Open-string vertex algebras, tensor categories and operads

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Abstract

We introduce notions of open-string vertex algebra, conformal open-string vertex algebra and variants of these notions. These are "open-string-theoretic," "noncommutative" generalizations of the notions of vertex algebra and of conformal vertex algebra. Given an open-string vertex algebra, we show that there exists a vertex algebra, which we call the "meromorphic center," inside the original algebra such that the original algebra yields a module and also an intertwining operator for the meromorphic center. This result gives us a general method for constructing open-string vertex algebras. Besides obvious examples obtained from associative algebras and vertex (super)algebras, we give a nontrivial example constructed from the minimal model of central charge $c = \frac{1}{2}$. We establish an equivalence between the associative algebras in the braided tensor category of modules for a suitable vertex operator algebra and the grading-restricted conformal open-string vertex algebras containing a vertex operator algebra isomorphic to the given vertex operator algebra. We also give a geometric and operadic formulation of the notion of grading-restricted conformal open-string vertex algebra, we prove two isomorphism theorems, and in particular, we show that such an algebra gives a projective algebra over what we call the "Swiss-cheese partial operad."

0 Introduction

In the present paper, we introduce and study "open-string-theoretic," "noncommutative" generalizations of ordinary vertex algebras and vertex operator algebras, which we call "open-string vertex algebras" and "conformal open-string vertex algebras." This is a first step in a program to establish the fundamental and highly nontrivial assumptions used by physicists in the study of boundary (or open-closed) conformal field theories as mathematical theorems and to construct such theories mathematically. See [H9] and [HK2] for definitions of open-closed conformal field theory in the spirit of the definition of closed conformal field theory first given by Segal [S1]–[S3] and Kontsevich in 1987 and further rigorized by Hu and Kriz [HK1] recently. More recently, Moore suggested in [M3] that in order to generalize a certain formula relating a nonlinear σ model and the K-theory on its target space to conformal field theories without obvious target space interpretation, one should define some kind of algebraic K-theory for "open string vertex operator algebras." We hope that the notions and results in the present paper will provide a solid foundation for the formulation and study of such a K-theory.

Vertex (operator) algebras were introduced in mathematics by Borcherds in [B]. They arose naturally in the vertex operator construction of representations of affine Lie algebras and in the construction and study of the "moonshine module" for the Monster finite simple group by Frenkel-Lepowsky-Meurman [FLM] and Borcherds [B]. The notion of vertex (operator) algebra corresponds essentially to the notion of what physicists call "chiral algebra" in (two-dimensional) conformal field theory, a fundamental physical theory studied systematically first by Belavin, Polyakov and Zamolodchikov [BPZ]. Vertex operator algebras can be viewed as "closed-string-theoretic" analogues of both Lie algebras and commutative associative algebras, and they play important roles in a range of areas of mathematics and physics.

Recently, in addition to the continuing development of (closed) conformal field theories, boundary conformal field theories (open-closed conformal field theories) have attracted much attention. Boundary conformal field theory was first developed by Cardy in [C1], [C2] and [C3] and play a fundamental role in many problems in condensed matter physics. It has also become one of the main tools in the study of open strings and D-branes (certain important nonperturbative objects in string theory). Besides the obvious problem of constructing and classifying open-closed conformal field theories, the study of D-branes in physics and their possible applications in geometry have led to exciting and interesting mathematical problems. If open-closed conformal field theories associated to Calabi-Yau manifolds or other geometric objects are constructed eventually, they will provide even more powerful tools in geometry than the corresponding closed conformal field theories (see, for example, the survey [D] by Douglas). The paper [M3] by Moore mentioned above gave another example of the exciting and interesting mathematical problems associated to open-closed conformal field theories.

In the framework of topological field theories, boundary topological field theories (open-closed topological field theories) have been studied in detail by Lazaroiu [L] and by Moore and Segal [M1] [M2] [S4]. In this topological case, an open-closed topological field theory is roughly speaking a (typically noncommutative) Frobenius algebra and a commutative Frobenius algebra equipped with some other data and satisfying suitable conditions. The commutative Frobenius algebra is the state space for the closed string part of the theory and the (typically noncommutative) Frobenius algebra is the state space for the open string part of the theory.

To construct and study open-closed conformal field theories, one first has to find the analogues in the conformal case of commutative and noncommutative associative algebras. Since the corresponding algebras in the conformal case must be infinite-dimensional, their construction and study are much more difficult than the topological ones. In the conformal case, one lesson we have learned from various methods used by physicists is that the construction and study of chiral theories are necessary and crucial steps. If chiral theories are constructed, full theories can be constructed using unitary bilinear forms on substructures of chiral theories called "modular functors." In fact, it is also the chiral theories which are more similar to topological theories than full theories. It is clear that analogues of commutative associative algebras in the chiral conformal case are vertex (operator) algebras. To construct and study open-closed conformal field theories, one first has to answer the following question: What are the analogues of noncommutative associative algebras in the conformal case?

Assuming the existence of the structure of a modular tensor category on the category of modules for a vertex operator algebra and the existence of conformal blocks with monodromies compatible with the modular tensor category, Felder, Fröhlich, Fuchs and Schweigert [FFFS] and Fuchs, Runkel and Schweigert [FRS1] [FRS2] studied open-closed conformal field theory using the theory of tensor categories and three-dimensional topological field theories. They showed the existence of consistent operator product expansion coefficients for boundary and bulk operators. In particular, special symmetric Frobenius algebras in the modular tensor categories of modules are proposed as analogues in the conformal case of (typically noncommutative) Frobenius algebras in the topological case. However, since these works are based on the fundamental assumptions mentioned above, even in the genus-zero case, the corresponding open-string-theoretic and noncommutative analogues of vertex operator algebras have not been fully constructed and studied, and even chiral open-closed conformal field theories on the disks (the simplest parts of open-closed conformal field theories) have not been fully constructed.

The present paper is a first step in a program for establishing the fundamental and highly nontrivial assumptions mentioned above as mathematical theorems, using the results on representations of vertex operator algebras and closed conformal field theories. In particular, we solve the problem of constructing open-closed conformal field theories on the disks satisfying certain differentiability and meromorphicity conditions by introducing, constructing and studying open-string vertex algebras, conformal open-string vertex algebras and some other variants. These algebras are the open-string-theoretic or noncommutative analogues of vertex (operator) algebras we are looking for and, as we shall discuss in future publications, D-branes can be formulated and studied as irreducible modules for suitable open-string vertex algebras. Given an open-string vertex algebra, we show that there exists a vertex algebra, which we call the "meromorphic center," inside the open-string vertex algebra such that the open-string vertex algebra yields a module and also an intertwining operator for the meromorphic center. This relation between open-string vertex algebras and the representation theory of vertex algebras gives us a general method for constructing open-string vertex algebras. Besides obvious examples obtained from associative algebras and vertex (super)algebras, we give a nontrivial one constructed from the minimal model of central charge $c = \frac{1}{2}$. We establish an equivalence between grading-restricted conformal open-string vertex algebras containing a suitable vertex operator algebra and associative algebras in the braided tensor category of modules for the vertex operator algebra. We also give a geometric and operadic formulation of the notion of grading-restricted conformal open-string vertex algebra, we prove two isomorphism theorems (establishing the equivalence of geometric notions and algebraic notions), and in particular, we show that such an algebra gives a projective algebra over what we call the "Swiss-cheese partial operad."

Here is the organization of the present paper: In Section 1, we introduce the notions of open-string vertex algebra, conformal open-string vertex algebra and other variants. The connection between open-string vertex algebras the representation theory of vertex (operator) algebras is given in Section 2. Examples of (conformal) open-string vertex algebras are presented in Section 3. In Section 4, we show that for a vertex operator algebra satisfying certain finiteness and complete reductivity properties, associative algebras in the braided tensor category of modules for the vertex operator algebra are equivalent to grading-restricted conformal open-string vertex algebras containing the vertex operator algebra in their meromorphic centers. The geometric and operadic formulation of the notion of grading-restricted conformal open-string vertex algebra, the construction of projective algebras over the Swiss-cheese partial operad and the proof of the corresponding isomorphism theorems are given in Section 5.

We shall use $\mathbb{C}$, $\mathbb{H}$, $\overline{\mathbb{H}}$, $\hat{\overline{\mathbb{H}}}$, $\mathbb{R}$, $\mathbb{R}^{\times}$, $\mathbb{R}_{+}$, $\mathbb{Z}$, $\mathbb{Z}_{+}$ and $\mathbb{N}$ to denote the sets (with structures) of the complex numbers, the open upper half plane, the closed upper half plane, the one point compactification of the closed upper half plane, the nonzero real numbers, the positive real numbers, the integers, the positive integers and the nonnegative integers, respectively. For any $z \in \mathbb{C}^{\times}$ and $n \in \mathbb{C}$, we shall always use $\log z$ and z^n to denote $\log |z| + \arg z$, $0 \leq \arg z < 2\pi$, and $e^{n \log z}$, respectively.

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1 Definitions and basic properties

We introduce the notion of open-string vertex algebra and its variants and discuss some basic properties of these algebras in this section. We assume that the reader is familiar with the basic notions and properties in the theory of vertex operator algebras as presented in [FLM] and [FHL].

In the present paper, all vector spaces are over the field $\mathbb{C}$. For a vector space V, we shall use V^{-} to denote its complex conjugate space, which is characterized by the fact that if $\sqrt{-1}$ acts as J on the underlying real vector space of V, then $\sqrt{-1}$ acts as $-J$ on the underlying real vector space of V^{-}. For an $\mathbb{R}$-graded vector space $V = \coprod_{n \in \mathbb{R}} V_{(n)}$ and any $n \in \mathbb{R}$, we shall use P_n to denote the projection from V or $\overline{V} = \prod_{n \in \mathbb{R}} V_{(n)}$ to $V_{(n)}$. We give V and its graded dual $V' = \coprod_{n \in \mathbb{R}} V_{(n)}^{*}$ the topology induced from the pairing between V and V'. We also give $\mathrm{Hom}(V, \overline{V})$ the topology induced from the linear functionals on $\mathrm{Hom}(V, \overline{V})$ given by $f \mapsto \langle v', f(v) \rangle$ for $f \in \mathrm{Hom}(V, \overline{V})$, $v \in V$ and $v' \in V'$.

Definition 1.1 An *open-string vertex algebra* is an $\mathbb{R}$-graded vector space $V=\coprod_{n\in\mathbb{R}} V_{(n)}$ (graded by *weights*) equipped with a *vertex map*

$$\begin{aligned} Y^O : V\times\mathbb{R}_+ &\to \mathrm{Hom}(V,\overline{V})\\ (u,r) &\mapsto Y^O(u,r)\end{aligned}$$

or equivalently,

$$\begin{aligned} Y^O : (V\otimes V)\times\mathbb{R}_+ &\to \overline{V}\\ (u\otimes v,r) &\mapsto Y^O(u,r)v,\end{aligned}$$

a *vacuum* $\mathbf{1}\in V$ and an operator $D\in \mathrm{End}\ V$ of weight 1, satisfying the following conditions:

1. *Vertex map weight property*: For $n_1,n_2\in\mathbb{R}$, there exist a finite subset $N(n_1,n_2)\subset\mathbb{R}$ such that the image of $\left(\coprod_{n\in n_1+\mathbb{Z}} V_{(n)}\otimes\coprod_{n\in n_2+\mathbb{Z}} V_{(n)}\right)\times\mathbb{R}_+$ under Y^O is in $\prod_{n\in N(n_1,n_2)+\mathbb{Z}} V_{(n)}$.

2. Properties for the vacuum: For any $r\in\mathbb{R}_+$, $Y^O(\mathbf{1},r)=\mathrm{id}_V$ (the *identity property*) and $\lim_{r\to 0} Y^O(u,r)\mathbf{1}$ exists and is equal to u (the *creation property*).

3. *Local-truncation property for D'*: Let $D':V'\to V'$ be the adjoint of D. Then for any $v'\in V'$, there exists a positive integer k such that $(D')^k v'=0$.

4. *Convergence properties*: For $v_1,\dots,v_n,v\in V$ and $v'\in V'$, the series

$$\begin{aligned}&\langle v', Y^O(v_1,r_1)\cdots Y^O(v_n,r_n)v\rangle\\ &\quad=\sum_{m_1,\dots,m_{n-1}\in\mathbb{R}}\langle v', Y^O(v_1,r_1)P_{m_1}Y^O(v_2,r_2)\cdots P_{m_{n-1}}Y^O(v_n,r_n)v\rangle\end{aligned}$$

converges absolutely when $r_1>\cdots>r_n>0$. For $v_1,v_2,v\in V$ and $v'\in V'$, the series

$$\langle v', Y^O(Y^O(v_1,r_0)v_2,r_2)v\rangle$$

converges absolutely when $r_2>r_0>0$.

5. *Associativity*: For $v_1, v_2, v \in V$ and $v' \in V'$,

$$\langle v', Y^O(v_1, r_1)Y^O(v_2, r_2)v\rangle = \langle v', Y^O(Y^O(v_1, r_1 - r_2)v_2, r_2)v\rangle$$

   for $r_1, r_2 \in \mathbb{R}$ satisfying $r_1 > r_2 > r_1 - r_2 > 0$.

6. $\mathbf{d}$-*bracket property*: Let $\mathbf{d}$ be the grading operator on V, that is, $\mathbf{d}u = mu$ for $m \in \mathbb{R}$ and $u \in V_{(m)}$. For $u \in V$ and $r \in \mathbb{R}_+$,

$$[\mathbf{d}, Y^O(u, r)] = Y^O(\mathbf{d}u, r) + r\frac{d}{dr}Y^O(u, r). \tag{1.1}$$

7. D-*derivative property*: We still use D to denote the natural extension of D to $\mathrm{Hom}(\overline{V}, \overline{V})$. For $u \in V$, $Y^O(u, r)$ as a map from $\mathbb{R}_+$ to $\mathrm{Hom}(V, \overline{V})$ is differentiable and

$$\frac{d}{dr}Y^O(u, r) = [D, Y^O(u, r)] = Y^O(Du, r). \tag{1.2}$$

Homomorphisms, *isomorphisms*, *subalgebras* of open-string vertex algebras are defined in the obvious way.

We shall denote the open-string vertex algebra by $(V, Y^O, \mathbf{1}, D)$ or simply V. For $u \in V$ and $r \in \mathbb{R}_+$, we call the map $Y^O(u, r) : V \to \overline{V}$ the vertex operator associated to u and r.

Remark 1.2 Note that in the definition above, the real number r in the vertex operator $Y^O(u, r)$ is positive, not in $\mathbb{R}^\times$. So a natural question is whether one has natural vertex operators associated to negative real numbers so that we have a vertex map Y^O from $(V \otimes V) \times \mathbb{R}^\times$ to $\overline{V}$. The answer is yes. For any $u, v \in V$ and $r \in -\mathbb{R}_+$, we define

$$Y^O(u, r)v = e^{rD}Y^O(v, -r)u. \tag{1.3}$$

(Note that $e^{rD}Y^O(v, -r)u$ is a well-defined element of $\overline{V}$ by the local-truncation property for D'.) Note that (1.3) resembles the skew-symmetry for vertex operator algebras. We know that the skew-symmetry is analogous to commutativity for commutative associative algebras. But (1.3) does not give a skew-symmetry property and is not an analogue of the commutativity mentioned above. Instead, (1.3) is an analogue of the relation between the product and the opposite product for an associative algebra. In fact, for an associative algebra V, we can define an opposite product

$$(uv)^{\mathrm{op}} = vu \tag{1.4}$$

for $u, v \in V$. We can also define an open-string vertex algebra in terms of a vertex map of the form $V \otimes V \times \mathbb{R}^{\times} \to \overline{V}$ and then (1.3) becomes an axiom. In applications, it is convenient to have vertex operators associated to negative numbers. For example, since

$$\begin{aligned}
\langle v', Y^O(Y^O(v_1, r_0)v_2, r_2)v\rangle &= \sum_{m\in\mathbb{R}} \langle v', Y^O(P_m Y^O(v_1, r_0)v_2, r_2)v\rangle \\
&= \sum_{m\in\mathbb{R}} \langle e^{r_2 D'}v', Y^O(v, -r_2)P_m Y^O(v_1, r_0)v_2\rangle,
\end{aligned}$$

the left-hand side (a matrix elements of an iterate of vertex operators) is absolutely convergent when $r_2 > r_0 > 0$ if and only if the right-hand side (a matrix element of a product of vertex operators) is. (Note that by the local-truncation property for D', $e^{r_2 D'}v' \in V'$.) In fact, one can prove that for $v_1, \ldots, v_n, v \in V$ and $v' \in V'$, the series

$$\begin{aligned}
&\langle v', Y^O(v_1, r_1)\cdots Y^O(v_n, r_n)v\rangle \\
&\quad = \sum_{m_1,\ldots,m_{n-1}\in\mathbb{R}} \langle v', Y^O(v_1, r_1)P_{m_1}Y^O(v_2, r_2)\cdots P_{m_{n-1}}Y^O(v_n, r_n)v\rangle
\end{aligned}$$

converges absolutely when $|r_1| > \cdots > |r_n| > 0$. One can also prove the absolute convergence of all the products and iterates of vertex operators associated to real numbers in natural regions. For the skew-symmetry for Y^O, see Remark 1.6.

We still use $\mathbf{d}$ to denote the natural extension of $\mathbf{d}$ to an element of $\mathrm{Hom}(\overline{V}, \overline{V})$.

Proposition 1.3 *The $\mathbf{d}$-bracket property (1.1) for all $u \in V$ and $r \in \mathbb{R}_+$, is equivalent to the $\mathbf{d}$-conjugation property*

$$a^{\mathbf{d}}Y^O(u, r)a^{-\mathbf{d}} = Y^O(a^{\mathbf{d}}u, ar) \tag{1.5}$$

for all $u \in V$, $r \in \mathbb{R}_+$ and $a \in \mathbb{R}_+$. We also have $\mathbf{1} \in V_{(0)}$ and $D\mathbf{1} = 0$.

Proof. If (1.5) holds for all $u \in V$, $r \in \mathbb{R}_+$ and $a \in \mathbb{R}_+$, then

$$e^{s\mathbf{d}} Y^O(u, r) e^{-s\mathbf{d}} = Y^O(e^{s\mathbf{d}} u, e^s r) \tag{1.6}$$

for all $u \in V$, $r \in \mathbb{R}_+$ and $s \in \mathbb{R}$. Taking the derivative with respect to s of both sides of (1.6) and then letting $s = 0$, we obtain (1.1).

Conversely, assume that (1.1) holds for all $u \in V$ and $r \in \mathbb{R}_+$. Let $u, v \in V$ and $v' \in V'$ be homogeneous. We have

$$\begin{aligned} \langle v', [\mathbf{d}, Y^O(u, r)]v\rangle &= \langle \mathbf{d}'v', , Y^O(u, r)v\rangle - \langle v', , Y^O(u, r)\mathbf{d}v\rangle \\ &= (\text{wt } v' - \text{wt } v)\langle w, Y^O(u, r)v\rangle, \end{aligned} \tag{1.7}$$

where $\mathbf{d}'$ is the adjoint of $\mathbf{d}$. On the other hand,

$$\left\langle v', \left(Y^O(\mathbf{d}u, r) + r\frac{d}{dr} Y^O(u, r) \right) v \right\rangle = \left(\text{wt } u + r\frac{d}{dr} \right) \langle v', Y^O(u, r)v\rangle. \tag{1.8}$$

By (1.1), (1.7) and (1.8), we see that $f(r) = \langle v', Y^O(u, r)v\rangle$ satisfies the differential equation

$$r\frac{df(r)}{dr} = (\text{wt } v' - \text{wt } u - \text{wt } v) f(r)$$

Any solution of this equation is of the form $Cr^{\text{wt } w - \text{wt } u - \text{wt } v}$ for some $C \in \mathbb{C}$. In particular, $\langle v', Y^O(u, r)v\rangle$ is of this form. Therefore,

$$\begin{aligned} \langle v', a^{\mathbf{d}} Y^O(u, r) a^{-\mathbf{d}} v\rangle &= \langle a^{\mathbf{d}} v', Y^O(u, r) a^{-\mathbf{d}} v\rangle \\ &= a^{\text{wt } v' - \text{wt } v} \langle v', Y^O(u, r)v\rangle \\ &= Ca^{\text{wt } v' - \text{wt } v} r^{\text{wt } w - \text{wt } u - \text{wt } v} \\ &= Ca^{\text{wt } u} (ar)^{\text{wt } w - \text{wt } u - \text{wt } v} \\ &= \langle v', Y^O(a^{\mathbf{d}} u, ar)v\rangle. \end{aligned}$$

Since such u, v span V and such v' spans V', we obtain (1.5).

The identity property, the creation property and (1.1) imply $\mathbf{d1} = 0$ which means $\mathbf{1} \in V_{(0)}$. The identity property and the D-derivative property imply $D\mathbf{1} = 0$. ∎

The $\mathbf{d}$-conjugation property also has the following very important consequence:

Proposition 1.4 *For $u \in V$, there exist $u_n^+ \in \mathrm{End}\, V$ of weights* wt $u-n-1$ *for $n \in \mathbb{R}$ such that for $r \in \mathbb{R}_+$,*

$$Y^O(u, r) = \sum_{n\in\mathbb{R}} u_n^+ r^{-n-1}. \tag{1.9}$$

Proof. For homogeneous $u \in V$ and $n \in \mathbb{R}$, let $u_n^+ \in \mathrm{End}\, V$ be defined by

$$u_n^+ v = P_{\mathrm{wt}\ u-n-1+\mathrm{wt}\ v} Y^O(u, 1)v$$

for homogeneous $v \in V$. Then by the **d**-conjugation property, for any homogeneous $u, v \in V$,

$$\begin{aligned} Y^O(u, r)v &= r^{\mathbf{d}} Y^O(r^{-\mathbf{d}}u, 1) r^{-\mathbf{d}} v \\ &= r^{-\mathrm{wt}\ u-\mathrm{wt}\ v} r^{\mathbf{d}} \sum_{n\in\mathbb{R}} P_{\mathrm{wt}\ u-n-1+\mathrm{wt}\ v} Y^O(u, 1)v \\ &= \sum_{n\in\mathbb{R}} P_{\mathrm{wt}\ u-n-1+\mathrm{wt}\ v} Y^O(u, 1)v r^{-n-1} \\ &= \sum_{n\in\mathbb{R}} u_n^+ v r^{-n-1}. \end{aligned}$$

∎

Remark 1.5 In the proposition above, (1.9) holds only for $r \in \mathbb{R}_+$. In fact, there are also $u_n^- \in \mathrm{End}\, V$ of weights wt $u - n - 1$ for $n \in \mathbb{R}$ such that for $r \in -\mathbb{R}_+$,

$$Y^O(u, r) = \sum_{n\in\mathbb{R}} u_n^- r^{-n-1}.$$

But in general $u_n^- \neq u_n^+$. In this paper, we shall not use u_n^-, $n \in \mathbb{R}$.

From Proposition 1.4, we see that for any $u \in V$, there is a formal-variable vertex operator

$$\mathcal{Y}^f(u, x) = \sum_{n\in\mathbb{R}} u_n^+ x^{-n-1} \in (\mathrm{End}\, V)[[x, x^{-1}]]$$

where x is a formal variable. We shall also use the notation $\mathcal{Y}^f(u, z)$ to denote the vertex operator associated to $u \in V$ and a nonzero complex number z, that is,

$$\mathcal{Y}^f(u, z) = \sum_{n\in\mathbb{R}} u_n^+ z^{-n-1}.$$

(Note that by our convention, for $z \in \mathbb{C}^\times$, $z^{-n-1} = e^{(-n-1)\log z}$ for $n \in \mathbb{R}$, where $\log z = \log|z| + i \arg z$, $0 \le \arg z < 2\pi$.) Thus for $u \in V$, $Y^O(u, r) = \mathcal{Y}^f(u, r)$ for $r \in \mathbb{R}_+$ but in general $Y^O(u, r) \neq \mathcal{Y}^f(u, r)$ for $r \in -\mathbb{R}_+$.

Remark 1.6 As we have discussed in Remark 1.2, (1.3) has nothing to do with skew-symmetry. In fact, if $\mathcal{Y}^f$ satisfies

$$\mathcal{Y}^f(u, x)v = e^{xL(-1)}\mathcal{Y}^f(v, y)u\Big|_{y^n = e^{n\pi i}x^n,\ n\in\mathbb{R}}$$

for $u, v \in V$, then we say that Y^O has *skew-symmetry*. (For simplicity, in the remaining part of this paper, we shall use $\mathcal{Y}^f(v, -x)u$ to denote $\mathcal{Y}^f(v, y)u|_{y^n = e^{n\pi i}x^n,\ n\in\mathbb{R}}$.) Note that skew-symmetry for Y^O gives a relation between $Y^O(u, r)v$ and its analytic extension to the negative real line for $u, v \in V$ and $r \in \mathbb{R}_+$ while (1.3) gives a relation between $Y^O(u, r)v$ and $Y^O(v, -r)u$ for $u, v \in V$ and $r \in \mathbb{R}_+$. Clearly, these two relations are in general different.

Proposition 1.7 *The* $\mathbf{d}$*-bracket and* D*-derivative properties hold for* $\mathcal{Y}^f$*, that is, (1.1) and (1.2) hold when* Y^O *is replaced by* $\mathcal{Y}^f$ *and* r *is replaced by the formal variable* x*. We also have the following* $\mathbf{d}$*- and* D*-conjugation properties: For* $u \in V$ *and* y *another formal variable,*

$$y^{\mathbf{d}}\mathcal{Y}^f(u, x)y^{-\mathbf{d}} = \mathcal{Y}^f(y^{\mathbf{d}}u, yx),$$

and

$$\mathcal{Y}^f(u, x + y) = e^{yD}\mathcal{Y}^f(u, x)e^{-yD} = \mathcal{Y}^f(e^{yD}u, x). \tag{1.10}$$

In particular, these conjugation formulas also hold when we substitute suitable complex numbers for x *and* y *such that both sides of these formulas make sense as (or converges to) maps from* V *to* $\overline{V}$*.*

Proof. The $\mathbf{d}$-bracket formula and D-derivative property follow from the definition of the formal-variable vertex operators and the corresponding properties for the defining vertex map. The $\mathbf{d}$-conjugation property for the formal vertex operator follows immediately from (1.5). The D-conjugation property follows from the D-derivative property. ∎

We have the following easy consequence of Proposition 1.7:

Corollary 1.8 *For any $u \in V$,*

$$\mathcal{Y}^f(u, x)\mathbf{1} = e^{xD}u. \tag{1.11}$$

Proof. By the creation property, we see that for any $r \in \mathbb{R}_+$ and any $u \in V$,

$$Y^O(u, r)\mathbf{1} = \sum_{n \in (-\mathbb{R}_+ - 1) \cup \{-1\}} u_n^+ \mathbf{1} r^{-n-1}$$

and $u_{-1}^+\mathbf{1} = u$. But by the D-derivative property,

$$\begin{aligned} \lim_{r \to 0} \frac{d^k}{dr^k} Y^O(u, r)\mathbf{1} &= \lim_{r \to 0} Y^O(D^k u, r)\mathbf{1} \\ &= D^k u \end{aligned}$$

for $k \in \mathbb{N}$. Thus we see that

$$Y^O(u, r)\mathbf{1} = \sum_{n \in -\mathbb{Z}_+} u_n^+ \mathbf{1} r^{-n-1}.$$

So $\mathcal{Y}^f(u, x)\mathbf{1}$ is a power series in x and $\lim_{x \to 0} \mathcal{Y}^f(u, x)\mathbf{1} = u$. By these properties, $D\mathbf{1} = 0$ and the D-conjugation property for $\mathcal{Y}^f$, we obtain

$$\begin{aligned} \mathcal{Y}^f(u, y)\mathbf{1} &= \lim_{x \to 0} \mathcal{Y}^f(u, x + y)\mathbf{1} \\ &= \lim_{x \to 0} e^{yD} \mathcal{Y}^f(u, x) e^{-yD} \mathbf{1} \\ &= \lim_{x \to 0} e^{yD} \mathcal{Y}^f(u, x)\mathbf{1} \\ &= e^{yD} u, \end{aligned}$$

proving (1.11). ∎

Proposition 1.9 *The formal vertex operator map $\mathcal{Y}^f$ has the following properties:*

1. *Convergence: The series*

$$\langle v', \mathcal{Y}^f(v_1, z_1)\mathcal{Y}^f(v_2, z_2)v\rangle, \tag{1.12}$$
$$\langle v', \mathcal{Y}^f(v_2, z_2)\mathcal{Y}^f(v_1, z_1)v\rangle, \tag{1.13}$$
$$\langle v', \mathcal{Y}^f(\mathcal{Y}^f(v_1, z_1 - z_2)v_2, z_2)v\rangle, \tag{1.14}$$
$$\langle v', \mathcal{Y}^f(\mathcal{Y}^f(v_2, z_2 - z_1)v_1, z_1)v\rangle \tag{1.15}$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, $|z_1| > |z_1 - z_2| > 0$, respectively.

2. *Associativity: For $v_1, v_2, v \in V$ and $v' \in V'$, (1.12) and (1.14) are equal in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$, and (1.13) and (1.15) are equal in the region $|z_2| > |z_1| > |z_1 - z_2| > 0$.*

Proof. By definition, (1.12), (1.13), (1.14) and (1.15) converge absolutely when $z_1, z_2 \in \mathbb{R}_+$ satisfying $z_1 > z_2 > 0$, $z_2 > z_1 > 0$, $z_2 > z_1 - z_2 > 0$ and $z_1 > z_1 - z_2 > 0$, respectively. Consequently, (1.12),(1.13), (1.14) and (1.15) converge absolutely for $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$ $|z_1| > |z_1 - z_2| > 0$, respectively. The convergence is proved.

In particular, (1.12) and (1.14) give (possibly multivalued) analytic functions defined on the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively. By associativity for Y^O, (1.12) and (1.14) are equal for $z_1, z_2 \in \mathbb{R}_+$ satisfying $z_1 > z_2 > z_1 - z_2 > 0$. By the basic properties of analytic functions, (1.12) and (1.14) are equal for $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$ (the intersection of the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$ on which the analytic functions (1.12) and (1.14) are defined). The second part of the associativity for $\mathcal{Y}^f$ can be obtained from the first part by substituting v_2, v_1, z_2 and z_1 for v_1, v_2, z_1 and z_2. ∎

Definition 1.10 A *grading-restricted open-string vertex algebra* is an open-string vertex algebra satisfying the following conditions:

8. The *grading-restriction conditions*: For all $n \in \mathbb{R}$, $\dim V_{(n)} < \infty$ (the *finite-dimensionality of homogeneous subspaces*) and $V_{(n)} = 0$ when n is sufficiently negative (the *lower-truncation condition for grading*).

A *conformal open-string vertex algebra* is an open-string vertex algebra equipped with a *conformal element* $\omega \in V$ satisfying the following conditions:

9. The *Virasoro relations*: For any $m, n \in \mathbb{Z}$,

$$[L(m), L(n)] = (m - n)L(m + n) - \frac{c}{12}(m^3 - m)\delta_{m+n,0},$$

where $L(n)$, $n \in \mathbb{Z}$ are given by

$$Y^O(\omega, r) = \sum_{n \in \mathbb{Z}} L(n)r^{-n-2}$$

and $c \in \mathbb{C}$.

10. The *commutator formula for Virasoro operators and formal vertex operators (or component operators)*: For $v \in V$, $\mathcal{Y}^f(\omega, x)v$ involves only finitely many negative powers of x and

$$[\mathcal{Y}^f(\omega, x_1), \mathcal{Y}^f(v, x_2)] = \mathrm{Res}_0 x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\mathcal{Y}^f(\mathcal{Y}^f(\omega, x_0)v, x_2).$$

11. The $L(0)$*-grading property* and $L(-1)$*-derivative property*: $L(0) = \mathbf{d}$ and $L(-1) = D$.

A *grading-restricted conformal open-string vertex algebra* or *open-string vertex operator algebra* is a conformal open-string vertex algebra satisfying the grading-restriction condition.

We shall denote the conformal open-string vertex algebra defined above by $(V, Y^O, \mathbf{1}, \omega)$ or simply V. The complex number c in the definition is called the *central charge* of the algebra. Note that the grading-restriction conditions imply the local-truncation property for D'.

Proposition 1.11 *Let V be a grading-restricted open-string vertex algebra. Then for $u, v \in V$, $u_n^+ v = 0$ if n is sufficiently negative.*

Proof. This follows immediately from the lower-truncation condition for grading and the fact that the weights of u_n^+ for $n \in \mathbb{R}$ is $\mathrm{wt}\ u - n - 1$. ∎

2 Intertwining operators and open-string vertex algebras

In this section, we establish a connection between open-string vertex algebras and intertwining operator algebras. We assume that the reader is familiar with the basic notions and properties in the representation theory of vertex operator algebras and we also assume that the reader is familiar with the notion of intertwining operator algebra. See [FHL], [H7] and [H8] for details.

Let V be an open-string vertex algebra and S a subset of V. Then the *open-string vertex subalgebra of V generated by S* is the smallest open-string vertex subalgebra of V containing S.

Proposition 2.1 *Let V be a conformal open-string vertex algebra and $\langle\omega\rangle$ the open-string vertex subalgebra of V generated by ω. Then $\langle\omega\rangle$ is in fact a vertex operator algebra. In particular, V is a module for the vertex operator algebra $\langle\omega\rangle$.*

Proof. All the axioms for a vertex operator algebra are satisfied by $\langle\omega\rangle$ obviously except for the commutativity or equivalently the commutator formula. But the Virasoro relations imply the commutator formula for the vertex operators for $\langle\omega\rangle$. ∎

More generally, we have the following generalization: Let V be an open-string vertex algebra and let

$$C_0(V) = \left\{ u \in \coprod_{n\in\mathbb{Z}} V_{(n)} \;\middle|\; \mathcal{Y}^f(u,x) \in (\mathrm{End}\, V)[[x,x^{-1}]], \right.$$
$$\left. \mathcal{Y}^f(v,x)u = e^{xD}\mathcal{Y}^f(u,-x)v,\ \forall v \in V \right\}.$$

In particular, for elements of $C_0(V)$, skew-symmetry holds. Clearly $C_0(V)$ is not zero since by (1.11), $\mathbf{1} \in C_0(V)$.

For an open-string vertex algebra V, the formal vertex operator map $\mathcal{Y}^f$ for V induces a map from $C_0(V)\otimes C_0(V)$ to $V[[x,x^{-1}]]$. We denote this map by $\mathcal{Y}^f|_{C_0(V)}$. We first need:

Proposition 2.2 *Let $v_1 \in C_0(V)$, $v_2, v \in V$ and $v' \in V'$. Then there exists a (possibly multivalued) analytic function on*

$$M^2 = \{(z_1,z_2) \in \mathbb{C}^2 \mid z_1, z_2 \neq 0, z_1 \neq z_2\}$$

such that it is single valued in z_1 and is equal to the (possibly multivalued) analytic extensions of (1.12), (1.13), (1.14) and (1.15) in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$ and $|z_1| > |z_1 - z_2| > 0$, respectively. Moreover, if v_2 is in $C_0(V)$, then this analytic function is single valued in both z_1 and z_2. If V satisfies the grading-restriction condition, then this analytic function is a rational function with the only possible poles $z_1, z_2 = 0$ and $z_1 = z_2$.

Proof. By Proposition 1.9, (1.12), (1.13) and (1.14) are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, and the associativity for $\mathcal{Y}^f$ holds.

Since $v_1 \in C_0(V)$, by definition, $\mathcal{Y}^f(v_1, x)v_2 \in V[[x, x^{-1}]]$ and we have the skew-symmetry

$$\begin{aligned}\mathcal{Y}^f(v_1, x)v_2 &= e^{xD}\mathcal{Y}^f(v_2, -x)v_1,\\ \mathcal{Y}^f(v_2, x)v_1 &= e^{xD}\mathcal{Y}^f(v_1, -x)v_2.\end{aligned}$$

In [H7] it was proved that commutativity for intertwining operators follows from associativity and skew-symmetry for intertwining operators. For reader's convenience, here we give a proof of commutativity in the special case in which we are interested.

By associativity, (1.12) and (1.14) are equal in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$. By associativity also, (1.13) and (1.15) converge absolutely to analytic functions defined on the regions $|z_2| > |z_1| > 0$ and $|z_1| > |z_1 - z_2| > 0$, respectively, and are equal in the region $|z_2| > |z_1| > |z_1 - z_2| > 0$. By skew-symmetry and the D-derivative property, for $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_1 - z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, we have

$$\begin{aligned}&\langle v', \mathcal{Y}^f(\mathcal{Y}^f(v_1, z_1 - z_2)v_2, z_2)v\rangle\\ &\quad = \langle v', \mathcal{Y}^f(e^{(z_1-z_2)D}\mathcal{Y}^f(v_2, -(z_1 - z_2))v_1, z_2)v\rangle\\ &\quad = \langle v', \mathcal{Y}^f(\mathcal{Y}^f(v_2, z_2 - z_1)v_1, z_2 + (z_1 - z_2))v\rangle\\ &\quad = \langle v', \mathcal{Y}^f(\mathcal{Y}^f(v_2, z_2 - z_1)v_1, z_1)v\rangle,\end{aligned}$$

that is, in the region given by $|z_1| > |z_1 - z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, (1.14) and (1.15) are equal. Since (1.12) is equal to (1.14) in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$, (1.14) is equal to (1.15) in the region given by $|z_1| > |z_1 - z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, and (1.15) is equal to (1.13) in the region $|z_2| > |z_1| > |z_1 - z_2| > 0$, we see that (1.12) and (1.13) are analytic extensions of each other. So commutativity is proved.

Now we prove the existence of the function stated in the proposition. By skew-symmetry, we have

$$\mathcal{Y}^f(v, z)\mathbf{1} = e^{zD}\mathcal{Y}^f(\mathbf{1}, -z)v = e^{zD}v$$

for any $v \in C_0(V)$. Thus by definition, for $v_1 \in C_0(V)$, $v_2, v \in V$ and $v' \in (C_0(V))'$,

$$\langle v', \mathcal{Y}^f(v_1, z_1)\mathcal{Y}^f(v_2, z_2)e^{z_3D}v\rangle = \langle v', \mathcal{Y}^f(v_1, z_1)\mathcal{Y}^f(v_2, z_2)\mathcal{Y}^f(v, z_3)\mathbf{1}\rangle$$

converges absolutely for $z_1, z_2, z_3 \in \mathbb{R}^\times$ satisfying $|z_1| > |z_2| > |z_3| > 0$. Consequently it also converges absolutely for $z_1, z_2, z_3 \in \mathbb{C}$ satisfying $|z_1| >$

$|z_2| > |z_3| > 0$. Now the same proof as the one for Lemma 4.1 in [H7] shows that there exists a (possibly multivalued) analytic functions on M^2 such that it is equal to (possibly multivalued) analytic extensions of (1.12), (1.13), (1.14) and (1.15) in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$ and $|z_1| > |z_1 - z_2| > 0$, respectively. Since (1.12), (1.13) and (1.14) give analytic functions which are all single valued in z_1, this function as the analytic extension of these functions must also be single valued in z_1.

If v_2 is in $C_0(V)$, then by definition, $\mathcal{Y}^f(v_2, x)v \in V[[x, x^{-1}]]$ and thus (1.12), (1.13) and (1.15) give analytic functions which are also single valued in z_2. So their analytic extension is also single valued in both z_1 and z_2. If V satisfies the grading-restriction condition, then the singularities $z_1, z_2 = 0, \infty$ and $z_1 = z_2$ of this analytic extension are all poles and is therefore a rational function in z_1 and z_2 with the only possible poles $z_1, z_2 = 0$ and $z_1 = z_2$. ∎

Theorem 2.3 *Let V be a grading-restricted open-string vertex algebra. Then the image of $C_0(V) \otimes C_0(V)$ under $\mathcal{Y}^f|_{C_0(V)}$ is in $C_0(V)[[x, x^{-1}]]$ and the image of $C_0(V)$ under D is in $C_0(V)$. Moreover,*

$$(C_0(V), \mathcal{Y}^f|_{C_0(V)}, \mathbf{1}, D)$$

is a grading-restricted vertex algebra, V is a $C_0(V)$-module and $\mathcal{Y}^f$ is an intertwining operator of type $\binom{V}{VV}$ for the vertex algebra $C_0(V)$.

Proof. Let v_1, v_2 be homogeneous elements of $C_0(V)$. We would like to show that $\mathcal{Y}^f(v_1, x)v_2 \in C_0(V)[[x, x^{-1}]]$. First of all, since $v_1 \in C_0(V)$, $\mathcal{Y}^f(v_1, x)v_2 \in V[[x, x^{-1}]]$. Since $v_1, v_2 \in C_0(V)$, $\mathrm{wt}\ v_1, \mathrm{wt}\ v_2 \in \mathbb{Z}$. Thus by Proposition 1.4, $\mathcal{Y}^f(v_1, x)v_2 \in \left(\coprod_{n\in\mathbb{Z}} V_{(n)}\right)[[x, x^{-1}]]$. By Proposition 2.2, the analytic extension of (1.14) to M^2 is a single-valued analytic function. In particular, (1.14) gives a single-valued analytic function in z_1 and z_2. Thus

$$\mathcal{Y}^f(\mathcal{Y}^f(v_1, x)v_2, x_2)v \in (V[[x_2, x_2^{-1}]])[[x, x^{-1}]].$$

For $v \in V$, $v' \in V'$ and $z_1, z_2 \in \mathbb{R}_+$ satisfying $z_1 > z_2 > z_1 - z_2 > 0$,

$$\begin{aligned}\langle v', \mathcal{Y}^f(\mathcal{Y}^f(v_1, z_1 - z_2)v_2, z_2)v\rangle &= \langle v', \mathcal{Y}^f(v_1, z_1)\mathcal{Y}^f(v_2, z_2)v\rangle \\ &= \langle v', \mathcal{Y}^f(v_1, z_1)e^{z_2 D}\mathcal{Y}^f(v, -z_2)v_2\rangle.\end{aligned} \tag{2.1}$$

The right-hand side of (2.1) is well defined when $z_1, z_2 \in \mathbb{C}$ and $|z_1| > |z_2| > 0$ and is equal to

$$\begin{aligned}
&\langle v', e^{z_2 D}\mathcal{Y}^f(v_1, z_1 - z_2)\mathcal{Y}^f(v, -z_2)v_2\rangle \\
&\quad = \langle v', e^{z_1 D}\mathcal{Y}^f(\mathcal{Y}^f(v, -z_2)v_2, -(z_1 - z_2))v_1\rangle \\
&\quad = \langle v', e^{z_1 D}\mathcal{Y}^f(v, -z_1)\mathcal{Y}^f(v_2, -(z_1 - z_2))v_1\rangle \\
&\quad = \langle v', e^{z_1 D}\mathcal{Y}^f(v, -z_1)e^{-(z_1 - z_2)D}\mathcal{Y}^f(v_1, z_1 - z_2)v_2\rangle
\end{aligned} \tag{2.2}$$

when $z_1, z_2 \in \mathbb{R}_+$ and $z_1 > z_1 - z_2 > z_2 > 0$. The right-hand side of (2.2) is well defined when $z_1, z_2 \in \mathbb{C}$ and $|z_1| > |z_1 - z_2| > 0$ and is equal to

$$\langle v', e^{z_2 D}\mathcal{Y}^f(v, -z_2)\mathcal{Y}^f(v_1, z_1 - z_2)v_2\rangle \tag{2.3}$$

when $z_1, z_2 \in \mathbb{C}$ and $|z_1| > |z_2| > |z_1 - z_2| > 0$. From (2.1)–(2.3), we see that the left-hand side of (2.1) and the right-hand side of (2.3) are analytic extensions of each other. Since both the left-hand side of (2.1) and the right-hand side of (2.3) are well defined single-valued analytic functions on the region $|z_2| > |z_1 - z_2| > 0$, they are equal when $|z_2| > |z_1 - z_2| > 0$. Thus we obtain

$$\mathcal{Y}^f(\mathcal{Y}^f(v_1, x)v_2, x_2)v = e^{x_2 D}\mathcal{Y}^f(v, -x_2)\mathcal{Y}^f(v_1, x)v_2$$

where x and x_2 are two commuting formal variables. So $\mathcal{Y}^f(v_1, x)v_2 \in C_0(V)[[x, x^{-1}]]$.

Let u be a homogeneous element of $C_0(V)$. Then wt $u \in \mathbb{Z}$. Since D has weight 1, $Du \in \coprod_{n\in\mathbb{Z}} V_{(n)}$. By the D-derivative property, we see that

$$\mathcal{Y}^f(Du, x) = \frac{d}{dx}\mathcal{Y}^f(u, x) \in (\text{End } V)[[x, x^{-1}]].$$

For any $v \in V$, using the D-derivative property and the D-bracket formula, we obtain

$$\begin{aligned}
\mathcal{Y}^f(Du, x)v &= \frac{d}{dx}\mathcal{Y}^f(u, x)v \\
&= \frac{d}{dx}e^{xD}\mathcal{Y}^f(v, -x)u \\
&= e^{xD}D\mathcal{Y}^f(v, -x)u - e^{xD}\mathcal{Y}^f(Dv, -x)u \\
&= e^{xD}\mathcal{Y}^f(v, -x)Du.
\end{aligned}$$

So $Du \in C_0(V)$.

To show that $C_0(V)$ is a vertex algebra, we need only verify commutativity, associativity and rationality since all the other axioms are clearly satisfied. But associativity, commutativity and rationality have been proved in Proposition 2.2. The proof of the fact that V is a $C_0(V)$-module and $\mathcal{Y}^f$ is an intertwining operator of type $\binom{V}{VV}$ for $C_0(V)$ is completely the same. ∎

Proposition 2.4 *Let V be a conformal open-string vertex algebra. Then $\omega \in C_0(V)$.*

Proof. By definition, $\omega \in \coprod_{n\in\mathbb{Z}} V_{(n)}$ and $\mathcal{Y}^f(\omega, x) \in (\mathrm{End}\ V)[[x, x^{-1}]]$. For any $v \in V$, the commutator formula for ω and formal vertex operators implies the commutativity for $\mathcal{Y}^f(\omega, z_1)$ and $\mathcal{Y}^f(v, z_2)$. In particular, for any $v' \in V'$,

$$\langle v', \mathcal{Y}^f(\omega, z_1)\mathcal{Y}^f(v, z_2)\mathbf{1}\rangle \tag{2.4}$$

and

$$\langle v', \mathcal{Y}^f(v, z_2)\mathcal{Y}^f(\omega, z_1)\mathbf{1}\rangle \tag{2.5}$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively, and are analytic extensions of each other. Also by associativity we know that

$$\langle v', \mathcal{Y}^f(\mathcal{Y}^f(\omega, z_1 - z_2)v, z_2)\mathbf{1}\rangle \tag{2.6}$$

and

$$\langle v', \mathcal{Y}^f(\mathcal{Y}^f(v, z_2 - z_1)\omega, z_1)\mathbf{1}\rangle \tag{2.7}$$

are absolutely convergent in the region $|z_2| > |z_1 - z_2| > 0$ and $|z_1| > |z_1 - z_2| > 0$, respectively, and are equal to (2.4) and (2.5), respectively, in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$ and $|z_2| > |z_1| > |z_1 - z_2| > 0$, respectively. Thus (2.6) and (2.7) are also analytic extensions of each other. Note that by (1.10),

$$\begin{aligned}&\langle v', \mathcal{Y}^f(e^{(z_1-z_2)L(-1)}\mathcal{Y}^f(v, z_2 - z_1)\omega, z_2)\mathbf{1}\rangle\\ &\quad = \langle e^{(z_1-z_2)L'(-1)}v', \mathcal{Y}^f(\mathcal{Y}^f(v, z_2 - z_1)\omega, z_2)\mathbf{1}\rangle\end{aligned} \tag{2.8}$$

is absolutely convergent in the region $|z_2| > |z_1 - z_2| > 0$ and is equal to (2.7) in the region $|z_1|, |z_2| > |z_1 - z_2| > 0$. So (2.6) and the left-hand side of (2.8) are analytic extensions of each other.

We know that both (2.6) and the left-hand side of (2.8) are convergent absolutely in the region $|z_2| > |z_1 - z_2| > 0$ and, moreover, we know that

(2.4), (2.5), (2.6) and (2.7) give single-valued analytic functions in z_1 and z_2. Thus in the region $|z_2| > |z_1 - z_2| > 0$, (2.6) and the left-hand side of (2.8) are equal, that is,

$$\langle v', \mathcal{Y}^f(\mathcal{Y}^f(\omega, z_1 - z_2)v, z_2)\mathbf{1}\rangle = \langle v', \mathcal{Y}^f(e^{(z_1-z_2)L(-1)}\mathcal{Y}^f(v, z_2 - z_1)\omega, z_2)\mathbf{1}\rangle. \tag{2.9}$$

By taking coefficients of $z_1 - z_2$ and z_2 in both sides of (2.9) and then taking the generating functions of these coefficients, we obtain

$$\langle v', \mathcal{Y}^f(\mathcal{Y}^f(\omega, x)v, y)\mathbf{1}\rangle = \langle v', \mathcal{Y}^f(e^{xL(-1)}\mathcal{Y}^f(v, -x)\omega, y)\mathbf{1}\rangle, \tag{2.10}$$

where x and y are commuting formal variables. Since $v' \in V'$ is arbitrary, (2.10) gives

$$\mathcal{Y}^f(\mathcal{Y}^f(\omega, x)v, y)\mathbf{1} = \mathcal{Y}^f(e^{xL(-1)}\mathcal{Y}^f(v, -x)\omega, y)\mathbf{1}. \tag{2.11}$$

Taking the formal limit $y \to 0$ (that is, taking the constant term of of series in y) of both sides of (2.11), we obtain

$$\mathcal{Y}^f(\omega, x)v = e^{xL(-1)}\mathcal{Y}^f(v, -x)\omega.$$

So we conclude that $\omega \in C_0(V)$. ∎

One immediate consequence of this result is the following:

Corollary 2.5 *Let V be a grading-restricted conformal open-string vertex algebra. Then the vertex operator algebra $\langle\omega\rangle$ is a subalgebra of $C_0(V)$.*

Recall the following main theorem in [H8]:

Theorem 2.6 *Let V be a vertex operator algebra satisfying the following conditions:*

1. *Every generalized V-module is a direct sum of irreducible V-modules.*
2. *There are only finitely many inequivalent irreducible V-modules and these irreducible V-modules are all $\mathbb{R}$-graded.*
3. *Every irreducible V-module satisfies the C_1-cofiniteness condition.*

Then the direct sum of all (inequivalent) irreducible V-modules has a natural structure of an intertwining operator algebra. In particular, the following associativity for intertwining operators holds: For any V-modules W_0, W_1, W_2, W_3 and W_4, any intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of types $\binom{W_0}{W_1 W_4}$ and $\binom{W_4}{W_2 W_3}$, respectively,

$$\langle w'_{(0)}, \mathcal{Y}_1(w_{(1)}, z_1)\mathcal{Y}_2(w_{(2)}, z_2)w_{(3)}\rangle \tag{2.12}$$

is absolutely convergent when $|z_1| > |z_2| > 0$ for $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$, and there exist V-module W_5 and intertwining operators $\mathcal{Y}_3$ and $\mathcal{Y}_4$ of types $\binom{W_5}{W_1 W_2}$ and $\binom{W_0}{W_5 W_3}$, respectively, such that

$$\langle w'_{(0)}, \mathcal{Y}_4(\mathcal{Y}_3(w_{(1)}, z_1 - z_2)w_{(2)}, z_2)w_{(3)}\rangle$$

is absolutely convergent when $|z_2| > |z_1 - z_2| > 0$ for $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$ and is equal to (2.12) when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

Theorems 2.3 and 2.6 suggest a method to construct conformal open-string vertex algebra: We start with a vertex operator algebra $(V, Y, \mathbf{1}, \omega)$ satisfying the conditions in Theorem 2.6 and look for a module W and an intertwining operator $\mathcal{Y}^f$ of type $\binom{W}{WW}$ such that if we define

$$\begin{aligned} Y^O : (W \otimes W) \times \mathbb{R}_+ &\to \overline{W} \\ (w_1 \otimes w_2, r) &\mapsto Y^O(w_1, r)w_2 \end{aligned}$$

by

$$Y^O(w_1, r)w_2 = \mathcal{Y}^f(w_1, r)w_2 \tag{2.13}$$

for $r \in \mathbb{R}_+$, then $(W, Y^O, \mathbf{1}, \omega)$ is a conformal open-string vertex algebra.

We give more details here. Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra satisfying the conditions in Theorem 2.6. For simplicity, we assume that V is simple. Let $\mathcal{A}$ be the set of equivalence classes of irreducible V-modules and, for $a \in \mathcal{A}$, let W^a be a representative in a. Then by Theorem 2.6, $\coprod_{a\in A} W^a$ has a natural structure of an intertwining operator algebra. Let $W = \coprod_{a\in A} E^a \otimes W^a$ where E^a for $a \in \mathcal{A}$ are vector spaces to be determined. We give W the obvious $V = \mathbb{C} \otimes V$-module structure. We also let

$$\begin{aligned} \mathcal{Y}^f &\in \mathrm{Hom}(W \otimes W, W\{x\}) \\ &= \coprod_{a_1,a_2,a_3\in\mathcal{A}} \mathrm{Hom}(E^{a_1} \otimes E^{a_2}, E^{a_3}) \otimes \mathrm{Hom}(W^{a_1} \otimes W^{a_2}, W^{a_3}\{x\}) \end{aligned}$$

be given by

$$\mathcal{Y}^f = \sum_{a_1,a_2,a_3\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a_3}} C_{a_1a_2}^{a_3;i} \otimes \mathcal{Y}_{a_1a_2}^{a_3;i}$$

where for $a_1, a_2, a_3 \in \mathcal{A}$, $\mathcal{N}_{a_1a_2}^{a_3}$ is the fusion rule of type $\binom{W^{a_3}}{W^{a_1}W^{a_2}}$, $C_{a_1a_2}^{a_3;i} \in \mathrm{Hom}(E^{a_1}\otimes E^{a_2}, E^{a_3})$ for $i = 1, \ldots, \mathcal{N}_{a_1a_2}^{a_3}$ are to be determined, and $\mathcal{Y}_{a_1a_2}^{a_3;i}$ for $i = 1, \ldots, \mathcal{N}_{a_1a_2}^{a_3}$ is a basis of the space $\mathcal{V}_{a_1a_2}^{a_3}$ of intertwining operators of type $\binom{W^{a_3}}{W^{a_1}W^{a_2}}$.

Let e be the equivalence class of irreducible V-modules containing V. Note that $\mathcal{N}_{ea}^{a}$ for $a \in \mathcal{A}$ are always one-dimensional. We choose the basis $\mathcal{Y}_{ea}^{a;1}$ for $a \in \mathcal{A}$ to be the vertex operator for the V-module W^a. In particular, $\mathcal{Y}_{ea}^{a;1}(\mathbf{1}, x)w^a = w^a$ for $a \in \mathcal{A}$ and $w^a \in W^a$. We also choose the basis $\mathcal{Y}_{ae}^{a;1}$ for $a \in \mathcal{A}$ to be the ones given by

$$\mathcal{Y}_{ae}^{a;1}(w^a, x)u = e^{xL(-1)}\mathcal{Y}_{ea}^{a;1}(u, -x)w$$

for $u \in V$ and $w^a \in W^a$. Thus we have $\lim_{x\to 0}\mathcal{Y}_{ae}^{a;1}(w^a, x)\mathbf{1} = w^a$ for $a \in \mathcal{A}$ and $w^a \in W^a$.

We would like to choose E^a for $a \in \mathcal{A}$ and $C_{a_1a_2}^{a_3;i}$ for $a_1, a_2, a_3 \in \mathcal{A}$ and $i = 1, \ldots, \mathcal{N}_{a_1a_2}^{a_3}$ such that the map Y^O given by (2.13) in terms of $\mathcal{Y}^f$ satisfies the associativity

$$Y^O(w_1, r_1)Y^O(w_2, r_2)w_3 = Y^O(Y^O(w_1, r_1 - r_2)w_2, r_2)w_3 \tag{2.14}$$

for $r_1, r_2 \in \mathbb{R}_+$ satisfying $r_1 > r_2 > r_1 - r_2 > 0$ and $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$. Note that both sides of (2.14) are well-defined since $\coprod_{a\in A} W^a$ is an intertwining operator algebra. The left-hand side of (2.14) gives

$$\begin{aligned}
&\sum_{\substack{a_1,a_2,a_3\\ a_4,a;i,j}} (C_{a_1a}^{a_4;i} \circ (\mathrm{id}_{E^{a_1}} \otimes C_{a_2a_3}^{a;j})) \otimes \mathcal{Y}_{a_1a}^{a_4;i}(w_1, r_1)\mathcal{Y}_{a_2a_3}^{a;j}(w_2, r_2)w_3\\
&\quad = \sum_{\substack{a_1,a_2,a_3\\ a_4,a;i,j}} (C_{a_1a}^{a_4;i} \circ (\mathrm{id}_{E^{a_1}} \otimes C_{a_2a_3}^{a;j}))\\
&\qquad\qquad \otimes \sum_{a_5;k,l} \mathcal{F}_{a;a_5}^{ij;kl}(a_1, a_2, a_3; a_4)\mathcal{Y}_{a_5a_3}^{a_4;l}(\mathcal{Y}_{a_1a_2}^{a_5;k}(w_1, r_1 - r_2)w_2, r_2)w_3
\end{aligned}$$

where for any $a \in \mathcal{A}$, id_{E^a} is the identity on E^a and $\mathcal{F}_{a;a_5}^{ij;kl}(a_1, a_2, a_3; a_4)$, for $a, a_1, \ldots, a_5 \in \mathcal{A}$, $i = 1, \ldots, \mathcal{N}_{a_1a}^{a_4}$, $j = 1, \ldots, \mathcal{N}_{a_2a_3}^{a}$, $k = 1, \ldots, \mathcal{N}_{a_1a_2}^{a_5}$

and $l = 1, \dots, \mathcal{N}_{a_5a_3}^{a_4}$, are the matrix elements of the corresponding fusing isomorphisms. (In the formulas above and below, for simplicity, we omit the ranges over which the sums are taken, since these are clear and some of them have been given above.) The right-hand side of (2.14) gives

$$\sum_{\substack{a_1, a_2, a_3\\ a_4, a_5; k, l}} (C_{a_5a_3}^{a_4;l} \circ (C_{a_1a_2}^{a_5;k} \otimes \mathrm{id}_{E^{a_3}})) \otimes \mathcal{Y}_{a_5a_3}^{a_4;l}(\mathcal{Y}_{a_1a_2}^{a_5;k}(w_1, r_1 - r_2)w_2, r_2)w_3.$$

It is clear that in this case $\mathcal{Y}_{a_5a_3}^{a_4;l}(\mathcal{Y}_{a_1a_2}^{a;k}(\cdot, r_1-r_2)\cdot, r_2)\,\cdot$ for $a_1, a_2, a_3, a_4, a_5 \in \mathcal{A}$ are linearly independent. Thus (2.14) gives

$$\sum_{a;i,j} \mathcal{F}_{a;a_5}^{ij;kl}(a_1, a_2, a_3; a_4)(C_{a_1a}^{a_4;i} \circ (\mathrm{id}_{E^{a_1}} \otimes C_{a_2a_3}^{a;j})) = C_{a_5a_3}^{a_4;l} \circ (C_{a_1a_2}^{a_5;k} \otimes \mathrm{id}_{E^{a_3}}) \quad (2.15)$$

for $a_1, a_2, a_3, a_4, a_5 \in \mathcal{A}$, $k = 1, \dots, \mathcal{N}_{a_1a_2}^{a_5}$ and $l = 1, \dots, \mathcal{N}_{a_5a_3}^{a_4}$.

We need a vacuum for W. Let $1^e \in E^e$. If we want the vacuum to be of the form $\mathbf{1}_W = 1^e \otimes \mathbf{1}$, then we must have the following identity property and creation property:

$$\begin{aligned} Y^O(\mathbf{1}_W, r)(\alpha^a \otimes w^a) &= \alpha^a \otimes w^a, && (2.16)\\ \lim_{r\to 0} Y^O((\alpha^a \otimes w^a), r)\mathbf{1}_W &= \alpha^a \otimes w^a && (2.17) \end{aligned}$$

for $a \in \mathcal{A}$, $\alpha^a \in E^a$ and $w^a \in W^a$. The equations (2.16) and (2.17) together with the properties of intertwining operators for V gives

$$\begin{aligned} C_{ea}^{a;1}(1^e \otimes \alpha^a) &= \alpha^a, && (2.18)\\ C_{ae}^{a;1}(\alpha^a \otimes 1^e) &= \alpha^a && (2.19) \end{aligned}$$

for $a \in \mathcal{A}$ and $\alpha^a \in E^a$.

Let $\mathbf{1}_W = 1^e \otimes \mathbf{1}$ and $\omega_W = 1^e \otimes \omega$. Then we have just proved the following:

Proposition 2.7 *Let V be a simple vertex operator algebra satisfying the conditions in Theorem 2.6 and let $\mathcal{A}$, e and W^a for $a \in \mathcal{A}$ be as above. If we choose the vector spaces E^a for $a \in \mathcal{A}$, $C_{a_1a_2}^{a_3;i} \in \mathrm{Hom}(E^{a_1} \otimes E^{a_2}, E^{a_3})$ for $a_1, a_2, a_3 \in \mathcal{A}$, $i = 1, \dots, \mathcal{N}_{a_1a_2}^{a_3}$, and $1^e \in E^e$ such that (2.15), (2.18) and (2.19) hold, then the quadruple $(W, Y^O, \mathbf{1}_W, \omega_W)$ is a grading-restricted conformal open-string vertex algebra.*

3 Examples

In this section, we give some examples of open-string vertex algebras. Examples can also be constructed using the main results in Sections 4 and 5.

First of all, we have the following examples for which the axioms are trivial to verify:

1. Associative algebras.

2. Vertex (super)algebras.

3. Tensor products of algebras above, for example, $A \otimes V$ where A is an associative algebra and V a vertex (super)algebra.

The examples above are trivial to construct because they satisfy some much stronger axioms than those in the definition of open-string vertex algebra. Nontrivial examples of open-string vertex algebras can be constructed from the direct sum of a vertex algebra and an $\mathbb{R}$-graded module for the vertex algebra in the same ways as in the construction of the example of vertex operator algebras in Example 3.4 in [H3] and as in the conceptual construction of the vertex operator algebra structure on the moonshine module in [H5], except that here the module does not have to be $\mathbb{Z}$-graded. Note that in the construction of the vertex operator algebra structure on the moonshine module in [H5], the hard part is to prove the duality properties, which follow from the duality properties of a larger intertwining operator algebra. If we start with a vertex operator algebra satisfying the conditions in Theorem 2.6, then the construction becomes very easy because the duality properties have been established by Theorem 2.6.

We now give an example constructed using a different method. It is an example constructed from modules for the minimal Virasoro vertex operator algebra of central charge $c = \frac{1}{2}$. This example is nontrivial because it is not an associative algebra, a vertex (super)algebra or a tensor product of these algebras. Here we describe the data. For the details, we refer the reader to the second author's thesis [K]. For the minimal Virasoro vertex operator algebras, their representations, intertwining operators and chiral correlation functions, see, for example, [DF], [BPZ], [W], [H4], [FRW] and [DMS].

Let $L(\frac{1}{2}, 0)$ be the minimal Virasoro vertex operator algebra of central charge $\frac{1}{2}$. It has three inequivalent irreducible modules $W_0 = L(\frac{1}{2}, 0)$, $W_1 =$

$L(\frac{1}{2}, \frac{1}{2})$ and $W_2 = L(\frac{1}{2}, \frac{1}{16})$. It is well known that the fusion rules $\mathcal{N}_{ij}^k = \mathcal{N}_{W_iW_j}^{W_k}$ for $i, j, k = 0, 1, 2$ are equal to 1 for

$$
\begin{aligned}
(i, j, k) \quad = \quad & (0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), (0, 2, 2), \\
& (2, 0, 2), (2, 2, 0), (1, 2, 2), (2, 1, 2), (2, 2, 1)
\end{aligned}
$$

and are equal to 0 otherwise. It was proved in [H4] that the direct sum of W_0, W_1 and W_3 has a structure of intertwining operator algebra. When $\mathcal{N}_{ij}^k = 1$, we choose a basis $\mathcal{Y}_{ij}^k$ of $\mathcal{V}_{ij}^k$. Given $i, j, k, l \in \{0, 1, 2\}$, $m \in \{0, 1, 2\}$ is said to be *coupled with* $n \in \{0, 1, 2\}$ *through* $(i, j, k; l)$ if $\mathcal{V}_{im}^l$, $\mathcal{V}_{jk}^m$, $\mathcal{V}_{ij}^n$ and $\mathcal{V}_{nk}^l$ are all nonzero. We use the notation $m \bowtie_{i,j,k}^l n$ to denote the fact that m is coupled with n through $(i, j, k; l)$.

For $i, j, k, l \in \{0, 1, 2\}$, the matrix elements $\mathcal{F}_{m;n}(i, j, k; l)$ for $m, n = 0, 1, 2$ of the fusing isomorphisms

$$
\mathcal{F}(i, j, k; l) : \coprod_{m=0}^{2} \mathcal{V}_{im}^l \otimes \mathcal{V}_{jk}^m \to \coprod_{n=0}^{2} \mathcal{V}_{ij}^n \otimes \mathcal{V}_{nk}^l
$$

are determined by the following associativity relations (see [H7])

$$
\begin{aligned}
&\langle w_l', \mathcal{Y}_{im}^l(w_i, z_1)\mathcal{Y}_{jk}^m(w_j, z_2)w_k\rangle \\
&\quad = \sum_{m \bowtie_{i,j,k}^l n} \mathcal{F}_{m;n}(i, j, k; l)\langle w_l', \mathcal{Y}_{nk}^l(\mathcal{Y}_{ij}^n(w_i, z_1 - z_2)w_j, z_2)w_k\rangle
\end{aligned}
$$

for $i, j, m = 0, 1, 2$, $z_1, z_2 \in \mathbb{R}$ satisfying $z_1 > z_2 > z_1 - z_2 > 0$ and $w_i \in W_i$, $w_j \in W_j$, $w_k \in W_k$, where the sum is over all $k, l, n = 0, 1, 2$ such that $m \bowtie_{i,j,k}^l n$. For simplicity, we use $\tilde{\mathcal{F}}(i, j, k; l)$ for $i, j, k, l = 0, 1, 2$ to denote matrices whose entries $\tilde{\mathcal{F}}_{mn}(i, j, k; l)$ for $m, n = 0, 1, 2$ is the symbol DC (meaning decoupled) if m is not coupled with n through $(i, j, k; l)$ and is $\mathcal{F}_{m;n}(i, j, k; l)$ if m is coupled with n through $(i, j, k; l)$. We call these matrices the *fusing-coupling* matrices. For $m, n = 0, 1, 2$, we use $\pm E_{mn}$ to denote the 3×3 matrices with the entry in the m-th row and the n-th column being ± 1 and the other entries being DC.

Proposition 3.1 *For $i, j, k = 0, 1, 2$ such that $\mathcal{N}_{ij}^k = 1$, there exist basis $\mathcal{Y}_{ij}^k$ of $\mathcal{V}_{ij}^k$ such that*

$$
\tilde{\mathcal{F}}(0, 0, 0, 0) = \tilde{\mathcal{F}}(1, 1, 1, 1) = E_{00},
$$

$$
\begin{aligned}
\tilde{\mathcal{F}}(1,1,0,0) &= \tilde{\mathcal{F}}(0,0,1,1) = E_{01},\\
\tilde{\mathcal{F}}(1,0,0,1) &= \tilde{\mathcal{F}}(0,1,1,0) = E_{10},\\
\tilde{\mathcal{F}}(1,0,1,0) &= \tilde{\mathcal{F}}(0,1,0,1) = E_{11},\\
\tilde{\mathcal{F}}(2,2,0,0) = \tilde{\mathcal{F}}(0,0,2,2) &= \tilde{\mathcal{F}}(1,1,2,2) = \tilde{\mathcal{F}}(2,2,1,1) = E_{02},\\
\tilde{\mathcal{F}}(0,2,2,0) = \tilde{\mathcal{F}}(2,0,0,2) &= \tilde{\mathcal{F}}(1,2,2,1) = \tilde{\mathcal{F}}(2,1,1,2) = E_{20},\\
\tilde{\mathcal{F}}(0,1,2,2) = \tilde{\mathcal{F}}(1,0,2,2) &= \tilde{\mathcal{F}}(2,2,0,1) = \tilde{\mathcal{F}}(2,2,1,0) = E_{12},\\
\tilde{\mathcal{F}}(0,2,2,1) = \tilde{\mathcal{F}}(1,2,2,0) &= \tilde{\mathcal{F}}(2,0,1,2) = \tilde{\mathcal{F}}(2,1,0,2) = E_{21},\\
\tilde{\mathcal{F}}(1,2,1,2) &= \tilde{\mathcal{F}}(2,1,2,1) = -E_{22},\\
\tilde{\mathcal{F}}(0,2,0,2) = \tilde{\mathcal{F}}(2,0,2,0) &= \tilde{\mathcal{F}}(0,2,1,2) = \tilde{\mathcal{F}}(1,2,0,2)\\
&= \tilde{\mathcal{F}}(2,0,2,1) = \tilde{\mathcal{F}}(2,1,2,0) = E_{22},\\
\tilde{\mathcal{F}}(2,2,2,2) &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & DC \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & DC \\ DC & DC & DC \end{pmatrix};
\end{aligned}
$$

all other fusing-coupling matrices have entries which are either 0 *or* DC*.*

The proposition above gives the complete information about the fusing isomorphisms for the minimal model of central charge $\frac{1}{2}$. Now consider the irreducible modules $W_i \otimes W_i$ for $i = 0,1,2$ for the tensor product vertex operator algebra $L(\frac{1}{2},0) \otimes L(\frac{1}{2},0)$. Let $W = \coprod_{i=0}^{2} W_i \otimes W_i$ and let

$$\mathcal{Y}^f : (W \otimes W) \to W\{x\}$$

be given by

$$\mathcal{Y}^f = \sum_{i,j,k=0}^{2} \mathcal{Y}_{ij}^k \otimes \mathcal{Y}_{ij}^k$$

where we have taken $\mathcal{Y}_{ij}^k = 0$ for $i,j,k \in \{0,1,2\}$ such that $\mathcal{V}_{ij}^k = 0$ and where $\mathcal{Y}_{ij}^k \otimes \mathcal{Y}_{ij}^k$ for $i,j,k \in \{0,1,2\}$ act on $W \otimes W$ in the obvious way. Let

$$
\begin{aligned}
Y^O : (W \otimes W) \times \mathbb{R}_+ &\to \overline{W}\\
(w_1 \otimes w_2, r) &\mapsto Y^O(w_1, r)w_2
\end{aligned}
$$

be given by

$$Y^O(w_1, r)w_2 = \mathcal{Y}^f(w_1, r)w_2$$

for $r \in \mathbb{R}_+$ and $w_1, w_2 \in W$. Let $\mathbf{1}$ and ω be the vacuum and conformal element of $L(\frac{1}{2},0)$. Then we have:

Proposition 3.2 *The quadruple* $(W, Y^O, \mathbf{1}\otimes\mathbf{1}, \omega\otimes\mathbf{1}+\mathbf{1}\otimes\omega)$ *is a grading-restricted conformal open-string vertex algebra with* $C_0(W) = W_0 \otimes W_0$.

The proof is a straightforward verification. See [K] for details.

Remark 3.3 In the construction above, $\mathcal{Y}^f$ and Y^O involve fractional powers. So W is not a vertex operator algebra.

4 Braided tensor categories and open-string vertex algebras

In this section, we show that an associative algebra in the braided tensor category of modules for a suitable vertex operator algebra V is equivalent to an open-string vertex algebra with V in its meromorphic center. The main result of this section (Theorem 4.3) is a straightforward generalization of the main result in [HKL]. In this section, we assume that the reader is familiar with the tensor product theory developed by Lepowsky and the first author. See [HL3]–[HL6] and [H3] for details.

First of all, we have the following result established in [H9]:

Theorem 4.1 *Let V be a vertex operator algebra satisfying the conditions in Theorem 2.6. Then the category of V-modules has a natural structure of vertex tensor category with V as its unit object. In particular, this category has a natural structure of braided tensor category.*

Given a braided tensor category $\mathcal{C}$, we use $1_{\mathcal{C}}$ to denote its unit object. We need the following concept:

Definition 4.2 Let $\mathcal{C}$ be a braided tensor category. An *associative algebra in* $\mathcal{C}$ (or *associative* $\mathcal{C}$*-algebra*) is an object $A \in \mathcal{C}$ along with a morphism $\mu : A \otimes A \to A$ and an injective morphism $\iota_A : 1_{\mathcal{C}} \to A$ such that the following conditions hold:

1. Associativity: $\mu\circ(\mu\otimes \mathrm{id}_A) = \mu\circ(\mathrm{id}_A\otimes\mu)\circ\mathcal{A}$ where $\mathcal{A}$ is the associativity isomorphism from $A\otimes(A\otimes A)$ to $(A\otimes A)\otimes A$.

2. Unit properties: $\mu\circ(\iota_A\otimes \mathrm{id}_A)\circ l_A^{-1} = \mu\circ(\iota_A\otimes \mathrm{id}_A)\circ r_A^{-1} = \mathrm{id}_A$ where $l_A : 1_{\mathcal{C}}\otimes A \to A$ and $r_A : A\otimes 1_{\mathcal{C}} \to A$ are the left and right unit isomorphism, respectively.

We say that the unit of an associative algebra A in $\mathcal{C}$ is *unique* if

$$\dim \mathrm{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, A) = 1.$$

We use (A, μ, ι_A) or simply A to denote the associative algebra in $\mathcal{C}$ just defined.

Let V be a vertex operator algebra satisfying the conditions in Theorem 2.6. Then we know that the direct sum of all irreducible V-modules is an intertwining operator algebra. We say that this intertwining operator algebra satisfies the *positive weight condition* if for any irreducible V-module W, the weights of nonzero elements of W are nonnegative, $W_{(0)} \neq 0$ if and only if W is isomorphic to V, and $V_{(0)} = \mathbb{C}\mathbf{1}$. We say that an open-string vertex algebra V satisfy the *positive weight condition* if the weights of elements of V are nonnegative and $V_{(0)} = \mathbb{C}\mathbf{1}$.

Theorem 4.3 *Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra satisfying the conditions in Theorem 2.6 and let $\mathcal{C}$ be the braided tensor category of V-modules. Then the categories of the following objects are isomorphic:*

1. *A grading-restricted conformal open-string vertex algebra V_e and an injective homomorphism of vertex operator algebras from V to the meromorphic center $C_0(V_e)$ of V_e.*

2. *An associative algebra V_e in $\mathcal{C}$.*

If the intertwining operator algebra on the direct sum of all irreducible V-modules satisfies the positive weight condition, then an algebra V_e in Category 1 above satisfy the positive weight condition if and only if the unit of the corresponding associative algebra V_e in $\mathcal{C}$ is unique.

Proof. Let V_e be a grading-restricted conformal open-string vertex algebra, $\mathbf{1}_e$ the vacuum of V_e and ι_{V_e} an injective homomorphism of vertex operator algebras from V to $C_0(V_e)$. Then we have $\iota_{V_e}(\mathbf{1}) = \mathbf{1}_e$. Then by Theorem 2.3, V_e is an $\iota_{V_e}(V)$-module and thus a V-module. So V_e is an object in $\mathcal{C}$. Since V_e is an open-string vertex algebra, we have a vertex operator map Y_e^O for V_e. By Theorem 2.3 again, the corresponding formal vertex operator map $\mathcal{Y}_e^f$ is in fact an intertwining operator for V of type $\binom{V_e}{V_e V_e}$. Let $\mu : V_e \boxtimes V_e \to V_e$ be the module map corresponding to the intertwining operator $\mathcal{Y}_e^f$. We claim that (V_e, μ, ι_{V_e}) is an associative algebra in $\mathcal{C}$. The proof is similar to the proof of the result in [HKL] that suitable commutative associative algebras in $\mathcal{C}$ are equivalent to vertex operator algebras extending V. For reader's convenience, we give a proof here.

For $r \in \mathbb{R}_+$, let μ_r be the morphism from $V_e \boxtimes_{P(r)} V_e$ to V_e corresponding to the intertwining operator Y_e^f and let $\overline{\mu}_r : \overline{V_e \boxtimes_{P(r)} V_e} \to \overline{V}_e$ be the natural extension of $\mu_{P(r)}$. Then by definition, $\mu = \mu_1$ and

$$\overline{\mu}_r(u \boxtimes_{P(r)} v) = \mathcal{Y}_e^f(u, r)v = Y_e^O(u, r)v.$$

for $u, v \in V_e$. For simplicity, we shall use id to denote id_{V_e} in this proof. Thus for $u, v, w \in V_e$ and $r_1, r_2 \in \mathbb{R}_+$ satisfying $r_1 > r_2 > r_1 - r_2 > 0$,

$$\begin{aligned} &\overline{(\mu_{r_1} \circ (\mathrm{id} \boxtimes_{P(r_1)} \mu_{r_2}))}(u \boxtimes_{P(r_1)} (v \boxtimes_{P(r_2)} w)) \\ &\qquad = Y_e^O(u, r_1)Y_e^O(v, r_2)w \qquad (4.1)\\ &\overline{(\mu_{r_2} \circ (\mu_{r_1 - r_2} \boxtimes_{P(r_2)} \mathrm{id}))}((u \boxtimes_{P(r_1 - r_2)} v) \boxtimes_{P(r_2)} w) \\ &\qquad = Y_e^O Y_e^O(u, r_1 - r_2)v, r_2)w, \qquad (4.2)\end{aligned}$$

where (and below) we use the notation that a linear map preserving gradings with a horizontal line over it always mean the natural extension of the map to a map between the algebraic completions of the original graded spaces. The associativity for Y_e^O gives

$$Y_e^O(u, r_1)Y_e^O(v, r_2)w = Y_e^O(Y_e^O(u, r_1 - r_2)v, r_2)w. \qquad (4.3)$$

The associativity isomorphism

$$\mathcal{A}_{P(r_1),P(r_2)}^{P(r_1-r_2),P(r_2)} : V_e \boxtimes_{P(r_1)} (V_e \boxtimes_{P(r_2)} V_e) \to (V_e \boxtimes_{P(r_1-r_2)} V_e) \boxtimes_{P(r_2)} V_e$$

is characterized by

$$\overline{\mathcal{A}}_{P(r_1),P(r_2)}^{P(r_1-r_2),P(r_2)}(u \boxtimes_{P(r_1)} (v \boxtimes_{P(r_2)} w)) = (u \boxtimes_{P(r_1-r_2)} v) \boxtimes_{P(r_2)} w \qquad (4.4)$$

for $u, v, w \in V_e$, where $\overline{\mathcal{A}}_{P(r_1),P(r_2)}^{P(r_1-r_2),P(r_2)}$ is the natural extension of $\mathcal{A}_{P(r_1),P(r_2)}^{P(r_1-r_2),P(r_2)}$.

Combining (4.1)–(4.4), we obtain

$$(\mu_{r_1} \circ (\mathrm{id} \boxtimes_{P(r_1)} \mu_{r_2})) = (\mu_{r_2} \circ (\mu_{r_1-r_2} \boxtimes_{P(r_2)} \mathrm{id})) \circ \mathcal{A}_{P(r_1),P(r_2)}^{P(r_1-r_2),P(r_2)}. \qquad (4.5)$$

From (4.5), we obtain

$$\begin{aligned} &(\mu_{r_1} \circ (\mathrm{id} \boxtimes_{P(r_1)} \mu_{r_2})) \circ (\mathrm{id} \boxtimes_{P(r_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1} \\ &\qquad = (\mu_{r_2} \circ (\mu_{r_1-r_2} \boxtimes_{P(r_2)} \mathrm{id})) \circ \mathcal{A}_{P(r_1),P(r_2)}^{P(r_1-r_2),P(r_2)} \circ (\mathrm{id} \boxtimes_{P(r_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1}. \end{aligned} \qquad (4.6)$$

where r_1, r_2 are real numbers satisfying $r_1 > r_2 > r_1 - r_2 > 0$, γ_1 and γ_2 are paths in $\mathbb{R}_+$ from 1 to r_1 and r_2, respectively, and $\mathcal{T}_{\gamma_1}$ and $\mathcal{T}_{\gamma_2}$ the parallel transport isomorphisms associated to γ_1 and γ_2, respectively. (For reader's convenience, we recall the definition of parallel transport isomorphism here. Let γ be a path from $z_1 \in \mathbb{C}^\times$ to $z_2 \in \mathbb{C}^\times$. The parallel isomorphism $\mathcal{T}_\gamma : W_1 \boxtimes_{P(z_1)} W_2 \to W_1 \boxtimes_{P(z_2)} W_2$ is given as follows: Let $\mathcal{Y}$ be the intertwining operator corresponding to the intertwining map $\boxtimes_{P(z_2)}$ and $l(z_1)$ the value of the logarithm of z_1 determined uniquely by $\log z_2$ (satisfying $0 \leq \Im(\log z_2) < 2\pi$) and the path γ. Then $\mathcal{T}_\gamma$ is characterized by

$$\overline{\mathcal{T}}_\gamma(w_1 \boxtimes_{P(z_1)} w_2) = \mathcal{Y}(w_1, x)w_2\Big|_{x^n = e^{nl(z_1)},\ n \in \mathbb{C}}$$

for $w_1 \in W_1$ and $w_2 \in W_2$, where $\overline{\mathcal{T}}_\gamma$ is the natural extension of $\mathcal{T}_\gamma$ to the algebraic completion $\overline{W_1 \boxtimes_{P(z_1)} W_2}$ of $W_1 \boxtimes_{P(z_1)} W_2$. The parallel isomorphism depends only on the homotopy class of γ.)

By definition, we have

$$(\mu_{r_1} \circ (\mathrm{id} \boxtimes_{P(r_1)} \mu_{r_2})) \circ (\mathrm{id} \boxtimes_{P(r_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1} = \mu \circ (\mathrm{id} \boxtimes \mu). \tag{4.7}$$

Similarly, we have

$$(\mu_{r_2} \circ (\mu_{r_1 - r_2} \boxtimes_{P(r_2)} \mathrm{id})) \circ (\mathcal{T}_{\gamma_3} \circ (\mathcal{T}_{\gamma_4} \boxtimes_{P(r_2)} \mathrm{id}))^{-1} = (\mu \circ (\mu \boxtimes \mathrm{id})), \tag{4.8}$$

where γ_3 and γ_4 are paths in $\mathbb{R}_+$ from r_2 and $r_1 - r_2$ to 1, respectively, and $\mathcal{T}_{\gamma_3}$ and $\mathcal{T}_{\gamma_4}$ the parallel transport isomorphisms associated to γ_3 and γ_4, respectively. Combining (4.6)–(4.8) with the definition

$$\mathcal{A} = \mathcal{T}_{\gamma_3} \circ (\mathcal{T}_{\gamma_4} \boxtimes_{P(z_2)} \mathrm{id}) \circ \mathcal{A}_{P(z_1), P(z_2)}^{P(z_1 - z_2), P(z_2)} \circ (\mathrm{id} \boxtimes_{P(z_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1}. \tag{4.9}$$

of the associativity isomorphism for the tensor product structure, we obtain the associativity

$$\mu \circ (\mathrm{id} \boxtimes \mu) = (\mu \circ (\mu \boxtimes \mathrm{id})) \circ \mathcal{A}.$$

For the unit property, we note that the inverse $l_{V_e}^{-1} : V_e \to V \boxtimes V_e$ of the left unit isomorphism is defined by $l_{V_e}^{-1}(u) = \mathbf{1} \boxtimes u$ for $u \in V_e$ and thus

$$\begin{aligned}
(\mu \circ (\iota_{V_e} \boxtimes \mathrm{id}_{V_e}) \circ l_{V_e}^{-1})(u) &= \mu((\iota_{V_e} \boxtimes \mathrm{id}_{V_e})(\mathbf{1} \boxtimes u)) \\
&= \mu(\mathbf{1}_e \boxtimes u) \\
&= Y_e(\mathbf{1}_e, 1)u \\
&= \mathrm{id}_{V_e}(u)
\end{aligned}$$

for $u \in V_e$. The other unit property is proved similarly.

Conversely, let (V_e, μ, ι_{V_e}) be an associative $\mathcal{C}$-algebra. In particular, V_e is a V-module. The module map $\mu : V_e \boxtimes V_e \to V_e$ corresponds to an intertwining operator $\mathcal{Y}_e^f$ of type $\binom{V_e}{V_e V_e}$ such that

$$\overline{\mu}(u \boxtimes v) = \mathcal{Y}_e^f(u, 1)v \tag{4.10}$$

for $u, v \in V_e$. Let $\mathbf{1}_e = \iota_{V_e}(\mathbf{1})$ and $\omega_e = \iota_{V_e}(\omega)$. We define

$$\begin{aligned} Y_e^O : (V_e \otimes V_e) \times \mathbb{R}_+ &\to \overline{V}_e \\ (u \otimes v, r) &\mapsto Y_e^O(u, r)v \end{aligned}$$

by

$$Y_e^O(u, r)v = \mathcal{Y}_e^f(u, r)v$$

for $r \in \mathbb{R}_+$, $u, v \in V_e$. Then we claim that $(V_e, Y_e^O, \mathbf{1}_e, \omega_e)$ is an grading-restricted conformal open-string vertex algebra satisfying the positive weight condition above and with V in its meromorphic center. Again, the proof is similar to the proof of the result in [HKL] mentioned above. For reader's convenience, we give a proof here.

The identity property for the vacuum follows immediately from the left unit property $\mu \circ (\iota_{V_e} \boxtimes \mathrm{id}_{V_e}) \circ l_{V_e}^{-1} = \mathrm{id}_{V_e}$. The creation property follows from the right unit property $\mu \circ (\iota_{V_e} \otimes \mathrm{id}_{V_e}) \circ r_{V_e}^{-1} = \mathrm{id}_{V_e}$. The Virasoro relations and the $L(0)$-grading property follows from the fact that V_e is a V-module. The $L(-1)$-derivative property and the commutator formula for the Virasoro operators and $\mathcal{Y}_e^f$ follow from the fact that $\mathcal{Y}_e^f$ is an intertwining operator.

We now prove associativity. As above, for any $r \in \mathbb{R}_+$, let

$$\mu_r : V_e \boxtimes_{P(r)} V_e \to V_e$$

be the module map corresponding to the intertwining operator $\mathcal{Y}_e^f$. By definition, we have

$$\mu_r(u \boxtimes_{P(r)} v) = \mathcal{Y}_e^f(u, r)v = (\mu \circ \mathcal{T}_\gamma)(u \boxtimes_{P(r)} v) \tag{4.11}$$

for $u, v \in V_e$ and $r \in \mathbb{R}_+$, where γ is a path from r to 1 in $\mathbb{R}_+$. By definition, for $r_1, r_2 \in \mathbb{R}_+$ satisfying $r_1 > r_2 > r_1 - r_2 > 0$, paths γ_1 and γ_2 in $\mathbb{R}_+$ from 1 to r_1, r_2, respectively, and paths γ_3 and γ_4 in $\mathbb{R}_+$ from r_2 and $r_1 - r_2$ to 1, respectively, (4.7)–(4.8) hold.

Compose both sides of the associativity

$$\mu \circ (\mathrm{id} \boxtimes \mu) = (\mu \circ (\mu \boxtimes \mathrm{id})) \circ \mathcal{A}$$

for the $\mathcal{C}$-algebra V_e with

$$((\mathrm{id} \boxtimes_{P(z_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1})^{-1},$$

where $r_1, r_2 \in \mathbb{R}_+$ satisfying $r_1 > r_2 > r_1 - r_2 > 0$ and γ_1 and γ_2, as above, are paths from 1 to r_1 and r_2, respectively, in $\mathbb{R}_+$. Then we obtain

$$\begin{aligned}
&\mu \circ (\mathrm{id} \boxtimes \mu) \circ ((\mathrm{id} \boxtimes_{P(r_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1})^{-1} \\
&\quad = (\mu \circ (\mu \boxtimes \mathrm{id})) \circ \mathcal{A} \circ ((\mathrm{id} \boxtimes_{P(r_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1})^{-1}.
\end{aligned} \tag{4.12}$$

Using (4.7)–(4.9) and (4.12), we obtain

$$\begin{aligned}
&\mu_{r_1} \circ (\mathrm{id} \boxtimes_{P(r_1)} \mu_{r_2}) \\
&\quad = \mu \circ (\mathrm{id} \boxtimes \mu) \circ ((\mathrm{id} \boxtimes_{P(r_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1})^{-1} \\
&\quad = (\mu \circ (\mu \boxtimes \mathrm{id})) \circ \mathcal{A} \circ ((\mathrm{id} \boxtimes_{P(r_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1})^{-1} \\
&\quad = (\mu_{r_2} \circ (\mu_{r_1-r_2} \boxtimes_{P(r_2)} \mathrm{id})) \circ (\mathcal{T}_{\gamma_3} \circ (\mathcal{T}_{\gamma_4} \boxtimes_{P(r_2)} \mathrm{id}))^{-1} \\
&\qquad\qquad \circ \mathcal{A} \circ ((\mathrm{id} \boxtimes_{P(r_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1})^{-1} \\
&\quad = (\mu_{r_2} \circ (\mu_{r_1-r_2} \boxtimes_{P(r_2)} \mathrm{id})) \circ \mathcal{A}_{P(r_1),P(r_2)}^{P(r_1-r_2),P(r_2)}.
\end{aligned} \tag{4.13}$$

For the next step, we use the convergence of products and iterates of intertwining operators for V. Because of the convergence, $\mathrm{id} \boxtimes_{P(r_1)} \overline{\mu_{r_2}}$ is well defined and it is clear that $\overline{\mu_{r_1}} \circ (\mathrm{id} \boxtimes_{P(r_1)} \overline{\mu_{r_2}})$ is equal to $\overline{\mu_{r_1} \circ (\mathrm{id} \boxtimes_{P(r_1)} \mu_{r_2})}$. Similarly, $\overline{\mu_{r_1-r_2}} \boxtimes_{P(r_2)} \mathrm{id}$ is well-defined and $\overline{\mu_{r_1}} \circ (\overline{\mu_{r_1-r_2}} \boxtimes_{P(r_2)} \mathrm{id})$ is equal to $\overline{\mu_{r_1} \circ (\mu_{r_1-r_2} \boxtimes_{P(r_2)} \mathrm{id})}$. Taking the natural completions of both sides of (4.13), we obtain

$$\overline{\mu_{r_1}} \circ (\mathrm{id} \boxtimes_{P(r_1)} \overline{\mu_{r_2}}) = \overline{\mu_{r_1}} \circ (\overline{\mu_{r_1-r_2}} \boxtimes_{P(r_2)} \mathrm{id}) \circ \overline{\mathcal{A}}_{P(r_1),P(r_2)}^{P(r_1-r_2),P(r_2)}. \tag{4.14}$$

Applying both sides of (4.14) to $u \boxtimes_{P(r_1)} (v \boxtimes_{P(r_2)} w)$ for $u, v, w \in V_e$, pairing the result with $v' \in V_e$ and using (4.11) and

$$\overline{\mathcal{A}}_{P(r_1),P(r_2)}^{P(r_1-r_2),P(r_2)}(u \boxtimes_{P(r_1)} (v \boxtimes_{P(r_2)} w)) = (u \boxtimes_{P(r_1-r_2)} v) \boxtimes_{P(r_2)} w,$$

we obtain the associativity

$$\langle v', Y_e^O(u, r_1) Y_e^O(v, r_2) w \rangle = \langle v', Y_e^O(Y_e^O(u, r_1 - r_2) v, r_2) w \rangle$$

for $u, v, w \in V_e$, $v' \in V_e'$ and $r_1, r_2 \in \mathbb{R}_+$ satisfying $r_1 > r_2 > r_1 - r_2 > 0$.

We now prove that $\iota_{V_e}(V)$ is in the meromorphic center of V_e. Clearly $\iota_{V_e}(V)$ is a vertex operator algebra isomorphic to V, ι_{V_e} is an isomorphism of vertex operator algebras from V to $\iota_{V_e}(V)$ and thus V_e is an $\iota_{V_e}(V)$-module. We know that the restriction $\mathcal{Y}_e^f|_{V_e \otimes \iota_{V_e}(V)}$ of $\mathcal{Y}_e^f$ to $V_e \otimes \iota_{V_e}(V)$ is in fact the intertwining operator of type $\binom{V_e}{V_e \iota_{V_e}(V)}$ for the vertex operator algebra $\iota_{V_e}(V)$ corresponding to the module map $\mu|_{V_e \boxtimes \iota_{V_e}(V)} : V_e \boxtimes \iota_{V_e}(V) \to V_e$ which is the restriction of μ to $V_e \boxtimes \iota_{V_e}(V)$. By the creation property for Y_e^O, we have

$$\lim_{r \to 0} \mathcal{Y}_e^f(u, r)\mathbf{1}_e = \lim_{r \to 0} Y_e^O(u, r)\mathbf{1}_e = u$$

for $u \in V_e$. Since the space of intertwining operators of type $\binom{V_e}{V_e \iota_{V_e}(V)}$ is isomorphic to the space of intertwining operators of type $\binom{V_e}{\iota_{V_e}(V) V_e}$, which in turn is isomorphic to the space of module maps from V_e to itself, any intertwining operator $\mathcal{Y}$ of this type satisfying the creation property

$$\lim_{r \to 0} \mathcal{Y}(u, r)\mathbf{1}_e = u$$

must be equal to $\mathcal{Y}_e^f|_{V_e \otimes \iota_{V_e}(V)}$. In fact, the intertwining operator $\mathcal{Y}$ of such type defined by

$$\mathcal{Y}(u, x)v = e^{xL(-1)} Y_{V_e}(v, -x)u$$

for $u \in V_e, v \in \iota_{V_e}(V)$, where Y_{V_e} is the vertex operator map for the $\iota_{V_e}(V)$-module V_e, is such an intertwining operator. Thus we have

$$\mathcal{Y}_e^f|_{V_e \otimes \iota_{V_e}(V)}(u, x)v = e^{xL(-1)} Y_{V_e}(v, -x)u \tag{4.15}$$

for $u \in V_e, v \in \iota_{V_e}(V)$. But both Y_{V_e} and $\mathcal{Y}_e^f|_{\iota_{V_e}(V) \otimes V_e}$ are intertwining operators of type $\binom{V_e}{\iota_{V_e}(V) V_e}$ satisfying the identity property and the space of intertwining operators of such type, as we mentioned above, is isomorphic to the space of module maps from V_e to itself. So Y_{V_e} and $\mathcal{Y}_e^f|_{\iota_{V_e}(V) \otimes V_e}$ must be equal. Thus (4.15) says that $\iota_{V_e}(V)$ is in the meromorphic center of V_e. So ι_{V_e} is an injective homomorphism from V to the meromorphic center of V_e.

The constructions above give two functors and it is easy to see that they are inverse to each other. Thus the two categories are isomorphic.

Finally we prove the last statement. We assume that the intertwining operator algebra on the direct sum of all irreducible V-modules satisfies the positive weight condition. In particular, as an open-string vertex algebra, V

itself satisfies the positive weight condition. Let V_e be a grading-restricted conformal open-string vertex algebra and ι_{V_e} an injective homomorphism of vertex operator algebras from V to $C_0(V_e)$. Since the weights of the nonzero elements of all the irreducible V-modules are nonnegative, the weights of the nonzero elements of the V-module V_e are also nonnegative. Assume that V_e satisfies the positive weight condition. Let $f \in \mathrm{Hom}_{\mathcal{C}}(V, V_e)$. Since f preserves the grading and since V and V_e both satisfy the positive weight condition, it is clear that f maps $\mathbf{1}$ to a scalar multiple of $\mathbf{1}_e$. Since V as a module is generated by $\mathbf{1}$, f is determined completely by the scalar above. On the other hand, given any scalar, we can also construct an element of $\mathrm{Hom}_{\mathcal{C}}(V, V_e)$ such that it maps $\mathbf{1}$ to the scalar times $\mathbf{1}_e$. Thus $\dim \mathrm{Hom}_{\mathcal{C}}(V, V_e) = 1$. Conversely, assume that $\dim \mathrm{Hom}_{\mathcal{C}}(V, V_e) = 1$. We already know that the weights of nonzero elements of the V-module V_e are also nonnegative. Assume that there is an element of $(V_e)_{(0)}$ which is not proportional to $\mathbf{1}_e$. Then this element generates a V-submodule of the V-module V_e. Since all V-modules are completely reducible, we can find an irreducible V-submodule of this V-submodule such that it is generated by an element of $(V_e)_{(0)}$ which is not proportional to $\mathbf{1}_e$. Since any irreducible V-module having a nonzero element of weight 0 must be isomorphic to V, this V-submodule is isomorphic to V. But this V-submodule is not equal to $\iota_e(V) \subset C_0(V_e)$ since its generator of weight 0 is not proportional to $\mathbf{1}_e$. Thus we see that $\dim \mathrm{Hom}_{\mathcal{C}}(V, V_e) > 1$. Contradiction. So V_e satisfies the positive weight condition. ∎

Remark 4.4 Recall that a *commutative associative algebra in a braided tensor category* $\mathcal{C}$ or a *commutative associative* $\mathcal{C}$*-algebra* is an associative $\mathcal{C}$-algebra satisfying $\mu \circ \mathcal{R} = \mu$ (*commutativity*), where $\mathcal{R}$ is the commutativity isomorphism from $A \otimes A$ to itself. Let V be a vertex operator algebra as in Theorem 4.3 and $\mathcal{C}$ the category of V-modules. Then an associative $\mathcal{C}$-algebra V_e is in general not commutative In fact, for the category $\mathcal{C}$ of modules for V, the commutativity isomorphism $\mathcal{R}$ is characterized by

$$\overline{\mathcal{R}}(u \boxtimes v) = e^{L(-1)} \overline{\mathcal{T}}_{\gamma_+}(v \boxtimes_{P(-1)} u) \tag{4.16}$$

where $u, v \in V_e$, γ_+ is a path from -1 to 1 in the closed upper half plane without passing through 0, $\mathcal{T}_{\gamma_+}$ is the corresponding parallel transport isomorphism and $\overline{\mathcal{T}}_{\gamma_+}$ is the natural extension of $\mathcal{T}_{\gamma_+}$ to the algebraic completion $\overline{V_e \boxtimes V_e}$ of $V_e \boxtimes V_e$. The natural extensions of the left- and right-hand sides of commutativity applied to $u \boxtimes v$ for $u, v \in V_e$ gives $\overline{\mu}(\overline{\mathcal{R}}(u \boxtimes v))$ and

$\overline{\mu}(u \boxtimes v)$, respectively. By the characterization (4.16) of $\mathcal{R}$ and the relation between μ and $\mathcal{Y}_e^f$, the left- and right-hand sides of commutativity are further equal to $e^{L(-1)}\mathcal{Y}_e^f(v,-1)u$ and $Y_e^f(u,1)v$, respectively. Note that in general $\mathcal{Y}_e^f(v,-1)u \neq Y_e^O(v,-1)u$. So $e^{L(-1)}\mathcal{Y}_e^f(v,-1)u$ and $Y_e^f(u,1)v$ are not equal in general. Thus commutativity is not true in general.

5 A geometric and operadic formulation

In this section, we give a geometric and operadic formulation of the notion of grading-restricted conformal open-string vertex algebra. For the notion of open-string vertex algebra and other variations, we have similar geometric and operadic formulations. In the present section, we discuss only grading-restricted conformal open-string vertex algebras. We assume that the reader is familiar with the geometric and operadic formulation of the notion of vertex operator algebra given by the first author. See [H1], [H2], [H6], [HL1] and [HL2] for details.

We first introduce a geometric partial operad. Note that $\hat{\mathbb{H}}$ is analytically diffeomorphic to the closed unit disk. We use Δ_a^r and $\bar{\Delta}_a^r$ to denote the relatively open upper-half disk in $\bar{\mathbb{H}}$ and the closed upper-half disk in $\bar{\mathbb{H}}$, respectively, centered at $a \in \mathbb{R}$ with radius $r \in \mathbb{R}_+$, that is, $\Delta_a^r = B_a^r \cap \overline{\mathbb{H}}$ and $\bar{\Delta}_a^r = \bar{B}_a^r \cap \overline{\mathbb{H}}$ where B_a^r and $\bar{B}_a^r$ are the open and closed disks centered at $a \in \mathbb{R}$ with radius $r \in \mathbb{R}_+$.

A *disk with strips of type* (m,n) $(m,n \in \mathbb{N})$ is a disk S (a genus-zero compact connected one-dimensional complex manifold with one connected component of boundary) with $m+n$ distinct, ordered points $p_1,\dots,p_{m+n}$ (called *boundary punctures*) on the boundary of S with $p_1,\dots,p_m$ negatively oriented and the other punctures positively oriented, and with local analytic coordinates

$$(U_1,\varphi_1),\dots,(U_{m+n},\varphi_{m+n})$$

vanishing at the boundary punctures $p_1,\dots,p_{m+n}$, respectively, where for each $i=1,\dots,m+n$, U_i is a local coordinate neighborhood at p_i and $\varphi_i : U_i \to \bar{\mathbb{H}}$, mapping the boundary part of U_i analytically to $\mathbb{R}$ and satisfying $\varphi_i(p_i)=0$, is a local analytic coordinate map vanishing at p_i. In the present paper, we consider only disks with strips of types $(1,n)$ for $n \in \mathbb{N}$. For such a disk with strips, we use the subscript 0 and the subscripts $1,\dots,n$ to indicate that the corresponding boundary punctures are negatively oriented and positively oriented, respectively.

Let S_1 and S_2 be disks with strips of type $(1, m)$ and of type $(1, n)$, respectively. Let $p_0, \dots, p_m$ be the boundary punctures of S_1, $q_0, \dots, q_n$ the boundary punctures of S_2, (U_i, φ_i) the local coordinate at p_i for some fixed i satisfying $0 < i \leq m$, and (V_0, ψ_0) the local coordinate at q_0. Note that in our convention discussed above, p_0 and q_0 are the negatively oriented boundary punctures on S_1 and S_2, respectively. Assume that there exists $r \in \mathbb{R}_+$ such that $\varphi_i(U_i)$ contains $\bar{\Delta}_0^r$ and $\psi_0(V_0)$ contains $\bar{\Delta}_0^{1/r}$. Assume also that p_i and q_0 are the only boundary punctures in $\varphi_i^{-1}(\bar{\Delta}_0^r)$ and $\psi_0^{-1}(\bar{\Delta}_0^{1/r})$, respectively. In this case we say that *the i-th boundary puncture of the first disk with strips can be sewn with the* 0*-th boundary puncture of the second disk with strips*. From these two disks with strips we obtain a disk with strips of type $(1, m+n-1)$ by cutting $\varphi_i^{-1}(\Delta_0^r)$ and $\psi_0^{-1}(\Delta_0^{1/r})$ from S_1 and S_2, respectively, and then identifying the new parts of the boundaries (the parts not on the boundaries of the original surfaces) of the resulting surfaces using the map $\varphi_i^{-1} \circ (-J) \circ \psi_0$ where J is the map from $\mathbb{C}^\times$ to itself given by $J(w) = 1/w$. The boundary punctures (with ordering) of this disk with strips are $p_0, \dots, p_{i-1}$, $q_1, \dots, q_n$, $p_{i+1}, \dots, p_m$. The local coordinates vanishing at these punctures are given in the obvious way. This sewing procedure gives a partial operation which we call the *sewing operation*. Note that we have to use $-J$ instead of J (as in [H6]) in the definition of the sewing operation.

We define the notion of conformal equivalence between two disks with strips in the obvious way. The space of equivalence classes of disks with strips is called the *moduli space of disks with strips*. Similar to the moduli spaces of spheres with tubes in [H6], the moduli space of disks with strips of type $(1, n)$ $(n \geq 1)$ can be identified with $\Upsilon(n) = \Lambda^{n-1} \times \Pi \times \Pi_{\mathbb{R}_+}^n$ where Π is the set of all sequences $A = \{A_j\}_{j \in \mathbb{Z}_+}$ of real numbers such that

$$\exp\left(\sum_{j>0} A_j x^{j+1} \frac{d}{dx}\right) x$$

is a convergent power series in some neighborhood of 0, $\Pi_{\mathbb{R}_+} = \mathbb{R}_+ \times \Pi$, and Λ^{n-1} is the set of elements of $\mathbb{R}^{n-1}$ with nonzero and distinct components. We think of each element of $\Upsilon(n)$, $n \geq 1$, as the disk $\hat{\mathbb{H}}$ equipped with ordered punctures $\infty, r_1, \dots, r_{n-1}, 0$, with an element of Π specifying the local coordinate at ∞ and with n elements of $\Pi_{\mathbb{R}_+}$ specifying the local coordinates at the other punctures. Analogously, the moduli space of disks with strips of type $(1, 0)$ can be identified with $\Upsilon(0) = \{A \in \Pi \mid A_1 = 0\}$. Then the moduli space of disks with strips can be identified with $\cup_{n \geq 0} \Upsilon(n)$. From now on we will refer to $\cup_{n\in\mathbb{N}}\Upsilon(n)$ as the moduli space of disks with strips. The sewing operation for disks with strips induces a partial operation on $\cup_{n\in\mathbb{N}}\Upsilon(n)$. It is still called the sewing operation.

Let $I_\Upsilon \in \Upsilon(1)$ be the equivalence class containing the standard disk $\hat{\mathbb{H}}$ with the negatively oriented puncture ∞, the only positively oriented puncture 0 , and with standard local coordinates vanishing at ∞ and 0. Here for $a \in \mathbb{R} \subset \hat{\mathbb{H}}$, the standard local coordinate vanishing at a is given by $w \mapsto w - a$. and for $\infty \in \hat{\mathbb{H}}$, the standard local coordinate vanishing at ∞ is given by $w \mapsto -\frac{1}{w}$. Note the minus sign in the definition of the standard local coordinate at ∞. For $n \in \mathbb{N}$, the symmetric group S_n acts on $\Upsilon(n)$ in an obvious way. Then by construction, the following result is clear:

Proposition 5.1 *The sequences $\Upsilon = \{\Upsilon(n) \mid n \in \mathbb{N}\}$ of moduli spaces, together with the sewing operation, the identity I_Υ and the actions of the symmetric groups, has a structure of an associative smooth $\mathbb{R}_+$-rescalable partial operad.*

We shall call the $\mathbb{R}_+$-rescalable partial operad Υ the *boundary disk partial operad*. Note that the boundary disk partial operad is very different from the so-called little disk operad which are constructed using the embeddings of disks in the unit disk. In fact, Υ can be viewed as a partial suboperad of the sphere partial operad K discussed in [H6]. Geometrically, any disk with strips of type $(1, n)$ is conformally equivalent to a disk with strips of type $(1, n)$ whose underlying disk is $\hat{\mathbb{H}}$ and whose negatively oriented puncture is ∞. But any such disk with strips of type $(1, n)$ corresponds to a sphere with tubes of type $(1, n)$ whose underlying sphere is $\hat{\mathbb{C}}$, whose punctures are the same as those on the disk with strips, whose local coordinates vanishing at positively oriented punctures are the analytic extensions of those on the disk with strips and whose local coordinate vanishing at the negatively oriented puncture is the analytic extension of the negation of that on the disk with strips. Thus we obtain a map from $\Upsilon(n)$ to $K(n)$ and this map is clearly injective. In fact the images of $\Upsilon(n)$ in $K(n)$ for $n \geq 2$ are

$$\{(r_1, \dots, r_{n-1}; A^{(0)}, (a_0^{(1)}, A^{(1)}), \cdots, (a_0^{(n)}, A^{(n)})) \in K(n) \mid \\ r_1, \dots, r_{n-1} \in \mathbb{R}, a_0^{(1)}, \dots, a_0^{(n)} \in \mathbb{R}_+, A^{(0)}, \dots, A^{(n)} \in \Pi\}.$$

The images of $\Upsilon(0)$ in $K(0)$ and of $\Upsilon(1)$ in $K(1)$ are

$$\{A^{(0)} \in K(0) \mid A^{(0)} \in \Pi,\ A_1^{(0)} = 0\}$$

and

$$\{(A^{(0)}, (a_0^{(1)}, A^{(1)})) \in K(1) \mid a_0^{(1)} \in \mathbb{R}_+, A^{(0)}, A^{(1)} \in \Pi\},$$

respectively. In addition, by the definitions of the maps from $\Upsilon(n)$ to $K(n)$ for $n \in \mathbb{N}$ and the sewing operations in Υ and K, it is clear that the maps from $\Upsilon(n)$ to $K(n)$ for $n \in \mathbb{N}$ respect the sewing operations, the identities and the actions of S_n and thus give an injective morphism of partial operads. From now on, we shall identify the partial operad Υ with its image in K under this injective morphism.

For any $c \in \mathbb{C}$, the restriction of the partial operad $\tilde{K}^c$ of the $\frac{c}{2}$-th power of the determinant line bundles over K to Υ gives a partial suboperad $\tilde{\Upsilon}^c$ of $\tilde{K}^c$. This partial operad is called the *$\mathbb{C}$-extension of Υ of central charge c.*

We now consider certain (pseudo-)algebras over the partial operad $\tilde{\Upsilon}^c$ for $c \in \mathbb{C}$. In the terminology of [HL1], [HL2] and [H6], we consider $\tilde{\Upsilon}^c$-associative (pseudo-)algebras satisfying an additional differentiability condition. Since the rescaling group of $\tilde{\Upsilon}^c$ is $\mathbb{R}_+$, we need to consider modules for $\mathbb{R}_+$. Since an equivalence class of irreducible modules for $\mathbb{R}_+$ is determined by a real number s such that $a \in \mathbb{R}_+$ acts on modules in this class as the scalar multiplication by a^{-s}, any completely reducible module for $\mathbb{R}_+$ is of the form $V = \coprod_{s\in\mathbb{R}} V_{(s)}$ where $V_{(s)}$ is the sum of the $\mathbb{R}_+$-submodules in the class determined by the real number s. We shall consider only those algebras over $\tilde{\Upsilon}^c$ whose underlying vector space is of the form $V = \coprod_{s\in\mathbb{R}} V_{(s)}$ such that $\dim V_{(s)} < \infty$. Recall from [HL1], [HL2] and [H6] that given any $\mathbb{R}_+$-submodule W of V, the endomorphism partial pseudo-operad associated to the pair (V, W) is the sequence $H_{V,W}^{\mathbb{R}_+} = \{H_{V,W}^{\mathbb{R}_+}(n)\}_{n\in\mathbb{N}}$. where $H_{V,W}^{\mathbb{R}_+}(n)$ is the set of all multilinear maps from $V^{\otimes n}$ to $\overline{V}$ such that $W^{\otimes n}$ is mapped to $\overline{W}$, equipped with natural operadic structures.

Definition 5.2 A *differentiable (or C^1) $\tilde{\Upsilon}^c$-associative pseudo-algebra* is a completely reducible $\mathbb{R}_+$-module $V = \coprod_{s\in\mathbb{R}} V_{(s)}$ satisfying the condition $\dim V_{(r)} < \infty$ for $s \in \mathbb{R}$ equipped with an $\mathbb{R}_+$-submodule W and a morphism Φ of $\mathbb{R}_+$-rescalable pseudo-partial operad from $\tilde{\Upsilon}^c$ to the endomorphism partial pseudo-operad $H_{V,W}^{\mathbb{R}_+}$ (that is, an $\tilde{\Upsilon}^c$-associative pseudo-algebra) satisfying the following conditions:

1. For s sufficiently negative, $V_{(s)} = 0$.
2. For any $n \in \mathbb{N}$, $\Phi_n : \tilde{\Upsilon}^c(n) \to H_{V,W}^{\mathbb{R}_+}(n)$ is linear on the fibers of $\tilde{\Upsilon}^c(n)$.

3. For any $s_1, \dots, s_n \in \mathbb{R}$, there exists a finite subset $R(s_1, \dots, s_n) \subset \mathbb{R}$ such that the image of $\coprod_{s\in s_1+\mathbb{Z}} V_{(s)} \otimes \cdots \otimes \coprod_{s\in s_n+\mathbb{Z}} V_{(s)}$ under $\Phi_n(\psi_n(Q))$ for any $Q \in \tilde{\Upsilon}^c(n)$ is in $\coprod_{s\in R(s_1,\dots,s_n)+\mathbb{Z}} V_{(s)}$.

4. For any $v' \in V'$, $v_1, \dots, v_n \in V$, $\langle v', \Phi_n(\psi_n(Q))(v_1 \otimes \ldots \otimes v_n)\rangle$ as a function of
$$Q = (r_1, \dots, r_{n-1}; A^{(0)}, (a_0^{(1)}, A^{(1)}), \cdots, (a_0^{(n)}, A^{(n)})) \in \tilde{\Upsilon}^c(n)$$
is of the form
$$\sum_{i=1}^{m} f_i(r_1, \dots, r_{n-1}) g_i(A^{(0)}, (a_0^{(1)}, A^{(1)}), \cdots, (a_0^{(n)}, A^{(n)}))$$
where $f_i(r_1, \dots, r_{n-1})$ for $i = 1, \dots, m$ are continuous differentiable functions of $r_1, \dots, r_{n-1}$ and $g_i(A^{(0)}, (a_0^{(1)}, A^{(1)}), \cdots, (a_0^{(n)}, A^{(n)}))$ for $i = 1, \dots, m$ are polynomials in $A^{(0)}, (a_0^{(1)})^{\pm 1}, A^{(1)}, \cdots, (a_0^{(n)})^{\pm 1}, A^{(n)}$.

Morphisms (respectively, *isomorphisms*) of differentiable $\tilde{\Upsilon}^c$-associative pseudo-algebras are morphisms (respectively, isomorphisms) of the underlying $\tilde{\Upsilon}^c$-associative pseudo-algebras.

We denote the differentiable $\tilde{\Upsilon}^c$-associative pseudo-algebra just defined by (V, W, Φ) or simply V. It is easy to see that a differentiable $\tilde{\Upsilon}^c$-associative pseudo-algebra is actually analytic in the sense that for any $v' \in V'$, $v_1, \dots, v_n \in V$, $\langle v', \nu(Q)(v_1, \dots, v_n)\rangle$ is analytic in Q because of the sewing axiom (that is, the sewing operation in Υ corresponds to the contraction in $H_{V,W}^{\mathbb{R}_+}$ under Φ). Using this fact and the fact that the expansion of analytic functions are always absolutely convergent in the domain of convergence, it is easy to obtain:

Proposition 5.3 *Any differentiable $\tilde{\Upsilon}^c$-associative pseudo-algebra (V, W, Φ) is an $\tilde{\Upsilon}^c$-associative algebra, that is, the image of $\tilde{\Upsilon}^c$ under Φ is a partial operad (the image of $\tilde{\Upsilon}^c$ under Φ satisfies the composition-associativity).*

We omit the proof of this result since it is the same as the proof of the corresponding result in [H6]. Because of this result, we shall call a differentiable $\tilde{\Upsilon}^c$-associative pseudo-algebra simply a *differentiable $\tilde{\Upsilon}^c$-associative algebra.*

Now we have the following main theorem which gives a geometric and operadic formulation of the notion of grading-restricted conformal open-string vertex algebras:

Theorem 5.4 *The category of grading-restricted conformal open-string vertex algebras of central charge c is isomorphic to the category of differentiable $\tilde{\Upsilon}^c$-associative algebras.*

Proof. The proof of this theorem is basically the same as that of the isomorphism theorem for the geometric and operadic formulation of vertex operator algebras in [H6]. Here we give a sketch. Some more details will be given in [K].

Let $(V, Y^O, \mathbf{1}, \omega)$ be a grading-restricted conformal open-string vertex algebra of central charge c. We construct a differentiable $\tilde{\Upsilon}^c$-associative algebras of central charge c as follows: The $\mathbb{R}$-graded vector space V is naturally a completely reducible $\mathbb{R}_+$-module. The module W for the Virasoro algebra generated by $\mathbf{1}$ is an $\mathbb{R}$-graded subspace of V and therefore is an $\mathbb{R}_+$-submodule of V. In [H2] and [H6], a section ψ of the line bundle $\tilde{K}^c$ over K is chosen. The restriction of this section to Υ is a section of $\tilde{\Upsilon}^c$ and, for simplicity, we still denote it by ψ. For an element

$$Q = (r_1, \dots, r_{n-1}; A^{(0)}, (a_0^{(1)}, A^{(1)}), \dots, (a_0^{(n)}, A^{(n)})) \tag{5.1}$$

of $\Upsilon(n)$, any element of the fiber of $\tilde{\Upsilon}^c$ over Q is of the form $\lambda\psi_n(Q)$ where $\lambda \in \mathbb{C}$. When $r_1 > \cdots > r_{n-1} > 0$, we define $\Phi_n(\lambda\psi_n(Q))$ by

$$\begin{aligned}(\Phi_n(\lambda\psi_n(Q)))(v_1 \otimes \cdots \otimes v_n) \\ = \lambda e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(0)} L(-j)} Y^O(e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(1)} L(j)} (a_0^{(1)})^{-L(0)} v_1, r_1) \cdots \\ \cdot Y^O(e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(n-1)} L(j)} (a_0^{(n-1)})^{-L(0)} v_{n-1}, r_{n-1}) \cdot \\ \cdot e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(n)} L(j)} (a_0^{(n)})^{-L(0)} v_n\end{aligned}$$

for $v_1, \dots, v_n \in V$. In general, for any $Q \in \Upsilon(n)$, we can always find $\sigma_Q \in S_n$ such that $\sigma_Q(Q)$ is of the form of the right-hand side of (5.1) such that $r_1 > \cdots > r_{n-1} > 0$. We define $\Phi_n(\lambda\psi_n(Q))$ by

$$(\Phi_n(\lambda\psi_n(Q)))(v_1 \otimes \cdots \otimes v_n) = \Phi_n(\lambda\psi_n(\sigma_Q(Q)))(v_{\sigma_Q^{-1}(1)} \otimes \cdots \otimes v_{\sigma_Q^{-1}(n)})$$

for $v_1, \dots, v_n \in V$. It can be verified in the same way as in [H6] that the triple (V, W, ν) is a differentiable $\tilde{\Upsilon}^c$-associative algebra of central charge c. This construction gives a functor from the category of grading-restricted conformal open-string vertex algebras of central charge c to the category of differentiable $\tilde{\Upsilon}^c$-associative algebras.

Conversely, given a differentiable $\tilde{\Upsilon}^c$-associative algebra (V, W, Φ), we construct a grading-restricted conformal open-string vertex algebra as follows: As in [H6], for $\varepsilon \in \mathbb{R}$ and $i \in \mathbb{Z}_+$, let $A(\varepsilon; i)$ be the element of Π whose i-th component is equal to ε and all other components are 0 and $\mathbf{0}$ the element of Π whose components are all 0, and for $r \in \mathbb{R}_+$, let

$$P(r) = (r; \mathbf{0}, (1, \mathbf{0}), (1, \mathbf{0})) \in \Upsilon(2) \subset K(2).$$

We define the vertex operator map

$$\begin{array}{rcl} Y^O : (V \otimes V) \times \mathbb{R}_+ & \to & \overline{V}, \\ (v_1 \otimes v_2, r) & \mapsto & Y^O(v_1, r)v_2 \end{array}$$

by

$$Y^O(v_1, r)v_2 = (\Phi_2(\psi_2((P(r))))(v_1 \otimes v_2)$$

for $v_1, v_2 \in V$ and $r \in \mathbb{R}_+$. The vacuum $\mathbf{1} \in V$ is given by

$$\mathbf{1} = \Phi_0(\psi_0(\mathbf{0})).$$

The conformal element ω_ν is given by

$$\omega = -\frac{d}{d\varepsilon}\Phi_0(\psi_0((A(\varepsilon; 2))))\bigg|_{\varepsilon=0}.$$

It can be proved in the same way as in [H6] that $(V, Y^O, \mathbf{1}, \omega)$ is a grading-restricted conformal open-string vertex algebra. This construction gives a functor from the category of differentiable $\tilde{\Upsilon}^c$-associative (pseudo-)algebras to the category of conformal open-string vertex algebras of central charge c.

It can be shown in the same way as in [H6] that these two functors constructed above are inverse to each other. Thus the conclusion of the theorem is true. ∎

The result above can actually be generalized to show that a grading-restricted conformal open-string vertex algebra of central charge c gives an algebra over a partial operad extending the operad of the c-th power of the determinant line bundles over the so-called "Swiss-cheese" operad (see [V]). We actually have a stronger isomorphism theorem than Theorem 5.4 involving meromorphic centers of grading-restricted conformal open-string vertex algebras. To formulate this result, we first introduce the underlying partial operads.

A *disk with strips and tubes of type* $(m, n; k, l)$ $(m, n, k, l \in \mathbb{N})$ is a disk S with $m+n$ distinct, ordered points $p_1^B, \dots, p_{m+n}^B$ (called *boundary punctures*) on the boundary of S and $k+l$ distinct, ordered points $p_1^I, \dots, p_{k+l}^I$ (called *interior punctures*) in the interior of S with $p_1^B, \dots, p_m^B$ and $p_1^I, \dots, p_k^I$ negatively oriented and the other (boundary or interior) punctures positively oriented, and with local analytic coordinates

$$(U_1^B, \varphi_1^B), \dots, (U_{m+n}^B, \varphi_{m+n}^B), (U_1^I, \varphi_1^I), \dots, (U_{k+l}^I, \varphi_{k+l}^i)$$

vanishing at the (boundary or interior) punctures $p_1^B, \dots, p_{m+n}^B$, $p_1^I, \dots, p_{k+l}^I$, respectively, where for each $i = 1, \dots, m+n$ (or $j = 1, \dots, k+l$), U_i^B (or U_j^I) is a local coordinate neighborhood at p_i^B (or p_j^I) and $\varphi_i^B : U_i^B \to \bar{\mathbb{H}}$ (or $\varphi_j^I : U_j^I \to \mathbb{C}$), mapping the boundary part of U_i^B (or mapping U_j^I) analytically to $\mathbb{R}$ (or $\mathbb{C}$) and satisfying $\varphi_i^B(p_i^B) = 0$ (or $\varphi_i^I(p_i^I) = 0$), is a local analytic coordinate map vanishing at p_i^B (or p_i^I). Note that when $k = l = 0$, we have a disk with strips of type (m, n). In the present paper, we consider only disks with strips and tubes of types $(1, n; 0, l)$ for $n, l \in \mathbb{N}$. For such a disk with strips, we use the subscript 0 and the subscripts $1, \dots, n$ to indicate that the corresponding boundary punctures are negatively oriented and positively oriented, respectively.

Similar to disks with strips, we have a sewing operation which sews two disks with strips and tubes at boundary punctures of opposite orientations. Here we shall call this sewing operation the *boundary sewing operation*. On the other hand, we can also sew the negatively oriented puncture of a sphere with tubes to an interior puncture of a disk with strips and tubes just as we sew two spheres with tubes in [H6]. We shall call this sewing operation the *interior sewing operation*.

The conformal equivalences for these disks with strips and tubes are defined in the obvious way. For $n \geq 1$ and $l \in \mathbb{N}$, the moduli space of disks with strips and tubes of type $(1, n; 0, l)$ can be identified with $\Upsilon(n; l) = \Lambda^{n-1} \times \Pi \times \Pi_{\mathbb{R}_+}^n \times M_{\mathbb{H}}^l \times H_c^l$ where Λ, Π and $\Pi_{\mathbb{R}_+}$ are defined above, H ands H_c are defined in [H6] and $M_{\mathbb{H}}^l$ is the set of elements of $\mathbb{H}^l$ with nonzero and distinct components. Analogously, for $l \in \mathbb{N}$, the moduli space of disks with strips and tubes of type $(1, 0; 0, l)$ can be identified with $\Upsilon(0; 1) = \{A \in \Pi \mid A_1 = 0\} \times H_c^l$. Note that $\Upsilon(n) = \Upsilon(n; 0)$ for $n \in \mathbb{N}$. In particular, the identity I_Υ is an element of $\Upsilon(1; 0)$. Also, for $n \in \mathbb{N}$, S_n acts on $\Upsilon(n; l)$ in the obvious way. Let $\mathfrak{S}(n) = \cup_{l \in \mathbb{N}} \Upsilon(n; l)$ for $n \in \mathbb{N}$. Then S_n acts on $\mathfrak{S}(n)$ for $n \in \mathbb{N}$. The following result is clear:

Proposition 5.5 *The sequences $\mathfrak{S} = \{\mathfrak{S}(n)\}_{n\in\mathbb{N}}$ together with the boundary sewing operation, the identity $I_\Upsilon \in \Upsilon(1) = \Upsilon(1;0)$ and the actions of the symmetric groups, has a structure of a smooth $\mathbb{R}_+$-rescalable partial operad. In addition, for each $n \in \mathbb{N}$, there is an action of the sphere partial operad K on $\mathfrak{S}(n)$ given by the interior sewing operation.*

Borrowing the terminology used by Voronov in [V], we shall call the $\mathbb{R}_+$-rescalable partial operad $\mathfrak{S}$ the *Swiss-cheese partial operad*. But note that our partial operad is much more larger than the Swiss cheese operad. In fact, Swiss cheese operad is an analogue of the little disk operad while our Swiss cheese partial operad is an analogue of the sphere partial operad in [H6].

For each pair $n, l \in \mathbb{N}$, we have an injective map from $\Upsilon(n;l)$ to $K(n+2l)$ obtained by doubling disks with strips and tubes as follows: For any disk with strips and tubes of type $(1, n; 0, l)$, by the uniformization theorem, we can find a conformally equivalent disk with strips and tubes of the same type such that its underlying disk is $\hat{\mathbb{H}}$. This latter disk with strips and tubes can be doubled to obtain a sphere with tubes of type $(1, n + 2l)$ such that its underlying sphere is the double $\mathbb{C} \cup \{\infty\}$ of $\hat{\mathbb{H}}$. By definition, we see that conformally equivalent disks with strips give conformally equivalent spheres with tubes. Thus we obtain a map from $\Upsilon(n;l)$ to $K(n + 2l)$. Clearly this map is injective.

It is clear from the definition that these maps respect the (boundary) sewing operations. In addition, these maps also intertwine the actions of K on $\mathfrak{S}(n)$ for $n \in \mathbb{N}$ and the actions of K on the images of $\mathfrak{S}(n)$ obtained by doubling the actions of K on $K(n)$. We shall identify $\Upsilon(n;l)$ with its image in $K(n + 2l)$.

For any $c \in \mathbb{C}$, the restriction of the partial operad $\tilde{K}^c$ of the $\frac{c}{2}$-th power of the determinant line bundles over K to $\mathfrak{S}$ has a natural structure of a partial operad. This partial operad is called the $\mathbb{C}$-*extension of* $\mathfrak{S}$ *of central charge* c and is denoted $\tilde{\mathfrak{S}}^c$. For any $n \in \mathbb{N}$, the action of K on $\mathfrak{S}(n)$ also induces an action of $\tilde{K}^c$ on $\tilde{\mathfrak{S}}^c(n)$. For any $n \in \mathbb{N}$, the restrictions of the sections ψ_{n+2l} of $\tilde{K}^c(n + 2l)$ for $l \in \mathbb{N}$ to $\Upsilon(n;l)$ gives a section of $\tilde{\mathfrak{S}}^c(n)$ and we shall use $\psi_n^{\mathfrak{S}}$ to denote this section.

We now consider a completely reducible $\mathbb{R}_+$-module or, equivalently, an $\mathbb{R}$-graded vector spaces $V^O = \coprod_{s\in\mathbb{R}} V^O_{(s)}$ and completely reducible $\mathbb{C}^\times$-modules or, equivalently, $\mathbb{Z}$-graded vector spaces $V^{LC} = \coprod_{m\in\mathbb{Z}} V^{LC}_{(m)}$ and $V^{RC} = \coprod_{m\in\mathbb{Z}} V^{RC}_{(m)}$. (Here O, LC and RC means open, left closed and right closed, respectively.) Let W^O, W^{LC} and W^{RC} be an $\mathbb{R}_+$-submodule of V^O, a $\mathbb{C}^\times$-submodule of V^{LC} and a $\mathbb{C}^\times$-submodule of V^{RC}, respectively. Associated to V^{LC}, W^{LC}, V^{RC}, W^{RC}, we have the endomorphism partial pseudo-operads $H^{\mathbb{C}^\times}_{V^{LC},W^{LC}}$, $H^{\mathbb{C}^\times}_{V^{RC},W^{RC}}$ and $H^{\mathbb{C}^\times}_{V^{LC}\otimes(V^{RC})^-,W^{LC}\otimes(W^{RC})^-}$ (see [HL1], [HL2] and [H6]), where $(V^{RC})^-$ and $(W^{RC})^-$ are the complex conjugate of V^{RC} and W^{RC}. We also need an endomorphism partial operad constructed from V^O, W^O, V^{LC}, W^{LC}, V^{RC} and W^{RC}. For $n, l \in \mathbb{N}$, let $H^{\mathbb{R}_+}_{V^O,W^O;V^{LC}\otimes(V^{RC})^-,W^{LC}\otimes(W^{RC})^-}(n;l)$ be the space of all linear maps from $(V^O)^{\otimes n}\otimes(V^{LC}\otimes(V^{RC})^-)^{\otimes l}$ to $\overline{V^O}$ such that $(W^O)^{\otimes n}\otimes(W^{LC}\otimes(W^{RC})^-)^{\otimes l}$ is mapped to $\overline{W^O}$ and for $n\in\mathbb{N}$, let

$$
\begin{aligned}
&H^{\mathbb{R}_+}_{V^O,W^O;V^{LC}\otimes(V^{RC})^-,W^{LC}\otimes(W^{RC})^-}(n)\\
&\quad=\coprod_{l\in\mathbb{N}} H^{\mathbb{R}_+}_{V^O,W^O;V^{LC}\otimes(V^{RC})^-,W^{LC}\otimes(W^{RC})^-}(n;l).
\end{aligned}
$$

Then it is clear that for $n\in\mathbb{N}$, the endomorphism partial pseudo-operad $H^{\mathbb{C}^\times}_{V^{LC}\otimes(V^{RC})^-,W^{LC}\otimes(W^{RC})^-}$ acts on $H^{\mathbb{R}_+}_{V^O,W^O;V^{LC}\otimes(V^{RC})^-,W^{LC}\otimes(W^{RC})^-}(n)$ and

$$
\begin{aligned}
&H^{\mathbb{R}_+}_{V^O,W^O;V^{LC}\otimes(V^{RC})^-,W^{LC}\otimes(W^{RC})^-}\\
&\quad=\{H^{\mathbb{R}_+}_{V^O,W^O;V^{LC}\otimes(V^{RC})^-,W^{LC}\otimes(W^{RC})^-}(n)\}_{n\in\mathbb{N}}
\end{aligned}
$$

is an $\mathbb{R}_+$-rescalable partial pseudo-operad. We call it the *endomorphism partial pseudo-operad for* $(V^O, W^O; V^{LC}\otimes(V^{RC})^-, W^{LC}\otimes(W^{RC})^-)$.

Notice that $H^{\mathbb{C}^\times}_{V^{LC},W^{LC}}\otimes(H^{\mathbb{C}^\times}_{V^{RC},W^{RC}})^-$ (here $(H^{\mathbb{C}^\times}_{V^{RC},W^{RC}})^-$ is the complex conjugate of $H^{\mathbb{C}^\times}_{V^{RC},W^{RC}}$) can be embedded naturally into the space $H^{\mathbb{C}^\times}_{V^{LC}\otimes(V^{RC})^-,W^{LC}\otimes(W^{RC})^-}$. Below we shall view $H^{\mathbb{C}^\times}_{V^{LC},W^{LC}}\otimes(H^{\mathbb{C}^\times}_{V^{RC},W^{RC}})^-$ as a partial pseudo-suboperad of $H^{\mathbb{C}^\times}_{V^{LC}\otimes(V^{RC})^-,W^{LC}\otimes(W^{RC})^-}$.

Let $\bar{c}$ be the complex conjugate of $c\in\mathbb{C}$. The complex conjugate $\overline{\tilde{K}^{\bar{c}}}$ of $\tilde{K}^{\bar{c}}$ is also a $\mathbb{C}^\times$-rescalable partial operad. Consequently the tensor product $\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}$ (the tensor product of line bundles) is also a $\mathbb{C}^\times$-rescalable partial operad. Interpreting the action of K on $\mathfrak{S}$ using the method of doubling disks, we see that $\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}$ acts naturally on $\tilde{\mathfrak{S}}^c$.

We are interested in certain algebras over $\tilde{\mathfrak{S}}^c$ for $c\in\mathbb{C}$.

Definition 5.6 A *pseudo-algebra over $\tilde{\mathfrak{S}}^c$ generated by a differentiable $\tilde{\Upsilon}^c$-associative pseudo-algebra and meromorphic actions of two $\tilde{K}^c$-associative algebras* or simply a *differentiable-meromorphic pseudo-algebra over $\tilde{\mathfrak{S}}^c$* consists of the following data:

1. A completely reducible $\mathbb{R}_+$-module $V^O = \coprod_{s\in\mathbb{R}} V^O_{(s)}$ satisfying the condition $\dim V^O_{(s)} < \infty$ for $s \in \mathbb{R}$ and completely reducible $\mathbb{C}^\times$-modules $V^{LC} = \coprod_{m\in\mathbb{Z}} V^{LC}_{(m)}$ and $V^{RC} = \coprod_{m\in\mathbb{Z}} V^{RC}_{(m)}$ satisfying the condition $\dim V^{LC}_{(m)} < \infty$ and $\dim V^{RC}_{(m)} < \infty$ for $m \in \mathbb{Z}$.

2. An $\mathbb{R}_+$-submodule W^O of V^O and $\mathbb{C}^\times$-submodules W^{LC} and W^{RC} of V^{LC} and V^{RC}, respectively.

3. A morphism Φ of $\mathbb{R}_+$-rescalable partial pseudo-operads from $\tilde{\mathfrak{S}}^c$ to the endomorphism partial pseudo-operad $H^{\mathbb{R}_+}_{V^O,W^O;V^{LC}\otimes(V^{RC})^-,W^{LC}\otimes(W^{RC})^-}$ and a morphism Ψ of $\mathbb{C}^\times$-rescalable partial pseudo-operads from $\tilde{K}^c \otimes \overline{\tilde{K}^{\bar{c}}}$ to the endomorphism partial pseudo-operad $H^{\mathbb{C}^\times}_{V^{LC}\otimes(V^{RC})^-,W^{LC}\otimes(V^{RC})^-}$.

These data satisfy the following conditions:

1. For $s \in \mathbb{R}$ sufficiently negative, $V^O_{(s)} = 0$ and for $m \in \mathbb{Z}$ sufficiently negative, $V^{LC}_{(m)} = V^{RC}_{(m)} = 0$.

2. For any $n \in \mathbb{N}$, $\Phi_n : \tilde{\mathfrak{S}}^c(n) \to H^{\mathbb{R}_+}_{V^O,W^O;V^{LC}\otimes V^{RC},W^{LC}\otimes W^{RC}}(n)$ is linear on the fibers of $\tilde{\mathfrak{S}}^c(n)$.

3. The morphism Ψ is equal to $\Psi^L \otimes \overline{\Psi^R}$ where Ψ^L (Ψ^R) is a morphism of $\mathbb{C}^\times$-rescalable partial pseudo-operads from $\tilde{K}^c$ ($\tilde{K}^{\bar{c}}$) to $H^{\mathbb{C}^\times}_{V^{LC},W^{LC}}$ ($H^{\mathbb{C}^\times}_{V^{RC},W^{RC}}$) and $\overline{\Psi^R}$ is the complex conjugate of Ψ^R. In addition, the triples (V^{LC}, W^{LC}, Ψ^L) and (V^{RC}, W^{RC}, Ψ^R) are meromorphic $\tilde{K}^c$-associative algebra and $\tilde{K}^{\bar{c}}$-associative algebra, respectively.

4. For any $n \in \mathbb{N}$, the map

$$\Phi_n : \tilde{\mathfrak{S}}^c(n) \to H^{\mathbb{R}_+}_{V^O,W^O;V^{LC}\otimes(V^{RC})^-,W^{LC}\otimes(W^{RC})^-}(n)$$

intertwines the action of the partial operad $\tilde{K}^c \otimes \overline{\tilde{K}^{\bar{c}}}$ on $\tilde{\mathfrak{S}}^c(n)$ and the action of the partial pseudo-operad $H^{\mathbb{C}^\times}_{V^{LC}\otimes(V^{RC})^-,W^{LC}\otimes(V^{RC})^-}$ on $H^{\mathbb{R}_+}_{V^O,W^O;V^{LC}\otimes(V^{RC})^-,W^{LC}\otimes(V^{RC})^-}(n)$.

5. For any $s_1, \dots, s_n \in \mathbb{R}$, there exists a finite subset $R(s_1, \dots, s_n) \subset \mathbb{R}$ such that the image of $\coprod_{s\in s_1+\mathbb{Z}} V^O_{(s)} \otimes \cdots \otimes \coprod_{s\in s_n+\mathbb{Z}} V^O_{(s)}$ under $\Phi_n(\psi^{\mathfrak{S}}_n(Q))$ for any $Q \in \tilde{\Upsilon}^c(n; 0)$ is in $\coprod_{s\in R(s_1,\dots,s_n)+\mathbb{Z}} V^O_{(s)}$.

6. For any $v' \in (V^O)'$, $u_1, \dots, u_n \in V^O$, $v_1^L \otimes \bar{v}_1^R, \dots, v_l^L \otimes \bar{v}_l^R \in V^{LC} \otimes (V^{RC})^-$,

$$\langle v', \Phi_n(\psi_n^{\mathfrak{S}}(Q))(u_1 \otimes \cdots \otimes u_n \otimes v_1^L \otimes \bar{v}_1^R \otimes \cdots \otimes v_l^L \otimes \bar{v}_l^R)\rangle$$

as a function of

$$\begin{aligned} Q \quad = \quad & (r_1, \dots, r_{n-1}; A^{(0)}, (a_0^{(1)}, A^{(1)}), \cdots, (a_0^{(n)}, A^{(n)}); \\ & \qquad z_1, \dots, z_l; (b_0^{(1)}, B^{(1)}), \cdots, (b_0^{(l)}, B^{(l)}); \\ & \qquad \bar{z}_1, \dots, \bar{z}_l; (\bar{b}_0^{(1)}, \bar{B}^{(1)}), \cdots, (\bar{b}_0^{(l)}, \bar{B}^{(l)})) \\ & \qquad \in \tilde{\Upsilon}^c(n; l) \subset K(n+2l) \end{aligned}$$

is of the form

$$\begin{aligned} & \sum_{i=1}^{k} f_i(r_1, \dots, r_{n-1}; z_1, \dots, z_l; \bar{z}_1, \dots, \bar{z}_l) \cdot \\ & \qquad \cdot g_i(A^{(0)}, (a_0^{(1)}, A^{(1)}), \cdots, (a_0^{(n)}, A^{(n)}); (b_0^{(1)}, B^{(1)}), \cdots, (b_0^{(l)}, B^{(l)}); \\ & \qquad\qquad (\bar{b}_0^{(1)}, \bar{B}^{(1)}), \cdots, (\bar{b}_0^{(l)}, \bar{B}^{(l)})) \end{aligned}$$

where the functions

$$f_i(r_1, \dots, r_{n-1}; z_1, \dots, z_l; \xi_1, \dots, \xi_l)$$

for $i = 1, \dots, k$ are continuous differentiable in $r_1, \dots, r_{n-1}$ and are meromorphic in $z_1, \dots, z_l$, $\xi_1, \dots, \xi_l$ with $z_i = 0, \infty$ and $z_i = z_k$, $i < k$, $z_i = r_j$, $\xi_i = 0, \infty$, $\xi_i = \xi_k$, $i < k$, $\xi_i = r_j$ and $z_i = \xi_k$ for $i, k = 1, \dots, l$ and $j = 1, \dots, n-1$ as the only possible poles, and

$$\begin{aligned} & g_i(A^{(0)}, (a_0^{(1)}, A^{(1)}), \cdots, (a_0^{(n)}, A^{(n)}); (b_0^{(1)}, B^{(1)}), \cdots, (b_0^{(l)}, B^{(l)}); \\ & \qquad\qquad (d_0^{(1)}, D^{(1)}), \cdots, (d_0^{(l)}, D^{(l)})) \end{aligned}$$

for $i = 1, \dots, k$ are polynomials in $A^{(0)}, (a_0^{(1)})^{\pm 1}, A^{(1)}, \cdots, (a_0^{(n)})^{\pm 1}, A^{(n)}$, $(b_0^{(1)})^{\pm 1}, B^{(1)}, \cdots, (b_0^{(l)})^{\pm 1}, B^{(l)}$ and $(d_0^{(1)})^{\pm 1}, D^{(1)}, \cdots, (d_0^{(l)})^{\pm 1}, D^{(l)}$.

Morphisms (respectively, *isomorphisms*) of such pseudo-algebras over $\tilde{\mathfrak{S}}^c$ are morphisms (respectively, isomorphisms) of the underlying pseudo-algebras over $\tilde{\mathfrak{S}}^c$ preserving all the structures.

We denote the differentiable-meromorphic pseudo-algebra over $\tilde{\mathfrak{S}}^c$ just defined by

$$(V^O, W^O, V^{LC}, W^{LC}, V^{RC}, W^{RC}, \Phi, \Psi)$$

or simply by (V^O, V^{LC}, V^{RC}). For these pseudo-algebras, we also have the following result whose proof is the same as those of the corresponding result in [H6] and for Proposition 5.3:

Proposition 5.7 *Any differentiable-meromorphic pseudo-algebra over* $\tilde{\mathfrak{S}}^c$

$$(V^O, W^O, V^{LC}, W^{LC}, V^{RC}, W^{RC}, \Phi, \Psi)$$

is an algebra over $\tilde{\mathfrak{S}}^c$*, that is, the image of* $\tilde{\mathfrak{S}}^c$ *under* Φ *is a partial operad (the image of* $\tilde{\mathfrak{S}}^c$ *under* Φ *satisfies the composition-associativity).*

Because of this result, we shall omit the word "pseudo" from now on.

Note that given a vertex operator algebra V, its complex conjugate space V^- has a natural vertex operator algebra structure $(V^-, Y^-, \overline{\mathbf{1}}, \overline{\omega})$ of central charge $\bar{c}$. We have the following generalization of Theorem 5.4:

Theorem 5.8 *The category of objects consisting of a grading-restricted conformal open-string vertex algebra of central charge* $c \in \mathbb{C}$*, two vertex operator algebras, one of central charge* c *and the other of central charge* $\bar{c}$*, and homomorphisms from the first vertex operator algebra and the complex conjugate of the second vertex operator algebra to the meromorphic center of the grading-restricted conformal open-string vertex algebra is isomorphic to the category of differentiable-meromorphic algebras over* $\tilde{\mathfrak{S}}^c$*.*

Proof. The proof of this theorem is basically the same as the proof of isomorphism theorem for the geometric and operadic formulation of vertex operator algebras in [H6] and the proof of Theorem 5.4. The main new ingredient is that we use those spheres with tubes which are obtained by doubling disks with strips and tubes. Here we give only a sketch. More details will be given in [K].

Given a grading-restricted conformal open-string vertex algebra V^O of central charge c, vertex operator algebras V^{LC} and V^{RC} of central charge c and $\bar{c}$, respectively, and homomorphisms $h^L : V^{LC} \to C_0(V^O)$ and $h^R : (V^{RC})^- \to C_0(V^O)$. Let W^O, W^{LC} and W^{RC} be the modules for the Virasoro algebra generated by $\mathbf{1}^O$, $\mathbf{1}^{LC}$ and $\mathbf{1}^{RC}$ (the vacuums for these algebras), respectively. By the isomorphism theorem in [H6], there are meromorphic $\tilde{K}^c$-associative algebra (V^{LC}, W^{LC}, Ψ^L) and meromorphic $\tilde{K}^{\bar{c}}$-associative algebra (V^{RC}, W^{RC}, Ψ^R) constructed from the vertex operator algebras V^{LC} and V^{RC}, respectively. Let $\Psi = \Psi^L \otimes \overline{\Psi^R}$.

By Theorem 5.4, there is a differentiable $\tilde{\Upsilon}^c$-associative algebra $(V^O, W^O, \Phi^\Upsilon)$ constructed from V^O. Note that $\tilde{\Upsilon}^c$ is actually a partial suboperad of $\tilde{\mathfrak{S}}^c$. So this differentiable $\tilde{\Upsilon}^c$-associative algebra gives us part of a differentiable-meromorphic algebra structure over $\tilde{\mathfrak{S}}^c$. In general, the construction of the differentiable-meromorphic algebra over $\tilde{\mathfrak{S}}^c$ can be obtained using the meromorphic $\tilde{K}^c$-associative algebra (V^{LC}, W^{LC}, Ψ^L) and meromorphic $\tilde{K}^c$-associative algebra (V^{RC}, W^{RC}, Ψ^R), the differentiable $\tilde{\Upsilon}^c$-associative algebra $(V^O, W^O, \Phi^\Upsilon)$ and the homomorphisms h^L and h^R. The action of the $\tilde{K}^c$-associative algebra $(V^{LC}\otimes (V^{RC})^-, W^{LC}\otimes (W^{RC})^-, \Psi)$ is also obtained using the homomorphisms h^L and h^R. Thus we have a functor from the category of objects of the form $(V^O, V^{LC}, V^{RC}, h^L, h^R)$ to the category of differentiable-meromorphic pseudo-algebra over $\tilde{\mathfrak{S}}^c$.

Now given a differentiable-meromorphic pseudo-algebra over $\tilde{\mathfrak{S}}^c$, by the definition and the isomorphism theorem in [H6], we know that there are vertex operator algebra structures of central charge c and $\bar{c}$ on V^{LC} and V^{RC}, respectively. In particular, we have vacuums $\mathbf{1}^{LC} \in V^{LC}$ and $\mathbf{1}^{RC} \in V^{RC}$. Since $\tilde{\Upsilon}^c$ is actually a partial suboperad of $\tilde{\mathfrak{S}}^c$, by Theorem 5.4, we also obtain a grading-restricted conformal open-string vertex algebra structure of central charge c on V^O. For $z \in \mathbb{H}$, we consider an element $\Omega(z)$ of $\Upsilon(0;1)$ which is the conformal equivalence class containing the following disk with strips and tubes of type $(1,0;0,1)$: The disk $\hat{\mathbb{H}}$ with the boundary puncture ∞ and the interior puncture z and with the standard local coordinates vanishing at these punctures. Then

$$\begin{aligned}\Phi_0(\psi_0^{\mathfrak{S}}(\Omega(z))) &\in H^{\mathbb{R}_+}_{V^O, W^O; V^{LC}\otimes (V^{RC})^-, W^{LC}\otimes (W^{RC})^-}(0;1)\\ &= \mathrm{Hom}(V^{LC}\otimes (V^{RC})^-, \overline{V^O}).\end{aligned}$$

By Condition 6 in Definition 5.6, we know that $\Phi_0(\psi_0^{\mathfrak{S}}(\Omega(z))(v^L \otimes \overline{\mathbf{1}^{RC}})$ is meromorphic in z with the only pole $z = \infty$. In particular,

$$\lim_{z\to 0} \Phi_0(\psi_0^{\mathfrak{S}}(\Omega(z)))(v^L \otimes \overline{\mathbf{1}^{RC}})$$

exists. We define

$$h^L(v^L) = \lim_{z\to 0} \Phi_0(\psi_0^{\mathfrak{S}}(\Omega(z))(v^L \otimes \overline{\mathbf{1}^{RC}}).$$

Thus we obtain a linear map h^L from V^{LC} to $\overline{V^O}$. It is easy to see that the image of h^L is in fact in $C_0(V^O)$ and h^L is a homomorphism from V^{LC} to $C_0(V^O)$. Similarly we can construct h^R. Now we have a functor from the category of differentiable-meromorphic pseudo-algebra over $\tilde{\mathfrak{S}}^c$ to the category of objects of the form $(V^O, V^{LC}, V^{RC}, h^L, h^R)$.

From the isomorphism theorem in [H6], Theorem 5.4 and the construction of the two functors above, we see that these two functors are inverse to each other. ∎

In particular, we have:

Corollary 5.9 *Let V be a grading-restricted conformal open-string vertex algebra of central charge c. Then V gives a natural structure of an algebra over $\tilde{\mathfrak{S}}^c$ in the sense that $(V, C_0(V), C_0(V)^-)$ has a natural structure of differentiable-meromorphic algebra over $\tilde{\mathfrak{S}}^c$.*

Proof. We have a grading-restricted conformal algebra V and a vertex operator algebra $C_0(V)$. Let $h^L, h^R : C_0(V) \to C_0(V)$ be the identity map. Then h^L and h^R are homomorphisms from $C_0(V)$ to the meromorphic center of V. Then Theorem 5.8 gives $(V, C_0(V), (C_0(V))^-)$ a natural structure of differentiable-meromorphic algebra over $\tilde{\mathfrak{S}}^c$. ∎

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