Orbifold conformal field theory: General conjectures, open probelms and initial results

Yi-Zhi Huang

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Outline

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1. The main problem in orbifold conformal field theory
2. Twisted intertwining operators
3. Conjectures and problems
4. Results on twisted intertwining operators and tensor product bifunctors

Conformal field theories

- A conformal field theory is a projective algebra over the PROP of the moduli space of Riemann surfaces with parametrized boundaries (in the sense of Kontsevich, Segal and possibly some generalizations to include pseudo-traces).
- The first main problem in conformal field theory is to give a construction.
- A program to construct conformal field theories using the representation theory of vertex operator algebras has been very successful in the past thirty years.
- Intertwining operators are the main objects to study and is the same as chiral conformal fields in conformal field theory.
- To construct and study a conformal field theory is the same thing as proving properties of intertwining operators among modules for a vertex operator algebras.

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Orbifold conformal field theories

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- In 1984 (exactly 40 years ago), the first example of orbifold conformal field theories, the moonshine module, was constructed in mathematics by Frenkel, Lepowsky and Meurman.
- In 1985, Dixon, Harvey, Vafa and Witten published their first paper in physics on orbifold conformal field theories.
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The main problem

- **Main problem**: Given a vertex operator algebra V and a group G of automorphisms of V, construct and classify all the conformal field theories whose chiral algebras (vertex operator algebras) contain the fixed point vertex operator subalgebra V^G of V as a subalgebra.
- It is difficult to study directly V^G-modules and intertwining operators among V^G-modules.
- The only results we have now is the results by Miyamoto and Carnahan-Miyamoto in the case that V is C_2-cofinite and G is finite cyclic.
- On the other hand, we expect that V^G-modules are all contained in some twisted V-modules.
- So another approach is to study intertwining operators among twisted modules for vertex operator algebras. This approach is even more natural than the approach of studying intertwining operators for V^G.

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- Intertwining operators among twisted modules are called twisted intertwining operators.
- When the automorphisms involved are of finite orders and commute with each other, Xu in 1995 generalized intertwining operators to twisted intertwining operators using a Jacobi identity.
- But in general, an orbifold conformal field theory is associated to a nonabelian group of automorphisms.
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Twisted intertwining operators

- In 2017, I introduced and studied one class of such twisted intertwining operators.
- But this class of twisted intertwining operators are still not general enough because the correlation functions defining such twisted intertwining operators are of a special form.
- Recently, JIshen Du and I introduced the most general possible twisted intertwining operators.
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Twsited intertwining operators

- For (generalized) g_1-, g_2- and g_3-twisted V-modules W_1, W_2 and W_3 , a twisted intertwining operator of type $\binom{W_3}{W_1 W_2}$ is a linear map $\mathcal{Y} : W_1 \otimes W_2 \to W_3\{x\}[\log x]$, $w_1 \otimes w_2 \mapsto \mathcal{Y}(w_1, x)w_2$ satisfying the following axioms:
- The $L(-1)$-derivative property:

$$\frac{d}{dx}\mathcal{Y}(w_1, x) = \mathcal{Y}(L(-1)w_1, x).$$

- The series

$$\langle w_3', Y_{W_3}^{g_3}(u_1, z_1) \cdots Y_{W_3}^{g_3}(u_{k-1}, z_{k-1})\mathcal{Y}(w_1, z_k)w_2\rangle$$

is absolutely convergent in the region $|z_1| > \cdots > |z_k| > 0$ and can be maximally extended to a multivalued analytic function on the region $z_i \neq 0$, $z_i \neq z_j$ for $i \neq j$.

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- The duality property: The series

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are absolutely convergent on the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively. Moreover, their sums are equal to a preferred single-valued branch $f^e(z_1, z_2; u, w_1, w_2, w_3')$ of a maximally extended multivalued analytic function $f(z_1, z_2; u, w_1, w_2, w_3')$ on the region given by $|z_1| > |z_2| > 0$ and $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$, the region given by $|z_2| > |z_1| > 0$ and $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$, the region given by $|z_2| > |z_1 - z_2| > 0$ and $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$, respectively.

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Orbifold theory conjectures for finite groups

Conjecture (H., 2020, 2024)

Let V be C_1-cofinite vertex operator algebra such that all V-modules are C_1-cofinite and the differences between the real parts of lowest weights of irreducible V-modules are bounded. Let G be a finite group of automorphisms of V. Then twisted intertwining operators among grading-restricted generalized g-twisted V-modules of finite lengths for $g \in G$ satisfy convergence, associativity and commutativity properties.

Conjecture (H., 2016)

The category of grading-restricted generalized g-twisted V-modules for all $g \in G$ has a natural structure of G-crossed braided tensor category.

- In the case of $G = \{1_V\}$, these conjectures are theorems (H. 2007).

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Orbifold theory conjectures for strongly rational vertex operator algebras and finite groups

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For a strongly rational vertex operator algebra V and a finite group G of automorphisms of V, twisted intertwining operators among g-twisted V-modules for all $g \in G$ satisfy the convergence, associativity, commutativity and modular invariance properties.

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- In the case that $G = \{1_V\}$ or that V is holomorphic and G is finite cyclic, these conjectures have been proved (H., 2002, 2003, 2005, Carnahan -Miyamoto 2016, van Ekeren-Möller-Scheithauer 2015, Möller 2016).

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- In the case that $G = \{1_V\}$ or that V is holomorphic and G is finite cyclic, these conjectures have been proved (H., 2002, 2003, 2005, Carnahan -Miyamoto 2016, van Ekeren-Möller-Scheithauer 2015, Möller 2016).

Orbifold theory conjectures for strongly rational vertex operator algebras and finite groups

Conjecture (H., 2020, 2024)

For a strongly rational vertex operator algebra V and a finite group G of automorphisms of V, twisted intertwining operators among g-twisted V-modules for all $g \in G$ satisfy the convergence, associativity, commutativity and modular invariance properties.

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Problems in the general case

Problem (H., 2020)

Let V be a vertex operator algebra and let G be a group of automorphisms of V. Under what conditions do the twisted intertwining operators among grading-restricted generalized g-twisted V-modules of finite lengths for all $g \in G$ satisfy the convergence, associativity, commutativity and modular invariance properties?

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Under what conditions does the category of grading-restricted generalized g-twisted V-modules of finite lengths for all $g \in G$ has a structure of a G-crossed braided tensor category or a G-crossed modular tensor category structure?

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Outline

Actions of automorphisms of V on twisted V-modules and $P(z)$-intertwining maps

- Let (W, Y_W^g) be a (generalized) g-twisted V-module.
- Let h be annother automorphism of V and let $\phi_h(Y_W^g) : V \times W \to W\{x\}[logx]$, $v \otimes w \mapsto \phi_h(Y^g)(v, x)w$ be the linear map defined by

$$\phi_h(Y_W^g)(v, x)w = Y_W^g(h^{-1}v, x)w.$$

- The pair $(W, \phi_h(Y_W^g))$ is a (generalized) hgh^{-1}-twisted V-module denoted by $\phi_h(W)$ and $W \mapsto \phi_h(W)$ gives an action of h on the category of (generalized) twisted V-modules.
- Let W_1, W_2, W_3 be g_1-, g_2-, g_1g_2-twisted V-modules, respectively, and $z \in \mathbb{C}^\times$. A (twisted) $P(z)$-intertwining map of type $\binom{W_3}{W_1 W_2}$ is a linear map $I : W_1 \otimes W_2 \to \overline{W}_3$ given by $I(w_1 \otimes w_2) = \mathcal{Y}(w_1, z)w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$, where $\mathcal{Y}$ is a twisted intertwining operator of type $\binom{W_3}{W_1 W_2}$.

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Definition of $P(z)$-tensor product of two twisted V-modules

- The definition and construction of $P(z)$-tensor product below is essentially the same as those in the untwisted case given by H.-Lepowsky.
- Let G be a group of automorphisms of V and $\mathcal{C}$ the category of grading-restricted generalized g-twisted V-modules for $g \in G$.
- Let W_1 and W_2 be g_1- and g_2-twisted V-modules, respectively, in $\mathcal{C}$. A $P(z)$-product of W_1 and W_2 is a pair (W_3, I) consisting of a g_1g_2-twisted V-module W_3 in $\mathcal{C}$ and a twisted $P(z)$-intertwining map I of type $\binom{W_3}{W_1 W_2}$.
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A construction of $P(z)$-tensor product of two twisted V-modules

- Given a $P(z)$-product (W_3, I) of W_1 and W_2, for $w_3' \in W_3'$, we have an element $\lambda_{I,w_3'} \in (W_1 \otimes W_2)^*$ defined by $\lambda_{I,w_3'}(w_1 \otimes w_2) = \langle w_3', I(w_1 \otimes w_2)\rangle$ for $w_1 \in W_1$ and $w_2 \in W_2$.
- Let $W_1 \boxdot_{P(z)} W_2$ be the subspace of $(W_1 \otimes W_2)^*$ consisting of $\lambda_{I,w_3'}$ for all $P(z)$-products (W_3, I) and $w_3' \in W_3'$.
- For $v \in V$ and $\lambda_{I,w_3'} \in W_1 \boxdot_{P(z)} W_2$, we define

$$Y_{W_1 \boxdot_{P(z)} W_2}(v, x)\lambda_{I,w_3'} = \lambda_{I, Y_{W_3'}^{(g_1 g_2)^{-1}}(v,x) w_3'}.$$

- We now assume that $W_1 \boxdot_{P(z)} W_2$ is in $\mathcal{C}$ (see a theorem below). Let

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A construction of $P(z)$-tensor product of two twisted V-modules

- We still need to give a $P(z)$-intertwining map $\boxtimes_{P(z)}$.
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Theorem (Jishen Du and H.)

The $P(z)$-product $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$ is a $P(z)$-tensor product of W_1 and W_2.

Theorem (Jishen Du and H.)

Assume that the following conditions are satisfied:

1. *There are only finitely many irreducible grading-restricted twisted V-modules.*
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Skew symmetry

- For a twisted intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$, let $\Omega(\mathcal{Y}) : W_2 \otimes W_1 \to W_3\{x\}[\log x]$, $w_2 \otimes w_1 \mapsto \Omega(\mathcal{Y})(w_2, x)w_1$ be the linear map defined by

$$\Omega(\mathcal{Y})(w_2, x)w_1 = e^{xL(-1)}\mathcal{Y}(w_1, y)w_2\Big|_{y^n = e^{n(\log z - \pi i)}, \log y = \log z - \pi i}.$$

Theorem (Jishen Du and H.)

$\Omega(\mathcal{Y})$ *is a twisted intertwining operator of type* $\binom{W_3}{\phi_{g_1}(W_2)\, W_1}$.

- Using the skew-symmetry isomorphism $\Omega(\mathcal{Y})$, we obtain a G-commutativity isomorphism $\mathcal{R}_{P(z)} : W_1 \boxtimes_{P(z)} W_2 \to \phi_{g_1}(W_2) \boxtimes_{P(-z)} W_1$ and a G-braiding isomorhism $\mathcal{B} : W_1 \boxtimes_{P(1)} W_2 \to \phi_{g_1}(W_2) \boxtimes_{P(1)} W_1$.

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$P(z)$-compatibility condition, $P(z)$-grading-restriction condition and another construction of $P(z)$-tensor products

- As in the untwisted case, the first major difficulty is the construction of associativity isomorphisms.
- The existence of the associativity isomorphism is in fact equivalent to the associativity of twisted intertwining operators.
- One of the main technique used in the proof of the associativity of intertwining operators in the untwisted case is another construction of the tensor product using the $P(z)$-compatibility condition and $P(z)$-grading-restriction condition.
- In our case, Jishen Du and I find a $P(z)$-compatibility condition, a $P(z)$-grading-restriction condition and give another construction of the tensor product using these conditions.

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Convergence and extension property, associativity and differential equations

- Under the assumption that the products of twisted intertwining operators satisfy a convergence and extension property, Jishen Du is making progress on the proof of the associativity of twisted intertwining operators.
- To prove the convergence and extension property, we need to show that the products of twisted intertwining operators satisfy differential equations with regular singular points.
- Dan Tan derived such differential equations in certain special cases under a suitable C_1-cofiniteness condition on twisted modules.
- In the case that the automorphisms involved are of infinite order, the singularities of these differential equations are not regular.
- This means that we will need new tools and techniques.

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THANK YOU!