Orbifold conformal field theory: General conjectures, open probelms and initial results

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Outline

1. The main problem in orbifold conformal field theory
2. Twisted intertwining operators
3. Conjectures and problems
4. Results on twisted intertwining operators and tensor product bifunctors

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Conformal field theories

- A conformal field theory is a projective algebra over the PROP of the moduli space of Riemann surfaces with parametrized boundaries (in the sense of Kontsevich, Segal and possibly some generalizations to include pseudo-traces).
- The first main problem in conformal field theory is to give a construction.
- A program to construct conformal field theories using the representation theory of vertex operator algebras has been very successful in the past thirty years.
- Intertwining operators are the main objects to study and is the same as chiral conformal fields in conformal field theory.
- To construct and study a conformal field theory is the same thing as proving properties of intertwining operators among modules for a vertex operator algebras.

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Orbifold conformal field theories

- Orbifold conformal field theories are conformal field theories constructed from known conformal field theories and their automorphisms.
- In 1984 (exactly 40 years ago), the first example of orbifold conformal field theories, the moonshine module, was constructed in mathematics by Frenkel, Lepowsky and Meurman.
- In 1985, Dixon, Harvey, Vafa and Witten published their first paper in physics on orbifold conformal field theories.
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The main problem

- **Main problem**: Given a vertex operator algebra V and a group G of automorphisms of V, construct and classify all the conformal field theories whose chiral algebras (vertex operator algebras) contain the fixed point vertex operator subalgebra V^G of V as a subalgebra.
- It is difficult to study directly V^G-modules and intertwining operators among V^G-modules.
- The only results we have now is the results by Miyamoto and Carnahan-Miyamoto in the case that V is C_2-cofinite and G is finite cyclic.
- On the other hand, we expect that V^G-modules are all contained in some twisted V-modules.
- So another approach is to study intertwining operators among twisted modules for vertex operator algebras. This approach is even more natural than the approach of studying intertwining operators for V^G.

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- Intertwining operators among twisted modules are called twisted intertwining operators.
- When the automorphisms involved are of finite orders and commute with each other, Xu in 1995 generalized intertwining operators to twisted intertwining operators using a Jacobi identity.
- But in general, an orbifold conformal field theory is associated to a nonabelian group of automorphisms.
- Also, the group might not be finite. So we also have to introduce and study intertwining operators among twisted modules associated to automorphisms of infinite orders.
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Twisted intertwining operators

- In 2017, I introduced and studied one class of such twisted intertwining operators.
- But this class of twisted intertwining operators are still not general enough because the correlation functions defining such twisted intertwining operators are of a special form.
- Recently, JIshen Du and I introduced the most general possible twisted intertwining operators.
- They are defined using commutativity and assocaitivity for products and iterates of twisted vertex operators and the twisted intertwining operator we want to define, but without assuming any type of generalized rationality for the multivalued correlation functions.

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Twsited intertwining operators

- For (generalized) g_1-, g_2- and g_3-twisted V-modules W_1, W_2 and W_3 , a twisted intertwining operator of type $\binom{W_3}{W_1 W_2}$ is a linear map $\mathcal{Y}: W_1 \otimes W_2 \to W_3\{x\}[\log x]$, $w_1 \otimes w_2 \mapsto \mathcal{Y}(w_1, x)w_2$ satisfying the following axioms:
- The $L(-1)$-derivative property:

$$\frac{d}{dx}\mathcal{Y}(w_1, x) = \mathcal{Y}(L(-1)w_1, x).$$

- The series

$$\langle w_3', Y_{W_3}^{g_3}(u_1, z_1) \cdots Y_{W_3}^{g_3}(u_{k-1}, z_{k-1})\mathcal{Y}(w_1, z_k)w_2\rangle$$

is absolutely convergent in the region $|z_1| > \cdots > |z_k| > 0$ and can be maximally extended to a multivalued analytic function on the region $z_i \neq 0$, $z_i \neq z_j$ for $i \neq j$.

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- The duality property: The series

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$$\langle w_3', \mathcal{Y}(Y_{W_1}^{g_1}(u, z_1 - z_2)w_1, z_2)w_2\rangle$$

are absolutely convergent on the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively. Moreover, their sums are equal to a preferred single-valued branch $f^e(z_1, z_2; u, w_1, w_2, w_3')$ of a maximally extended multivalued analytic function $f(z_1, z_2; u, w_1, w_2, w_3')$ on the region given by $|z_1| > |z_2| > 0$ and $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$, the region given by $|z_2| > |z_1| > 0$ and $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$, the region given by $|z_2| > |z_1 - z_2| > 0$ and $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$, respectively.

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Orbifold theory conjectures for finite groups

Conjecture (H., 2020, 2024)

Let V be C_1-cofinite vertex operator algebra such that all V-modules are C_1-cofinite and the differences between the real parts of lowest weights of irreducible V-modules are bounded. Let G be a finite group of automorphisms of V. Then twisted intertwining operators among grading-restricted generalized g-twisted V-modules of finite lengths for $g \in G$ satisfy convergence, associativity and commutativity properties.

Conjecture (H., 2016)

The category of grading-restricted generalized g-twisted V-modules for all $g \in G$ has a natural structure of G-crossed braided tensor category.

- In the case of $G = \{1_V\}$, these conjectures are theorems (H. 2007).

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Orbifold theory conjectures for strongly rational vertex operator algebras and finite groups

Conjecture (H., 2020, 2024)

For a strongly rational vertex operator algebra V and a finite group G of automorphisms of V, twisted intertwining operators among g-twisted V-modules for all $g \in G$ satisfy the convergence, associativity, commutativity and modular invariance properties.

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The category of g-twisted V-modules for all $g \in G$ has a natural structure of G-crossed modular tensor category.

- In the case that $G = \{1_V\}$ or that V is holomorphic and G is finite cyclic, these conjectures have been proved (H., 2002, 2003, 2005, Carnahan -Miyamoto 2016, van Ekeren-Möller-Scheithauer 2015, Möller 2016).

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For a strongly rational vertex operator algebra V and a finite group G of automorphisms of V, twisted intertwining operators among g-twisted V-modules for all $g \in G$ satisfy the convergence, associativity, commutativity and modular invariance properties.

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The category of g-twisted V-modules for all $g \in G$ has a natural structure of G-crossed modular tensor category.

- In the case that $G = \{1_V\}$ or that V is holomorphic and G is finite cyclic, these conjectures have been proved (H., 2002, 2003, 2005, Carnahan -Miyamoto 2016, van Ekeren-Möller-Scheithauer 2015, Möller 2016).

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Problems in the general case

Problem (H., 2020)

Let V be a vertex operator algebra and let G be a group of automorphisms of V. Under what conditions do the twisted intertwining operators among grading-restricted generalized g-twisted V-modules of finite lengths for all $g \in G$ satisfy the convergence, associativity, commutativity and modular invariance properties?

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Outline

Actions of automorphisms of V on twisted V-modules and $P(z)$-intertwining maps

- Let (W, Y_W^g) be a (generalized) g-twisted V-module.
- Let h be annother automorphism of V and let $\phi_h(Y_W^g) : V \times W \to W\{x\}[logx]$, $v \otimes w \mapsto \phi_h(Y^g)(v, x)w$ be the linear map defined by

$$\phi_h(Y_W^g)(v, x)w = Y_W^g(h^{-1}v, x)w.$$

- The pair $(W, \phi_h(Y_W^g))$ is a (generalized) hgh^{-1}-twisted V-module denoted by $\phi_h(W)$ and $W \mapsto \phi_h(W)$ gives an action of h on the category of (generalized) twisted V-modules.
- Let W_1, W_2, W_3 be g_1-, g_2-, g_1g_2-twisted V-modules, respectively, and $z \in \mathbb{C}^\times$. A (twisted) $P(z)$-intertwining map of type $\binom{W_3}{W_1 W_2}$ is a linear map $I : W_1 \otimes W_2 \to \overline{W}_3$ given by $I(w_1 \otimes w_2) = \mathcal{Y}(w_1, z)w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$, where $\mathcal{Y}$ is a twisted intertwining operator of type $\binom{W_3}{W_1 W_2}$.

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Definition of $P(z)$-tensor product of two twisted V-modules

- The definition and construction of $P(z)$-tensor product below is essentially the same as those in the untwisted case given by H.-Lepowsky.
- Let G be a group of automorphisms of V and $\mathcal{C}$ the category of grading-restricted generalized g-twisted V-modules for $g \in G$.
- Let W_1 and W_2 be g_1- and g_2-twisted V-modules, respectively, in $\mathcal{C}$. A $P(z)$-product of W_1 and W_2 is a pair (W_3, I) consisting of a $g_1 g_2$-twisted V-module W_3 in $\mathcal{C}$ and a twisted $P(z)$-intertwining map I of type $\binom{W_3}{W_1 W_2}$.
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A construction of $P(z)$-tensor product of two twisted V-modules

- Given a $P(z)$-product (W_3, I) of W_1 and W_2, for $w_3' \in W_3'$, we have an element $\lambda_{I,w_3'} \in (W_1 \otimes W_2)^*$ defined by $\lambda_{I,w_3'}(w_1 \otimes w_2) = \langle w_3', I(w_1 \otimes w_2)\rangle$ for $w_1 \in W_1$ and $w_2 \in W_2$.
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- For $v \in V$ and $\lambda_{I,w_3'} \in W_1 \boxtimes_{P(z)} W_2$, we define

$$Y_{W_1 \boxtimes_{P(z)} W_2}(v, x)\lambda_{I,w_3'} = \lambda_{I, Y_{W_3'}^{(g_1 g_2)^{-1}}(v,x) w_3'}.$$

- We now assume that $W_1 \boxtimes_{P(z)} W_2$ is in $\mathcal{C}$ (see a theorem below). Let

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A construction of $P(z)$-tensor product of two twisted V-modules

- We still need to give a $P(z)$-intertwining map $\boxtimes_{P(z)}$.
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Theorem (Jishen Du and H.)

The $P(z)$-product $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$ is a $P(z)$-tensor product of W_1 and W_2.

Theorem (Jishen Du and H.)

Assume that the following conditions are satisfied:

1. *There are only finitely many irreducible grading-restricted twisted V-modules.*
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Skew symmetry

- For a twisted intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$, let $\Omega(\mathcal{Y}) : W_2 \otimes W_1 \to W_3\{x\}[\log x]$, $w_2 \otimes w_1 \mapsto \Omega(\mathcal{Y})(w_2, x)w_1$ be the linear map defined by

$$\Omega(\mathcal{Y})(w_2, x)w_1 = e^{xL(-1)}\mathcal{Y}(w_1, y)w_2\Big|_{y^n = e^{n(\log z - \pi i)}, \log y = \log z - \pi i}.$$

Theorem (Jishen Du and H.)

$\Omega(\mathcal{Y})$ *is a twisted intertwining operator of type* $\binom{W_3}{\phi_{g_1}(W_2) W_1}$.

- Using the skew-symmetry isomorphism $\Omega(\mathcal{Y})$, we obtain a G-commutativity isomorphism $\mathcal{R}_{P(z)} : W_1 \boxtimes_{P(z)} W_2 \to \phi_{g_1}(W_2) \boxtimes_{P(-z)} W_1$ and a G-braiding isomorhism $\mathcal{B} : W_1 \boxtimes_{P(1)} W_2 \to \phi_{g_1}(W_2) \boxtimes_{P(1)} W_1$.

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$P(z)$-compatibility condition, $P(z)$-grading-restriction condition and another construction of $P(z)$-tensor products

- As in the untwisted case, the first major difficulty is the construction of associativity isomorphisms.
- The existence of the associativity isomorphism is in fact equivalent to the associativity of twisted intertwining operators.
- One of the main technique used in the proof of the associativity of intertwining operators in the untwisted case is another construction of the tensor product using the $P(z)$-compatibility condition and $P(z)$-grading-restriction condition.
- In our case, Jishen Du and I find a $P(z)$-compatibility condition, a $P(z)$-grading-restriction condition and give another construction of the tensor product using these conditions. (See Jishen Du's talk.)

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Convergence and extension property, associativity and differential equations

- Under the assumption that the products of twisted intertwining operators satisfy a convergence and extension property, Jishen Du is making progress on the proof of the associativity of twisted intertwining operators.
- To prove the convergence and extension property, we need to show that the products of twisted intertwining operators satisfy differential equations with regular singular points.
- Dan Tan derived such differential equations in certain special cases under a suitable C_1-cofiniteness condition on twisted modules.
- In the case that the automorphisms involved are of infinite order, the singularities of these differential equations are not regular.
- This means that we will need new tools and techniques.

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