abelian group (an abelian group whose operation is written as a multiplication instead of an addition), K a commutative ring with identity and $c : G \times G \to K^*$ a bilinear form (K^* being the set of invertible elements of K), that is, for $g, g', h, h' \in G$, we have

$$\begin{aligned} c(gg', h) &= c(g, h)c(g', h), \\ c(g, hh') &= c(g, h)c(g, h'). \end{aligned}$$

We construct a ribbon category as follows: The objects of the category are elements of G. For any $g, h \in G$, $\mathrm{Hom}(g, h)$ is K if $g = h$ and 0 if $g \neq h$. The composition of two morphisms $g \to h \to\to f$ is the product of the two elements of K is $g = h = f$ and 0 otherwise. The tensor product of two objects $g, h \in G$ is their product gh. The tensor product $gg' \to hh'$ of two morphisms $g \to g'$ and $h \to h'$ is the product of the two elements in K if $g = h$ and $g' = h'$ and is 0 otherwise. The unit object is the identity of G. The associativity and left and right unit isomorphisms are the identity natural isomorphisms. For $g, h \in G$, the briading $gh \to hg = gh$ is defined to be $c(g, h)$. For $g \in G$, the twist $g \to g$ is defined to be $c(g, g)$. For $g \in G$, the (left and right) dual of g is g^{-1}. The morphisms ev_g, coev_g, ev'_g and coev'_g are the indentity of K. Then we have a ribbon category.

Exercise 3.35. Verify that the example above is indeed a ribbon category.

We now consider ribbon tensor categories, that is, rigid braided tensor categories with twists.

Let $\mathcal{C}$ be a ribbon tensor category. Then $K = \mathrm{Hom}(\mathbf{1}, \mathbf{1})$ acts on $\mathrm{Hom}(A, B)$ for any objects A and B by $kf = l_B \circ (k \otimes f) \circ l_A^{-1}$ for $k \in K$ and $f \in \mathrm{Hom}(A, B)$. This action gives $\mathrm{Hom}(A, B)$ a K-module structure.

Definition 3.36. An object A of a ribbon tensor category is said to be *irreducible* if $\mathrm{Hom}(A, A)$ is a free K-module of rank 1. A ribbon tensor category is said to be *semisimple* if the following conditions are satisfied:

1. For any simple objects A and B, $\mathrm{Hom}(A, B) = 0$ if A is not isomorphic to B.

2. Every object is a direct sum of finitely many irreducible objects.

Example 3.37. The unit object is an irreducible object.

Example 3.38. The ribbon tensor category of finite-dimensional representations over a field of a finite group such that the characteristic of the field does not divide the order of the group and the ribbon tensor category of finite-dimensional modules for a finite-dimensional semisimple Lie algebra are semisimple.

Definition 3.39. A *modular tensor category* is a semisimple ribbon tensor category $\mathcal{C}$, with finitely many equivalence classes of irreducible objects satisfying the following *nondegeneracy property*: Let $\{A_i\}_{i=1}^n$ be a set of representatives of the equivalence classes of irreducible objects of $\mathcal{C}$. Then the matrix (S_{ij}) where

$$S_{ij} = \mathrm{Tr}\ R_{A_j,A_i} \circ R_{A_i,A_j}$$

for $i, j = 1, \ldots, n$ is invertible.

Let I be the set of equivalence classes of irreduible objects in a modular tensor category. We shall use 0 to denote the equivalence class in I containing the unit object.

Proposition 3.40. *The dual object of an irreducible object is also irreducible.*

We omit the proof.

From this proposition, we see that there is a map $*: I \to I$ such that for any $i \in I$, i^* is the equivalence class in I such that objects in i^* are duals of objects in i.

We now choose one object A_i for each equivalence class $i \in I$. Then by definition, we have

$$S_{0,i} = S_{i,0} = \dim A_i$$

for $i \in I$.

Definition 3.41. Let $\mathcal{C}$ be a modular tensor category. Assuming that there exists $\mathcal{D} \in K$ such that

$$\mathcal{D}^2 = \sum_{i \in I} (\dim A_i)^2.$$

We call $\mathcal{D}$ the *rank* of $\mathcal{C}$.

If there is no such $\mathcal{D}$ in K, we can always enlarge K and the sets of morphisms such that in the new category, there exists such a $\mathcal{D}$.

For $i \in I$, the twist θ_{A_i} as an element of $\mathrm{Hom}(A_i, A_i)$ must be proportional to 1_{A_i}, that is, there exists $A_i \in K$ such that $\theta_i = A_i 1_{A_i}$. Since θ_{A_i} is an isomorphism, A_i is invertible. Let $\Delta = \sum_{i \in I} v_i^{-1} (\dim A_i)^2$, $T = (\delta_i^j v_i)$ and $J = (\delta_{i^*}^j)$. Then we have

$$\begin{aligned} (\mathcal{D}^{-1} S)^4 &= I, \\ (\mathcal{D}^{-1} T^{-1} S)^3 &= \Delta \mathcal{D}^{-1} (\mathcal{D}^{-1} S)^2. \end{aligned}$$

Let

$$\begin{aligned} s &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ t &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Then s and t are the generators of the modular group

$$SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}$$

satisfying the relations

$$s^4 = I,\ (ts)^3 = s^2.$$

Thus we see that $s \mapsto \mathcal{D}^{-1} S$ and $t \mapsto T^{-1}$ give a projective matrix representation of $SL(2, \mathbb{Z})$.

Since $\mathcal{C}$ is semisimple and I is the set of equivalence classes of irreducible objects in $\mathcal{C}$, we see that $A_i \otimes A_j$ for $i, j \in I$ must be isomorphic to $\oplus_{k \in I} N_{ij}^k A_k$, where N_{ij}^k are nonnegative integers giving the numbers of copies of A_k. These numbers N_{ij}^k afre called *fusion rules.*

Theorem 3.42. *For $i, l, m \in I$, we have*

$$\sum_{j,k\in I} S^{-1}_{mj} N^k_{ij} S_{kl} = (\dim A_m)^{-1} S_{il}\delta_{lm}.$$

In fact, if we let

$$N_i = (N^k_{ij})$$

for $i \in I$, then the theorem above says that the matrix S diagonalizes N_i for $i \in I$ simultaneously.

Corollary 3.43. *For $i, j, k \in I$, we have*

$$N^k_{ij} = \mathcal{D}^{-2} \sum_{l\in I} (\dim A_l)^{-1} S_{il} S_{jl} S_{k^* l}.$$

We omit the proofs of these results.

4 Vertex tensor category and braided tensor category of grading-restricted generalized V-modules

Let V be a vertex operator algebra such that for some $z \in \mathbb{C}^\times$, $W_1 \boxtimes_{P(z)} W_2$ for grading-restricted generalized V-modules W_1 and W_2 is grading-restricted and the associativity of intertwining operators hold in the category of grading-restricted generalized V-modules. From the results in the preceding section, a C_2-cofinite vertex operator algebra of positive energy is such a vertex operator algebra. Let $\mathcal{C}$ be the category of grading-restricted generalized V-modules. We give $\mathcal{C}$ a vertex tensor category structure and a braided tensor category structure in this section.

The material in this section is from [H6] and [HLZ2].

4.1 The vertex tensor category structure

We have constructed a tensor product bifunctor $\boxtimes_{P(z)} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ for $z \in \mathbb{C}^\times$. To construct a tensor category, we need only one tensor product bifunctor. We shall choose this bifunctor to be $\boxtimes_{P(1)}$ and denote it simply as $\boxtimes$. But we will first work with the tensor product functor $\boxtimes_{P(z)}$ for general z since these are parts of a structure called called "vertex tensor category" (see [HL1]).

We take the unit object to be V. To obtain a tensor category, we also have to give the left and right unit isomorphisms and the associativity isomorphism. Given a grading-restricted generalized V-module W, let $\mathcal{Y}$ be the intertwining operator of type $\binom{V\boxtimes_{P(z)}W}{VW}$ given in the construction of the tensor product $V \boxtimes_{P(z)} W$. Then $V \boxtimes_{P(z)}$ is spanned by the homogeneous components of $\mathcal{Y}(v, z)w$ for $v \in V$ and $w \in W$. Using $v = \mathrm{Res}_x x^{-1} Y_W(v, x)\mathbf{1}$ and the associator formula for the intertwining operator $\mathcal{Y}$, we see that homogeneous components of $\mathcal{Y}(v, z)w$ for $v \in V$ and $w \in W$ are in fact spanned by elements of the form $\mathcal{Y}(\mathbf{1}, z)w$

for $w \in W$. But by the $L(-1)$-derivative property, $\mathcal{Y}(\mathbf{1}, z)w$ is independent of z and by the $L(0)$-commutator formulas, it is homogeneous of wieght $\mathrm{wt}\, w$ if w is homogeneous. In particular, it is a well defined element of $V \boxtimes_{P(z)} W$. Then $\mathcal{Y}(\mathbf{1}, z)$ is a linear map from W to $V \boxtimes_{P(z)} W$. We denote this map by ψ. For $v \in V$ and $w \in W$, using the commutator formula for $\mathcal{Y}$ and $Y_V(v, x)\mathbf{1} \in V[[x_0]]$, we obtain

$$\begin{aligned} Y_{V\boxtimes_{P(z)}W}(v, x)\psi(w) &= Y_{V\boxtimes_{P(z)}W}(v, x)\mathcal{Y}(\mathbf{1}, z)w \\ &= \mathcal{Y}(\mathbf{1}, z)Y_W(v, x)w + \mathrm{Res}_x x^{-1}\delta\left(\frac{z+x}{x}\right)\mathcal{Y}(Y_V(v, x)\mathbf{1}, z)w \\ &= \mathcal{Y}(\mathbf{1}, z)Y_W(v, x)w \\ &= \psi(Y_W(v, x)w). \end{aligned}$$

So ψ is a V-module map. Since Y_W is an intertwining operator of type $\binom{W}{VW}$, $(W, Y_W(\cdot, z)\cdot)$ is a $P(z)$-product $(W, Y_W(\cdot, z)\cdot)$ of V and W. By the universal property of the tensor product $(V \boxtimes_{P(z)} W, \mathcal{Y}(\cdot, z)\cdot)$, there exists a unique V-module map $\phi : V \boxtimes_{P(z)} W \to W$ such that $Y_W(v, z)w = \overline{\phi}(\mathcal{Y}(v, z)w)$ for $v \in V$ and $w \in W$. In particular,

$$w = Y_W(\mathbf{1}, z)w = \overline{\phi}(\mathcal{Y}(\mathbf{1}, z)w) = \phi(\psi(w)).$$

So ϕ and ψ are inverse to each other and thus are equivalences. We define the left $P(z)$-unit isomorphism $l_{W;z} : V \boxtimes_{P(z)} W \to W$ to be ϕ.

We can also define the right $P(z)$-unit isomorphism $r_{W;z} : W \boxtimes_{P(z)} V \to W$ similarly. We omit the details here.

Let W_1, W_2 and W_3 be grading-restricted generalized V-modules. Let $z_1, z_2 \in \mathbb{C}^\times$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$. We now construct an associativity isomorphism

$$\alpha^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)} : (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \to W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3).$$

To construct $\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}$, we need only construct

$$(\alpha^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)})' : W_1 \boxdot_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \to (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxdot_{P(z_2)} W_3.$$

Let $\mathcal{Y}_1$, $\mathcal{Y}_2$ be intertwining operators of types $\binom{W_1\boxtimes_{P(z_1)}(W_2\boxtimes_{P(z_2)}W_3)}{W_1(W_2\boxtimes_{P(z_2)}W_3)}$ and $\binom{W_2\boxtimes_{P(z_2)}W_3}{W_2W_3}$, respectively, for the tensor product $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ and $W_2 \boxtimes_{P(z_2)} W_3$, respectively. Let $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$ and $w' \in W_1 \boxdot_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$. By the associativity of intertwining operators, there exist grading-restricted generalized V-modules W_4 and intertwining operators $\mathcal{Y}_3$ and $\mathcal{Y}_4$ of types $\binom{(W_1\boxtimes_{P(z_1-z_2)}W_2)\boxtimes_{P(z_2)}W_3}{W_4W_3}$ and $\binom{W_4}{W_1W_2}$, respectively, such that

$$\langle w', \mathcal{Y}_1(w_1, z_2)\mathcal{Y}_2(w_2, z_2)w_3\rangle = \langle w', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3\rangle.$$

We now have a product $(W_4, \mathcal{Y}_4)$ of W_1 and W_2. Let $\mathcal{Y}^2$ be the intertwining operator associated to the tensor product $W_1 \boxtimes_{P(z_1-z_2)} W_2$. Then by the universal property of the tensor product, there exists a unique V-module map $f : W_1 \boxtimes_{P(z_1-z_2)} W_2 \to W_4$ such that

$$\mathcal{Y}_4(w_1, z_1 - z_2)w_2 = f(Y^2(w_1, z_1 - z_2)w_2).$$

Since f is a V-module map, $\mathcal{Y}^1 = \mathcal{Y}_3 \circ (f \otimes 1_{W_3})$ is an intertwining operator of type $\binom{(W_1\boxtimes_{P(z_1-z_2)}W_2)\boxtimes_{P(z_2)}W_3}{(W_1\boxtimes_{P(z_1-z_2)}W_2)\ W_3}$. It can be shown that $\mathcal{Y}^1$ is in fact the intertwining operator associated to the tensor product $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$. Then we have

$$\begin{aligned}
&\langle w', \mathcal{Y}_1(w_1, z_2)\mathcal{Y}_2(w_2, z_2)w_3\rangle \\
&\quad = \langle w', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3\rangle \\
&\quad = \langle w', \mathcal{Y}^1(\mathcal{Y}^2(w_1, z_1 - z_2)w_2, z_2)w_3\rangle.
\end{aligned}$$

By the definition of $(W_1 \boxtimes_{P(z_1-z_2)} W_2)\boxbackslash_{P(z_2)}W_3$, we know that for

$$\langle w', \mathcal{Y}^1(\cdot, z_2)\cdot\rangle \in ((W_1 \boxtimes_{P(z_1-z_2)} W_2) \otimes W_3)^*$$

is in fact in $(W_1 \boxtimes_{P(z_1-z_2)} W_2)\boxbackslash_{P(z_2)}W_3$. We define $(\alpha^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)})'$ by

$$(\alpha^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)})'(w') = \langle w', \mathcal{Y}^1(\cdot, z_2)\cdot\rangle.$$

It can be proved that $(\alpha^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)})'$ is a V-module map and is invertable.

We now define the associativity isomorphism

$$\alpha^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)} : (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \to W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$$

to be the adjoint of $(\alpha^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)})'$. From the definition of $(\alpha^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)})'$ and $\alpha^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}$, we have

$$\begin{aligned}
&\langle w', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle \\
&\quad = \langle w', \mathcal{Y}^1(\mathcal{Y}^2(w_1, z_1 - z_2)w_2, z_2)w_3\rangle \\
&\quad = ((\alpha^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)})'(w'))(\mathcal{Y}^2(w_1, z_1 - z_2)w_2 \otimes w_3) \\
&\quad = \langle(\alpha^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)})'(w'), \mathcal{Y}^1(\mathcal{Y}^2(w_1, z_1 - z_2)w_2, z_2)w_3\rangle.
\end{aligned}$$

Thus we obtain

$$\overline{\alpha^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}}(\mathcal{Y}^1(\mathcal{Y}^2(w_1, z_1 - z_2)w_2, z_2)w_3) = \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3.$$

We want to rewrite this formula using tensor products of elements. Let W_1 and W_2 be two grading-restricted generalized V-modues and $z \in \mathbb{C}^\times$. Let $\mathcal{Y}$ be the intertwining operator associated to the $P(z)$-tensor product $W_1 \boxtimes_{P(z)} W_2$. For $w_1 \in W_1$ and $w_2 \in W_2$, we define the tensor product $w_1 \boxtimes_{P(z)} w_2$ of w_1 and w_2 by

$$w_1 \boxtimes_{P(z)} w_2 = \mathcal{Y}(w_1, z)w_2.$$

Note that $w_1 \boxtimes_{P(z)} w_2 \in \overline{W_1 \boxtimes_{P(z)} W_2}$ but in general $w_1 \boxtimes_{P(z)} w_2 \notin W_1 \boxtimes_{P(z)} W_2$. Though $w_1 \boxtimes_{P(z)} w_2$ is not in the tensor product $W_1 \boxtimes_{P(z)} W_2$, it is a (in general infinite) sum of homogeneous elements of $W_1 \boxtimes_{P(z)} W_2$. It is not difficult to show that these homogeneous componenets for all $w_1 \in W_1$ and $w_2 \in W_2$ span $W_1 \boxtimes_{P(z)} W_2$. This is the reason why these elements are useful and important.

Using this notation, in the setting of the construction of the associativity isomorphism above, we have

$$\mathcal{Y}^1(\mathcal{Y}^2(w_1, z_1 - z_2)w_2, z_2)w_3 = (w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3,$$
$$\mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3 = w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3).$$

The we have

$$\overline{\alpha_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}}((w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3) = w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3).$$

We often use the inverse

$$\mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)} : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \to (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$$

of $\alpha_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}$ and we have

$$\overline{\mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}}(w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)) = (w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3.$$

To prove the pentagon diagram is commutative, we also need the tensor product element of four elements. For example, we also have

$$w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} (w_3 \boxtimes_{P(z_3)} w_4)) \in \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4))}$$

when $|z_1| > |z_2| > |z_3| > 0$ and

$$((w_1 \boxtimes_{P(z_{12})} w_2) \boxtimes_{P(z_{23})} w_3) \boxtimes_{P(z_3)} w_4 \in \overline{((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4}$$

when $|z_3| > |z_2 - z_3| > |z_1 - z_2| > 0$. These are also given by intertwining operators evaluated at the corresponding complex numbers. Moreover, the natural extensions of the associativity isomorphisms to the algebriac completions of the corresponding modules in $\mathcal{C}$ send such an element to another such element. Also the homogeneous componenets of these elements span the tensor product modules.

We first prove the commutativity of the pentagon diagram involving these z's. This is in fact the prntagon diagram for vertex tensor categories. Let W_1, W_2, W_3 and W_4 be V-modules and let $z_1, z_2, z_3 \in \mathbb{C}$ satisfying

$$|z_1| > |z_2| > |z_3| > |z_1 - z_3| > |z_2 - z_3| > |z_1 - z_2| > 0,$$
$$|z_1| > |z_2 - z_3| + |z_3|,$$
$$|z_2| > |z_1 - z_2| + |z_3|,$$
$$|z_2| > |z_1 - z_2| + |z_2 - z_3|.$$

For example, we can take $z_1 = 7$, $z_2 = 6$ and $z_3 = 4$. We wang to prove the commutativity of the diagram:

$$
\begin{tikzcd}
& W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)) \arrow[dl] \arrow[dr] & \\
(W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4) \arrow[d] & & W_1 \boxtimes_{P(z_1)} ((W_2 \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4)) \arrow[d] \\
((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4 & & (W_1 \boxtimes_{P(z_{13})} (W_2 \boxtimes_{P(z_{23})} W_3)) \boxtimes_{P(z_3)} W_4 \arrow[ll]
\end{tikzcd}
\tag{4.26}
$$

where $z_{12} = z_1 - z_2$ and $z_{23} = z_2 - z_3$. For $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$ and $w_4 \in W_4$, we consider

$$w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} (w_3 \boxtimes_{P(z_3)} w_4)) \in \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4))}.$$

Since the natural extensions of the associativity isomorphisms send tensor products of elements to tensor products of elements, we see that the compositions of the natural extensions of the V-module maps in the two routes in (4.26) applying to this element both give

$$((w_1 \boxtimes_{P(z_{12})} w_2) \boxtimes_{P(z_{23})} w_3) \boxtimes_{P(z_3)} w_4 \in \overline{((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4}.$$

Since the homogeneous components of

$$w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} (w_3 \boxtimes_{P(z_3)} w_4))$$

for $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$ and $w_4 \in W_4$ span

$$W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)),$$

the diagram (4.26) above is commutative.

We also need parallel transport isomorphisms. Let W_1 and W_2 be grading-restricted generalized V-modules. Let $z_1, z_2 \in \mathbb{C}^\times$ and γ is a path in $\mathbb{C}^\times$ from z_1 to z_2. Let $\mathcal{Y}$ be the intertwining operator associated to the tensor product $W_1 \boxtimes_{P(z_2)} W_2$. Using the value $\log z_2 = \log |z_2| + i \arg z_2$ (satisfying $0 \leq \arg z_2 < 2\pi$) and the path γ, we obtain a unique value $l(z_1$ determined uniquely by $\log z_2$ and the homotopy class of the path γ. We define a linear map

$$\mathcal{T}_\gamma : W_1 \boxtimes_{P(z_1)} W_2 \to W_1 \boxtimes_{P(z_2)} W_2$$

by

$$\mathcal{T}_\gamma(w_1 \boxtimes_{P(z_1)} w_2) = \mathcal{Y}(w_1, x)w_2\Big|_{x^n = e^{nl(z_1)},\ \log x = l(z_1)}$$

for $w_1 \in W_1$ and $w_2 \in W_2$. Since the homogeneous components of the tensor products $w_1 \boxtimes_{P(z_1)} w_2$ of elements $w_1 \in W_1$ and $w_2 \in W_2$ span $W_1 \boxtimes_{P(z_1)} W_2$ and the image is given by an intertwining operator associated to w_1 evaluated at z_1 and acting on w_2, this linear map is well defined and is a V-module map. Clearly this is invertable and is therefore an isomorphism. The parallel transport isomorphism $\mathcal{T}_\gamma$ depends only on the homotopy class of γ.

We also have a commutativity isomorphism. Let $z \in \mathbb{C}^\times$. For grading-restricted generalized V-modules W_1 and W_2, let $\mathcal{Y}$ be the intertwining operator associated to the $P(z)$-tensor product $W_2 \boxtimes_{P(-z)} W_1$. The we have an intertwining operator $\Omega(\mathcal{Y})$ of type $\binom{W_2 \boxtimes_{P(-z)} W_1}{W_2 W_1}$, where $\Omega(\mathcal{Y})$ is defined by

$$\Omega(\mathcal{Y})(w_2, x)w_1 = e^{xL_{W_2 \boxtimes_{P(-z)} W_1}(-1)} \mathcal{Y}(w_1, y)w_2 \Big|_{y^n = e^{n\pi i}x^n,\ \log y = \log x + \pi i}.$$

Then the pair $(W_2 \boxtimes_{P(-z)} W_1, \Omega(\mathcal{Y})(\cdot, z)\cdot)$ is a $P(z)$-product opf W_1 and W_2. By the universal property of the tensor product $W_1 \boxtimes_{P(z)} W_2$, there exists a unique V-module map

$$\mathcal{C}_{P(z)} : W_1 \boxtimes_{P(z)} W_2 \to W_2 \boxtimes_{P(-z)} W_1$$

such that

$$\Omega(\mathcal{Y})(\cdot, z)\cdot = \overline{\mathcal{C}}_{P(z)} \circ \boxtimes_{P(z)},$$

where $\overline{\mathcal{C}}_{P(z)}$ is the natrual extension of $\mathcal{C}_{P(z)}$ and $\boxtimes_{P(z)}$ is the value at z of the intertwining operator associated to the tensor product $W_1 \boxtimes_{P(z)} W_2$.

4.2 The braided tensor category structure

We now discuss the tensor category structure. As we mentioned in the beginning of the preceding subsection, we choose the tensor product bifunctor to be $\boxtimes = \boxtimes_{P(1)}$. The unit object is still V and the unit isomorphisms are $l_W = l_{W;1}$ and $r_W = r_{W;1}$.

To construct the associativity isomorphism

$$\mathcal{A} : W_1 \boxtimes (W_2 \boxtimes W_3) \to (W_1 \boxtimes W_2) \boxtimes W_3$$

for the braided tensor category structure, we need to use certain parallel isomorphisms. Let z_1 and z_2 be real numbers satisfying $z_1 > z_2 > z_1 - z_2 \geq 0$. Let γ_1 and γ_2 be paths in $(0, \infty)$ from 1 to z_1 and z_2, respectively, and γ_3 and γ_4 be paths in $(0, \infty)$ from z_2 and $z_1 - z_2$ to 1, respectively. Then the associativity isomorphism for the braided tensor category structure on the module category for V is given by

$$\mathcal{A} = \mathcal{T}_{\gamma_3} \circ (\mathcal{T}_{\gamma_4} \boxtimes_{P(z_2)} I_{W_3}) \circ \mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)} \circ (I_{W_1} \boxtimes_{P(z_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1},$$

that is, given by the commutative diagram

$$\begin{tikzcd}
W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \arrow[r, "\mathcal{A}_{P(z_2),P(z_2)}^{P(z_2-z_3),P(z_3)}"] & (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \arrow[d, "\mathcal{T}_{\gamma_3} \circ (\mathcal{T}_{\gamma_4} \boxtimes_{P(z_2)} I_{W_3})"] \\
W_1 \boxtimes (W_2 \boxtimes W_3) \arrow[u, "(I_{W_1} \boxtimes_{P(z_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1}"] \arrow[r, "\mathcal{A}"] & (W_1 \boxtimes W_2) \boxtimes W_3
\end{tikzcd}$$

On the other hand, by the definition of $\mathcal{A}$, the diagrams

$$\begin{tikzcd}
W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)) \arrow[r] \arrow[d] & (W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4) \arrow[d] \\
W_1 \boxtimes (W_2 \boxtimes (W_3 \boxtimes W_4)) \arrow[r] & (W_1 \boxtimes W_2) \boxtimes (W_3 \boxtimes W_4)
\end{tikzcd} \tag{4.27}$$

$$\begin{tikzcd}
(W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4) \arrow[r] \arrow[d] & ((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4 \arrow[d] \\
(W_1 \boxtimes W_2) \boxtimes (W_3 \boxtimes W_4) \arrow[r] & ((W_1 \boxtimes W_2) \boxtimes W_3) \boxtimes W_4
\end{tikzcd} \tag{4.28}$$

$$\begin{tikzcd}
W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)) \arrow[r] \arrow[d] & W_1 \boxtimes_{P(z_1)} ((W_2 \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4)) \arrow[d] \\
W_1 \boxtimes (W_2 \boxtimes (W_3 \boxtimes W_4)) \arrow[r] & W_1 \boxtimes ((W_2 \boxtimes W_3) \boxtimes W_4)
\end{tikzcd} \tag{4.29}$$

$$\begin{tikzcd}
W_1 \boxtimes_{P(z_1)} ((W_2 \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4)) \arrow[r] \arrow[d] & (W_1 \boxtimes_{P(z_{13})} (W_2 \boxtimes_{P(z_{23})} W_3)) \boxtimes_{P(z_3)} W_4 \arrow[d] \\
W_1 \boxtimes ((W_2 \boxtimes W_3) \boxtimes W_4) \arrow[r] & (W_1 \boxtimes (W_2 \boxtimes W_3)) \boxtimes W_4
\end{tikzcd} \tag{4.30}$$

$$\begin{tikzcd}
(W_1 \boxtimes_{P(z_{13})} (W_2 \boxtimes_{P(z_{23})} W_3)) \boxtimes_{P(z_3)} W_4 \arrow[r] \arrow[d] & ((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4 \arrow[d] \\
(W_1 \boxtimes (W_2 \boxtimes W_3)) \boxtimes W_4 \arrow[r] & ((W_1 \boxtimes W_2) \boxtimes W_3) \boxtimes W_4
\end{tikzcd} \tag{4.31}$$

are all commutative. Combining all the diagrams (4.26)–(4.31) above, we see that the pentagon diagram

$$
\begin{tikzcd}
& W_1 \boxtimes (W_2 \boxtimes (W_3 \boxtimes W_4)) \arrow[dl] \arrow[dr] & \\
(W_1 \boxtimes W_2) \boxtimes (W_3 \boxtimes W_4) \arrow[d] & & W_1 \boxtimes ((W_2 \boxtimes W_3) \boxtimes W_4) \arrow[d] \\
((W_1 \boxtimes W_2) \boxtimes W_3) \boxtimes W_4 & & (W_1 \boxtimes (W_2 \boxtimes W_3)) \boxtimes W_4 \arrow[ll]
\end{tikzcd}
$$

is also commutative.

Let γ be a path from -1 to 1 in the closed upper half plane with 0 deleted. Let W_1 and W_2 be grading-restricted generalized V-modules. We define the brading isomorphsim $\mathcal{R} : W_1 \boxtimes W_2 \to W_2 \boxtimes W_1$ by

$$\mathcal{R} = \mathcal{T}_\gamma \circ \mathcal{C}_{P(1)}.$$

We still need to prove the commutativity of the hexagon diagrams for the braiding isomorphsim. To prove this, we need to introduce tensor products $w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)$ and $(w_1 \boxtimes_{P(z_1-z_1)} w_2) \boxtimes_{P(z_2)} w_3)$ for $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$ and $z_1, z_2 \in \mathbb{C}^\times$ satisfying $|z_1| = |z_2| = |z_1 - z_2|$. Here we introduce these tensor products of elements for $z_1, z_2 \in \mathbb{C}^\times$ such that $z_1 \neq z_2$.

Let $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Y}_3$, $\mathcal{Y}_4$ be intertwining operators of types

$$
\binom{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}{W_1 \;\; W_2 \boxtimes_{P(z_2)} W_3}, \;\; \binom{W_2 \boxtimes_{P(z_2)} W_3}{W_2 \;\; W_3},
$$
$$
\binom{(W_1 \boxtimes_{P(z_4)} W_2) \boxtimes_{P(z_3)} W_3}{W_1 \boxtimes_{P(z_4)} W_2 \;\; W_3}, \;\; \binom{W_1 \boxtimes_{P(z_4)} W_2}{W_1 \;\; W_2}
$$

respectively, associated to the tensor products $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$, $W_2 \boxtimes_{P(z_2)} W_3$, $(W_1 \boxtimes_{P(z_4)} W_2) \boxtimes_{P(z_3)} W_3$, $W_1 \boxtimes_{P(z_4)} W_2$, respectively. Then

$$\langle w', \mathcal{Y}_1(w_{(1)}, \zeta_1)\mathcal{Y}_2(w_{(2)}, \zeta_2)w_{(3)}\rangle$$

and

$$\langle \tilde{w}', \mathcal{Y}_3(\mathcal{Y}_4(w_{(1)}, \zeta_4)w_{(2)}, \zeta_3)w_{(3)}\rangle$$

are absolutely convergent for

$$w' \in (W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))'$$

and

$$\tilde{w}' \in ((W_1 \boxtimes_{P(z_4)} W_2) \boxtimes_{P(z_3)} W_3)',$$

when $|\zeta_1| > |\zeta_2| > 0$ and when $|\zeta_3| > |\zeta_4| > 0$, respectively, and can be analytically extended to multivalued analytic functions in the regions given by $\zeta_1, \zeta_2 \neq 0$ and $\zeta_1 \neq \zeta_2$ and by $\zeta_3, \zeta_4 \neq 0$ and $\zeta_3 \neq -\zeta_4$, respectively. Cutting these regions along $\zeta_1, \zeta_2 \geq \mathbb{R}_+$ and $\zeta_3, \zeta_4 \in \mathbb{R}_+$, respectively, we obtain simply-connected regions and we can choose single-valued branches of these multivalued analytic functions. In particular, we have the branches of these two multivalued analytic functions such that their values at points satisfying $|\zeta_1| > |\zeta_2| > 0$ and $|\zeta_3| > |\zeta_4| > 0$ are

$$\langle w', \mathcal{Y}_1(w_{(1)}, \zeta_1)\mathcal{Y}_2(w_{(2)}, \zeta_2)w_{(3)}\rangle$$

and

$$\langle \tilde{w}', \mathcal{Y}_3(\mathcal{Y}_4(w_{(1)}, \zeta_4)w_{(2)}, \zeta_3)w_{(3)}\rangle,$$

respectively.

Let $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. Then for any $z_1, z_2, z_3, z_4 \in \mathbb{C}^\times$ satisfying $z_1 \neq z_2$ and $z_3 \neq -z_4$, there exist unique elements

$$w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \in \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}$$

and

$$(w_{(1)} \boxtimes_{P(z_4)} w_{(2)}) \boxtimes_{P(z_3)} w_{(3)} \in \overline{(W_1 \boxtimes_{P(z_4)} W_2) \boxtimes_{P(z_3)} W_3}$$

such that for any

$$w' \in (W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))'$$

and

$$\tilde{w}' \in ((W_1 \boxtimes_{P(z_4)} W_2) \boxtimes_{P(z_3)} W_3)',$$

the numbers

$$\langle w', w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})\rangle$$

and

$$\langle \tilde{w}', (w_{(1)} \boxtimes_{P(z_4)} w_{(2)}) \boxtimes_{P(z_3)} w_{(3)}\rangle$$

are the values at $(\zeta_1, \zeta_2) = (z_1, z_2)$ and $(\zeta_3, \zeta_4) = (z_3, z_4)$, respectively, of the branches of the multivalued analytic functions above of ζ_1 and ζ_2 and of ζ_3 and ζ_4 above, respectively.

From the definition of

$$w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})$$

and

$$(w_{(1)} \boxtimes_{P(z_4)} w_{(2)}) \boxtimes_{P(z_3)} w_{(3)},$$

we see that when $|z_1| = |z_2|$ (with $z_1 \neq z_2$) or $|z_3| = |z_4|$ (with $z_3 \neq z_4$), they are uniquely determined by

$$\begin{aligned}&\langle w', w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})\rangle\\ &\quad = \lim_{\zeta_1 \to z_1,\, \zeta_2 \to z_2,\, |\zeta_1| > |\zeta_2| > 0} \langle w', \mathcal{Y}_1(w_{(1)}, \zeta_1)\mathcal{Y}_2(w_{(2)}, \zeta_2)w_{(3)}\rangle\end{aligned}$$

and

$$\begin{aligned}&\langle \tilde{w}', (w_{(1)} \boxtimes_{P(z_4)} w_{(2)}) \boxtimes_{P(z_3)} w_{(3)}\rangle\\ &\quad = \lim_{\zeta_3 \to z_4,\, \zeta_4 \to z_4,\, |\zeta_3| > |\zeta_4| > 0} \langle \tilde{w}', \mathcal{Y}_3(\mathcal{Y}_4(w_{(1)}, \zeta_4)w_{(2)})\zeta_3)w_{(3)}\rangle\end{aligned}$$

for

$$w' \in (W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))'$$

and

$$\tilde{w}' \in ((W_1 \boxtimes_{P(z_4)} W_2) \boxtimes_{P(z_3)} W_3)',$$

where the limits take place in the complex plane with a cut along $\mathbb{R}_+$.

For any $z_1, z_2, z_3, z_4 \in \mathbb{C}^\times$ satisfying $z_1 \neq z_2$ and $z_3 \neq -z_4$, the homogeneous componenets of the elements of the form $w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})$, $(w_{(1)} \boxtimes_{P(z_4)} w_{(2)}) \boxtimes_{P(z_3)} w_{(3)}$ for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ span $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$, $(W_1 \boxtimes_{P(z_4)} W_2) \boxtimes_{P(z_3)} W_3$, respectively.

Here we sketch the proof of the commutativity of the hexagon diagram. We first consider the following diagram:

$$\begin{tikzcd}
 & (W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} W_3 \arrow[dl, "\mathcal{R}_{P(z_{12})} \boxtimes_{P(z_2)} 1_{W_3}"'] \arrow[dr, "\left(\mathcal{A}^{P(z_{12}),P(z_2)}_{P(z_1),P(z_2)}\right)^{-1}"] & \\
(W_2 \boxtimes_{P(-z_{12})} W_1) \boxtimes_{P(z_2)} W_3 \arrow[d, "\mathcal{T}_{\gamma_1}"'] & & W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \arrow[dd, "\mathcal{R}_{P(z_1)}"] \\
(W_2 \boxtimes_{P(-z_{12})} W_1) \boxtimes_{P(z_1)} W_3 \arrow[d, "\left(\mathcal{A}^{P(-z_{12}),P(z_1)}_{P(z_2),P(z_1)}\right)^{-1}"'] & & \\
W_2 \boxtimes_{P(z_2)} (W_1 \boxtimes_{P(z_1)} W_3) \arrow[ddr, "1_{W_2} \boxtimes_{P(z_2)} \mathcal{R}_{P(z_1)}"'] & & (W_2 \boxtimes_{P(z_2)} W_3) \boxtimes_{P(-z_1)} W_1 \arrow[d, "\left(\mathcal{A}^{P(z_2),P(-z_1)}_{P(-z_{12}),P(-z_1)}\right)^{-1}"] \\
 & & W_2 \boxtimes_{P(-z_{12})} (W_3 \boxtimes_{P(-z_1)} W_1) \arrow[dl, "\mathcal{T}_{\gamma_2}"] \\
 & W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(-z_1)} W_1) &
\end{tikzcd} \tag{4.32}$$

where γ_1 and γ_2 are paths from z_2 to z_1 and from $-z_{12}$ to z_2, respectively, in $\mathbb{C}$ with a cut along the nonnegative real line.

For $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$, the images of the element

$$\left(w_{(1)} \boxtimes_{P(z_{12})} w_{(2)}\right) \boxtimes_{P(z_2)} w_{(3)}$$

under the natural extension to $\overline{(W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} W_3}$ of the compositions of the maps in both the left and right routes in (4.32) from $(W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} W_3$ to $W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(-z_1)} W_1)$ are

$$w_{(2)} \boxtimes_{P(z_2)} (e^{z_1 L(-1)}(w_{(3)} \boxtimes_{P(-z_1)} w_{(1)})).$$

Since the homogeneous components of $(w_{(1)} \boxtimes_{P(z_{12})} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}$ for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$ span $(W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} W_3$, the diagram (4.32) commutes.

To prove the commutativity of the hexagon diagram for the braiding isomorphism $\mathcal{R}$, we need to consider the following diagrams:

$$\begin{array}{ccc} (W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} W_3 & \longrightarrow & (W_1 \boxtimes W_2) \boxtimes W_3 \\ \downarrow & & \downarrow \\ (W_2 \boxtimes_{P(-z_{12})} W_1) \boxtimes_{P(z_2)} W_3 & \longrightarrow & (W_2 \boxtimes W_1) \boxtimes W_3 \end{array} \tag{4.33}$$

$$\begin{array}{ccc} (W_2 \boxtimes_{P(-z_{12})} W_1) \boxtimes_{P(z_2)} W_3 & \longrightarrow & (W_2 \boxtimes W_1) \boxtimes W_3 \\ \downarrow & & \downarrow \\ (W_2 \boxtimes_{P(-z_{12})} W_1) \boxtimes_{P(z_1)} W_3 & \longrightarrow & (W_2 \boxtimes W_1) \boxtimes W_3 \end{array} \tag{4.34}$$

$$\begin{array}{ccc} (W_2 \boxtimes_{P(-z_{12})} W_1) \boxtimes_{P(z_1)} W_3 & \longrightarrow & (W_2 \boxtimes W_1) \boxtimes W_3 \\ \downarrow & & \downarrow \\ W_2 \boxtimes_{P(z_2)} (W_1 \boxtimes_{P(z_1)} W_3) & \longrightarrow & W_2 \boxtimes (W_1 \boxtimes W_3) \end{array} \tag{4.35}$$

$$\begin{array}{ccc} W_2 \boxtimes_{P(z_2)} (W_1 \boxtimes_{P(z_1)} W_3) & \longrightarrow & W_2 \boxtimes (W_1 \boxtimes W_3) \\ \downarrow & & \downarrow \\ W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(-z_1)} W_1) & \longrightarrow & W_2 \boxtimes (W_3 \boxtimes W_1) \end{array} \tag{4.36}$$

$$\begin{array}{ccc} (W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} W_3 & \longrightarrow & (W_1 \boxtimes W_2) \boxtimes W_3 \\ \downarrow & & \downarrow \\ W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) & \longrightarrow & W_1 \boxtimes (W_1 \boxtimes W_3) \end{array} \tag{4.37}$$

$$\begin{array}{ccc} W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) & \longrightarrow & W_1 \boxtimes (W_1 \boxtimes W_3) \\ \downarrow & & \downarrow \\ (W_2 \boxtimes_{P(z_2)} W_3) \boxtimes_{P(-z_1)} W_1 & \longrightarrow & (W_2 \boxtimes W_3) \boxtimes W_1 \end{array} \tag{4.38}$$

$$\begin{array}{ccc} (W_2 \boxtimes_{P(z_2)} W_3) \boxtimes_{P(-z_1)} W_1 & \longrightarrow & (W_2 \boxtimes W_3) \boxtimes W_1 \\ \downarrow & & \downarrow \\ W_2 \boxtimes_{P(-z_{12})} (W_3 \boxtimes_{P(-z_1)} W_1) & \longrightarrow & W_2 \boxtimes (W_3 \boxtimes W_1) \end{array} \tag{4.39}$$

$$
\begin{array}{ccc}
W_2 \boxtimes_{P(-z_{12})} (W_3 \boxtimes_{P(-z_1)} W_1) & \longrightarrow & W_2 \boxtimes (W_3 \boxtimes W_1) \\
\downarrow & & \downarrow \\
W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(-z_1)} W_1) & \longrightarrow & W_2 \boxtimes (W_3 \boxtimes W_1)
\end{array}
\tag{4.40}
$$

The commutativity of the diagrams (4.33), (4.36) and (4.38) follows from the definition of the commutativity isomorphism for the braided tensor category structure and the naturality of the parallel transport isomorphisms. The commutativity of (4.35), (4.37) and (4.39) follows from the definition of the associativity isomorphism for the braided tensor product structure. The commutativity of (4.34) and (4.40) follows from the facts that compositions of parallel transport isomorphisms are equal to the parallel transport isomorphisms associated to the products of the paths and that parallel transport isomorphisms associated to homotopically equivalent paths are equal. The commutativity of the hexagon diagram involving $\mathcal{R}$ follows from (4.32)–(4.40).

Similarly, we can prove the commutativity of the triangle diagram for the unit isomorphisms. We omit the proof here.

We have proved the following result:

Theorem 4.1. *Let V be a vertex operator algebra such that for some $z \in \mathbb{C}^\times$, $W_1 \boxslash_{P(z)} W_2$ for grading-restricted generalized V-modules W_1 and W_2 is grading-restricted and the associativity of intertwining operators hold in the category of grading-restricted generalized V-modules. Then the category of grading-restricted generalized V-modules with the tensor product bifunctor $\boxtimes$, the unit object V, the left and right unit isomorphisms l and r, the associativity isomorphism $\mathcal{A}$ and the brading isomorphism $\mathcal{R}$ is a braided tensor category.*

5 Modular invariance, Verlinde formula, and modular tensor category structure

In this section, we discuss the modular invariance of intertwining operators, the Verlinde formula and their applications to the construction of tensor category structure on the category of mofules for a vertex operator algebra satisfying suitable finiteness and reductive conditions.

5.1 Modular invariance

In this subsection, let V be a vertex operator algebra satisfying the folliwing conditions:

1. For $n < 0$, $V_{(n)} = 0$ and $V_{(0)} = \mathbb{C}\mathbf{1}$.

2. Every lower-bounded generalized V-module is completely reducible.

3. V is C_2-cofinite.

In this case, every irreducible lower-bounded generalized V-module is an ordinary V-module, that is, it is grading-restricted and $L(0)$ acts semisimply. We shall consider only ordinary V-modules in this section.

We first recall geometrically-modified intertwining operators from [H4] (see also [H8]). Given an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1\, W_2}$ and $w_1 \in W_1$, we have an operator (actually a series with linear maps from W_2 to W_3 as coefficients) $\mathcal{Y}_1(w_1, z)$. The corresponding geometrically-modified operator is

$$\mathcal{Y}_1(\mathcal{U}(q_z)w_1, q_z),$$

where $q^z = e^{2\pi iz}$, $\mathcal{U}(q_z) = (2\pi i q_z)^{L(0)}e^{-L^+(A)}$ and $A_j \in \mathbb{C}$ for $j \in \mathbb{Z}_+$ are defined by

$$\frac{1}{2\pi i}\log(1 + 2\pi i y) = \left(\exp\left(\sum_{j\in\mathbb{Z}_+} A_j y^{j+1}\frac{\partial}{\partial y}\right)\right) y.$$

See [**?**] for details.

First, we have the convergence and extension property of q-traces of products of n geometrically-modified intertwining operators:

Theorem 5.1 ([H4]). *Let W_i and $\tilde{W}_i$ for $i = 1, \dots, n$ be (ordinary) V-modules, and $\mathcal{Y}_i$ for $i = 1, \dots, n$ intertwining operators of types $\binom{\tilde{W}_{i-1}}{W_i\tilde{W}_i}$, respectively, where we use the convention $\tilde{W}_0 = \tilde{W}_n$. For $w_i \in W_i$, $i = 1, \dots, n$,*

$$\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q_\tau^{L(0)-\frac{c}{24}}$$

is absolutely convergent in the region $1 > |q_{z_1}| > \ldots > |q_{z_n}| > |q_\tau| > 0$ and can be extended to a multivalued analytic function

$$\overline{F}_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1, \dots, w_n; z_1, \dots, z_n; \tau).$$

in the region $\Im(\tau) > 0$, $z_i \neq z_j + l + m\tau$ for $i \neq j$, $l, m \in \mathbb{Z}$.

The space of all these multivalued functions have the following modular invariance property:

Theorem 5.2 ([H4]). *For (ordinary) V-modules W_i and $w_i \in W_i$ for $i = 1, \dots, n$, let $\mathcal{F}_{w_1,\dots,w_n}$ be the vector space spanned by functions of the form*

$$\overline{F}^{\phi}_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1, \dots, w_n; z_1, \dots, z_n; \tau)$$

for all (ordinary) V-modules $\tilde{W}_i$ for $i = 1, \dots, n$, all intertwining operators $\mathcal{Y}_i$ of types $\binom{\tilde{W}_{i-1}}{W_i\tilde{W}_i}$ for $i = 1, \dots, n$, respectively. Then for

$$\left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}\right) \in SL(2, \mathbb{Z}),$$

$$\overline{F}_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}\left(\left(\frac{1}{\gamma\tau+\delta}\right)^{L(0)} w_1,\dots,\left(\frac{1}{\gamma\tau+\delta}\right)^{L(0)} w_n; \frac{z_1}{\gamma\tau+\delta},\dots,\frac{z_n}{\gamma\tau+\delta}; \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)$$

is in $\mathcal{F}_{w_1,\dots,w_n}$.

When $W_i = V$ for $i = 1,\dots,n$ and $\tilde{W}_i$ for $i = 1,\dots,n$ are irreducible, intertwining operators of type $\binom{\tilde{W}_{i-1}}{V\tilde{W}_i}$ must be 0 if $\tilde{W}_{i-1}$ is not equivalenet to $\tilde{W}_i$ and are proportional to the vertex operator map Y_{W_i} if $\tilde{W}_{i-1}$ is equivalenet to $\tilde{W}_i$. Thus in the case that $W_i = V$ for $i = 1,\dots,n$, we need only consider all the analytic functions of the form

$$\overline{F}_{Y_W,\dots,Y_W}(v_1,\dots,v_n; z_1,\dots,z_n;\tau)$$

for a V-module W, $v_1,\dots,v_n \in V$. In particular, Theorem 5.2 states that the space of all such analytic functions are invariant under the modular transformations. This is the modular invariance theorem proved first by Zhu [Z]. But the method Zhu used cannot be used to prove Theorems 5.1 and 5.2 when $n \geq 2$.

5.2 The Verlinde formula

In the case that $n = 1$, $W_1 = V$, $w_1 = \mathbf{1}$ $\tilde{W} = W$, we see that Theorem 5.2 says that the space of vacuum characters or shifted graded dimensions

$$\mathrm{Tr}_W q_\tau^{L_W(0)-\frac{c}{24}}$$

of V-modules is invariant under the modular transformation. Moreover, if W_i for $i = 1,\dots,m$ are all the inequivalent irreducible V-modules, then

$$\mathrm{Tr}_{W_i} q_\tau^{L_{W_i}(0)-\frac{c}{24}}$$

for $a \in \mathcal{A}$ in fact form a basis of this space of vacuum characters or shifted graded dimensions. For the WZW models, minimal models and lattice theories, this result was well known before Zhu's theorem.

Note that the modular group $SL(2,\mathbb{Z})$ is generated by two elements

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The action of T on the vacuum character of an irreducible V-module is in fact given by a number. So the only nontrivial action is given by S. Let S_{ij} for $i, j = 1,\dots,m$ be the entries of the matrix of the action of S under the basis

$$\mathrm{Tr}_{W_i} q_\tau^{L_{W_i}(0)-\frac{c}{24}}$$

for $i=1,\dots,m$, that is,

$$\mathrm{Tr}_{W_i} q_{-\frac{1}{\tau}}^{L_{W_i}(0)-\frac{c}{24}} = \sum_{j=1}^{m} S_{ij}\mathrm{Tr}_{W_j} q_{\tau}^{L_{W_i}(0)-\frac{c}{24}}.$$

For simplicity, we shall still denote this matrix by S.

For $i,j,k=1,\dots,n$, we also have the fusion rules $N_{ij}^{k}=\dim \mathcal{V}_{W_iW_j}^{W_k}$, where $\mathcal{V}_{W_iW_j}^{W_k}$ is the space of intertwining operators of type $\binom{W_k}{W_iW_j}$. Let $N_j=(N_{ij}^{k})$ for $j=1,\dots,m$. Then Verlinde conjectured in [V] that for rational conformal field theories, the matrix S diagonalizes the matrices N_j for $j=1,\dots,m$ simultaneously. Using this conjecture, Verlinde also derived in [V] a formula for the fusion rules, called Verlinde formula. Moore and Seiberg derived a set of polynomial equations in [MS1] and [MS2] from the operator product expansion and modular invariance of intertwining operators. The Verlinde conjecture and Verlinde formula were derived in [MS1] and [MS2] from this set of Moore-Seiberg equations.

Mathematically, the operator product expansion and modular invariance of intertwining operators were proved in [H3] and [H4] under the conditions stated in the preceding subsection, as we have discussed above. Then in [H5], the Verlinde conjecture and Verlinde formula were proved under the three conditions on V in the preceding subsection together with the additional condition that V is simple (meaning irreducible as V-module) and as a V-module is equivalent to its contragredient V-module V'. When V is simple, it is equivalent to one of the irreducible V-modules W_i. We shall assume that the irreducible V-module equivalent to V is W_1. Also since the contragredient of an irreducible V-module is still irreducible, given $1\le i\le m$, there exists $1\le i'\le m$ such that $W_{i'}$ is equivalent to W_i'. If V' is equivalent to V as a V-module, then we have $1'=1$.

Theorem 5.3 ([H5]). *Let V be a simple vertex operator algebra staisfying the following condition:*

1. *For $n<0$, $V_{(n)}=0$ and $V_{(0)}=\mathbb{C}\mathbf{1}$ and as a V-module, V is equivalent to its contragredient V-module V'.*
2. *Every lower-bounded generalized V-module is completely reducible.*
3. *V is C_2-cofinite.*

Then the matrix S diagonalizes the matrix N_j for $j=1,\dots,m$ and we have

$$N_{ij}^{k}=\sum_{l=1}^{m}\frac{S_{li}S_{lj}S_{k'l}}{S_{l1}}, \tag{5.41}$$

for $i,j,k=1,\dots,m$.

In fact, the dioganoal entries of the diagonal matrices obtained from N_i after diagonalization were calculated explicitly (see [MS1], [MS2] and [H5]). We now describe these diagonal entries.

For $i = 1, \dots, m$, we use $\mathcal{Y}_{1i}^1$ to denote the vertex operator map for the V-module W_i. Let $\mathcal{Y}_{i1}^i$, $\mathcal{Y}_{ii'}^1$, $\mathcal{Y}_{i'i}^1$ and $\mathcal{Y}_{ii'}^1$ be the intertwining operators obtained from $\mathcal{Y}_{1i}^1$ using the skew-symmetry, from $\mathcal{Y}_{i1}^i$ using contragredient and from $\mathcal{Y}_{ii'}^1$ using skew-symmetry, respectively. Then for $w_i \in W_i$, $w_j \in W_j$, $w_i' \in W_i'$ and $w_j' \in W_j'$, we have a multivalued analytic function in z_1 and z_2 obtained by analytically extending

$$\langle w_i', \mathcal{Y}_{i1}^i(w_i, z_1)\mathcal{Y}_{j'j}^1(w_j', z_2)w_j\rangle.$$

Starting from the value

$$\langle w_i', \mathcal{Y}_{i1}^i(w_i, 2)\mathcal{Y}_{j'j}^1(w_j', 1)w_j\rangle$$

of this multivalued analytic function, we obtain another value of the multivalued analytic function at the same point $(2, 1)$ along the path

$$t \mapsto \left(\frac{3}{2} + \frac{e^{-2\pi it}}{2}, \frac{3}{2} - \frac{e^{-2\pi it}}{2}\right).$$

But such a value can also be written as the value of the product of two intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ in the form

$$\langle w_i', \mathcal{Y}_1(w_i, 2)\mathcal{Y}_2(w_j', 1)w_j\rangle.$$

Since every V-module is a direct sum of irreducible V-modules and $\mathcal{Y}_1$ and $\mathcal{Y}_2$ can always be written as the linear combinations of bases of intertwining operators, we have

$$\langle w_i', \mathcal{Y}_1(w_i, 2)\mathcal{Y}_2(w_j', 1)w_j\rangle = (B^{(-1)})_{ij}^2\langle w_i', \mathcal{Y}_1(w_i, 2)\mathcal{Y}_2(w_j', 1)w_j\rangle + \cdots,$$

where $(B^{(-1)})_{ij}^2 \in \mathbb{C}$ and $\cdots$ are sums of products of other basis elements of intertwining operators. The number $(B^{(-1)})_{ij}^2$ is in fact one entry of a matrix $(B^{(-1)})^2$. We use the notation $(B^{(-1)})^2$ because this matrix is in fact the square of a matrix called braiding matrix obtained from the commutativity of intertwining operators, which is in fact a consequence of the associativity and skew-symmetry of intertwining operators.

Also, by the associativity of intertwining operators, for $w_i, \tilde{w}_i \in W_i$ and $w_i', \tilde{w}_i \in W_i'$,

$$\langle \tilde{w}_i', \mathcal{Y}_{i1}^i(w_i, z_1)\mathcal{Y}_{j'j}^1(w_i', z_2)\tilde{w}_i\rangle = \langle \tilde{w}_i', \mathcal{Y}_3(\mathcal{Y}_4(w_i, z_1 - z_2)w_i', z_2)\tilde{w}_j\rangle.$$

Agian, since every V-module is a direct sum of irreducible V-modules and $\mathcal{Y}_3$ and $\mathcal{Y}_4$ can always be written as the linear combinations of bases of intertwining operators, we have

$$\langle \tilde{w}_i', \mathcal{Y}_3(\mathcal{Y}_4(w_i, z_1 - z_2)w_i', z_2)\tilde{w}_j\rangle = F_i\langle \tilde{w}_i', \mathcal{Y}_{1i}^i(\mathcal{Y}_{ii'}^1(w_i, z_1 - z_2)w_i', z_2)\tilde{w}_j\rangle + \cdots,$$

where $F_i \in \mathbb{C}$ and $\cdots$ sums of iterates of other basis elements of intertwining operators. The number F_i is also one entry of a matrix F called fusing matrix.

In fact, the proof of the theorem is to derive first the formula

$$\sum_{l=1}^{m} S_{li}(B^{(-1)})_{lj}^2(S^{-1})_{kl} = N_{ij}^k F_j \tag{5.42}$$

using the associativity and modular invariance of intertwining operators. Then it is easy to see that F_j cannot be 0. In fact, if $F_j = 0$, then $(B^{(-1)})^2_{lj} = 0$ for $l = 1, \ldots, m$. This means that the matrix $(B^{(-1)})^2$ is not invertable. But $(B^{(-1)})^2$ is the square of the braiding matrix and thus is always invertable. Then from (5.43), we obtain

$$\sum_{i,k=1}^{m} (S^{-1})_{li} N_{ij}^{k} S_{kn} = \delta_{ln} \frac{(B^{(-1)})^2_{lj}}{F_j}. \tag{5.43}$$

One then prove

$$(S^{-1})_{li} = S_{l'j}, \tag{5.44}$$

$$\frac{(B^{(-1)})^2_{lj}}{F_j} = \frac{S_{lj}}{S_{l1}}. \tag{5.45}$$

From (5.43), (5.44) and (5.45), we obtain (5.41). Another important formula is

$$S_{ij} = \frac{S_{11}(B^{(-1)})^2_{ij}}{F_i F_j}. \tag{5.46}$$

5.3 Modular tensor category structure

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