Representation theory of vertex operator algebras and orbifold conformal field theory

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Abstract

We discuss some basic problems and conjectures in a program to construct general orbifold conformal field theories using the representation theory of vertex operator algebras. We first review a program to construct conformal field theories. We also clarify some misunderstandings on vertex operator algebras, modular functors and intertwining operator algebras. Then we discuss some basic open problems and conjectures in mathematical orbifold conformal field theory. Generalized twisted modules and their variants, their constructions and some existence results are reviewed. Twisted intertwining operators and their basic properties are also reviewed. The conjectural properties in the basic open problems and conjectures mentioned above are then formulated precisely and explicitly. Some thoughts of the author on further developments of orbifold conformal field theory are also discussed.

1 Introduction

Two-dimensional orbifold conformal field theories are two-dimensional conformal field theories constructed from known conformal field theories and their automorphisms. The first example of two-dimensional orbifold conformal field theories, the moonshine module, was constructed by Frenkel, Lepowsky and Meurman [FLM1] [FLM2] [FLM3] in mathematics. In this construction, twisted vertex operators studied by Lepowsky [Le] played an important role. The systematic study of two-dimensional orbifold conformal field theories in string theory was started by Dixon, Harvey, Vafa and Witten in [DHVW1] and [DHVW2]. (Since we do not discuss higher-dimensional conformal field theories in this paper, as usual we shall omit the word "two-dimensional" unless we want to emphasize the special feature of two-dimensional theories.) Since then, orbifold conformal field theory has been developed in mathematics as an important part of mathematical conformal field theory.

Orbifold conformal field theory is not just a mathematical procedure to obtain new examples of conformal field theories. More importantly, we expect that it will provide a powerful approach to solve mathematical problems and prove mathematical conjectures after it is fully developed. For example, one of the most important conjecture in the theory of vertex operator algebras and mathematical conformal field theory is the uniqueness of the moonshine module formulated by Frenkel, Lepowsky and Meurman [FLM3]. The moonshine module is constructed as an orbifold conformal field theory from the Leech lattice vertex operator algebra and its automorphism induced from the point reflection in the origin of the Leech lattice (see [FLM3]). From the viewpoint of orbifold conformal field theory, the uniqueness means that every vertex operator algebra obtained as an orbifold conformal field theory satisfying the three conditions in the uniqueness conjecture must be isomorphic to the moonshine module as a vertex operator algebra. In particular, we have to study general orbifold conformal field theories satisfying the three conditions in this uniqueness conjecture. Such a study in turn means that we have to develop a general orbifold conformal field theory.

We would like to emphasize that to solve mathematical problems and prove mathematical conjectures, it is important to develop mathematical techniques that might not be most interesting in physics. For example, existence of orbifold conformal field theories must be proved, not assumed. Since the state spaces of orbifold conformal field theories are always infinite dimensional, it is important to prove, not assume, the convergence of series appearing in mathematical constructions and proofs.

On the other hand, the ideas from physics will undoubtedly play a crucial role in this development of general orbifold conformal field theory. In the last thirty years, mathematicians have witnessed the power of ideas coming from quantum field theory. Quantum field theory not only provides beautiful interpretations of many deep mathematical results, but also suggests new approaches to many difficult mathematical problems. Since topological quantum field theory has been understood quite well in mathematics, the approaches developed using topological quantum field theory have been very successful in solving mathematical problems. On the other hand, although the ideas and conjectures from nontopological quantum field theories such as two-dimensional conformal field theory and four-dimensional Yang-Mills theory have led to solutions of some important problems in mathematics, the nontopological quantum-field-theoretic approaches themselves have not been completely developed and some related mathematical problems were solved using approaches different from the stillto-be-developed quantum-field-theoretic approaches. In the author's opinion, to understand completely the beautiful mathematical structures suggested by nontopological quantum field theories and to solve completely the deep mathematical problems coming from nontopological quantum-field-theoretic ideas, it is necessary to construct these nontopological quantum field theories mathematically and to develop them into new mathematical approaches and tools for mathematical problems.

Conformal field theory is the main nontopological quantum field theory that has been developed substantially in mathematics. We expect that after the mathematical constructions are completed, conformal field theory can be further developed to have the power to solve a number of difficult mathematical problems. Orbifold conformal field theory as an important part of conformal field theory will be crucial in this further development.

In the construction of conformal field theories using the representation theory of vertex operator algebras, chiral conformal field theories are in fact the same as the theory of intertwining operators. Intertwining operators among modules for a vertex operator algebra together with their properties can be used as a working definition of chiral conformal field theory. See Section 2 for a discussion about a program to construct conformal field theories using the representation theory of vertex operator algebras. This program is very successful for rational conformal field theories and their logarithmic generalizations.

We know much less about orbifold conformal field theories. In fact, only the rational orbifold conformal field theories associated to finite cyclic groups have been fully constructed by van Ekeren, Möller and Scheithauer [EMS] and by Möller [Mo] based on the basic constructions and results on rational conformal field theories obtained by the author in [H11], [H12], [H13] and [H14]. Although there are examples of nonabelian orbifold conformal field theories (see for example the one constructed from a lattice of rank 72 and a nonabelian finite group of its automorphisms by Gemünden and Keller [GK]), the full development of mathematical orbifold conformal theory, especially those associated to nonabelian automorphism groups, is still in the beginning stage.

In the present paper, we discuss some basic problems and conjectures in a program to construct general orbifold conformal field theories using the representation theory of vertex operator algebras. Most of the problems and conjectures discussed in this paper were proposed but were not formulated explicitly in the paper [H19]. These problems and conjectures are the foundation for future mathematical developments of orbifold conformal field theory. As in the case of conformal field theories mentioned above, our approach is to view chiral orbifold conformal field theories as the theory of twisted intertwining operators. We use twisted intertwining operators among twisted modules for a vertex operator algebra and a group of its automorphisms and their (conjectural) properties as a working definition of chiral orbifold conformal field theory. Constructing a chiral orbifold conformal field theory is the same as proving the conjectural properties of twisted intertwining operators.

At the end of Section 2, we also clarify some misunderstandings on vertex operator algebras, modular functors and intertwining operator algebras, which for many years, in the author's opinion, have often become an obstruction to the mathematical development of conformal field theory and the representation theory of vertex operator algebras.

The present paper is organized as follows: In Section 2, we review a program to construct conformal field theories using the representation theory of vertex operator algebras. The basic problems and conjectures on a general construction of orbifold conformal field theories are given in Section 3. These problems and conjectures are formulated using suitable twisted modules and twisted intertwining operators. So in Sections 4 and 5, we recall the basic notions and general results about suitable twisted modules and twisted intertwining operators, respectively. In Section 6, we formulate and discuss in details the conjectural properties of twisted intertwining operators stated in the problems and conjectures in Section 3. Although there is no result in Section 6 and the formulations are simple generalizations of those in the untwisted case, the material in this section has not appeared in any other publications. In Section 7, we discuss some thoughts of the author on further developments based on these problems and conjectures.

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2 A program to construct conformal field theories

Vertex operator algebras were introduced in mathematics by Borcherds [Bo] and Frenkel, Lepowsky and Meurman [FLM3] in the study of the moonshine module underlying the Monster finite simple group. They are also algebraic structures appearing in physics in the study of conformal field theory in the work of Belavin, Polyakov and Zamolodchikov [BPZ] and Moore and Seiberg [MS]. Mathematically, Kontsevich and G. Segal [S] formulated a definition of conformal field theory. A conformal field theory is, roughly speaking, an algebra over the operad of Riemann surfaces with parametrized boundaries satisfying certain additional conditions (see [S], [H3], [H4] and [H5]). Conformal field theories can be constructed and studied using intertwining operators (or chiral vertex operators in [MS]) among modules for vertex operator algebras satisfying suitable conditions.

The first main problem in the mathematical study of conformal field theory is to give a mathematical construction. A program to construct conformal field theories using the representation theory of vertex operator algebras has been quite successful in the past thirty years. In this program, the construction of a chiral conformal field theory can be divided into the following steps: (i) Construct a vertex operator algebra and study their modules. (ii) Prove the associativity (or operator product expansion) of intertwining operators among modules for this vertex operator algebra. (iii) Prove the modular invariance of products of intertwining operators. (iv) Prove a higher-genus convergence property (this is still a conjecture even for a rational chiral conformal field theory). To construct a full conformal field theory, one needs to further construct a nondegenerate bilinear form or Hermitian form on the space of intertwining operators satisfying certain conditions so that a chiral conformal field theory. See [H3], [H4], [H8] and [H17] for expositions on this program and [H19] for some of the remaining main open problems in this program.

Many results in the representation theory of vertex operator algebras also hold for more general grading-restricted vertex algebras and Möbius vertex algebras. So here we recall the the notion of vertex operator algebra by first recalling these notions. See [FHL], and [HLZ2], [H18] and [H24] for details. Roughly speaking, a grading-restricted vertex algebra is a \mathbb{Z} graded vector space $V = \coprod_{n \in \mathbb{Z}} V_{(n)}$ satisfying the grading-restriction conditions dim $V_{(n)} < \infty$ for $n \in \mathbb{Z}$ and $V_{(n)} = 0$ for n sufficiently negative, equipped with a vertex operator map

$$Y_V: V \otimes V \to V[[z, z^{-1}]],$$
$$u \otimes v \mapsto Y(u, z)v.$$

(analogous to the multiplication for an associative algebra but parametrized by z), a vacuum $\mathbf{1} \in V_{(0)}$ (analogous to the identity in an associative algebra) satisfying axioms similar to those for commutative associative algebras and also axioms on the derivatives of vertex operators with respect to z and the meromorphicity of the correlation functions obtained from products of vertex operators. A Möbius vertex algebra is a grading-restricted vertex algebra V equipped with a compatible structure of \mathfrak{sl}_2 -module. A vertex operator algebra is a Möbius vertex algebra equipped with a compatible conformal vector such that the

components of the vertex operator of the conformal vector satisfy the Virasoro relations and some other properties. These algebras are in fact analogues of commutative associative algebras with their multiplications controlled by Riemann spheres with three punctures and local coordinates vanishing at the punctures. See [H4] for details.

We now discuss axioms for these algebras. For example, we have two axioms for the vacuum: For $u \in V$, $Y(\mathbf{1}, z)u = u$ and $\lim_{z\to 0} Y(u, z)\mathbf{1} = u$. These are analogues of the axioms for the identity in an associative algebra. The main axiom is the associativity (the analogue of the associativity for an associative algebra). It says that for $u_1, u_2, u_3 \in V$ and $u' \in V'$ (the graded dual of V),

$$\langle u', Y(u_1, z_1)Y(u_2, z_2)u_3 \rangle$$

and

$$\langle u', Y(Y(u_1, z_1 - z_2)u_2, z_2)u_3 \rangle$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1, z_2 with the only possible poles at $z_1 = 0, z_2 = 0$ and $z_1 = z_2$. Another main axiom is the commutativity (the analogue of the commutativity for a commutative associative algebra in a more subtle sense). Roughly speaking, it says that for $u_1, u_2 \in V$, we require that the rational functions obtained by analytically extending

$$\langle u', Y(u_1, z_1)Y(u_2, z_2)u_3 \rangle$$

and

$$\langle u', Y(u_2, z_2)Y(u_1, z_1)u_3 \rangle$$

are the same. In addition, we have the $L_V(-1)$ -derivative property

$$\frac{d}{dz}Y(u,z) = Y(L_V(-1)u,z)$$

for $v \in V$, where

$$L_V(-1)v = \lim_{x \to 0} Y_V(v, z)\mathbf{1}$$

for $v \in V$. This property means that $L_V(-1)$ corresponding to the derivative or infinitesimal translation of the variable z) For a Möbius vertex algebra, let $L_V(1), L_V(0), L_V(-1)$ be the operators giving the \mathfrak{sl}_2 -module structure. Then

$$L_V(-1)v = \lim_{z \to 0} Y_V(v, z)\mathbf{1}$$

for $v \in V$, $L_V(0)$ is the operator giving the grading of V and

$$[L_V(1), Y_V(v, z)] = Y_V(L_V(1)v, z) + 2zY_V(L_V(0)v, z) + z^2Y_V(L_V(-1)v, z).$$

For a vertex operator algebra, we have the axioms for the conformal symmetry. For example, if we let $L_V(n) = \text{Res}_z z^{n+1} Y_V(\omega, z)$, then

$$[L_V(m), L_V(n)] = (m-n)L_V(m+n) + \frac{c}{12}(m^3 - m)\delta_{m+b,0}$$

(the Virasoro relation). Also the Virasoro operators $L_V(1)$, $L_V(0)$ and $L_V(-1)$ should be the same as those operators for the underlying Möbius vertex algebra.

A module for a vertex operator algebra V is roughly speaking a \mathbb{C} -graded vector space and a vertex operator map $Y_W : V \otimes W \to W[[z, z^{-1}]]$ satisfying all the axioms that still make sense. For three modules W_1, W_2 and W_3 , an intertwining operator of type $\binom{W_3}{W_1W_2}$ is a linear map $\mathcal{Y} : W_1 \otimes W_2 \to W_3\{z\}[\log z]$ (here $W_3\{z\}$ means series in complex powers of zwith coefficients in W_3) satisfying all axioms for modules that still make sense.

The intertwining operators of type $\binom{W_3}{W_1W_2}$ form a vector space. This is in fact the space of conformal blocks on the Riemann sphere with three marked points labeled with the equivalence classes of the modules W_1 , W_2 and W'_3 . Its dimension is called a fusion rule. Intertwining operators were called chiral vertex operators in [MS] and were introduced mathematically in [FHL]. If a set of modules for a vertex operator algebra equipped with subspaces of intertwining operators among modules in this set satisfying the associativity and commutativity, we call it an intertwining operator algebra (see [H3], [H5] and [H7] and see also [DL] for a special type of intertwining operators, we mean, roughly, for any intertwining operators \mathcal{Y}_1 and \mathcal{Y}_2 , there exist intertwining operators \mathcal{Y}_3 and \mathcal{Y}_4 such that for w_1, w_2, w_3 and w'_4 in suitable modules,

$$\langle w_4', \mathcal{Y}_1(w_1, z_1) \mathcal{Y}_2(w_2, z_2) w_3 \rangle$$

and

$$\langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3 \rangle$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively. Moreover, these functions can be analytic extended to a common multivalued analytic function with the only possible singular points at $z_1 = 0$, $z_2 = 0$ and $z_1 = z_2$. The associativity of intertwining operators is equivalent to the operator product expansion of chiral vertex operators, one of the two major assumptions or conjectures in the important work [MS] of Moore and Seiberg. Mathematically it was first introduced and proved under suitable conditions in [H1]. By commutativity of intertwining operators, we mean, roughly, for any intertwining operators \mathcal{Y}_1 and \mathcal{Y}_2 , there exist intertwining operators \mathcal{Y}_3 and \mathcal{Y}_4 such that

$$\langle w_4', \mathcal{Y}_1(w_1, z_1) \mathcal{Y}_2(w_2, z_2) w_3 \rangle$$

and

$$\langle w_4', \mathcal{Y}_3(w_2, z_2) \mathcal{Y}_4(w_1, z_1) w_3 \rangle$$

are analytic extensions of each other. The commutativity of intertwining operators is an easy consequence of the associativity and skew-symmetry of intertwining operators (see [H7]).

Intertwining operator algebras give vertex tensor category structures which in turn give braided tensor category structures. Vertex tensor categories can be viewed as analogues of symmetric tensor categories with their tensor product bifunctors controlled by Riemann surfaces with three punctures and local coordinates vanishing at the punctures. See [HL2], [H14] and [HLZ9] for details. Intertwining operator algebras are equivalent to chiral genus-zero conformal field theories (see [H5]). In the program to construct conformal field theories using the representation theory of vertex operator algebras, after the first step of constructing a vertex operator algebra and studying its modules are finished, the second step of proving the associativity (or operator product expansion) of intertwining operators is equivalent to constructing an intertwining operator algebra. To prove the associativity, the main properties that need to be established first are a convergence and extension property for products of an arbitrary number of intertwining operators (see [H11] in the rational case, [HLZ8] for the adaption of the proof in [H11] in the logarithmic case and [Y] for a generalization to vertex algebras with infinite-dimensional homogeneous subspaces with respect to conformal weights but with finite-dimensional homogeneous subspaces with respect to an additional horizontal grading) and a property stating that suitable lower-bounded generalized modules are in the category of modules that we start with (see [H1] for the rational case, [HLZ8] and [H20] for the generalization to the logarithmic case and [HLZ7] for the generalization to the logarithmic case).

The next step is to construct the chiral genus-one conformal field theories. The main properties that need to be established are the convergence and analytic extensions of q-traces or pseudo-q-traces of products of geometrically-modified intertwining operators (see [H12] for the rational case and [F1] and [F2] for the generalization to the logarithmic case) and the modular invariance of the analytic extensions of these q-traces or pseudo-q-traces (see [H12] for the rational case). The genus-one associativity and commutativity are easy consequences of the convergence and analytic extensions of these q-traces or pseudo-q-traces (see [H12] and [F1] and [F2] for the generalization to the logarithmic case).

The main open problem in the construction of chiral higher-genus rational conformal field theories is the convergence of multi q-traces of products of geometrically-modified intertwining operators. The invariance under the action of the mapping class groups is an easy consequence of this convergence, the associativity of intertwining operators and the modular invariance of the q-traces of products of geometrically-modified intertwining operators. The future solution of this problem will depend on the further study of the moduli space of Riemann surfaces with parametrized boundaries, including in particular the study of a conjecture by the author on meromorphic functions on this moduli space. See [RS1]–[RS3] and [RSS1]–[RSS5] for results on this moduli space.

Another problem is the construction of locally convex topological completions of modules for the vertex operator algebra such that intertwining operators and higher-genus correlation functions give maps between these completions. Such completions of vertex operator algebras and their modules using only the correlation functions given by the algebras and modules were given in [H6] and [H9]. If we assume the convergence of multi *q*-traces of products of geometrically-modified intertwining operators discussed above, then the same method as in [H6] and [H9] works when we add those elements coming from genus-zero and genus-one correlation functions obtained using intertwining operators. The author conjectured that these completions obtained using all genus-zero and genus-one correlation functions are the same as the Hilbert space completions if the chiral conformal field theory is unitary (see [H25]).

The discussions above are about the construction of chiral conformal field theories. We also need to construct full conformal field theories and open-closed conformal field theories. To construct a genus-zero full rational conformal field theory, the main property that needs to be proved is the nondegeneracy of an invariant bilinear from on the space of intertwining operators. This nondegeneracy in fact needs a formula used by the author to prove the Verlinde formula in [H13] or equivalently the rigidity of the braided tensor category of modules for the vertex operator algebra proved in [H14]. The construction of genus-one and higher-genus full conformal field theories can then be obtained easily from genus-zero full conformal field theories and chrial genus-one and higher-genus conformal field theories (see [HK2] and [HK3]).

Finally, to construct open-closed conformal field theories, one needs to construct openstring vertex operator algebras from intertwining operator algebras (see [HK1]). Then one has to prove that with the choices of the open-string vertex algebra (the open part) and the full conformal field theory (the closed part), Cardy's condition on the compatibility between the closed part and open part is satisfied (see [Ko1] and [Ko2]).

From the discussion above, we see that intertwining operators are the main objects to study in conformal field theory and also in the representation theory of vertex operator algebras. In fact, conformal field theory is essentially the theory of intertwining operators. Therefore intertwining operators with all their properties can be viewed as a working definition of chiral conformal field theory. Constructing a chiral conformal field theory is the same as proving all the properties of intertwining operators. This will also be our approach in this paper to the construction of orbifold conformal field theories.

Before we end this section, the author would like to correct some misunderstandings about conformal field theories and vertex operator algebras because for a long time, opinions formed based on these and other misunderstandings have often been used mistakenly by journals, organizations and the mathematical community to evaluate researches in this area.

The first misunderstanding is about the role of vertex operator algebras in conformal field theory. A vertex operator algebra in general is certainly not even a chiral conformal field theory. This can be seen easily from the modular invariance of the space of characters of integrable highest weight modules for affine Lie algebras (see [Ka]) and from Zhu's modular invariance theorem (see [Z]) on q-traces of vertex operators acting on modules for a vertex operator algebra. Also many poweful methods used to study vertex operator algebras do not work for intertwining operators. For example, since vertex operator algebras involve only rational functions, the method of multiplying a polynomial to cancel the denominator of a rational function works very well. But this method in general does not work for products of at least two intertwining operators. Also for rational functions, one can use the global expressions of rational functions instead of analytic extensions but for multivalued functions obtained from products of at least two intertwining operators, one has to carefully use analytic extensions to obtain the correct results. In fact, fatal mistakes occurred in papers published in major mathematical journals claiming to simplify major results on intertwining operators without using complex analysis exactly because the methods that work only for vertex operator algebras were applied to the study of products of two intertwining operators. One mistake is to assume that intertwining operators involve only integral powers of the variable, which is not even true for non-self-dual lattice vertex operator algebras, the simplest minimal model of central charge $\frac{1}{2}$ and the simplest Wess-Zumino-Witten models for the Lie algebra \mathfrak{sl}_2 . Another mistake is to define maps without using analytic extensions. When working with multivalued analytic functions (not rational functions), one has to use analytic extensions to define a number of maps. Without a careful use of analytic extensions, one cannot even prove that an arbitrarily defined map is linear, not to mention many other properties that these maps should satisfy.

Another more subtle but also more important fact is that a vertex operator algebra in general does not even determine a chiral conformal field theory uniquely. Instead, it is in fact the choices of modules and intertwining operators that determine uniquely such a theory. Therefore to construct a chiral conformal field theory, though we need to start with a vertex operator algebra, the more crucial part is to choose a category of modules and spaces of intertwining operators. One simple example is the chiral conformal field theory associated to irrational tori. The vertex operator algebra for such a chiral conformal field theory is a Heisenberg vertex operator algebra, which is also the vertex operator algebra for the conformal field theory associated to the corresponding Euclidean space. This vertex operator algebra alone does not lead us to a unique chiral conformal field theory since all different irrational tori and the Euclidean space in the same dimension has the same vertex operator algebra. We have to choose the category of modules for this Heisenberg vertex operator algebra to be the category of finite direct sums of irreducible modules generated by the eigenfunctions of the Laplacian on the given irrational torus and the space of intertwining operators among the irreducible modules to be the spaces of all intertwining operators among these modules. Then we obtain the chiral conformal field theory associated to this particular irrational torus.

The second misunderstanding is about modular functors and modular tensor categories. Modular functors are operads formed by holomorphic vector bundles over the moduli space of Riemann surfaces with parametrized boundaries satisfying certain additional conditions and chiral conformal field theories are algebras over such operads satisfying additional conditions (see [S] and also [H3] and [H5] for the genus zero case). But modular functors themselves are not conformal field theories. To construct a chiral conformal field theory, one has to construct a modular functor with a nondegenrate bilinear form. But a modular functor with a nondegenrate bilinear form alone does not give a conformal field theory. Instead, a modular functor with a nondegenrate bilinear form gives a three-dimensional topological field theory. Similarly, modular tensor categories also give only three-dimensional topological field theories and is equivalent to modular functors with nondegenerate bilinear forms. They are far from conformal field theories. Certainly modular functors and modular tensor categories are very useful in the study of conformal-field-theoretic structures. But this happens only when we already proved a lot of results, for example, the convergence, associativity (operator product expansion) and modular invariance, about intertwining operators or some equivalent structures. For example, the modular tensor categories for the Wess-Zumino-Witten models can be constructed using representations of quantum groups. The fusion coefficients of these tensor categories are indeed given by the Verlinde formula. But these modular tensor categories do not give us the convergence, associativity (operator product expansion) and modular invariance for intertwining operators.

The third misunderstanding is about the history of intertwining operator algebras. Original papers, books and proposals on intertwining operator algebras have often been rejected based on wrong claims that some much later or nonexistent mathematical works already had this notion or some results before intertwining operator algebras were actually introduced and studied. To correct this misunderstanding, we give a brief history of intertwining operator algebras here. (To be accurate, some of the years below are the years that the papers appeared in the arXiv, not the years that they were published.)

In 1984, operator product expansion of chiral conformal fields was studied by Belavin, Polyakov and Zamolodchikov in [BPZ]. In 1988, by assuming that two major conjecturesthe operator product expansion and modular invariance (certainly including implicitly the corresponding convergence) of chiral vertex operators (equivalent to intertwining operators in mathematics)—hold, Moore and Seiberg derived a set of polynomial equations which corresponds to a modular tensor category in the sense of Turaev [T1] and obtained the Verlinde formula [V] as a consequence. In 1992, Dong and Lepowsky in [DL] introduced a special type of intertwining operator algebras called abelian intertwining algebras for which the corresponding braid group representations are one dimensional and gave examples constructed from lattices. In 1995, the author in [H1] formulated and proved the associativity of intertwining operators assuming that a convergence and extension property for intertwining operators and another algebraic condition hold. In particular, the operator product expansion of intertwining operators was proved assuming these conditions. At the same time in 1995, the author in [H2] proved the convergence and extension property and the other algebraic condition needed in [H1] for minimal models. In the same year, the author introduced in [H3] the mathematical notion of intertwining operator algebra using the associativity of intertwining operators and discussed its role in the construction of conformal field theories in the sense of Kontsevich and Segal [S].

In 1997 the author proved a generalized rationality and a Jacobi identity for intertwining operator algebras. In the same year, Lepowsky and the author in [HL6] proved the convergence and extension property and the other algebraic condition needed in [H1] for the Wess-Zumino-Witten models. Also in the same year, the author constructed genus-zero modular functors from intertwining operator algebras and proved that intertwining operator algebras are algebras over the partial operads of such genus-zero modular functors. In 1999 and 2000, Milas and the author in [HM1] and [HM2] proved the convergence and extension property and the other algebraic condition needed in [H1] for the Neveu-Schwarz sectors of N=1 and N=2 superconformal minimal models, respectively. In 2001, the author in [H10] introduced a notion of dual of an intertwining operator algebra analogous to the dual of a lattice and the dual of a code such that the dual of a vertex operator algebra satisfying suitable conditions is the intertwining operator algebra obtained from all intertwining operators among all irreducible modules. In 2002, the author proved the convergence and extension property and the other condition needed in [H1] for vertex operator algebras for which irreducible modules are C_1 -cofinite (weaker than C_2 -cofnite) and N-gradable weak modules are completely reducible. The convergence and extension property proved in this paper in fact also holds for vertex operator algebras for which grading-restricted generalized modules are C_1 -cofinite but might not be completely reducible (see [HLZ8] for a discussion about this fact). In 2003, the author proved the modular invariance for intertwining operator algebras on the direct sum of all (inequivalent representatives of) irreducible modules for a vertex operator algebra satisfying natural finite reductivity conditions, including in particular the C_2 -cofniteness condition and the complete reducibility of N-gradable weak modules. A number of these results on intertwining operator algebras were generalized to the logarithmic and other cases by Lepowsky, Zhang and the author in [HLZ1]–[HLZ9], by the author in [H15] and [H20], by Fiordalisi in [F1] and [F2] and by Yang in [Y].

3 Basic open problems and conjectures in orbifold conformal field theory

In this section, we discuss some basic open problems and conjectures in orbifold conformal field theory.

Roughly speaking, given a conformal field theory and a group of automorphisms of this theory, we would like to construct a conformal field theory that can be viewed as the original conformal field theory divided by this group of automorphisms. To formulate this notion precisely, we assume that the given conformal field theory is constructed using the representation theory of vertex operator algebras, as is discussed in the preceding section, and the group of automorphisms is a group of automorphisms of the vertex operator algebra.

Here is the main open problem on the construction of orbifold conformal field theories:

Problem 3.1 Given a vertex operator algebra V and a group G of automorphisms of V, construct and classify all the conformal field theories whose vertex operator algebras contain the fixed point vertex operator algebra V^G as subalgebras.

It is in fact very difficult to study V^G -modules and intertwining operators among V^G modules. On the other hand, twisted V-modules and twisted intertwining operators among twisted V-modules are analogues of V-modules and intertwining operators among V-modules, it is easier to study these than V^G -modules and intertwining operators among them. We expect that V^G -modules can all be obtained from twisted V-modules (see Theorem 7.1 in [H24] and Theorem 4.8 below for lower-bounded V^G -modules in the case that G is the cyclic group generated by an automorphism g of V). Thus our conjectures and problems below will be mainly on twisted intertwining operators among twisted modules.

Now we state the first main conjecture on orbifold conformal field theories (see [H19]):

Conjecture 3.2 Assume that V is a simple vertex operator algebra satisfying the following conditions:

- 1. $V_{(0)} = \mathbb{C}\mathbf{1}$, $V_{(n)} = 0$ for n < 0 and the contragredient V', as a V-module, is equivalent to V.
- 2. V is C₂-cofinite, that is, dim $V/C_2(V) < \infty$, where $C_2(V)$ is the subspace of V spanned by the elements of the form $\operatorname{Res}_x x^{-2}Y(u, x)v$ for $u, v \in V$ and $Y : V \otimes V \to V[[x, x^{-1}]]$ is the vertex operator map for V.
- 3. Every grading-restricted generalized V-module is completely reducible.

Let G be a finite group of automorphisms of V. Then the twisted intertwining operators among the g-twisted V-modules for all $g \in G$ satisfy the associativity, commutativity and modular invariance property.

The following conjecture (see also [H19]) is a consequence of Conjecture 3.2 (cf. Example 5.5 in [Ki]):

Conjecture 3.3 Let V be a vertex operator satisfying the three conditions in Conjecture 3.2 and let G be a finite group of automorphisms of V. The the category of g-twisted V-modules for all $g \in G$ is a G-crossed modular tensor category (see Turaev [T2] for G-crossed tensor categories).

These two conjectures given in [H19] are both under the complete reducibility assumption and are also about finite groups of automorphisms of V. In the case that grading-restricted generalized V-modules are not complete reducible or G is not finite, we have the following conjectures and problems:

Conjecture 3.4 Let V be a vertex operator algebra satisfying the first two conditions in Conjecture 3.2 and let G be a finite group of automorphisms of V. Then the twisted intertwining operators among the grading-restricted generalized g-twisted V-modules for all $g \in G$ satisfy the associativity, commutativity and modular invariance properties.

Conjecture 3.5 Let V be a vertex operator algebra satisfying the first two conditions in Conjecture 3.2 and let G be a finite group of automorphisms of V. Then the category of grading-restricted generalized g-twisted V-modules for all $g \in G$ has a natural structure of G-crossed tensor category satisfying additional properties.

Problem 3.6 Let V be a vertex operator algebra and let G be a group of automorphisms of V. If G is an infinite group, under what conditions do the twisted intertwining operators among the grading-restricted generalized g-twisted V-modules for all $g \in G$ satisfy the associativity, commutativity and modular invariance properties? Under what conditions is the category of g-twisted V-modules for all $g \in G$ a G-crossed (tensor) category?

Remark 3.7 In the case of $G = \{1\}$, Conjecture 3.2 is a theorem (see [H11] and [H12]). From this theorem, the author constructed a modular tensor category (see [H14]) and thus Conjecture 3.3 is also a theorem in this case.

In the remaining part of this paper, we describe in details the construction and study of twisted modules, the basic and conjectural properties of twisted intertwining operators and some thoughts of the author on further developments of orbifold conformal field theory.

4 Twisted modules, a general construction and existence results

To construct orbifold conformal field theories, we first have to understand the structures and properties of twisted modules. Twisted modules associated to automorphisms of finite orders of vertex operator algebras appeared first in the works of Frenkel-Lepowsky-Meurman [FLM3] and Lepowsky [Le]. In [H16], the author introduced twisted modules associated to automorphisms of arbitrary orders (including in particular, infinite orders). In the case of automorphisms of infinite orders, the logarithm of the variable might appear in twisted vertex operators. Here we recall g-twisted modules and their variants from [H16].

Definition 4.1 Let V be a grading-restricted vertex algebra or a vertex operator algebra. A generalized g-twisted V-module is a \mathbb{C} -graded vector space $W = \coprod_{n \in \mathbb{C}} W_{[n]}$ (graded by *weights*) equipped with a linear map

$$\begin{array}{rcl} Y^g_W : V \otimes W & \to & W\{x\}[\log x], \\ & v \otimes w & \mapsto & Y^g_W(v,x)w \end{array}$$

satisfying the following conditions:

1. The equivariance property: For $p \in \mathbb{Z}$, $z \in \mathbb{C}^{\times}$, $v \in V$ and $w \in W$,

$$(Y_W^g)^{p+1}(gv,z)w = (Y_W^g)^p(v,z)w,$$

where for $p \in \mathbb{Z}$, $(Y_W^g)^p(v, z)$ is the *p*-th analytic branch of $Y_W^g(v, x)$.

- 2. The *identity property*: For $w \in W$, $Y^g(\mathbf{1}, x)w = w$.
- 3. The duality property: For any $u, v \in V, w \in W$ and $w' \in W'$, there exists a multivalued analytic function with preferred branch of the form

$$f(z_1, z_2) = \sum_{i,j,k,l=0}^{N} a_{ijkl} z_1^{m_i} z_2^{n_j} (\log z_1)^k (\log z_2)^l (z_1 - z_2)^{-t}$$

for $N \in \mathbb{N}$, $m_1, \ldots, m_N, n_1, \ldots, n_N \in \mathbb{C}$ and $t \in \mathbb{Z}_+$, such that the series

$$\langle w', (Y_W^g)^p(u, z_1)(Y_W^g)^p(v, z_2)w \rangle, \langle w', (Y_W^g)^p(v, z_2)(Y_W^g)^p(u, z_1)w \rangle, \langle w', (Y_W^g)^p(Y_V(u, z_1 - z_2)v, z_2)w \rangle$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1-z_2| > 0$, respectively, and their sums are equal to the branch

$$f^{p,p}(z_1, z_2) = \sum_{i,j,k,l=0}^{N} a_{ijkl} e^{m_i l_p(z_1)} e^{n_j l_p(z_2)} l_p(z_1)^k l_p(z_2)^l (z_1 - z_2)^{-t}$$

of $f(z_1, z_2)$ in the region $|z_1| > |z_2| > 0$, the region $|z_2| > |z_1| > 0$, the region given by $|z_2| > |z_1 - z_2| > 0$ and $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$, respectively.

- 4. The L(0)-grading condition and g-grading condition: Let $L_W^g(0) = \operatorname{Res}_x X_W^g(\omega, x)$. Then for $n \in \mathbb{C}$ and $\alpha \in \mathbb{C}/\mathbb{Z}$, $w \in W_{[n]}^{[\alpha]}$, there exists $K, \Lambda \in \mathbb{Z}_+$ such that $(L_W^g(0) - n)^K w = (g - e^{2\pi\alpha i})^\Lambda w = 0$.
- 5. The L(-1)-derivative property: For $v \in V$,

$$\frac{d}{dx}Y_W^g(v,x) = Y_W^g(L_V(-1)v,x).$$

A lower-bounded generalized g-twisted V-module is a generalized g-twisted V-module satisfying the condition $W_{[n]} = 0$ for $\Re(n)$ sufficiently negative. A grading-restricted generalized g-twisted V-module is a lower-bounded generalized g-twisted V-module satisfying in addition the condition dim $W_{[n]} < \infty$ for $n \in \mathbb{C}$. An ordinary g-twisted V-module or simply a g-twisted V-module is a grading-restricted generalized g-twisted V-module such that $L_W(0)$ acts on W semisimply.

In [Li], Li studied twisted modules associated to automoprhisms of finite order of a vertex operator algebra using the method of weak commutativity and applied this method to such twisted modules for vertex operator (super)algebras obtained from infinite-dimensional Lie (super)algebras. In [Ba], Bakalov gave a Jacobi identity for twisted modules associated to automorphisms of possibly infinite order, which can be used to replace the duality property in the definition above. See [HY] for a proof.

In the representation theories of associative algebras and of Lie algebras, free modules and Verma modules, respectively, play fundamental roles. For vertex operator algebras, though there had been constructions of twisted modules for some examples of vertex operator algebras, for more than thirty years, no twisted modules analogous to free modules and Verma modules were constructed. In 2019, the author successfully constructed such analogues. These are lower-bounded generalized g-twisted V-modules satisfying a universal property. For the construction in the case that V is a grading-restricted vertex algebra, we refer the reader to [H23]. A minor modification of the construction in [H23] gives such a universal lower-bounded generalized twisted V-modules in the case that V is a vertex operator algebra (see [H26]). Here we state and discuss the existence results and some properties, including in particular the universal property, of lower-bounded and grading-restricted generalized g-twisted V-modules from [H23] and [H24].

Theorem 4.2 ([H23]) Let M be a vector space with actions of g and an operator $L_M(0)$. Assume that there exist operators \mathcal{L}_g , \mathcal{S}_g , \mathcal{N}_g such that on M, $g = e^{2\pi i \mathcal{L}_g}$ and \mathcal{S}_g and \mathcal{N}_g are the semisimple and nilpotent, respectively, parts of \mathcal{L}_g . Also assume that $L_M(0)$ can be decomposed as the sum of its semisimple part $L_M(0)_S$ and nilpotent part $L_M(0)_N$ and that the real parts of the eigenvalues of $L_M(0)$ has a lower bound. Let $B \in \mathbb{R}$ such that B is less than or equal to the real parts of the eigenvalues of $L_M(0)$ on M. Then there exists a lower-bounded generalized g-twisted V-module $\widehat{M}_B^{[g]}$ satisfying the following universal property: Let (W, Y_W^g) be a lower-bounded generalized g-twisted V-module such that $W_{[n]} = 0$ when $\Re(n) < B$ and let M_0 be a subspace of W invariant under the actions of g, \mathcal{S}_g , \mathcal{N}_g , $L_W(0), L_W(0)_S$ and $L_W(0)_N$. Assume that there is a linear map $f: M \to M_0$ commuting with the actions of $g, S_g, \mathcal{N}_g, L_M(0)$ and $L_W(0)|_{M_0}, L_M(0)_S$ and $L_W(0)_S|_{M_0}$ and $L_M(0)_N$ and $L_W(0)_N|_{M_0}$. Then there exists a unique module map $\hat{f}: \widehat{M}_B^{[g]} \to W$ such that $\hat{f}|_M = f$. If f is surjective and (W, Y_W^g) is generated by the coefficients of $(Y^g)_{WV}^W(w_0, x)v$ for $w_0 \in M_0$ and $v \in V$, where $(Y^g)_{WV}^W$ is the twist vertex operator map obtained from Y_W^g , then \hat{f} is surjective.

An explicit construction, not just the existence, of $\widehat{M}_{B}^{[g]}$ was given in [H23] for a gradingrestricted vertex algebra V. One crucial ingredient in this construction is the twist vertex operators introduced and studied in [H22]. In the case that V is a vertex operator algebra, see Subsection 4.1 of [H26]. The lower-bounded generalized g-twisted V-module $\widehat{M}_{B}^{[g]}$ is also unique up to equivalence by the universal property.

One immediate consequence of Theorem 4.2 is the following result:

Corollary 4.3 ([H23]) Let (W, Y_W^g) be a lower-bounded generalized g-twisted V-module generated by the coefficients of $(Y^g)_{WV}^W(w, x)v$ for $w \in M$, where $(Y^g)_{WV}^W$ is the twist vertex operator map obtained from Y_W^g and M is a \mathbb{Z}_2 -graded subspace of W invariant under the actions of g, S_g , \mathcal{N}_g , $L_W(0)$, $L_W(0)_S$ and $L_W(0)_N$. Let $B \in \mathbb{R}$ such that $W_{[n]} = 0$ when $\Re(n) < B$. Then there is a generalized g-twisted V-submodule J of $\widehat{M}_B^{[g]}$ such that W is equivalent as a lower-bounded generalized g-twisted V-module to the quotient module $\widehat{M}_B^{[g]}/J$.

The construction of $\widehat{M}_{B}^{[g]}$ has been used in [H24] to solve some open problems of more than twenty years. We now discuss these results. These results are proved in [H24] when Vis a grading-restricted vertex algebra or a Möbius vertex algebra. But by using the minor modification in [H26] of the construction of universal lower-bounded generalized g-twisted V-modules in the case that V is a vertex operator algebra, the same proofs also work in this case.

The first such open problem is the existence of nonzero lower-bounded generalized gtwisted V-modules. Note that since V itself is a V-module, the existence of nonzero Vmodules is obvious. But it is highly nontrivial why nonzero lower-bounded generalized g-twisted V-modules exist. Assuming that the vertex operator algebra is simple and C_2 cofinite and the automorphism is of finite order, Dong, Li and Mason proved the existence of an irreducible twisted module [DLM2]. But no progress has been made in the general case for more than twenty years. On the other hand, some classes of vertex operator algebras that are not C_2 -cofinite have a very rich and exciting representation theory. For example, vertex operator algebras associated to affine Lie algebras at admissible levels are not C_2 -cofinite. But the category of ordinary modules for such a vertex operator algebra has a braided tensor category structure with a twist (see [CHY]), which is also rigid and in many cases even has a modular tensor category structure (see [CHY] for the conjectures and a proof in the case of \mathfrak{sl}_2 and see [C] for a proof of the rigidity in the simply-laced case). It is important to study vertex operator algebras that are not C_2 -cofinite. From Theorem 4.2, fora general grading-restricted vertex algebra V (not necessarily C_2 -cofinite) we indeed have constructed lower-bounded generalized g-twisted V-modules. But it is not obvious from the construction in [H23] why $\widehat{M}_{B}^{[g]}$ is not 0. In [H24], the author solved this problem completely.

Theorem 4.4 ([H24]) The lower-bounded generalized g-twisted V-module $\widehat{M}_{B}^{[g]}$ is not 0. In particular, there exists nonzero lower-bounded generalized g-twisted V-modules.

In [DLM1], Dong, Li and Mason generalized Zhu's algebra A(V) (see [Z]) to a twisted Zhu's algebra $A_g(V)$ for a vertex operator algebra V and an automorphism g of V of finite order. In [HY], Yang and the author introduced twisted zero-mode algebra $Z_g(V)$ associated to V and an automorphism g of V not necessarily of finite order and also generalized the twisted Zhu's algebra $A_g(V)$ to the case that g is not of finite order. These two associative algebras are in fact isomorphic (see [HY]). In the case that g is of finite order, $A_g \neq 0$ is stated explicitly as a conjecture in the beginning of Section 9 of the arXiv version of [DLM1]. In the case that V is C_2 -cofinite and g is of finite order, Dong, Li and Mason proved this conjecture in [DLM2]. But in general, this conjecture had been open until it was proved in [H24] as an immediate consequence of Theorem 4.4.

Corollary 4.5 ([H24]) The twisted Zhu's algebra $A_g(V)$ or the twisted zero-mode algebra $Z_g(V)$ is not 0.

Another application of Theorem 4.4 is on the following existence of irreducible lowerbounded generalized g-twisted V-module:

Theorem 4.6 ([H24]) Let W be a lower-bounded generalized g-twisted V-module generated by a nonzero element w (for example, $\widehat{M}_B^{[g]}$ when M is a one dimensional space spanned by an element w and B is less than or equal to the real part of the weight of w). Then there exists a maximal submodule J of W such that J does not contain w and the quotient W/J is irreducible.

Though lower-bounded generalized g-twisted V-modules are important in our study of orbifold conformal field theories, we are mainly interested in grading-restricted generalized gtwisted V-modules and ordinary g-twisted V-modules, because their double contragredients are equivalent to themselves. One important problem is the existence of irreducible gradingrestricted generalized g-twisted V-modules and irreducible ordinary g-twisted V-modules. In the case that V is simple, C_2 -cofinite and g is of finite order, Dong, Li and Mason proved in [DLM2] the existence of an irreducible ordinary g-twisted V-module. Their proof used genus-one 1-point functions. Thus the simplicity and C_2 -cofiniteness of V and the finiteness of the order of g are necessary in their approach. Using our construction of the universal lower-bounded generalized g-twisted V-modules, the author proved in [H24] the existence of irreducible grading-restricted generalized g-twisted V-modules and irreducible ordinary g-twisted V-modules under some very weak conditions. In particular, the simplicity and C_2 -cofininess of V and the finiteness of the order of g are not needed. **Theorem 4.7 ([H24])** Let V be a Möbius vertex superalgebra and g an automorphism of V. Assume that the set of real parts of the numbers in P(V) has no accumulation point in \mathbb{R} . If the twisted Zhu's algebra $A_g(V)$ or the twisted zero-mode algebra $Z_g(V)$ is finite dimensional, then there exists an irreducible grading-restricted generalized g-twisted V-module. Such an irreducible grading-restricted generalized g-twisted V-module is an irreducible ordinary gtwisted V-module if g acts on it semisimply. In particular, if g is of finite order, there exists an irreducible ordinary g-twisted V-module.

The author also proved in [H24] that a lower-bounded generalized module for the fixedpoint subalgebra V^g of V can be extended to a lower-bounded generalized g-twisted Vmodule.

Theorem 4.8 ([H24]) Let V be a grading-restricted vertex algebra and W_0 a lower-bounded generalized V^g -module (in particular, W_0 has a lower-bounded grading by \mathbb{C}). Assume that g acts on W_0 and there are semisimple and nilpotent operators S_g and \mathcal{N}_g , respectively, on W_0 such that $g = e^{2\pi i \mathcal{L}_g}$ where $\mathcal{L}_g = \mathcal{S}_g + \mathcal{N}_g$. Then W_0 can be extended to a lower-bounded generalized g-twisted V-module, that is, there exists a lower-bounded generalized g-twisted V-module W and an injective module map $f : W_0 \to W$ of V^g -modules.

We have the following open problem:

Problem 4.9 Finding conditions on the vertex operator algebra such that under these conditions, the universal lower-bounded generalized g-twisted V-modules have irreducible quotients whose homogeneous subspaces are finite dimensional.

Remark 4.10 In the case that V is a grading-restricted vertex algebra or vertex operator algebra associated to an affine Lie algebra, the author solved this problem (see [H26]). Since the results in [H26] are only for special types of examples of grading restricted vertex algebras or vertex operator algebras, we shall not discuss the details here. The interested reader is referred to the paper [H26] for details.

5 Twisted intertwining operators

To construct orbifold conformal field theories, one approach is to study the representation theory of fixed point vertex operator algebras. If one can prove that intertwining operators among suitable modules for the fixed-point vertex operator algebra of a vertex operator algebra under a group of automorphisms satisfy the convergence and extension property, the associativity (operator product expansion), the modular invariance property and the highergenus convergence property as is decribed in Section 2, then one can construct and study the corresponding orbifold conformal field theories using the steps described in Section 2. But it is itself a difficult problem to prove these properties using the properties of intertwining operators among modules for the larger vertex operator algebra. On the other hand, as is mentioned in Section 3, since twisted modules for the larger vertex operator algebra are analogues of (untwisted) modules for the vertex operator algebra, we expect that the properties of intertwining operators among twisted modules can be studied by generalizing the results and approach for the intertwining operators among modules. Moreover, we expect that every module for the fixed-point vertex operator algebra can be obtained from a twisted module for the larger vertex operator algebra. In particular, intertwining operators among modules for the fixed-point vertex operator algebra can also be obtained from intertwining operators among twisted modules for the larger vertex operator algebra. Therefore instead of studying intertwining operators among modules for the fixedpoint vertex operator algebra, we study intertwining operators among twisted modules for the larger vertex operator algebra. For simplicity, as in [H21], we call intertwining operators among twisted modules twisted intertwining operators.

We still need a precise definition of twisted intertwining operator. By generalizing the Jacobi identity for intertwining operators, Xu introduced the notion of intertwining operators among twisted modules associated to commuting automorphisms of finite orders (see [X]). But in general, an orbifold conformal field theory might be associated to a nonabelian group of automorphisms. In particular, we have to introduce and study intertwining operators among twisted modules associated to not-necessarily-commuting automorphisms. Also, the group might not be finite. So we also have to introduce and study intertwining operators among twisted modules associated to automorphisms of infinite orders. The formulation used in [FHL] and [X] cannot be generalized directly to give a definition of intertwining operators among twisted modules associated to not-necessarily-commuting automorphisms. For more than twenty years, no such definition was given in the literature. This is the reason why orbifold conformal field theories associated to nonabelian groups had not been studied much mathematically in the past.

In [H21], the author found a formulation of such a notion of twisted intertwining operators associated to not-necessarily-commuting automorphisms of possibly infinite orders and proved their basic properties. The general theory and construction of orbifold conformal field theories associated to nonabelian groups (including infinite groups) can now be started from such operators.

We first give the precise definition of twisted intertwining operators.

Definition 5.1 Let g_1, g_2, g_3 be automorphisms of V and let W_1, W_2 and W_3 be g_1 -, g_2 - and g_3 -twisted V-modules, respectively. A twisted intertwining operator of type $\binom{W_3}{W_1W_2}$ is a linear map

$$\mathcal{Y}: W_1 \otimes W_2 \to W_3\{x\}[\log x]$$
$$w_1 \otimes w_2 \mapsto \mathcal{Y}(w_1, x)w_2 = \sum_{k=0}^K \sum_{n \in \mathbb{C}} \mathcal{Y}_{n,k}(w_1)w_2 x^{-n-1}(\log x)^k$$

satisfying the following conditions:

1. The lower truncation property: For $w_1 \in W_1$ and $w_2 \in W_2$, $n \in \mathbb{C}$ and $k = 0, \ldots, K$, $\mathcal{Y}_{n+l,k}(w_1)w_2 = 0$ for $l \in \mathbb{N}$ sufficiently large. 2. The duality property: For $u \in V$, $w_1 \in W_1$, $w_2 \in W_2$ and $w'_3 \in W'_3$, there exists a multivalued analytic function with preferred branch

$$f(z_1, z_2; u, w_1, w_2, w'_3) = \sum_{i,j,k,l,m,n=1}^N a_{ijklmn} z_1^{r_i} z_2^{s_j} (z_1 - z_2)^{t_k} (\log z_1)^l (\log z_2)^m (\log(z_1 - z_2))^n$$

for $N \in \mathbb{N}$, $r_i, s_j, t_k, a_{ijklmn} \in \mathbb{C}$, such that for $p_1, p_2, p_{12} \in \mathbb{Z}$, the series

$$\langle w_3', (Y_{W_3}^{g_3})^{p_1}(u, z_1)\mathcal{Y}^{p_2}(w_1, z_2)w_2 \rangle, \langle w_3', \mathcal{Y}^{p_2}(w_1, z_2)(Y_{W_2}^{g_2})^{p_1}(u, z_1)w_2 \rangle, \langle w_3', \mathcal{Y}^{p_2}((Y_{W_1}^{g_1})^{p_{12}}(u, z_1 - z_2)v, z_2)w \rangle$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1-z_2| > 0$, respectively. Moreover, their sums are equal to the branches

$$\begin{split} f^{p_1,p_2,p_1}(z_1,z_2;u,w_1,w_2,w_3') \\ &= \sum_{i,j,k,l,m,n=1}^{N} a_{ijklmn} z_1^{r_i} e^{r_i l_{p_1}(z_1)} e^{s_j l_{p_2}(z_2)} e^{t_k l_{p_1}(z_1-z_2)} (l_{p_1}(z_1))^l (l_{p_2}(z_2))^m (l_{p_1}(z_1-z_2))^n, \\ f^{p_1,p_2,p_2}(z_1,z_2;u,w_1,w_2,w_3') \\ &= \sum_{i,j,k,l,m,n=1}^{N} a_{ijklmn} z_1^{r_i} e^{r_i l_{p_1}(z_1)} e^{s_j l_{p_2}(z_2)} e^{t_k l_{p_2}(z_1-z_2)} (l_{p_1}(z_1))^l (l_{p_2}(z_2))^m (l_{p_2}(z_1-z_2))^n, \\ f^{p_2,p_2,p_{12}}(z_1,z_2;u,w_1,w_2,w_3') \\ &= \sum_{i,j,k,l,m,n=1}^{N} a_{ijklmn} z_1^{r_i} e^{r_i l_{p_2}(z_1)} e^{s_j l_{p_2}(z_2)} e^{t_k l_{p_{12}}(z_1-z_2)} (l_{p_2}(z_1))^l (l_{p_2}(z_2))^m (l_{p_{12}}(z_1-z_2))^n, \end{split}$$

respectively, of $f(z_1, z_2; u, w_1, w_2, w'_3)$ in the region given by $|z_1| > |z_2| > 0$ and $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$, the region given by $|z_2| > |z_1| > 0$ and $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$, the region given by $|z_2| > |z_1 - z_2| > 0$ and $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$, respectively.

3. The L(-1)-derivative property:

$$\frac{d}{dx}\mathcal{Y}(w_1,x) = \mathcal{Y}(L(-1)w_1,x).$$

The correct notion of twisted intertwining operator should have some basic properties. These properties were proved in [H21]. In particular, the notion of twisted intertwining operator introduced in [H21] is indeed the correct one.

The first property is the following:



Figure 1: The braiding graphs corresponding to g_3 (left) and g_1g_2 (right)

Theorem 5.2 ([H21]) Let g_1, g_2, g_3 be automorphisms of V and let W_1, W_2 and W_3 be g_1 -, g_2 - and g_3 -twisted V-modules, respectively. Assume that the vertex operator map for W_3 given by $u \mapsto Y_{W_3}^{g_3}(u, x)$ is injective. If there exists a twisted intertwining operator \mathcal{Y} of type $\binom{W_3}{W_1W_2}$ such that the coefficients of the series $\mathcal{Y}(w_1, x)w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$ span W_3 , then $g_3 = g_1g_2$.

For the proof of this theorem, we refer the reader to [H21]. Here we give a geometric explanation of the theorem in terms of a picture (Figure 1). From the equivariance property for twisted modules, the monodromy of the twisted vertex operators corresponds to the action of the automorphism on V. Given a twisted intertwining operator of type $\binom{W_3}{W_1W_2}$, the monodromy of the twisted vertex operator for W_3 gives g_3 . This is described in the left braiding graph in Figure 1. But this braiding graph is topologically the same as the right braiding graph in Figure 1. It is clear that the right braiding graph in Figure 1 is equal to the product of the monodromy of the twisted vertex operator for W_1 and the monodromy of the twisted vertex operator for W_2 . So the right braiding graph in Figure 1 gives g_1g_2 . Thus we see from this geometric picture, g_3 should be equal to g_1g_2 .

For the other properties, we first need to recall an action of an automorphism h of V on a g-twisted V-module. Let (W, Y_W^g) be a g-twisted V-module. Let h be an automorphism of V and let

$$\begin{aligned}
\phi_h(Y^g) &: V \times W \quad \to \quad W\{x\}[\log x] \\
& v \otimes w \quad \mapsto \quad \phi_h(Y^g)(v, x)w
\end{aligned}$$

be the linear map defined by

$$\phi_h(Y^g)(v,x)w = Y^g(h^{-1}v,x)w$$

Then the pair $(W, \phi_h(Y^g))$ is an hgh^{-1} -twisted V-module. We shall denote the hgh^{-1} -twisted V-module in the proposition above by $\phi_h(W)$.

We now discuss the skew-symmetry isomorphism for twisted intertwining operators. Let g_1, g_2 be automorphisms of V, W_1, W_2 and $W_3 g_1$ -, g_2 - and g_1g_2 -twisted V-modules and \mathcal{Y} a twisted intertwining operator of type $\binom{W_3}{W_1W_2}$. We define linear maps

$$\Omega_{\pm}(\mathcal{Y}) : W_2 \otimes W_1 \to W_3\{x\}[\log x]$$
$$w_2 \otimes w_1 \mapsto \Omega_{\pm}(\mathcal{Y})(w_2, x)w_1$$

by

$$\Omega_{\pm}(\mathcal{Y})(w_2, x)w_1 = e^{xL(-1)}\mathcal{Y}(w_1, y)w_2\Big|_{y^n = e^{\pm n\pi i}x^n, \log y = \log x \pm \pi i}$$

for $w_1 \in W_1$ and $w_2 \in W_2$.

Theorem 5.3 ([H21]) The linear maps $\Omega_+(\mathcal{Y})$ and $\Omega_-(\mathcal{Y})$ are twisted intertwining operators of types $\binom{W_3}{W_2 \phi_{g_2^{-1}}(W_1)}$ and $\binom{W_3}{\phi_{g_1}(W_2) W_1}$, respectively.

From Theorem 5.3, we see that Ω_+ and Ω_- are indeed ismorphisms between spaces of twisted intertwining operators.

Corollary 5.4 ([H21]) The maps $\Omega_+ : \mathcal{V}_{W_1W_2}^{W_3} \to \mathcal{V}_{W_2\phi_{g_2^{-1}}(W_1)}^{W_3}$ and $\Omega_- : \mathcal{V}_{W_1W_2}^{W_3} \to \mathcal{V}_{\phi_{g_1}(W_2)W_1}^{W_3}$ are linear isomorphisms. In particular, $\mathcal{V}_{W_1W_2}^{W_3}$, $\mathcal{V}_{\phi_{g_1}(W_2)W_1}^{W_3}$ and $\mathcal{V}_{W_2\phi_{g_2^{-1}}(W_1)}^{W_3}$ are linearly isomorphic.

Finally we discuss the contragredient isomorphism for twisted intertwining operators. We first recall contragredient twisted V-modules. Let (W, Y_W^g) be a g-twisted V-module relative to G. Let W' be the graded dual of W. Define a linear map

$$\begin{array}{rcl} (Y^g_W)': V \otimes W' & \to & W'\{x\}[\log x], \\ & v \otimes w' & \mapsto & (Y^g_W)'(v,x)w' \end{array}$$

by

$$\langle (Y_W^g)'(v,x)w',w\rangle = \langle w', Y_W^g(e^{xL(1)}(-x^{-2})^{L(0)}v,x^{-1})w\rangle$$

for $v \in V$, $w \in W$ and $w' \in W'$. Then the pair $(W', (Y_W^g)')$ is a g^{-1} -twisted V-module. We call $(W', (Y_W^g)')$ the contragredient twisted V-module of (W, Y_W^g) .

Let g_1, g_2 be automorphisms of V, W_1, W_2 and $W_3 g_{1^-}, g_{2^-}$ and g_1g_2 -twisted V-modules and \mathcal{Y} a twisted intertwining operator of type $\binom{W_3}{W_1W_2}$. We define linear maps

$$\begin{array}{rcl} A_{\pm}(\mathcal{Y}): W_1 \otimes W'_3 & \to & W'_2\{x\}[\log x] \\ & w_1 \otimes w'_3 & \mapsto & A_{\pm}(\mathcal{Y})(w_1, x)w'_3 \end{array}$$

by

$$\langle A_{\pm}(\mathcal{Y})(w_1, x)w_3', w_2 \rangle = \langle w_3', \mathcal{Y}(e^{xL(1)}e^{\pm \pi iL(0)}(x^{-L(0)})^2w_1, x^{-1})w_2 \rangle$$

for $w_1 \in W_1$ and $w_2 \in W_2$ and $w'_3 \in W'_3$.

Theorem 5.5 ([H21]) The linear maps $A_+(\mathcal{Y})$ and $A_-(\mathcal{Y})$ are twisted intertwining operators of types $\binom{\phi_{g_1}(W'_2)}{W_1W'_3}$ and $\binom{W'_2}{W_1\phi_{q_1}^{-1}(W'_3)}$, respectively.

From Theorem 5.5, we see that A_+ and A_- are indeed ismorphisms between spaces of twisted intertwining operators.

Corollary 5.6 ([H21]) The maps $A_+ : \mathcal{V}_{W_1W_2}^{W_3} \to \mathcal{V}_{W_1W_3'}^{\phi_{g_1}(W_2')}$ and $A_- : \mathcal{V}_{W_1W_2}^{W_3} \to \mathcal{V}_{W_1\phi_{g_1^{-1}}(W_3')}^{W_2}$ are linear isomorphisms. In particular, $\mathcal{V}_{W_1W_2}^{W_3}$, $\mathcal{V}_{W_1W_3'}^{\phi_{g_1}(W_2')}$ and $\mathcal{V}_{W_1\phi_{g_1^{-1}}(W_3')}^{W_2}$ are linearly isomorphic.

6 Main conjectural properties of twisted intertwining operators

In Section 3, we have recalled some main conjectures on the construction of orbifold conformal field theories using the representation theory of vertex operator algebras. In this section, we formulate precisely and discuss in details the conjectural properties in these conjectures.

In several conjectures and problems in Section 3, only associativity, commutativity and modular invariance of twisted intertwining operators are stated. But as in the untwisted case, the statements of these properties make sense only when the corresponding convergence properties hold. In fact, the proofs of these convergence properties are one of the main difficult parts of the proofs of these properties in the untwisted case. The twisted case will certainly be the same. So below we shall first give the formulation of these convergence together with the analytic extensions of the convergent series before the formulations of these properties themselves.

We first formulate the conjectural convergence and extension property, associativity and commutativity of twisted intertwining operators. We formulate them for lower-bounded generalized twisted V-modules so that they are more flexible. But we warn the reader that these properties in general will not be true for such general twisted modules. Usually the twisted modules should be grading-restricted and satisfy some additional conditions. In the case that the vertex operator algebra is finite reductive and the group of automorphisms is finite, the category of twisted modules for which these properties holds is conjectured to be the category of ordinary twisted modules. In general the correct categories of twisted modules will be given in precise conjectures in the future and are also an important part of the research on the construction of orbifold conformal field theories.

Convergence and extension property of products of n twisted intertwining operators: Let g_1, \ldots, g_{n+1} be automorphisms of V. Let $W_0, W_1, \ldots, W_{n+1}, \widetilde{W}_1, \ldots, \widetilde{W}_{n-1}$ be lowerbounded generalized $(g_1 \cdots g_{n+1})$ -, g_1 -, \ldots , g_{n+1} -, $(g_2 \cdots g_{n+1})$ -, \ldots , $(g_n g_{n+1})$ -twisted Vmodules, respectively, and $\mathcal{Y}_1, \ldots, \mathcal{Y}_i, \ldots, \mathcal{Y}_n$ twisted intertwining operators of types $\binom{W_0}{W_1 \widetilde{W}_1}$, $\ldots, \binom{\widetilde{W}_{i-1}}{W_i \widetilde{W}_i}$, $\ldots, \binom{\widetilde{W}_{n-1}}{W_n W_{n+1}}$, respectively. For $w_1 \in W_1, \ldots, w_{n+1} \in W_{n+1}$ and $w'_0 \in W'_0$, the series

$$\langle w_0', \mathcal{Y}_1(w_1, z_1) \cdots \mathcal{Y}_n(w_n, z_n) w_{n+1} \rangle$$

in complex variables $z_1, \ldots z_n$ is absolutely convergent in the region $|z_1| > \cdots > |z_n| > 0$ and its sum can be analytically continued to a multivalued analytic function

$$F(\langle u_1', \mathcal{Y}_1(w_1, z_1) \cdots \mathcal{Y}_n(w_n, z_n) u_{n+1} \rangle)$$

on the region

$$\{(z_1,\ldots,z_n) \mid z_i \neq 0, z_i - z_j \neq 0 \text{ for } i \neq j\} \subset \mathbb{C}^n$$

and the only possible singular points $z_i = 0, \infty$ and $z_i = z_j$ are regular singular points.

To formulate the associativity of twisted intertwining operators, we need the following result:

Proposition 6.1 Assume that the convergence and extension property of products of 2 twisted intertwining operators holds. Let g_1, \ldots, g_4, g be automorphisms of V. Let W_1, W_2, W_3, W_4, W be lower-bounded generalized g_1 -, g_2 -, g_3 -, $(g_1g_2g_3)$ -, (g_1g_2) -twisted V-modules and \mathcal{Y}_3 and \mathcal{Y}_4 twisted intertwining operators of types $\binom{W_4}{WW_3}$ and $\binom{W}{W_1W_2}$, respectively. Then for $w_1 \in W_1, w_2 \in W_2, w_3 \in W_3$ and $w'_4 \in W'_4$, the series

$$\langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3
angle$$

is absolutely convergent in the region $|z_2| > |z_1 - z_2| > 0$ and its sum can be analytically continued to a multivalued analytic function

$$F(\langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3 \rangle)$$

on the region

$$\{(z_1, z_2) \mid z_1, z_2, z_1 - z_2 \neq 0\} \subset \mathbb{C}^2$$

and the possible singular points $z_1, z_2, z_1 - z_2 = 0, \infty$ are regular singular points.

The proof of this proposition is completely the same as the proofs of Proposition 14.1 in [H1] and Proposition 7.3 in [HLZ6].

We are ready to state precisely the conjectural associativity or the operator product expansion of twisted intertwining operators.

Associativity of twisted intertwining operators Let g_1, g_2, g_3 be automorphisms of V. Let W_1, W_2, W_3, W_4, W_5 be lower-bounded generalized g_1 -, g_2 -, g_3 -, $(g_1g_2g_3), (g_2g_3)$ -twisted V-modules and \mathcal{Y}_1 and \mathcal{Y}_2 intertwining operators of types $\binom{W_4}{W_1 W_5}$ and $\binom{W_5}{W_2 W_3}$, respectively. There exist a lower-bounded (g_1g_2) -twisted generalized V-module W_6 and intertwining operators \mathcal{Y}_3 and \mathcal{Y}_4 of the types $\binom{W_4}{W_6 W_3}$ and $\binom{W_6}{W_1 W_2}$, respectively, such that for $w_1 \in W_1, w_2 \in W_2$, $w_3 \in W_3$ and $w'_4 \in W'_4$,

$$F(\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle) = F(\langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3\rangle).$$

Another important conjectural property following immediately from the associativity of twisted intertwining operators and Theorem 5.3 is the commutativity of twisted intertwining operators:

Commutativity of twisted intertwining operators Let g_1, g_2, g_3 be automorphisms of V. Let W_1, W_2, W_3, W_4, W_5 be lower-bounded generalized g_1 -, g_2 -, g_3 -, $(g_1g_2g_3)$, (g_2g_3) -twisted V-modules and \mathcal{Y}_1 and \mathcal{Y}_2 intertwining operators of types $\binom{W_4}{W_1 W_5}$ and $\binom{W_5}{W_2 W_3}$, respectively. There exist a lower-bounded g_1g_3 -twisted generalized V-module W_6 and intertwining operators \mathcal{Y}_3 and \mathcal{Y}_4 of the types $\binom{W_4}{\phi_{g_1}(W_2) W_6}$ and $\binom{W_6}{W_1 W_3}$, respectively, or a lower-bounded $(g_2^{-1}g_1g_2g_3)$ -twisted generalized V-module W_6 and intertwining operators \mathcal{Y}_3 and \mathcal{Y}_4 of the types $\binom{W_4}{\phi_{g_2^{-1}}(W_1) W_3}$, respectively, such that for $w_1 \in W_1, w_2 \in W_2$, $w_3 \in W_3$ and $w'_4 \in W'_4$,

$$F(\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle) = F(\langle w_4', \mathcal{Y}_3(w_2, z_2)\mathcal{Y}_4(w_1, z_1)w_3\rangle)$$

The convergence and extension property, the associativity and commutativity of twisted intertwining operators discussed above are genus-zero properties. We now discuss the conjectural genus-one properties. For a conformal field theory, genus-one correlation functions should be equal to the analytic extensions of q-traces or more generally pseudo-q-traces of products of geometrically-modified intertwining operators. So in our case, we have to consider q-trace or pseudo-q-traces of products of geometrically-modified intertwining operators.

We first have to recall briefly geometrically-modified twisted intertwining operators and pseudo-q-traces. Given a twisted intertwining operator \mathcal{Y} of type $\binom{W_3}{W_1 W_2}$ and $w_1 \in W_1$, we have an operator (actually a series with linear maps from W_2 to W_3 as coefficients) $\mathcal{Y}_1(w_1, z)$. The corresponding geometrically-modified operator is

$$\mathcal{Y}_1(\mathcal{U}(q_z)w_1, q_z),$$

where $q^z = e^{2\pi i z}$, $\mathcal{U}(q_z) = (2\pi i q_z)^{L(0)} e^{-L^+(A)}$ and $A_j \in \mathbb{C}$ for $j \in \mathbb{Z}_+$ are defined by

$$\frac{1}{2\pi i}\log(1+2\pi iy) = \left(\exp\left(\sum_{j\in\mathbb{Z}_+} A_j y^{j+1} \frac{\partial}{\partial y}\right)\right) y.$$

See [H12] for details.

For pseudo-traces, we need to consider grading-restricted twisted V-modules equipped with a projective right module structure for an finite-dimensional associative algebra P over \mathbb{C} . We first define pseudo-traces for a finitely generated projective right P-module M. For such a right P-module, there exists a projective basis, that is, a pair of sets $\{w_i\}_{i=1}^n \subseteq M$, $\{w_i'\}_{i=1}^n \subseteq \operatorname{Hom}_P(M, P)$ such that for all $w \in M$, $w = \sum_{i=1}^n w_i w_i'(w)$. A linear function $\phi: P \to \mathbb{C}$ is said to be symmetric if $\phi(pq) = \phi(qp)$ for all $p, q \in P$. For a symmetric linear function ϕ , the pseud-trace $\operatorname{Tr}_{M}^{\phi} \alpha$ for $\alpha \in \operatorname{End}_{P}(M)$ associated to ϕ is the function $\operatorname{Tr}_{M}^{\phi}$ defined by

$$\operatorname{Tr}_{M}^{\phi} \alpha = \phi\left(\sum_{i=1}^{n} w_{i}'(\alpha(w_{i}))\right).$$

For a grading-restricted twisted V-module W equipped with a projective right P-module structure, its homogeneous subspaces $W_{[n]}$ for $n \in \mathbb{C}$ are finitely generated projective right P-modules. Then for a given symmetric linear function ϕ on P, we have the pseudo-trace $\operatorname{Tr}_{M}^{\phi}\alpha_{n}$ of $\alpha_{n} \in \operatorname{End}_{P}(W_{[n]})$. For $\alpha \in \operatorname{End}_{P}(W)$, we have $\alpha_{n} = \pi_{n}\alpha|_{W_{[n]}} \in \operatorname{End}_{P}(W_{[n]})$. We define

$$\operatorname{Tr}_{W}^{\phi} \alpha = \sum_{n \in \mathbb{C}} \operatorname{Tr}_{W_{[n]}}^{\phi} \alpha_{n}.$$

Note that $\operatorname{Tr}_W^{\phi} \alpha$ a series of complex numbers, not a complex number. If we want to get a pseudo-trace in \mathbb{C} , we have to prove its convergence.

Convergence and extension property of pseudo-q-traces of products of n geometrically-modified twisted intertwining operators Let g_i for $i = 1, \ldots, n+1$ be automorphisms of V. Let W_i for $i = 1, \ldots, n$ be grading-restricted generalized g_i -twisted V-modules, \tilde{W}_i for $i = 1, \ldots, n$ $(g_{i+1} \cdots g_{n+1})$ -twisted V-modules, and \mathcal{Y}_i for $i = 1, \ldots, n$ twisted intertwining operators of types $\binom{\tilde{W}_{i-1}}{W_i \tilde{W}_i}$, respectively, where we use the convention $\tilde{W}_0 = \tilde{W}_n$. Let P be a finite-dimensional associative algebra and ϕ a symmetric linear function on P. Assume that $\tilde{W}_0 = \tilde{W}_n$ is also a projective right P-module and its twisted vertex operators commute with the action of P. Assume in addition that the product $\mathcal{Y}_1(w_1, x_1) \cdots \mathcal{Y}_n(w_n, x_n)$ for $w_1 \in W_1, \ldots, w_n \in W_n$ commutes with the action of P. For $w_i \in W_i$, $i = 1, \ldots, n$,

$$\operatorname{Tr}_{\tilde{W}_n}^{\phi} \mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1}) \cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n}) q_{\tau}^{L(0) - \frac{c}{24}}$$

is absolutely convergent in the region $1 > |q_{z_1}| > \ldots > |q_{z_n}| > |q_{\tau}| > 0$ and can be extended to a multivalued analytic function

$$\overline{F}^{\phi}_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1,\ldots,w_n;z_1,\ldots,z_n;\tau).$$

in the region $\Im(\tau) > 0$, $z_i \neq z_j + l + m\tau$ for $i \neq j, l, m \in \mathbb{Z}$.

These multivalued analytic functions are also conjectured to have associativity and commutativity. These properties are consequences of the convergence and extension property of pseudo-q-traces of products of n geometrically-modified twisted intertwining operators and the associativity and commutativity of twisted intertwining operators.

Genus-one associativity In the setting of the convergence and extension property of pseudo-q-traces of products of n geometrically-modified twisted intertwining operators, for $1 \le k \le n-1$, there exist a $(g_k g_{k+1})$ -twisted V-module \hat{W}_k and twisted intertwining operators

 $\hat{\mathcal{Y}}_k$ and $\hat{\mathcal{Y}}_{k+1}$ of types $\begin{pmatrix} \hat{W}_k \\ W_k W_{k+1} \end{pmatrix}$ and $\begin{pmatrix} \tilde{W}_{k-1} \\ \hat{W}_k \tilde{W}_{k+1} \end{pmatrix}$, respectively, such that

$$\overline{F}^{\phi}_{\mathcal{Y}_{1},\dots,\mathcal{Y}_{k-1},\hat{\mathcal{Y}}_{k+1},\mathcal{Y}_{k+2},\dots,\mathcal{Y}_{n}}(w_{1},\dots,w_{k-1},\hat{\mathcal{Y}}_{k}(w_{k},z_{k}-z_{k+1})w_{k+1},\\w_{k+2},\dots,w_{n};z_{1},\dots,z_{k-1},z_{k+1},\dots,z_{n};\tau)$$

is absolutely convergent in the region $1 > |q_{z_1}| > \cdots > |q_{z_{k-1}}| > |q_{z_{k+1}}| > \cdots > |q_{z_n}| > |q_{\tau}| > 0$ and $1 > |q_{(z_k-z_{k+1})} - 1| > 0$ and is convergent to

$$\overline{F}^{\phi}_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1,\ldots,w_n;z_1,\ldots,z_n;\tau)$$

in the region $1 > |q_{z_1}| > \cdots > |q_{z_n}| > |q_{\tau}| > 0$ and $|q_{(z_k - z_{k+1})}| > 1 > |q_{(z_k - z_{k+1})} - 1| > 0$.

Genus-one commutativity In the setting of the convergence and extension property of pseudo-q-traces of products of n geometrically-modified twisted intertwining operators, for $1 \leq k \leq n-1$, there exist a grading-restricted generalized $(g_k g_{k+2} \cdots g_{n+1})$ -twisted V-module \hat{W}_k and twisted intertwining operators $\hat{\mathcal{Y}}_k$ and $\hat{\mathcal{Y}}_{k+1}$ of types $\begin{pmatrix} \hat{W}_k \\ W_k \tilde{W}_{k+1} \end{pmatrix}$ and $\begin{pmatrix} \tilde{W}_{k-1} \\ \phi_{g_k}(W_{k+1}) \tilde{W}_k \end{pmatrix}$, respectively, or a grading-restricted generalized $(g_{k+1}^{-1}g_k g_{k+1}g_{k+2} \cdots g_{n+1})$ -twisted V-module \hat{W}_k and twisted intertwining operators $\hat{\mathcal{Y}}_k$ and $\hat{\mathcal{Y}}_{k+1}$ of types $\begin{pmatrix} \hat{W}_k \\ \phi_{g_{k+1}} \end{pmatrix}$ and $\begin{pmatrix} \tilde{W}_{k-1} \\ \phi_{g_{k+1}} \tilde{W}_k \end{pmatrix}$, respectively, such that

$$\overline{F}^{\phi}_{\mathcal{Y}_{1},\dots,\mathcal{Y}_{n}}(w_{1},\dots,w_{n};z_{1},\dots,z_{n};\tau) = \overline{F}^{\phi}_{\mathcal{Y}_{1},\dots,\mathcal{Y}_{k-1},\hat{\mathcal{Y}}_{k+1},\hat{\mathcal{Y}}_{k},\mathcal{Y}_{k+2}\dots,\mathcal{Y}_{n}}(w_{1},\dots,w_{k-1},w_{k+1},w_{k},w_{k+2},\dots,w_{n};z_{1},\dots,z_{k-1},z_{k+1},z_{k},z_{k+2},\dots,z_{n};\tau).$$

The most important conjectural property of twisted intertwining operators in the genusone case is the modular invaraince.

Modular invariance of twisted intertwining operators For automorphisms g_i of V, grading-restricted generalized g_i -twisted V-modules W_i and $w_i \in W_i$ for i = 1, ..., n, let $\mathcal{F}_{w_1,...,w_n}$ be the vector space spanned by functions of the form

$$\overline{F}^{\phi}_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1,\ldots,w_n;z_1,\ldots,z_n;\tau)$$

for all finite-dimensional associative algebras P, all symmetric linear functions ϕ , all $(g_{i+1} \cdots g_{n+1})$ twisted V-modules \tilde{W}_i for $i = 1, \ldots, n+1$, all projective right P-module structures on \tilde{W}_{n+1} commuting with its twisted vertex operators, all twisted intertwining operators \mathcal{Y}_i of types $\begin{pmatrix} \tilde{W}_{i-1} \\ W_i \tilde{W}_i \end{pmatrix}$ for $i = 1, \ldots, n$, respectively, such that their product commutes with the right action of P on \tilde{W}_{n+1} . Then for

$$\left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}\right) \in SL(2,\mathbb{Z}),$$

$$\overline{F}_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}^{\phi}\left(\left(\frac{1}{\gamma\tau+\delta}\right)^{L(0)}w_1,\dots,\left(\frac{1}{\gamma\tau+\delta}\right)^{L(0)}w_n;\frac{z_1}{\gamma\tau+\delta},\dots,\frac{z_n}{\gamma\tau+\delta};\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)$$

is in $\mathcal{F}_{w_1,\ldots,w_n}$.

Since the commutativity, genus-one associativity and genus-one commutativity are consequences of the other properties, the main properties that need to be proved are the convergence and extension property of products of twisted intertwining operators, associativity of twisted intertwining operators, convergence and extension property of pseudo-q-traces of products of n geometrically-modified twisted intertwining operators and the modular invariance of twisted intertwining operators.

7 Some thoughts on further developments

We discuss in this section briefly some thoughts of the author on further developments based on the conjectural properties in the preceding section.

Assuming that the conjectural properties in Section 6 hold for twisted intertwining operators among suitable twisted modules associated to a group of automorphisms of a vertex operator algebra. Then we can generalize the results described in Section 2 to results on what can be called "equivariant chiral and full conformal field theories." In particular, we should be able to construct equivariant modular functors, equivariant genus-zero and genusone chrial conformal field theories. In Section 3, we mentioned that to construct higher-genus conformal field theories, the problem of the convergence of multi pseudo-q-traces of products of geometrically-modified intertwining operators is still open. In the case of orbifold conformal field theory, there is also a conjectural convergence of multi q-traces of products of geometrically-modified twisted intertwining operators. If this convergence holds, then we can construct "equivariant (all-genus) chiral and full conformal field theories."

We will also be able to obtain G-crossed tensor category structures. As is mentioned in Section 3, Conjecture 3.3 is a consequence of Conjecture 3.2. In fact, without assuming that Conjecture 3.2 holds, tensor product bifunctors for suitable twisted modules can be defined in the same way as in [HL3], [HL5] and [HLZ3] with intertwining operators replaced by twisted intertwining operators in [H21]. On the other hand, though we do have a Jacobi identity for twisted vertex operators and one twisted intertwining operator obtained as a special case for the Jacobi identity for intertwining operators in [H7], it is not as nice as the one for vertex operators and one intertwining operator in [FHL]. Thus the construction of tensor product modules using the compatibility condition and local grading-restriction condition has to be modified by using the method of formulating and studying twisted intertwining operators in [H21] instead of the method based on the Jacobi identity in [HL3], [HL4], [HL5] and [HLZ4]. Now assuming that the convergence and extension property of products of twisted intertwining operators and associativity of twisted intertwining operators hold. Then the associativity of the tensor product bifunctors can be proved in the same way as in [H1] and [HLZ7]. With this construction, we actually obtain what should be called a "vertex G-crossed tensor category" structure and the G-crossed tensor category structure can be derived from this structure in the same way as how the braided tensor category structure is derived from the vertex tensor category structure in [H14] and [HLZ9]. We also expect that if the convergence and extension property of q-traces of products of n geometrically-modified twisted intertwining operators and the modular invariance of twisted intertwining operators hold, then the rigidity can be proved in the same way as in [H14] in the case that the category of lower-bounded generalized twisted modules are semisimple.

The conjectural properties in Section 6 can be proved if we know that the fixed point subalgebra V^G of V satisfies conditions in the papers [H11], [H12], [H13] and [H14] or some other conditions. See [Mi] and [CM] for proofs of the conditions for V^{G} in the case that G is finite cyclic and [Mc1] and [Mc2] for some results one can obtain when V^G is assumed to satisfy certain conditions, including in particular suitable conditions on suitable categories of V^G -modules. But proving these conditions to hold for V^G seems to be at least as difficult as proving the conjectural properties in Section 6. Also note that the reason why we want to prove these conditions for V^G is exactly that these conditions can be used to prove the the conjectural properties in Section 6. In fact, the proof of the reductivity property of V^G in [CM] uses heavily the theory of intertwining operators established in [H11] and [H12]. The author believes that these conditions for V^G follow from the conjectures in Section 3. Therefore in the author's opinion, if possible, we should prove the conjectural properties in Section 6 directly, without using the fixed point subalgebra V^{G} . Then we should be able to derive the algebraic conditions for V^G as consequences. In order to derive the conditions on V^G as consequences, we need to prove that for a general vertex operator algebra V. the properties of (untwisted) intertwining operators (such as the associativity and modular invariance) imply the algebraic conditions on V. This problem is interesting even without the study of orbifold conformal field theory since it provides a deep understanding of the connection between algebraic properties of the vertex operator algebra V and geometric properties of genus-zero and genus-one correlation functions of the corresponding conformal field theory.

In [H10], the author introduced a precise notion of dual of an intertwining operator algebra. Since vertex operator algebra is also an intertwining operator algebra, we also have a dual of a vertex operator algebra. In fact, the dual of a vertex operator algebra is simply the intertwining operator algebra constructed using all intertwining operators among all modules for the vertex operator algebras. In particular, a self-dual vertex operator algebra in this sense means that the only irreducible module is the vertex operator algebra itself and is called in many papers a holomorphic vertex operator algebra. The moonshine module constructed in [FLM3] is a self-dual vertex operator algebra and the uniqueness conjecture of the moonshine module states that a self-dual vertex operator algebra of central charge 24 and without nonzero weight 1 elements must be isomorphic to the moonshine module as a vertex operator algebra. Let V be a self-dual vertex operator algebra of central charge 24 and without nonzero weight 1 elements. It is desirable if we can embed V into a largest possible intertwining operator algebra. Then we can perform all types of operations and constructions in this largest intertwining operator algebra containing V. But taking the dual of V does not work since it is self dual. On the other hand, if we let G be the full automorphism group of V, then we can take the dual of the vertex operator subalgebra V^G of V. This dual of V^G must contain V and is the largest among the duals of all the fixed point subalgebras of V. The construction of the dual of V^G is in fact part of Problem 3.1. As is mentioned after Problem 3.1 and in the beginning of Section 5, instead of studying directly V^G -modules and intertwining operators among V^G -modules, we study twisted V-modules and twisted intertwining operators among twisted V-modules. Thus the construction and study of the dual of V^G are reduced to the construction and study of the orbifold conformal field theory obtained from the vertex operator algebra V and its automorphism group G. We hope that if this orbifold conformal field theory is constructed, we can use it to find a strategy to prove the uniqueness of the moonshine module. Note that the construction of such an orbifold conformal field theory from such an arbitrary vertex operator algebra satisfying three conditions must be general and abstract and cannot be worked out for a particular example such as the moonshine module itself. This is in fact one important reason why we have to develop a general orbifold conformal field theory instead of just explicit examples.

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