Representation theory of vertex operator algebras and conformal field theory

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The Mathematics of Conformal Field Theory II

Outline

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1. A definition of conformal field theory
2. The representation theory of vertex operator algebras and a program to construct conformal field theories
3. Modular functors, Verlinde formula and modular tensor categories
4. Towards a construction of higher-genus theories
5. The current status of the program

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Early mathematical formulations before a mathematical definition

- In this talk, by conformal field theory, I always mean two-dimensional conformal field theory.
- A formulation of conformal field theory on the complex plane in terms of operator product expansion was first given by Belavin, Polyakov and Zamolodchikov in 1984.
- Starting from 1986, under the direction of I. Frenkel, Tsukada constructed the lattice vertex operator algebras using the path integral approach. In the lattice case, there is no interaction term. So the path integral approach works.
- From this construction, it is clear that classical vertex operators correspond to certain geometric objects. Based on this geometric meaning, I. Frenkel gave a geometric interpretation of vertex operators and their associativity property.

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- In 1987, Verlinde formulated the Verlinde conjecture and derived the Verlinde formula from the Verlinde conjecture.
- In 1987, Moore and Seiberg showed that the Verlinde conjecture can be derived from more fundamental conjectures on rational conformal field theories: The operator product expansion and modular invariance of chiral vertex operators.
- Moore and Seiberg derived a set of polynomial equations from these two conjectures on chiral vertex operators. The Verlinde conjecture is a property of the solutions of these equations.

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The definition of (full) conformal field theory by Kontsevich and G. Segal

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- Kontsevich gave a talk on the definition. G. Segal gave a number of talks and circulated a manuscript "The definition of conformal field theory," which was published later in 2004.
- Segal introduced a category formed by Riemann surfaces with parametrized, oriented and ordered boundary components.
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The definition using the language of PROPs

- To avoid the complications caused by category theory, it is better to use the language of PROPs.
- A PROP is a sequence of sets of elements with m "inputs" and n "outputs" for $m, n \in \mathbb{N}$ equipped with composition operations satisfying natural conditions.
- An algebra over a PROP is a homomorphism from the PROP to the PROP formed by maps between tensor powers of a vector space or a Hilbert space or a locally convex topological vector space.
- The tensor product of a power of the determinant line bundle over the moduli spaces of Riemann surfaces with parametrized, oriented and ordered boundary components with a suitable power of its conplex conjugation forms a PROP.
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Segal's modular functors

- Conformal field theories are studied using their chiral and anti-chiral parts.
- To describe chiral and antichiral parts of conformal field theories, G. Segal introduced in 1988 the notions of modular functor and weakly conformal field theory.
- To describe modular functors and weakly conformal field theories, we need to consider Riemann surfaces with parametrized, ordered and labeled (by a finite set $\mathcal{A}$) boundary components.
- A modular functor is a functor from the category of Riemann surfaces with parametrized, ordered and labeled boundary components to the category of finite-dimensional vector spaces satisfying natural properties.
- One can also define a modular functor to be a finite-rank holomorphic vector bundle over the moduli space of such Riemann surfaces satisfying natural properties.

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- Given a modular functor, we can construct a PROP which is in particular a finite-rank holomorphic vector bundle with a projectively flat connection over the PROP of the moduli space of Riemann surfaces with parametrized, labeled and ordered boundary components.
- A weakly conformal field theory is an algebra over such a PROP satisfying some additional natural axioms.
- Given two weakly conformal field theories with nondegenerate bilinear forms on the fibers of the modular functors satisfying suitable conditions, one can put them together to give a (full) conformal field theory.
- This reduces the construction of (full) conformal field theories to the constructions of modular functors and weakly conformal field theories satisfying suitable conditions.

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Vertex operator algebras

- The formulation of Belavin, Polyakov and Zamolodchikov is for general chiral conformal fields, including multivalued conformal fields. When one considers only meromorphic fields, one obtain chiral algebras which are equivalent to vertex operator algebras.
- The precise definition of vertex operator algebras was given by Borcherds in 1986 and by Frenkel-Lepowsky-Meurman in a slightly different form in 1988.
- Inspired by the work of Frenkel, Segal and Vafa, a geometric definition of vertex operator algebra was given by H. in 1990.
- This geometric definition was later reformulated using the language of operads. The underlying (partial) operad is formed by a complex power of the determinant line bundle over the moduli space of genus-zero Riemann spheres with one negatively oriented and n positively oriented punctures and local coordinates vanishing at the punctures for $n \in \mathbb{N}$.

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A program to construct conformal field theories in the sense of Segal

- The geometry formulation of vertex operator algebras led naturally to an attempt to construct conformal field theories from vertex operator algebras.
- In the case that the only irreducible module for a vertex operator algebra is itself, it is possible to construct a conformal field theory using only the vertex operator algebra.
- It is in general impossible to construct genus-one conformal field theories without using modules and intertwining operators among modules.
- Thus the program to construct conformal field theory should start with modules and intertwining operators among modules.
- If this program is successful, then the representation theory of vertex operator algebra is in some sense equivalent to conformal field theory.

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- It was known that the character of an integrable highest irreducible module for an affine Lie algebra is not invariant under the modular transformations. Instead, it is the space spanned by such characters that is invariant under the modular transformation.
- In 1990, Zhu proved the modular invariance of the space spanned by the characters of irreducible modules for a suitable vertex operator algebra. In fact, Zhu proved a special case of the Moore-Seiberg conjecture on the modular invariance of chiral vertex operators in which the chiral vertex operators are vertex operators for modules.
- In particular, the character of a vertex operator algebra is in general not invariant under the modular transformations.
- This means that in general it is impossible to construct genus-one conformal field theories using vertex operator algebras only.
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Intertwining operators (chiral vertex operators)

- Since we must consider modules, we also have to consider conformal fields between modules.
- Intertwining operators without logarithms were studied in the work of Belavin-Polyakov- Zamolodchikov based on the assumption that they have operator product expansion. They were called chiral vertex operators by Moore-Seiberg.
- These are called intertwining operators in mathematics and were introduced in mathematics by Frenkel-H.-Lepowsky in 1989.
- Intertwining operators with logarithms were studied in physics first by Rozansky-Saleur and Gurarie and others in early and mid 1990's. In mathematics, they are studied first by Milas in 2000.
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Operator product expansion or associativity for intertwining operators

- Operator product expansion for intertwining operators was in fact one of most fundamental conjectures in the work of Moore-Seiberg.
- At this moment, the most general result on the associativity or the operator product expansion is the following:
- Assume that (i) every grading-restricted generalized module is C_1-cofinite and (ii) every lower-bounded generalized V-module generated by a generalized eigenvector for $L(0)$ in the dual space of the tensor product of two grading-restricted generalized modules is grading restricted. Then we have the associativity (or operator product expansion for intwertwining operators among grading-restricted generalized V-modues. (H., 2002, 2009, 2014.)

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- Assume that (i) the vertex operator algebra has only elements of nongegative weights and the the space of weight 0 is one dimensional, (ii) the vertex operator algebra is C_2-cofinite and (iii) every lower-bounded generalized V-module is a direct sum of irreducible modules. Then the spaces of q-traces of products of intertwining operators are invariant under the modular transformations. (H., 2003)

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Outline

Modular functors

- The associativity for intertwining operators allows us to construct chiral multipoint correlation functions using analytic extensions of products of intertwining operators.
- The modular invariance for intertwining operators allows us to construct chiral genus-one correlation functions using analytic extensions of q-traces of products of intertwining operators.
- The genus-zero and genus-one modular correlation functions gave holomorphic vector bundles over the genus-zero and genus-one moduli spaces of Riemann surfaces with parametrized, labeled and ordered boundary components.
- These vector bundles give us modular functors.
- One next step is to construct genus-zero and genus-one full conformal field theories.

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Verlinde formula

- To do this, we need to construct nondegenerate bilinear or Hermitian form on genus-zero modular functors.
- Surprisingly, the proof of the nondegeneracy of such bilinear or Hermitian forms on genus-zero modular functors needs the Verlinde formula or formulas used to prove it.
- The Verlinde formula states that the fusion rules can be expressed using the matrix of the transformation of the vacuum characters under $\tau \mapsto -\frac{1}{\tau}$.
- Under the assumption that the conjectures on operator product expansion and modular invariance for intertwining operators are true, Moore and Seiberg proved the Verlinde formula.
- As is discussed above, these two conjectures were proved in the rational case in 2002 and 2003 (H.). The Verlinde formula indeed follows (H. 2004).

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Bilinear forms on the space of intertwining operators

- There is a natural bilinear form on the space of intertwining operators. If there is a natural anti-linear involution, we will have a natural Hermitian form on this space.
- The main property to be proved is its nondegeneracy.
- Since intertwining operators are three point genus-zero correlation functions, in the beginning we expected that the nondegeneracy should be related only to those properties involving genus-zero Riemann surfaces.
- Surprisingly, we found that the proof of the nondegeneracy needs the Verlinde formula or formulas used to prove it.
- This nondegenracy was proved in 2005 (H.-Kong) and using this nondegenracy, (diagonal) genus-zero and genus-one full rational conformal field theories are constructed in 2005 and 2006, respectively (H.-Kong).

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- in 1992, Turaev formulated a mathematical notion of modular tensor category. Then the work of Moore-Seiberg in fact led to precise conjectures that suitable module categories for affine Lie algebras, the Virasoro algebra, possibly some $\mathcal{W}$-algebras and so on have natural structures of modular tensor categories.
- The construction of braided tensor category structures on suitable module categories for suitable vertex operator algebras were given in 1994 (H.-Lepowsky) in the semisimple case and in 2013 (H.-Lepowsky-Zhang) in the nonsemisimple case.
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Outline

Parametrizations of boundary components of Riemann surfaces

- The moduli space of Riemann surface with parametrized boundary components is the geometric structure underlying conformal field theories. It should be an infinite-dimensional complex manifold. But the manifold structure depends on how we choose the boundary parametrizations.
- The simplest boundary parametrizations are analytic parametrizations. But the limits of analytic parametrizations are not analytic. So these are not the general parametrizations we should use.
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Higher-genus convergence

- In the mathematical construction and study of nontopological quantum field theories, convergence is always a major problem to be solved.
- The convergence of suitable series are the main problems solved in the proofs of operator product expansion and modular invariance for intertwining operators.
- To construct higher-genus weakly conformal field theories, one can easily write down higher-genus correlation functions as series obtained from genus-zero and genus-one correlation functions.
- Assuming their convergence, it is easy to show that the space of these correlation functions are invariant under the action of the mapping class group using the associativity and modular invariance for intertwining operators.
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Gui's proof of the higher-genus convergence conjecture

- The higer-genus convergence in the case of rational conformal field theories was formulated as a conjecture (H. 2015).
- In the genus-zero and genus-one cases, the convergence was proved by using the well known decompositions of meromorphic functions on genus-zero or genus-one moduli spaces into infinite sums of meromorphic fuctions on moduli spaces of less points or lower-genus (genus-zero).
- In fact the q-expansion of the Weierstrass $\wp$-function is such a decomposition.
- The main obstruction is that we were not aware of such a decomposition of meromorphic functions on the moduli space in higher genus.
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Outline

The construction of rational conformal field theories

- In the case of rational conformal field theory, most of the problems have been solved. A long paper by Gui, H., Radnell, Schippers and Staubach to give a complete construction is in preparation.
- But there are still some relatively minor problems to be solved.
- The first is the sewing isomorphisms of determinant lines and their analyticity. Radnell, Schippers and Staubach are making progress on this.
- The second is the topological completions of all modules. Since the higher-genus convergence has been proved, one can use (i) the elements obtained from the series proved to be convergent, (ii) the results on moduli spaces and (iii) the method developed in 1998 and 2000 by H. to construct such completions.
- Finally in the case that the chiral rational conformal field theory is unitary, there is a conjecture (H., 2019) stating that the topological completions are the same as the Hilbert space completions.

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- Operator product expansion for intertwining operators were proved under suitable conditions including C_1-cofiniteness. The braided tensor category structures is a consequence.
- Under the C_2-cofiniteness condition, Fiordalisi proved in 2015 that the pseudo q-traces of products of intertwining operators are absolutely convergent and can be analytically extended to genus-one correlation functions.
- The proof of modular invariance still needs a construction of intertwining operators from suitable symmetric linear functionals.
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Some main open problems in this framework

- One important conformal field theory is the Liouville theory.
- In 2016, David, Kupiainen, Rhodes and Vargas constructed the Liouville theory using probability theory and in 2020, Guillarmou, Kupiainen, Rhodes and Vargas proved the conformal bootstrap hypothesis for Liouville theory using a probabilistic approach.
- Question: What is the correct vertex operator algebra and the correct category of modules for the Liouville theory?
- An important class of conformal field theories are the $N = 2$ superconformal field theories associated to Calabi-Yau manifolds.
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