Some open problems in mathematical two-dimensional conformal field theory

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Dedicated to Jim Lepowsky and Robert Wilson for their 70th birthday

ABSTRACT. We discuss some open problems in a program of constructing and studying two-dimensional conformal field theories using the representation theory of vertex operator algebras.

1. Introduction

Quantum field theory has become an active research area in mathematics in the last forty years. Among all the quantum field theories, topological quantum field theory is the most successful in mathematics mainly because the state space of a topological quantum field theory is typically finite dimensional. Compared with topological quantum field theory, nontopological quantum field theories and the deep mathematical conjectures derived from these theories are still quite distant from a complete mathematical understanding.

One of the most famous but also one of the most difficult problems on nontopological quantum field theories is the existence of four-dimensional quantum Yang-Mills theory and the mass gap problem. On the other hand, two-dimensional conformal field theory as a best understood nontopological quantum field theory has in fact been greatly developed and has also directly provided ideas and methods for the successful solutions of mathematical conjectures and problems. The study of two-dimensional conformal field theory will certainly also shed light on the other more difficult nontopological quantum field theories such as the four-dimensional Yang-Mills theory.

In a program of constructing and studying two-dimensional conformal field theories using the representation theory of vertex operator algebras, the mathematical foundation of two-dimensional conformal field theory has been essentially established. In this note, we discuss some open problems in this program. Some of the problems are well known. The others are what the author is interested in and believes to be important. Most of the problems here are enhanced and modified versions of some problems in the slides that the author prepared for the problem session of this conference.

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These problems emphasize mostly the general theory rather than special examples. For many years, compared with the study of special examples, the development of the general theory in this area has not received the attention that it should have received. Though it is important to verify that special examples satisfy suitable finiteness and/or complete reducibility conditions, it is at least equally important to give a construction of conformal field theories, prove various conjectures proposed by physicists and mathematicians and to solve open problems that will lead to future development of the theory and future interaction with other branches of mathematics and physics. In fact, many results in the general theory including the solutions of longstanding open problems, for example, the operator product expansion of (logarithmic) intertwining operators [H1] [H4] [HLZ6]-[HLZ8] [H8], modular invariance of (logarithmic) intertwining operators [H5] [Fi1] [Fi2] [FH], the Verlinde formula [H6], vertex tensor category structures [HL1]-[HL5] [H1] [HLZ1]-[HLZ9] [H8] and modular tensor category structures [H7], have been known to form the foundation of conformal field theory and are necessary for a deep understanding of even familiar examples such as Wess-Zumino-Witten models, minimal models and orbifold theories obtained from such familiar examples.

Recently, the general theory is starting to show its power in the solutions of some important problems. We expect that more problems will be solved soon using the general theory. On the other hand, since uniqueness and classification results can only be obtained using the general theory, the lack of major uniqueness and classification results in this area is an indication that the general theory is still far from well established. The author hopes that the problems given here will help to encourage the further development of the general theory.

For simplicity, by conformal field theories, we shall always mean two-dimensional conformal field theories.

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2. The construction of rational conformal field theories satisfying the axioms of Kontsevich-Segal-Moore-Seiberg

PROBLEM 2.1. Construct rational conformal field theories satisfying the axioms of Kontsevich, G. Segal [Se] and Moore-Seiberg [MS1] [MS2], or at least prove the existence of such rational conformal field theories. In particular, construct the Wess-Zumino-Witten models and minimal models or at least prove their existence.

In fact, the genus-zero and genus-one parts of the problem has been essentially solved. For a vertex operator algebra satisfying three conditions corresponding to the assumption that the conformal field theory is rational, the operator product expansion or associativity of intertwining operators and the modular invariance of the spaces of q-traces of products of intertwining operators have been proved [H4] [H5]. In fact, correlation functions of genus-zero and genus-one chiral rational conformal

field theories and their properties were given by these results. The genus-zero and genus-one correlation functions for the full rational conformal field theories and their properties were also obtained [**HK1**] [**HK2**]. But there is still no construction of the higher-genus correlation functions. Also, the state space obtained in these constructions is only a graded space with a nondegenerate symmetric bilinear or Hermitian form. To construct the conformal field theories completely, we also need to construct a locally convex complete topological vector space with a nondegenerate Hermitian form.

The construction of the higher-genus correlation functions is the main unsolved problem. From the axioms for conformal field theories, one can see that if a conformal field theory is constructed, then higher-genus correlation functions can be expanded as series obtained using intertwining operators. Since we have not constructed conformal field theories satisfying all the axioms, even though we can still write down these series using intertwining operators, we cannot use the axioms for conformal field theories to derive the convergence of these series. If the convergence of these series can be proved, then the sums of these series give functions on the Teichmüller spaces. Using the associativity of intertwining operators and the modular invariance for intertwining operators that have been proved as theorems in [H4] and [H5], one can prove that these functions on the Teichmüller spaces in fact give flat sections of holomorphic vector bundles with flat connections over the moduli spaces. These are higher-genus chiral correlation functions. Then the construction of higher-genus full correlation functions follows trivially. Thus the problem of constructing the higher-genus correlation functions is reduced to the problem of proving the convergence of the series above. This convergence problem is the generalization in the higher-genus case of the convergence proved in the associativity theorem for intertwining operators in $[\mathbf{H4}]$ and the modular invariance theorem for intertwining operators in [H5].

PROBLEM 2.2. Prove the convergence of these series obtained using intertwining operators to construct higher-genus correlation functions.

To prove this convergence, we need to prove a conjecture on certain functions on the Teichmüller spaces and moduli spaces of Riemann surfaces with parametrized boundaries. Since the conjecture is technical, we shall not state it here. In recent years, Radnell, Schippers and Staubach [**RS1**]-[**RS3**] [**RSS1**]-[**RSS5**] have made important progress in the study of these Teichmüller spaces and moduli spaces. They have found the correct class of Riemann surfaces with parametrized boundaries underlying the definition of conformal field theory by Kontsevich and G. Segal. We expect that the conjecture mentioned above on certain functions on the Teichmüller spaces and moduli spaces will be proved in the near future. Then it can be used to prove the convergence needed in the construction of higher-genus correlation functions.

The construction of locally convex complete topological vector spaces with nondegenerate Hermitian forms is related to the construction of higher-genus correlation functions. From the axioms of Kontsevich-Segal, we see that if we have indeed constructed a conformal field theory, then a Riemann surface S with one connected boundary component corresponds to a map from the field of complex numbers to the state space of the conformal field theory. Such a map is equivalent to an element



h(S) in the state space. Thus we see that the state space of a conformal field theory must contain elements corresponding to Riemann surfaces of arbitrary genera. Only after we have a construction of higher-genus correlation functions, we will be able to understand the structure of the state space of the conformal field theory completely. The genus-zero full conformal field theory constructed in [**HK1**] also has a state space. But this space is a graded vector space obtained from the tensor products and direct sums of modules for the vertex operator algebra and does not contain those elements corresponding to Riemann surfaces of arbitrary genera. The author in 1998 [**H2**] and 2000 [**H3**] constructed a locally convex topological completion of a vertex operator algebra. If we have a construction of higher-genus correlation functions, we can apply the method in [**H2**] and [**H3**] to add elements corresponding to Riemann surfaces of arbitrary genera to the graded state space of the genus-zero full conformal field theory in [**HK1**] to obtain the complete state space of the conformal field theory.

If the graded state space of a genus-zero full conformal field theory has an inner product invariant under the conformal fields constructed from intertwining operators (we call it a compatible inner product), then this graded state space also has a Hilbert space completion. We have the following conjecture:

CONJECTURE 2.3. If the graded state space of a genus-zero full conformal field theory has a compatible inner product, then the locally convex topological completion obtained by adding elements corresponding to higher-genus Riemann surfaces and the Hilbert space completion are isomorphic as locally convex topological vector spaces with grading structures and nondegenerate symmetric Hermitian forms.

3. Cohomology theory for graded vertex algebras and complete reducibility of their modules

In representation theory, complete reducibility of modules is a basic problem. For an associative algebra, if the algebra as a module for itself is completely reducible, then all modules for the algebra are completely reducible and all the irreducible modules appear in the decomposition of the algebra as a direct sum of irreducible modules. But for a vertex operator algebra, even if as a module for itself it is irreducible, there might still be many other irreducible modules and modules that are not completely reducible. For concrete examples of vertex operator algebras, the existing proofs of the complete reducibility of modules were reduced to the complete reducibility of modules for some other algebras or some other properties, not deduced from a general theorem on complete reducibility with easy-to-verify conditions.

An associative algebra is semisimple (equivalent to all modules being completely reducible) if and only if its Hochschild cohomology with any bimodule as coefficients is 0. For a vertex operator algebra, the conformal element is irrelevant to the complete reducibility of modules. Thus we need only consider grading-restricted vertex algebras, which are the same as vertex operator algebras except that they do not have conformal elements. On the other hand, the complete reducibility theorem for associative algebras also applies to commutative associativity algebras. There is no need to replace the Hochschild cohomology by Harrison cohomology, since the commutativity is irrelevant to the complete reducibility of modules. Grading-restricted vertex algebras also have a commutativity property, which is also irrelevant to the complete reducibility of modules. So if there is a similar complete reducibility theorem, it should be for more general algebras that do not have to satisfy commutativity. In 2012, the author [H12] introduced a notion of meromorphic open-string vertex algebra, which should be viewed as a noncommutative generalization of the notion of grading-restricted vertex algebra. In 2010, the author [H10] [H11] introduced a cohomology theory for grading-restricted vertex algebras and proved the basic properties that a cohomology theory must have. The construction of this cohomology in fact contains two steps. The first step is a construction of a cochain complex that is similar to the Hochschild cochain complex. From this cochain complex, we also have a cohomology. The second step is to define a cochain subcomplex of the cochain complex above that is similar to the Harrison cochain complex. The cohomology of this cochain subcomplex is the cohomology of the grading-restricted vertex algebra. The cohomology constructed in the first step is in fact the cohomology when we do not consider the commutativity of the grading-restricted vertex algebra, that is, the cohomology when we view the grading-restricted vertex algebra as a meromorphic open-string vertex algebra. The cochain complex and cohomology in this step can all be generalized to meromorphic open-string vertex algebras. Thus for such an algebra, we also have a cohomology theory.

If we regard grading-restricted vertex algebras as analogues of commutative associative algebras, then meromorphic open-string vertex algebras are analogues of associative algebras. Using the same analogy and what we know about gradingrestricted vertex algebras, we have the following conjecture:

CONJECTURE 3.1. Every grading-restricted generalized module of finite length for a meromorphic open-string vertex algebra is completely reducible if and only if for $n \in \mathbb{Z}_+$, the n-th cohomology of the meromorphic open-string vertex algebra with coefficients in any bimodule is 0. In particular, every grading-restricted generalized module of finite length for a grading-restricted vertex algebra is completely reducible if and only if for $n \in \mathbb{Z}_+$, the n-th cohomology of the grading-restricted vertex algebra viewed as a meromorphic open-string vertex algebra with coefficients in any bimodule (whose left and right actions can be different) is 0.

In 2015, Qi and the author [HQ] proved, under a convergence assumption, that if the first cohomology of a meromorphic open-string vertex algebra with coefficients in any bimodule is 0, then every grading-restricted generalized V-module of

finite length is completely reducible. Thus the conjecture above has been reduced to the convergence assumption and the conjecture that if every grading-restricted generalized module of finite length for a meromorphic open-string vertex algebra is completely reducible, then for $n \in \mathbb{Z}_+$, the *n*-th cohomology of the meromorphic open-string vertex algebra with coefficients in any bimodule is 0.

We know that grading-restricted generalized modules of finite length for the vertex operator algebras associated to the Wess-Zumino-Witten models, the minimal models, the lattice vertex operator algebras, and the moonshine module vertex operator algebra are all completely reducible. If Conjecture 3.1 is true, then their *n*-th cohomologies for $n \in \mathbb{Z}_+$ when they are viewed as meromorphic open-string vertex algebras should be 0. Thus to convince ourselves that Conjecture 3.1 is true, we should first verify that for $n \in \mathbb{Z}_+$, the *n*-th cohomology of such a vertex operator algebra as a meromorphic open-string vertex algebra is 0. But the calculation of the cohomology of such a vertex operator algebra is also a nontrivial open problem because the calculation involves a convergence problem.

CONJECTURE 3.2. Let V be a vertex operator algebra associated to a Wess-Zumino-Witten model, a minimal model, a lattice vertex operator algebra or the moonshine module vertex operator algebra. Then for $n \in \mathbb{Z}_+$, the n-th cohomology of V as a meromorphic open-string vertex algebra with coefficients in any V-bimodule is 0.

If Conjecture 3.1 is proved, we would obtain a criterion for the complete reducibility using the cohomology theory. In addition, the work **[HQ]** also simplifies this criterion to the criterion that the 0-th cohomology with coefficients in any bimodule is 0. But this criterion must hold for all bimodules. This is still very difficult to verify. Our hope is that this necessary and sufficient condition for the complete reducibility can help us to find a criterion that is easy to verify.

PROBLEM 3.3. Find a criterion on a grading-restricted vertex algebra itself for the complete reducibility of modules.

The author thinks that the connection between the cohomology and the Killing form of a Lie algebra might give some hint for the solution of this problem.

4. The moduli space of conformal field theories

Moduli spaces always play an important role in mathematics. From the moduli space of Riemann surfaces to the moduli space of self-dual or anti-self-dual solutions of Yang-Mills equations, many important mathematical results were obtained from the studies of them. The moduli space of conformal field theories is also an important mathematical structure. In mathematics, it is closely related to the moduli space of Calabi-Yau manifolds. In physics, it is closely related to the solution space of string theory and it might also be closely related to topological order. But at this moment, it is even not clear how the topology on this moduli space should be defined.

PROBLEM 4.1. Study the moduli space of conformal field theories. Give topological and geometric structures on the moduli space. Prove that rational conformal field theories are isolated points in the moduli space.

One of the approaches to study the moduli space is to develop a deformation theory of conformal field theories. A deformation theory will at least help us to

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understand the topology of the moduli space. In [H11], the author proved that the first order deformations of a grading-restricted vertex algebra correspond to the second cohomology of this vertex algebra with coefficients in itself. In fact, the author also proved that the third cohomology and a convergence in this cohomology theory is the obstruction for a first order deformation of the vertex algebra to be lifted to a formal deformation. But this proof involved very complicated and technical calculations. The author hopes that after introducing new concepts and methods, the calculations can be simplified. So this work has not been published yet.

We have the following problem:

PROBLEM 4.2. Find the conditions for the formal deformations of gradingrestricted vertex algebras to converge to analytic deformations. Generalize the deformation theory of grading-restricted vertex algebras to a deformation theory of genus-zero full conformal field theories.

5. The construction and study of logarithmic conformal field theories

Though we still need to construct higher-genus correlation functions and prove that the axioms of Kontsevich-Segal-Moore-Seiberg are satisfied, we already have a lot of important results on rational conformal field theories, including the operator product of intertwining operators [H1] [H4], modular invariance of intertwining operators [H5], the Verlinde formula [H6], modular tensor category structures [H7], three-dimensional topological quantum field theories and invariants of knots and 3manifolds (combining [H7] and [T1]), and genus-zero and genus-one rational chiral and full conformal field theories [HK1] [HK2]. But for logarithmic conformal field theories, many of the generalizations of these results still have not been proved and in some cases, even the precise formulations of some of the conjectures are still not known.

Starting from 2001, Lepowsky, Zhang and the author [HLZ1]-[HLZ9] developed a tensor category theory for logarithmic conformal field theories under suitable natural assumptions and have constructed vertex tensor category structures and braided tensor category structures from suitable categories of modules for suitable vertex algebras. The main results are the proof of the associativity of logarithmic intertwining operators (or the logarithmic operator product expansion) and the construction of vertex tensor category and braided tensor category structures. Even though they are natural, the assumptions in this theory are not always trivial to verify. In 2007, the author [H8] proved that for a vertex operator algebra satisfying the two conditions in the conjecture below, the assumptions in the work [HLZ1]-[HLZ9] are satisfied and thus in this case, the category of grading-restricted generalized modules for the vertex operator algebra has natural structures of a vertex tensor category and of a braided tensor category.

In 2009, the author [H9] proposed the following conjecture:

CONJECTURE 5.1. Assume that V is a simple vertex operator algebra satisfying the following conditions:

(1) $V_{(0)} = \mathbb{C}\mathbf{1}, V_{(n)} = 0$ for n < 0 and the contragredient V', as a V-module, is equivalent to V.

(2) V is C_2 -cofinite, that is, $\dim V/C_2(V) < \infty$, where $C_2(V)$ is the subspace of V spanned by elements of the form $\operatorname{Res}_x x^{-2}Y(u,x)v$ for $u, v \in V$ and $Y: V \otimes V \to V[[x, x^{-1}]]$ is the vertex operator map for V.

Then the braided tensor category given in [H8] based on the work [HLZ1]-[HLZ9] is rigid.

In the case of rational conformal field theories, the proof of the rigidity [H7] used the modular invariance [H5]. For a C_2 -cofinite vertex operator algebra without elements of negative weight, the work of Fiordalisi [Fi1] [Fi2] and a joint paper in preparation by Fiordalisi and the author [FH] have proved that modular invariance for logarithmic intertwining operators also holds. We expect that this modular invariance invariance can be used to prove the rigidity conjecture above.

For a rational conformal field theory, the corresponding category of modules for the vertex operator algebra is not only a rigid braided tensor category, but also satisfies a nondegeneracy property, so that together with some other minor properties that it has, it is a modular tensor category. But this nondegeneracy property is formulated using irreducible objects (irreducible modules) in the category, since for a rational conformal field theory, the category is semisimple, that is, every object is a direct sum of irreducible objects. For a logarithmic conformal field theory, since the category is not semisimple, it is not sufficient to consider only irreducible modules. So giving the correct definition of modular tensor category in the case of logarithmic conformal field theories is also an important unsolved problem.

PROBLEM 5.2. Give a definition of modular tensor category in the nonsemisimple case and prove that the braided tensor category in Conjecture 5.1 has such a modular tensor category structure. From such a nonsemisimple modular tensor category, can we still construct a 3-dimensional topological quantum field theory and invariants of knots and 3-manifolds?

The associativity of logarithmic intertwining operators and the modular invariance of logarithmic intertwining operators discussed above gave the genus-zero and genus-one chiral correlation functions in the corresponding logarithmic conformal field theory. But we do not know how to construct the genus-zero and genus-one full correlation functions for the logarithmic full conformal field theory.

PROBLEM 5.3. For the vertex operator algebra satisfying the conditions in Conjecture 5.1, is it possible to construct genus-zero and genus-one full correlation functions from the genus-zero and genus-one chiral correlation functions? From these genus-zero and genus-one correlation functions, is it possible to construct a conformal field theory satisfying the axioms of Kontsevich-Segal, except for the axioms involving unitarity.

6. Orbifold conformal field theories

Orbifold conformal field theories play an important role in the construction and application of conformal field theory. The moonshine module vertex operator algebra constructed by Frenkel, Lepowsky and Meurman [**FLM**] is in fact the first example of orbifold conformal field theories. To study orbifold conformal field theories, we need to first study twisted modules for vertex operator algebras. But the construction and study of twisted modules are still far away from the complete construction of orbifold conformal field theories. First, the author has the following conjecture: CONJECTURE 6.1. Let V be a vertex operator algebra satisfying the two conditions in Conjecture 5.1. Assume that in addition every grading-restricted generalized V-module is completely reducible. Let G be a finite group of automorphisms of V. Then the twisted intertwining operators among the g-twisted V-modules for all $g \in G$ satisfy the associativity, commutativity and modular invariance property.

If we replace the condition requiring that the vertex operator algebra is rational in a conjecture on the category of twisted modules (see Example 5.5 in [Ki]) by the three conditions in Conjecture 6.1, then we obtain the following conjecture:

CONJECTURE 6.2. Let V be a vertex operator satisfying the three conditions in Conjecture 6.1 and let G be a finite group of automorphisms of V. The the category of g-twisted V-modules for all $g \in G$ is a G-crossed (tensor) category in the sense of Turaev [**T2**].

These two conjectures are both under the complete reducibility assumption and are also about finite groups of automorphisms of V. In the case that gradingrestricted generalized V-modules are not complete reducible and G is not finite, we have the following conjecture and problem:

CONJECTURE 6.3. Let V be a vertex operator algebra satisfying the two conditions in Conjecture 5.1 and let G be a finite group of automorphisms of V. Then the twisted logarithmic intertwining operators among the g-twisted V-modules for all $g \in G$ satisfy the associativity, commutativity and modular invariance property.

PROBLEM 6.4. Let V be a vertex operator algebra and let G be a group of automorphisms of V. If G is an infinite group, under what conditions do the twisted logarithmic intertwining operators among the g-twisted V-modules for all $g \in G$ satisfy the associativity, commutativity and modular invariance property? Under what conditions is the category of g-twisted V-modules for all $g \in G$ a G-crossed (tensor) category?

7. The uniqueness of the moonshine module vertex operator algebra and the classification of meromorphic rational conformal field theories of central charge 24

An important problem in the study of conformal field theories is the classification of rational conformal field theories. But for general rational conformal field theories, since this problem might be related to the classification of finite simple groups, it is expected to be very difficult. A relatively practical problem is the classification of meromorphic rational conformal field theories. Here by a meromorphic rational conformal field theory we mean a rational conformal field theory whose chiral vertex operator algebra is the only irreducible module for itself. The classification of meromorphic rational conformal field theories is equivalent to the classification of the corresponding chiral vertex operator algebras.

Meromorphic rational conformal field theories of central charge 24 are especially important. In 1988, Frenkel, Lepowsky and Meurman proposed the following wellknown uniqueness conjecture for the moonshine module vertex operator algebra:

CONJECTURE 7.1. Let V be a vertex operator algebra satisfying the following three conditions:

(1) The central charge of V is 24.

- (2) V has no nonzero elements of weight 1.
- (3) Any irreducible V module is equivalent to V as a V-module and every V-module is completely reducible.

Then V must be isomorphic to the moonshine module vertex operator algebra.

If Conjecture 7.1 is true, then the Monster group can be defined as the automorphism group of a vertex operator algebra satisfying the three conditions in this conjecture.

In 1992, based on the study and conjectures on the Lie algebras obtained from the homogeneous subspaces of weight 1 of meromorphic rational conformal field theories, Schellekens [Sc] proposed a classification conjecture of meromorphic rational conformal field theories with nonzero homogeneous subspaces of weight 1 and of central charge 24. The conjecture says that there are 70 such meromorphic rational conformal field theories. Together with the moonshine module vertex operator algebra, conjecturally there is a total of 71 meromorphic rational conformal field theories of central charge 24. When Schellekens proposed the conjectures, there were only 39 vertex operator algebras for such meromorphic rational conformal field theories constructed. Also, until 2016, even the statement that there are only 70 possible nonzero Lie algebras obtained from the homogeneous subspaces of weight 1 of meromorphic rational conformal field theories of central charge 24 was a conjecture.

In recent years, important progress has been made on the existence part of Schellekens' conjecture. Lam [L], Lam-Shimakura [LS1] [LS2] [LS3], Miyamoto [M], Shimakura-Sagaki [SS], van Ekeren-Möller-Scheithauer [EMS] have obtained 68 of the 70 vertex operator algebras of central charge 24 with nonzero homogeneous subspaces of weight 1 in Schellekens' classification conjecture. There are still two of them that need to be constructed. In [EMS], van Ekeren, Möller and Scheithauer also proved that the Lie algebra obtained from the homogeneous subspace of weight 1 of such a vertex operator algebra must be one of the 70 Lie algebras in the list of Schellekens.

On the other hand, there is no real progress towards the uniqueness part of the Schellekens' classification conjecture, except for the uniqueness, proved in [**EMS**], of the Lie algebras obtained from these vertex operator algebras. Here we state this uniqueness part as the main remaining conjecture in this classification problem.

CONJECTURE 7.2. Let V be a vertex operator algebra satisfying the first and the third conditions in Conjecture 7.1. Prove that V is determined uniquely by the Lie algebra structure on its homogeneous subspace of weight 1.

These two conjectures might need stronger complete reducibility conditions. It is very likely that we might need to assume all weak modules are completely reducible.

In the case that the homogeneous subspace of weight 1 of such a vertex operator algebra is not 0, we can use all the possible vertex operator algebras generated by the corresponding Lie algebra in the list of Schellekens to study Conjecture 7.2. The difficulty of the uniqueness of the moonshine module vertex operator algebra is that there is no such Lie algebra that one can use. The author believes that the proofs of these conjectures need more powerful theories and sophisticated tools of vertex operator algebras that have been developed and are being developed.

8. Calabi-Yau superconformal field theories

In 1985, physicists, including Friedan, Candelas, Horowitz, Strominger, Witten, Alvarez-Gaumé, Coleman, Ginsparg [**Fr**] [**CHSW**] [**ACG**] (see also the paper [**NS**] by Nemeschansky and Sen), proposed a conjecture that the quantum nonlinear σ model with a Calabi-Yau manifold as the target space is an N = 2 superconformal field theory, that is, a conformal field theory with N = 2 superconformal structures.

In 1987, Gepner [**G**] obtained an N = 2 superconformal field theory from N = 2 superconformal minimal models (now called Gepner model). He also conjectured that this N = 2 superconformal field theory should be isomorphic to the quantum nonlinear σ -model with the Fermat quintic threefold in the four-dimensional complex projective space.

To study these conjectures using the representation theory of vertex operator algebras, the first step is to construct the corresponding N = 2 vertex operator superalgebras, that is, vertex operator algebras with N = 2 superconformal structures.

PROBLEM 8.1. Construct a functor from the category of Calabi-Yau manifolds to the category of N = 2 vertex operator superalgebras such that the Fermat quintic threefold corresponds to the N = 2 vertex operator superalgebra for the Gepner model and such that deformations of Calabi-Yau manifolds correspond to deformations of N = 2 superconformal field theories.

Although there have been constructions of N = 2 vertex operator superalgebras from some Calabi-Yau manifolds, there is no general construction that gives the functor in the problem above.

In [H13], the author constructed a sheaf \mathcal{V} of meromorphic open-string vertex algebras on a Riemannian manifold and a sheaf of left modules for \mathcal{V} generated by the sheaf of smooth functions. The Laplacian on the Riemannian manifold was shown in [H13] to be a component of a vertex operator acting on the space of smooth functions. In particular, we have sheaves of left modules for \mathcal{V} generated by eigenfunctions for the Laplacian. Since the state space of the nonlinear σ -models with a Riemannian manifold as the target must contain eigenfunctions for the Laplacian, this construction must be related to the nonlinear σ -model. Certainly for a Riemannian manifold, the corresponding nonlinear σ -model is in general not a conformal field theory. But this corresponds exactly to the fact that in the general case, we obtain sheaves of left modules for \mathcal{V} generated by eigenfunctions instead of modules for a sheaf of vertex operator algebras. We expect that when the Riemannian manifold is a Calabi-Yau manifold, we shall be able to construct sheaves of modules generated by eigenfunctions, eigenforms and eigenspinors for a sheaf of N = 2 superconformal vertex operator superalgebra, not just for \mathcal{V} . We hope that this construction will eventually lead to a construction of the functor discussed in Problem 8.1.

PROBLEM 8.2. If N = 2 vertex operator superalgebras corresponding to Calabi-Yau manifolds can be constructed, study the representation theory of these N = 2vertex operator superalgebras. Use this representation theory to construct the corresponding N = 2 superconformal field theories, including the proof of the operator product expansion of intertwining operators, the proof of modular invariance, the construction of modular tensor categories (in a suitable sense), and the construction of full N = 2 superconformal field theories.

For Calabi-Yau manifolds, the construction of full N = 2 superconformal field theories is very important because many conjectures given by physicists (for example, quantum cohomology and mirror symmetry) are obtained from full N = 2superconformal field theories but cannot be obtained from only chiral N = 2 superconformal field theories.

PROBLEM 8.3. Suppose that N = 2 vertex operator superalgebras corresponding to Calabi-Yau manifolds can be constructed, that the basic results in the representation theory of these algebras can be obtained and that the full N = 2 superconformal field theories can be constructed. Formulate and prove the conjectures (for example, quantum cohomology and mirror symmetry) by physicists on Calabi-Yau manifolds by turning physicists' intuition into mathematical methods using these results and constructions.

Recently, there have been some very interesting developments in the study of the N = 2 vertex operator superalgebras corresponding to K3 surfaces. Early in 2001, Wendland [W] studied the full N = 2 superconformal field theory corresponding to a special K3 surface. She found that it has a very large but finite automorphism group. In 2013, Gaberdiel, Taormina, Volpato and Wendland [GTVW] used the different descriptions of this particular full N = 2 superconformal field theory to determine completely its huge finite automorphism group. In 2015, Duncan and Mack-Crane [DM-C2] discovered that the state space of the Neveu-Schwarz sector of this full N = 2 superconformal field theory as a module for the Virasoro algebra is equivalent to the moonshine module vertex operator superalgebra $V^{S\natural}$ for the Conway group constructed by themselves in 2014 [DM-C1]. They also proved that the state space of the Ramond sector of this full N = 2 superconformal field theory as a module for the Virasoro algebra is equivalent to a twisted $V^{S\natural}$ -module. Using the connections established by these works, Taormina and Wendland have gone further to use the vertex operator superalgebra $V^{S^{\natural}}$ to describe the full N = 2superconformal field theory corresponding to this special K3 surface. We hope that these studies will provide important mathematical ideas and examples for the future construction of full N = 2 superconformal field theories corresponding to Calabi-Yau manifolds, including hints and ideas for the solutions of the problems above.

9. The relation between the approaches of vertex operator algebras and conformal nets

There are many different methods for the study of conformal field theories. But they can all be classified as belonging to one of the two types of methods. The first type is the method of the representation theory of vertex operator algebras. The other is the method of conformal nets. The problems discussed above are all problems using the first type of methods. For the method of conformal nets, the author recommends the survey [Ka] by Kawahigashi. For some results on the connection between the two methods, see [CKLW] by Carpi, Kawahigashi, Longo and Weiner. For a functional-analytic study of vertex operator algebras and their modules, see [H2] and [H3]. Both methods gave modular tensor category structures corresponding to rational conformal field theories. For the study of other problems, they have different advantages.

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PROBLEM 9.1. Find the connection between the method of the representation theory of vertex operator algebras and the method of conformal nets. Prove that at least for rational conformal field theories, they are equivalent.

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