C_1-cofiniteness and vertex tensor categories

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Abstract

We first generalize the logarithmic tensor category theory of Huang-Lepowsky-Zhang to the more general case that the module category for a vertex operator algebra V (more generally a Möbius vertex algebra) might not be closed under the contragredient functor. Then by verifying the assumptions to use this generalization, we obtain that (logarithmic) intertwining operators among C_1-cofinite grading-restricted generalized V-modules satisfy the associativity property (operator product expansion) and the category of C_1-cofinite grading-restricted generalized V-modules has a natural vertex tensor category structure. In particular, this category has a natural braided tensor category structure with a twist.

1 Introduction

In the study of logarithmic conformal field theories and the nonsemisimple representation theory of vertex operator algebras, Lepowsky, Zhang and the author developed a logarithmic tensor category theory in [HLZ1]–[HLZ9] by generalizing the early semisimple tensor category theory developed by Lepowsky and the author in [HL1]–[HL5] and by the author in [H1]. Under suitable assumptions on a vertex operator algebra (more generally, a Möbius vertex algebra) V and a module category for V, the associativity (operator product expansion) of (logarithmic) intertwining operators among objects of the category is proved and a natural vertex tensor category structure and consequently a natural braided tensor category structure with a twist are constructed in [HLZ1]–[HLZ9]. To use this theory, one needs to verify the assumptions in [HLZ1]–[HLZ9]. See Assumption 10.1 in [HLZ7] and Assumptions 12.1 and 12.2 in [HLZ9] and also Section 2 below for a complete list of assumptions.

A notion of quasi-rational module for a $\mathcal{W}$-algebra was first introduced by Nahm in [N]. In mathematics, $\mathcal{W}$-algebras are formulated rigorously as vertex operator algebras and quasi-rational modules correspond to C_1-cofinite modules. For a suitable module W with the vertex operator map Y_W for a vertex operator algebra V, we let $C_1(W)$ be the subspace spanned by $\mathrm{Res}_x x^{-1} Y_W(v, x)w$ for $v \in V_+ = \coprod_{n\in\mathbb{Z}_+} V_{(n)}$ and $w \in W$. W is C_1-cofinite means $\dim W/C_1(W) < \infty$. Nahm in [N] argued that if a suitable fusion product W_{12} of two V-modules W_1 and W_2 exists, then

$$\dim((W_{12})/C_1(W_{12})) \leq \dim(W_1/C_1(W_1)) \dim(W_2/C_1(W_2)).$$

Nahm did not give a construction of a fusion product in [N]. Instead, Nahm's result is derived from some basic assumptions in the physical study of conformal field theory.

In mathematics, Abe and Nagatomo showed in [AN] that for suitable vertex operator algebras, the spaces of conformal blocks on the Riemann sphere among C_1-cofinite modules are finite-dimensional. The importance of C_1-cofiniteness of suitable modules for vertex operator algebras in the study of the associativity of intertwining operators and in the construction of vertex and braided tensor category structures was first noticed by the author in [H3]. In [H3], the author proved that products of intertwining operators (without logarithms) among C_1-cofinite modules for a vertex operator algebra satisfy differential equations of regular singular points. In fact, the differential equations are also satisfied by products of logarithmic intertwining operators among C_1-cofinite grading-restricted generalized modules for a Möbius vertex algebra (see [HLZ8]). Using these differential equations, the convergence and extension property for products of (logarithmic) intertwining operators are proved in [H3] and [HLZ8]. It was also proved in [HLZ8] that the expansion condition (one of the assumptions in [HLZ1]–[HLZ9]) follows from the convergence and extension property for products of (logarithmic) intertwining operators.

In [H4], the author verified the assumptions in [HLZ1]–[HLZ9] for a C_2-cofinite vertex operator algebra of positive energy (CFT type) and the category of grading-restricted generalized V-modules. In fact, the author in [H4] verified these assumptions for a more general vertex operator algebra V and the category of C_1-cofinite grading-restricted generalized V-modules satisfying additional conditions. But since the additional conditions are quite strong, the results in [H4] do not apply to many interesting classes of vertex operator algebras and many interesting module categories. In [Mi], Miyamoto proved a weak version of the inequality of Nahm mentioned above and as a consequence, he obtained that the category of C_1-cofinite N-gradable weak V-modules (or equivalently, C_1-cofinite grading-restricted generalized V-modules) for a general vertex operator algebra V is closed under a tensor product bifunctor. In [MS], McRae and Sopin formulated a generalization of the inequality of Nahm mentioned above (see Proposition 3.6 in [MS]) and observed that one can also obtain the inequality of Nahm using the number of independent solutions of the differential equations in [Mi].

In [CHY], the results of [H3] for C_1-cofinite V-modules was first applied to the case that V is an affine vertex operator algebra at an admissible level and $\mathcal{C}$ is a semisimple category of C_1-cofinite ordinary V-modules. In [CJORY], Creutzig, Jiang, Orosz Hunziker, Ridout and Yang verified the assumptions in [HLZ1]–[HLZ9] for a Virasoro vertex operator algebra V and the category of C_1-cofinite grading-restricted generalized V-modules. In [CY], Creutzig and Yang showed that if all the irreducible V-modules are C_1-cofinite and the category of grading-restricted generalized V-modules are the same as the category of finite-length grading-restricted generalized V-modules, then the assumptions in [HLZ1]–[HLZ9] hold. In fact, the following result was implicitly proved in [CJORY] and [CY], explicitly stated and proved by McRae in [Mc] (see Theorem 2.3 in [Mc]), and explicitly stated and generalized to vertex operator superalgebras by Creutzig, McRae, Orosz Hunziker and Yang in [CMOY] (see Theorem 2.25 in [CMOY]):

Theorem 1.1 ([CJORY], [CY], [Mc], [CMOY])**.** *Let V be a vertex operator algebra. If the category of C_1-cofinite grading-restricted generalized V-modules is closed under the contragredient functor, then it has a natural braided tensor category structure.*

Note that the results in [CJORY], [CY], [Mc], and [CMOY] are obtained by applying the theory in [HLZ1]–[HLZ9]. So the theorem above is also true for a grading-restricted Möbius vertex algebra V. In addition, under the same condition, the associativity of (logarithmic) intertwining operator among the category of C_1-cofinite grading-restricted generalized V-modules also holds and there is a natural vertex tensor category structure on this category.

Theorem 1.1 applies to C_1-cofinite module categories for many interesting vertex operator algebras, including, for example, simple affine vertex operator algebras at all admissible and many non-admissible levels ([CHY], [CY]), Virasoro vertex operator algebras of all central charges ([CJORY]), the affine vertex operator superalgebra of $\mathfrak{gl}(1|1)$ ([CMY3]), the singlet vertex operator algebras ([CMY1], [CMY4]), the universal affine vertex operator algebra of $\mathfrak{sl}_2$ at admissible levels ([MY]), $N = 1$ super Virasoro vertex operator superalgebras of all central charges ([CMOY]), and $N = 2$ super Virasoro vertex operator algebras at central charge $\frac{3k}{k+2}$ ([C]). Theorem 1.1 also applies indirectly to some non-C_1-cofinite module categories for some vertex operator algebras. In the case of the simple affine vertex operator algebra V of $\mathfrak{sl}_2$ at an admissible level k, objects of the category of finitely-generated weight V-modules are in general not C_1-cofinite. But its Kazama-Suzuki dual (see [KS]) is the $N = 2$ super Virasoro vertex operator algebra of the central charges $\frac{3k}{k+2}$. In [C], Creutzig used the vertex tensor category structure for this $N = 2$ super Virasoro vertex operator algebra to obtain a vertex tensor category structure on the category of finitely-generated weight V-modules.

Despite the great progress discussed above, whether the category of C_1-cofinite grading-restricted modules for a general vertex operator algebra V has a natural vertex tensor category structure is still an open problem. In this paper, we solve this problem. Here is the main theorem of the present paper:

Theorem 1.2. *Let V be a grading-restricted Möbius vertex algebra and $\mathcal{C}$ the category of C_1-cofinite grading-restricted generalized V-modules. Then the associativity of (logarithmic) intertwining operators among objects of $\mathcal{C}$ holds and the category $\mathcal{C}$ has a natural vertex tensor category structure. In particular, the category $\mathcal{C}$ has a natural braided tensor category structure with a twist.*

This theorem is proved in Section 4. The proof is based on the generalizations and modifications given in this paper of the results and methods in [N], [H3], [HLZ1]–[HLZ9], [Mi], [CJORY], [CY], and [Mc]. We also note that Theorem 1.2 above and Theorem 2.25 in [CMOY] can be generalized easily to a corresponding result for a grading-restricted Möbius vertex superalgebra V.

The main reason why Theorem 1.1 still needs the condition that the category is closed under the contragredient functor is because this condition is one of the assumptions in [HLZ1]–[HLZ9]. We cannot apply directly the logarithmic tensor category theory [HLZ1]–[HLZ9] to the category of C_1-cofinite grading-restricted modules since in general, this category might not be closed under the contragredient bifunctor. In this paper, we generalize the theory in [HLZ1]–[HLZ9] by removing this assumption. We obtain several assumptions in this generalization such that when these assumptions are satisfied, the category that we are considering has natural vertex and braided tensor category structures.

To verify these assumptions in this generalization, we show that the arguments and proofs in [H3] used to derive the differential equations still works in the case that the contragredient of a C_1-cofinite grading-restricted V-module might not be C_1-cofinite. We also recall some results of Miyamoto in [Mi] and give a proof of the generalization formulated in [MS] of the inequality of Nahm in [N] without using the differential equations in [Mi]. Using these results, we verify that for any grading-restricted Möbius vertex algebra (in particular, a vertex operator algebra) V and the category of C_1-cofinite grading-restricted V-modules, the assumptions to use our generalization of the theory in [HLZ1]–[HLZ9] are satisfied. In fact, the proofs of the relevant results in [CJORY], [CY], [Mc] and [CMOY] can be modified to verify some of these assumptions. For the reader's convenience, we still give the full details of the verification. Then by our generalization, we obtain the associativity (operator product expansion) of (logarithmic) intertwining operators among C_1-cofinite grading-restricted V-modules and natural vertex and braided tensor category structures on this category.

Theorem 1.2 above confirms the conjecture that C_1-cofiniteness is a sufficient condition for the existence of genus-zero chiral (logarithmic) conformal field theories. A consequence of the C_1-cofiniteness condition is that the genus-zero conformal blocks are finite-dimensional since spaces of solutions of differential equations in finite-dimensional spaces are finite-dimensional. To go beyond conformal field theories with finite-dimensional genus-zero conformal blocks, it is necessary to study vertex operator algebras and modules that do not satisfy the C_1-confiniteness condition. The Liouville conformal field theory is an important example of such non-C_1-cofinite conformal field theories. The mathematical construction of the Liouville conformal field theory by Kupiainen, Rhodes and Vargas in [KRV] and by Guillarmou, Kupiainen, Rhodes and Vargas in [GKRV1] and [GKRV2] uses the probability approach, which is heavily based on analysis in infinite-dimensional spaces. The chiral part of the Liouville conformal field theory can be used to study the representation theory of Virasoro vertex operator algebras and their modules. The author conjectures that we might be able to reformulate some part of the infinite-dimensional analysis used in the works [KRV], [GKRV1] and [GKRV2] as a theory of differential equations in infinite-dimensional spaces. If such a theory can be developed in the future, non-C_1-cofinite modules for vertex operator algebras can also be studied using the representation theory of vertex operator algebras together with this to-be-developed theory of differential equations in infinite-dimensional spaces.

This paper is organized as follows: In the next section, we generalize the logarithmic tensor category in [HLZ1]–[HLZ9]. The assumptions for applying this generalization are given in this section. In Section 3, we show that the arguments and proofs used to derive in [H3] the differential equations still works in the case that the generalized V-modules placed at 0 and ∞ are quasi-finite-dimensional but not C_1-cofinite. We also recall some results of Miyamoto on C_1-cofinite $\mathbb{N}$-gradable V-modules and give a proof of a generalization of the inequality of Nahm in this section. We verify the assumptions of our generalization in

Section 2 for a grading-restricted Möbius vertex algebra V and the category of C_1-cofinite grading-restricted V-modules in Section 4. In particular, we obtain the main result Theorem 1.2 of the present paper.

Acknowledgment The author is grateful to Thomas Creutzig, Robert McRae and Jinwei Yang for helpful comments and discussions. I am especially grateful to Robert McRae for discussions on the inequality of Nahm.

2 A generalization of the logarithmic tensor category

In this section, we first review briefly the assumptions and results in the logarithmic tensor category theory developed in [HLZ1]–[HLZ9]. The construction and results in [HLZ1]–[HLZ9] work for a suitable vertex algebra V much more general than a vertex operator algebra. In general, V can be a strongly A-graded Möbius vertex algebra (see [FHL] and [HLZ2] for the precise definition of Möbius vertex algebra), where A is an abelian group. In this paper, we are interested only in the case that A is trivial and V is a grading-restricted Möbius vertex algebra. So for simplicity, we do not discuss the general case that A is not trivial. But it is clear that the generalization obtained in this section still works in the general case.

One of the main assumption in [HLZ1]–[HLZ9] is that the module category for V should be closed under the contragredient functor. In this section, we show that the construction and results in [HLZ1]–[HLZ9] can be generalized to the case that the module category might not be closed under the contragredient functor.

In this paper, we fix a grading-restricted Möbius vertex algebra V. (Using the terminology in [HLZ2], V is a Möbius vertex algebra strongly graded with respect to the weight-grading and the trivial abelian group grading.) So when we recall the material in [HLZ1]–[HLZ9], the abelian groups A and $\tilde{A}$ giving the horizontal gradings of V and its modules, respectively, are all trivial.

Though we are mainly interested in grading-restricted generalized V-modules, other types of V-modules always appear in our proofs and intermediate steps. This is an important feature of the representation theory of vertex (operator) algebras. So we first briefly discuss our terminology for these different types of V-modules and their differences. For the precise definitions, see [HLZ2] and [H4].

A generalized V-module is a $\mathbb{C}$-graded vector space W equipped with a vertex operator map $Y_W : V \otimes W \to W[[x, x^{-1}]]$ and operators $L_W(-1)$, $L_W(0)$, $L_W(1)$ satisfying all the axioms for a Möbius vertex algebra that still make sense. Note that though W has a $\mathbb{C}$-grading, it does not have to be lower bounded or grading restricted. A lower-bounded generalized V-module is a generalized V-module $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ such that $W_{[n]} = 0$ for $\Re(n) < N$ for some $N \in \mathbb{Z}$. A grading-restricted generalized V-module is a lower-bounded generalized V-module $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ such that $\dim W_{[n]} < \infty$ for $n \in \mathbb{C}$. An ordinary V-module is a grading-restricted generalized V-module such that $L(0)$ acts on the generalized V-module semisimply. A quasi-finite-dimensional generalized V-module is a generalized V-module $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ such that $\dim \coprod_{\Re(n)<N} W_{[n]} < \infty$ for $N \in \mathbb{Z}$.

A weak V-module is a vector space W and a vertex operator map $Y_W : V \otimes W \to W[[x, x^{-1}]]$ satisfying all the axioms, including the Jacobi identity, for generalized V-modules except for those involving the grading of W. For a weak V-module W, an $\mathbb{N}$-grading $W = \coprod_{n\in\mathbb{N}} W_{\llbracket n\rrbracket}$ on W is said to be compatible if for $v \in V_{(m)}$, $k \in \mathbb{Z}$, and $w \in W_{\llbracket n\rrbracket}$, $v_k w \in W_{\llbracket m-k-1+n\rrbracket}$, where $v_k = \mathrm{Res}_x x^k Y_W(v, x)$. An $\mathbb{N}$-gradable weak V-module is a weak V-module for which there exists a compatible $\mathbb{N}$-grading.

Intertwining operators among any types of V-modules above are well defined. Since $L(0)$ might act nonsemisimply on generalized V-modules, they might contain logarithms of the variables. These are called logarithmic intertwining operators. For simplicity, we shall simply call them intertwining operators also unless it is necessary to emphasize there are logarithms of the variables.

Let $\mathcal{C}$ be a full subcategory of generalized V-modules. The construction of the vertex tensor category and braided tensor category structures in [HLZ1]–[HLZ9] are obtained based the following assumptions on $\mathcal{C}$ (see Assumptions 10.1 in [HLZ7], Assumptions 12.1 and 12.2 in [HLZ9]):

1. Objects of $\mathcal{C}$ are grading-restricted.
2. For any object W in $\mathcal{C}$, the weights of homogeneous elements are real numbers and there exists $K \in \mathbb{Z}_+$ such that $L_W(0)_N^K = 0$ where $L_W(0)_N$ is the nilpotent part of $L_W(0)$.
3. $\mathcal{C}$ is closed under images, under the contragredient functor, under taking finite direct sums, and under $P(z)$-tensor products $\boxtimes_{P(z)}$ for some $z \in \mathbb{C}^\times$.
4. V is an object of $\mathcal{C}$.
5. The convergence and expansion conditions for intertwining maps in $\mathcal{C}$ hold.
6. The products of more than two intertwining operators are absolutely convergent in the corresponding region and can be analytically extended to multivalued analytic functions defined on the region where the complex variables in the intertwining operators are not equal 0 and each other.

It is proved in [HLZ1]–[HLZ9] that if Assumptions 1–6 hold, then intertwining operators among objects of $\mathcal{C}$ satisfy the associativity property and $\mathcal{C}$ has a natural vertex tensor category structure. In particular, $\mathcal{C}$ has a natural braided tensor category structure with a twist.

Theorem 11.4 in [HLZ8] states that Assumption 5 above holds if the following Assumptions hold:

7. Every finitely-generated lower-bounded generalized V-module is an object of $\mathcal{C}$.
8. The convergence and extension property for either products or iterates holds in $\mathcal{C}$.

By the properties of intertwining operators and Proposition 2.1 In [H6], the first part of Assumption 2 above, that the weights of homogeneous elements are real numbers, can be replaced by the assumption that for any object W of $\mathcal{C}$, the weights of homogeneous elements of W are congruent to finitely many complex numbers modulo $\mathbb{Z}$. So Assumption 2 can be replaced by the following weaker assumption:

9. For any object W of $\mathcal{C}$, the weights of homogeneous elements of W are congruent to finitely many complex numbers modulo $\mathbb{Z}$ and there exists $K \in \mathbb{Z}_+$ such that $L_W(0)_N^K = 0$ where $L_W(0)_N$ is the nilpotent part of $L_W(0)$.

Remark 2.1. If an object of $\mathcal{C}$ satisfies Assumption 9, then certainly the contragredient of this object also satisfies Assumption 9. This fact is needed in our generalization of [HLZ1]–[HLZ9]. For example, (9.139) in [HLZ7] is proved using the fact that $W_1 \boxbackslash_{P(z)} W_2$ is an object of $\mathcal{C}$. In our generalization, $\mathcal{C}$ is closed under $P(z)$-tensor product, $W_1 \boxbackslash_{P(z)} W_2$ as the contragredient of the $P(z)$-tensor product is in general not an object of $\mathcal{C}$. But since its contragredient is an object of $\mathcal{C}$, $W_1 \boxbackslash_{P(z)} W_2$ still satisfies Assumption 9.

Also in [H6], it is proved that Assumption 7 above can be replaced by the following weaker assumption:

10. For any objects W_1 and W_2 of $\mathcal{C}$ and any $z \in \mathbb{C}^\times$, if the generalized V-module W_λ generated by a generalized eigenvector $\lambda \in (W_1 \otimes W_2)^*$ for $L_{P(z)}(0)$ satisfying the $P(z)$-compatibility condition is lower bounded, then W_λ is an object of $\mathcal{C}$.

By these results on the assumptions, we see that for a Möbius vertex algebra V and a full subcategory of generalized V-modules $\mathcal{C}$ satisfying Assumptions 1, 3, 4, 6, 8, 9, and10, associativity of intertwining operators among objects of $\mathcal{C}$ is satisfied and $\mathcal{C}$ has a natural vertex tensor category structure and thus a braided tensor category structure with a twist. But this result still cannot be applied to the case that $\mathcal{C}$ is the category of C_1-cofinite grading-restricted generalized V-modules because in general, the contragredient of a C_1-cofinite grading-restricted generalized V-module might not be C_1-cofinite. Now we generalize the construction in [HLZ1]–[HLZ9] to the case that $\mathcal{C}$ might not be closed under the contragredient functor.

We need to replace Assumption 3 by the following weaker assumption:

11. $\mathcal{C}$ is closed under images, under taking finite direct sums, and under $P(z)$-tensor products $\boxtimes_{P(z)}$ for some $z \in \mathbb{C}^\times$.

Let $\mathcal{C}$ be a full subcategory of generalized V-modules satisfying Assumptions 1 and closed under images and finite direct sums. Even though $\mathcal{C}$ might not be closed under the contragredient functor, contragredients of objects of $\mathcal{C}$ are grading-restricted generalized V-modules.

In this case, the definition of $W_1 \boxbackslash_{P(z)} W_2$ in the category $\mathcal{C}$ is in fact the same as the one in [HLZ5] (see Definition 5.31). For the reader's convenience, we recall Definition 5.31 in [HLZ5] here.

Definition 2.2. For objects W_1, W_2 in $\mathcal{C}$, define the subset

$$W_1 \boxbackslash_{P(z)} W_2 \subset (W_1 \otimes W_2)^*$$

of $(W_1 \otimes W_2)^*$ to be the union of the images

$$I'(W') \subset (W_1 \otimes W_2)^*$$

as $(W; I)$ ranges through all the $P(z)$-products of W_1 and W_2 with objects W in $\mathcal{C}$. Equivalently, $W_1 \boxbackslash_{P(z)} W_2$ is the union of the images $I'(W')$ as W ranges through objects in $\mathcal{C}$ and I' ranges through the space of linear maps

$$W' \to (W_1 \otimes W_2)^*$$

intertwining the actions of

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$$

and $L(-1)$, $L(0)$ and $L(1)$ on both spaces.

Remark 2.3. Note that in Definition 5.31 in [HLZ5], we have the phrase "as W (or W') ranges through $\mathrm{ob}\,\mathcal{C}$" in the second equivalent formulation of the definition. But in the definition above we have deleted "(or W')" because in this case, W' might not be an object of $\mathcal{C}$.

The first result in [HLZ5] that needs to be generalized is Proposition 5.37. We also need to generalize Proposition 5.36 in [HLZ5]. But we need to use the generalization of Proposition 5.37 in [HLZ5] to generalize Proposition 5.36 in [HLZ5].

Proposition 2.4. *Assume that $\mathcal{C}$ satisfies Assumptions 1 and is closed under images and finite direct sums. Let W_1, W_2 be objects of $\mathcal{C}$. If the contragredient of $W_1 \boxbackslash_{P(z)} W_2$ is an object of $\mathcal{C}$, then the $P(z)$-tensor product of W_1 and W_2 in $\mathcal{C}$ exists and is $(W_1 \boxtimes_{P(z)} W_2, i')$, where $W_1 \boxtimes_{P(z)} W_2 = (W_1 \boxbackslash_{P(z)} W_2)'$ and i is the natural inclusion from $W_1 \boxbackslash_{P(z)} W_2$ to $(W_1 \otimes W_2)^*$. Conversely, if the $P(z)$-tensor product of W_1 and W_2 in $\mathcal{C}$ exists, then the contragredient of $W_1 \boxbackslash_{P(z)} W_2$ is an object of $\mathcal{C}$.*

Proof. The proof of the first part is the same as in [HLZ5]. For the converse, without using contragredients of objects of $\mathcal{C}$, the proof of Proposition 5.37 in [HLZ5] shows that $W_1 \boxbackslash_{P(z)} W_2 = I_0'(W_0')$, where (W_0, I_0) is the $P(z)$-tensor product of W_1 and W_2. By Proposition 4.23 in [HLZ4] (which is a straightforward generalization of Lemma 4.9 in [H1]), we know that the homogeneous components of elements of $\overline{W}_0$ of the form $I_0(w_1 \otimes w_2)$ for $w_1 \in W_1$ and $w_2 \in W_2$ span W_0. Then I_0' is injective. This means that as a V-module map from W_0' to $I'(W_0')$, I_0' is an equivalence. Hence its adjoint from the contragredient of $I_0'(W_0')$ to the contragredient W_0'' of W_0' is also an equivalence. Since objects of $\mathcal{C}$ are grading-restricted, W_0'' is equivalent to W_0, which is an object of $\mathcal{C}$. Thus W_0'' and the contragredient of $I'(W_0')$ are also objects of $\mathcal{C}$ since $\mathcal{C}$ is closed under images. Since $W_1 \boxbackslash_{P(z)} W_2 = I_0'(W_0')$, we see that the contragredient of $W_1 \boxbackslash_{P(z)} W_2$ is also an object of $\mathcal{C}$. $\square$

We now assume that $\mathcal{C}$ satisfies Assumptions 1 and 11. In this case, since the $P(z)$-tensor product of W_1 and W_2 in $\mathcal{C}$ exists, by Proposition 2.4, the contragredient $(W_1 \boxbackslash_{P(z)} W_2)'$ of $W_1 \boxbackslash_{P(z)} W_2$ is an object of $\mathcal{C}$. Then we let $W_1 \boxtimes_{P(z)} W_2 = (W_1 \boxbackslash_{P(z)} W_2)'$ and take the $P(z)$-tensor product of W_1 and W_2 to be $(W_1 \boxtimes_{P(z)} W_2, i')$, where $i : W_1 \boxbackslash_{P(z)} W_2 \to (W_1 \otimes W_2)^*$ is the inclusion map and $i' : W_1 \otimes W_2 \to W_1 \boxtimes_{P(z)} W_2$ is the $P(z)$-intertwining map corresponding to i.

We are ready to generalize Proposition 5.36 [HLZ5] now. The proof of Proposition 5.36 in [HLZ5] uses the assumption that the category there is closed under contragredient functor. Since $\mathcal{C}$ might not be closed under the contragredient functor, we need to use the converse part of Proposition 2.4, which in turn uses the assumption that $\mathcal{C}$ is closed under $P(z)$-tensor products $\boxtimes_{P(z)}$ for some $z \in \mathbb{C}^\times$.

Proposition 2.5. *Assume that $\mathcal{C}$ satisfies Assumptions 1 and 11. Let W_1, W_2 be objects of $\mathcal{C}$. Then the subspace $W_1 \boxbackslash_{P(z)} W_2$ of $(W_1 \otimes W_2)^*$ is equal to the union and also to the sum of grading-restricted generalized V-modules contained in $(W_1 \otimes W_2)^*$ (equipped with the action $Y'_{P(z)}(\cdot, x)$ of V and the corresponding actions of $L(-1)$, $L(0)$ and $L(1)$ on $(W_1 \otimes W_2)^*$) such that their contragredients are objects of $\mathcal{C}$.*

Proof. The only difference between Proposition 5.36 in [HLZ5] and this proposition is that here the sum is over grading-restricted generalized V-modules which might not be objects of $\mathcal{C}$, but whose contragredients are objects of $\mathcal{C}$. Since by Assumption 1, an object W of $\mathcal{C}$ is grading-restricted, W' is also grading-restricted. Then for an object W of $\mathcal{C}$ and a $P(z)$-intertwining map I of type $\binom{W}{W_1 W_2}$, $I'(W')$ is also grading restricted. So the only thing we need to prove is that the contragredient of $I'(W')$ is an object of $\mathcal{C}$.

Since $I'(W')$ is a generalized V-submodule of $W_1 \boxbackslash_{P(z)}$, we have the inclusion map i from $I'(W')$ to $W_1 \boxbackslash_{P(z)} W_2$. The inclusion map i is an injective V-module map. By Proposition 2.4, $W_1 \boxbackslash_{P(z)} W_2$ is grading-restricted. Then the adjoint i' of i is a surjective V-module map from $W_1 \boxtimes_{P(z)} W_2 = (W_1 \boxbackslash_{P(z)} W_2)'$ to the contragredient of $I'(W')$. Since $\mathcal{C}$ is closed under images, the contragredient of $I'(W')$ is also an object of $\mathcal{C}$. $\square$

In the work [HLZ5], one crucial result is that elements of $W_1 \boxbackslash_{P(z)} W_2$ can be characterized by two conditions called the $P(z)$-compatibility condition and $P(z)$-local grading restriction condition. See Section 5 in [HLZ5] for these conditions and Theorem 5.50 in [HLZ5] for the theorem. But the condition of Theorem 5.50 in [HLZ5] is not satisfied for our category $\mathcal{C}$, which is not closed under the contragredient functor. In fact, the condition of Theorem 5.50 in [HLZ5] states that every element of $(W_1 \otimes W_2)^*$ satisfies the $P(z)$-compatibility condition and the $P(z)$-local grading-restriction condition must contain in a generalized V-submodule of some object of $\mathcal{C}$. But for an object W of $\mathcal{C}$ and a $P(z)$-intertwining map I of type $\binom{W}{W_1 W_2}$, we obtain an element $I'(w') \in (W_1 \otimes W_2)^*$ for each $w' \in W'$ satisfying both the $P(z)$-compatibility condition and the $P(z)$-local grading-restriction condition. But since W' might not be an object of $\mathcal{C}$, $I'(w')$ also might not be in some object of $\mathcal{C}$. So we need to generalize Theorem 5.50 in [HLZ5].

Theorem 2.6. *Suppose that for every element $\lambda \in (W_1 \otimes W_2)^*$ satisfying both the $P(z)$-compatibility condition and the $P(z)$-local grading restriction condition, the generalized module W_λ generated by λ as given in Theorem 5.49 in [HLZ5] is a generalized V-submodule of a grading-restricted generalized V-module in $(W_1 \otimes W_2)^*$ whose contragredient is an object of $\mathcal{C}$. Then $W_1 \boxdot_{P(z)} W_2$ is equal to the space of all such λ.*

Proof. We denote the space of all such λ by W_3. We need to prove that $W_1 \boxdot_{P(z)} W_2 = W_3$. We prove $W_1 \boxdot_{P(z)} W_2 \subset W_3$ and $W_3 \subset W_1 \boxdot_{P(z)} W_2$.

From the calculations and discussions before the statements of $P(z)$-compatibility condition and the $P(z)$-local grading restriction condition in Section 5 of [HLZ5], we know that for an object W of $\mathcal{C}$, a $P(z)$-intertwining map of type $\binom{W}{W_1 W_2}$ and an element $w' \in W'$, the element $I'(w')$ of $(W_1 \otimes W_2)^*$ satisfies these two conditions. Also, the generalized V-module $W_{I(w')}$ is a generalized V-submodule of the grading-restricted generalized V-module $I'(W')$. By Proposition 2.5, the contragredient of $I'(W')$ is an object of $\mathcal{C}$. We have proved $I(w') \in W_3$. Thus we have $W_1 \boxdot_{P(z)} W_2 \subset W_3$.

Let $\lambda \in W_3$. By Theorem 5.49 in [HLZ5], $W_\lambda \subset (W_1 \otimes W_2)^*$ is a grading-restricted generalized V-module. By assumption, W_λ is the generalized V-submodule of a grading-restricted generalized V-module W in $(W_1 \otimes W_2)^*$ whose contragredient is an object of $\mathcal{C}$. In particular, $\lambda \in W$. Since the contragredient of W is an object of $\mathcal{C}$, by Proposition 2.5, $W \subset W_1 \boxdot_{P(z)} W_2$. In particular, $\lambda \in W_1 \boxdot_{P(z)} W_2$. Thus we have $W_3 \subset W_1 \boxdot_{P(z)} W_2$. □

Another important result that we need to generalize is Theorem 3.1 in [H6], which in turn is a generalization of Theorem 11.4 in [HLZ8]. To generalize this result, we first need to reformulate Condition 1 in this result, that is, Assumption 10 above. The following assumption is what we use to replace Assumption 10:

12. For any objects W_1 and W_2 of $\mathcal{C}$ and any $z \in \mathbb{C}^\times$, if the generalized V-module W_λ generated by a generalized eigenvector $\lambda \in (W_1 \otimes W_2)^*$ for $L'_{P(z)}(0)$ satisfying the $P(z)$-compatibility condition is lower bounded, then W_λ is grading-restricted and its contragredient W'_λ is an object of $\mathcal{C}$.

Theorem 2.7. *Suppose that in addition to Assumptions 1 and 11, Assumption 8, 9 and 12 also hold. Then the convergence and expansion conditions for intertwining maps in $\mathcal{C}$ both hold.*

Proof. This proof is based on the proof of Theorem 11.4 in [HLZ8]. The reader should read that proof there to see what is modified and generalized here.

Note that the convergence condition for intertwining maps in the category $\mathcal{C}$ holds since Assumption 8 in particular gives this condition.

As mentioned in the proof of Theorem 3.1 in [H6], the condition in Theorem 11.4 in [HLZ8] that every finitely-generated lower bounded generalized V-module is in $\mathcal{C}$ (Condition 1 in that theorem) is only used in the last paragraph showing that $W_{\lambda_n^{(2)}(w'_{(4)}, w_{(3)})}$ is an object of $\mathcal{C}$. In our case, $W_{\lambda_n^{(2)}(w'_{(4)}, w_{(3)})}$ might not be an object of $\mathcal{C}$. Instead, we need to show that the contragredient of $W_{\lambda_n^{(2)}(w'_{(4)},w_{(3)})}$ is an object of $\mathcal{C}$. But it is proved in the proof of Theorem 11.4 in [HLZ8] that $\lambda_n^{(2)}(w'_{(4)}, w_{(3)})$ is a generalized eigenvector of $L'_{P(z)}(0)$ with eigenvalue n and, by using Theorem 9.17 in [HLZ7], that $\lambda_n^{(2)}(w'_{(4)}, w_{(3)})$ satisfies the $P(z)$-compatibility condition. It is also shown in the proof of Theorem 11.4 in [HLZ8] that $W_{\lambda_n^{(2)}(w'_{(4)},w_{(3)})}$ is lower-bounded. Then by Assumption 12, $W_{\lambda_n^{(2)}(w'_{(4)},w_{(3)})}$ is grading-restricted and its contragredient is an object of $\mathcal{C}$. But an object of $\mathcal{C}$ is grading-restricted and its contragredient is also grading-restricted. So $W_{\lambda_n^{(2)}(w'_{(4)},w_{(3)})}$ is grading-restricted. This shows that $I_1 \circ (1_{W_2} \otimes I_2))'(w'_{(4)})$ of $(W_1 \otimes W_2 \otimes W_3)^*$ satisfies the $P^{(2)}(z_1 - z_2)$-local grading-restriction condition. Thus the expansion condition for intertwining maps in the category $\mathcal{C}$ holds. □

Theorem 2.7 in fact says that Assumptions 1, 8, 9, 11, and 12 implies Assumption 5.

Besides the generalizations above of the corresponding results in [HLZ6], we also need to modify all the statements in [HLZ5]–[HLZ9] involving statements on the contragredients of objects of $\mathcal{C}$, especially the statements on $W_1 \boxtimes_{P(z)} W_2$ and on generalized V-modules in $(W_1 \otimes W_2)^*$. Even under the assumption that the $P(z)$-tensor product of objects W_1 and W_2 of $\mathcal{C}$ exist, $W_1 \boxtimes_{P(z)} W_2$ and its generalized V-submodules are in general not objects of $\mathcal{C}$. So when we see a statement saying that $W_1 \boxtimes_{P(z)} W_2$ or a generalized V-module in $(W_1 \otimes W_2)^*$ is an object of $\mathcal{C}$, we need to change the statement to say that $W_1 \boxtimes_{P(z)} W_2$ or a generalized V-module in $(W_1 \otimes W_2)^*$ is a grading-restricted generalized V-module whose contragredient is an object of $\mathcal{C}$. The proofs are completely the same except that we use the generalizations above instead of the corresponding results in [HLZ5]. In addition to the generalizations above of the results related to $P(z)$-tensor products in [HLZ5]–[HLZ9], we also need to generalize of the results on $Q(z)$-tensor products in [HLZ5]. But since these generalizations are completely the same as those we have given above for $P(z)$-tensor products, we omit the detailed discussions here.

More specifically, the proofs of the main results in [HLZ7] use Propositions 5.36, 5.37 and Theorem 5.50 in [HLZ5]. Since we have generalized these results to Propositions 2.5, 2.4 and Theorem 2.6, the corresponding statements of some main results in [HLZ7] also need to be generalized as discussed above. As mentioned above, the proofs of these generalizations are completely the same as the proofs of the corresponding results in [HLZ7] except for the corresponding statements about the objects of $\mathcal{C}$. So here we do not give the full statements and the proofs of these generalizations. In fact, only one phrase needs to be changed in all these results. Here is a list of results in [HLZ7] and the change of one phrase needed to generalize these results on this list: In the statements of Propositions 9.13, Remark 9.20, Corollary 9.21, Theorem 9.23, Corollary 9.24, Theorem 9.27, and the proof of Theorem 10.3 in [HLZ7], replace the phrase "a generalized V-submodule (or a V-submodule) of some object of $\mathcal{C}$" by "a generalized V-submodule (or a V-submodule) of the contragredient of some object of $\mathcal{C}$."

Using the generalizations above, we see that the main results in [HLZ7] and [HLZ9] still hold. In summary, we obtain the following main theorem of this section:

Theorem 2.8. *Let V be a Möbius vertex algebra and $\mathcal{C}$ a full subcategory of generalized V-modules. Assume that Assumptions 1, 4, 6, 8, 9, 11, and 12 above hold. Then the associativity of intertwining operators among objects of $\mathcal{C}$ holds and $\mathcal{C}$ has a natural vertex tensor category structure. In particular, $\mathcal{C}$ has a natural braided tensor category structure with a twist.*

Remark 2.9. For the definition of vertex tensor category based on spheres with punctures and local coordinates vanishing at punctures, see [HL2]. If V is a vertex operator algebra, a vertex tensor category structure is essentially constructed in [HLZ9] but is not stated explicitly. The tensor product bifunctors associated to conformal classes of general spheres with three punctures and local coordinates and the corresponding associated isomorphisms, commutativity isomorphisms and so on can be obtained using the $P(z)$-tensor product bifunctors and the corresponding associativity isomorphisms, commutativity isomorphisms and so on given in [HLZ9] and the work [H2]. The substitution isomorphisms associated to conformal equivalence classes of spheres with two punctures and local coordinates can be constructed directly using the work [H2]. But for a grading-restricted Möbius vertex algebra, since there are no Virasoro algebra actions, we have only functors and isomorphisms associated to conformal classes of sphere with punctures and $SL(2, \mathbb{C})$ local coordinates vanishing at punctures. Such a vertex tensor category can be called a Möbius vertex tensor category. The Möbius vertex tensor category can be obtained easily using the results in [HLZ9].

We shall use Theorem 2.8 to prove Theorem 1.2.

3 C_1-cofiniteness and intertwining operators

To apply Theorem 2.8 to the case that $\mathcal{C}$ is the category of C_1-cofinite grading-restricted generalized V-modules, we need to verify Assumptions 1, 4, 6, 8, 9, 11, and 12. To verify Assumptions 6 and 8, we need to generalize the results on the differential equations derived by the author for intertwining operators without logarithms in [H3] and observed in [HLZ8] to hold also for logarithmic intertwining operators. To verify Assumptions 9, 11, and 12, we need some results of Miyamoto in [Mi] on C_1-cofinite $\mathbb{N}$-gradable weak V-modules and also a weak version proved by Miyamoto in [Mi] of the inequality of Nahm in [N].

In this section, we first recall and give proofs of the results of Miyamoto in [Mi] on C_1-cofinite $\mathbb{N}$-gradable weak V-modules. Then we generalize the results on the differential equations in [H3] by showing that the differential equations in [H3] are still satisfied even when two of the generalized V-modules are quasi-finite-dimensional but not C_1-cofinite. Finally, instead of recalling the weak version of the inequality of Nahm proved in [Mi], we give a proof of the generalization of the inequality of Nahm.

Before we give these results, we recall again the C_1-cofiniteness for weak V-modules and other types of V-modules. Let W be a weak V-module. We say that W is C_1-cofinite if $\dim W/C_1(W) < \infty$, where $C_1(W)$ is the subspace of W spanned by elements of the form $v_{-1}w$ for $v \in V_+ = \coprod_{n\in\mathbb{Z}_+} V_{(n)}$ and $w \in W$. Since any other classes of V-modules are all weak modules, this definition also defines C_1-cofiniteness for these other classes of V-modules.

We first give a result of Miyamoto in [Mi] on C_1-cofinite $\mathbb{N}$-gradable weak V-modules. This result combines several results in different places in [Mi]. For the reader's convenience, we also provide a proof of this result.

Proposition 3.1. *Let W be a C_1-cofinite $\mathbb{N}$-gradable weak V-module.*

1. *Let $W = \coprod_{n\in\mathbb{N}} W_{[\![n]\!]}$ be a compatible $\mathbb{N}$-grading. Then for $n \in \mathbb{N}$, $\dim W_{[\![n]\!]} < \infty$.*

2. *There exists a finite-dimensional subspace M of W such that W is spanned by elements of the form $v_{-1}^{(1)} \cdots v_{-1}^{(i)} w$ for $i \in \mathbb{N}$, $v^{(1)}, \dots, v^{(i)} \in V$ and $w \in M$. In particular, W is finitely generated.*

3. *There exists $h_1, \dots, h_k \in \mathbb{C}$ such that they are not congruent to each other modulo $\mathbb{Z}$ and $W = \coprod_{i=1}^{k} \coprod_{n\in h_i+\mathbb{N}} W_{[n]}$, where $W_{[n]}$ for $n \in h_i + \mathbb{N}$ are finite-dimensional generalized eigenspaces for $L_W(0)$ with eigenvalues n. In particular, W is a quasi-finite-dimensional generalized V-module.*

Proof. We prove first $\dim W_{[\![n]\!]} < \infty$ for $n \in \mathbb{N}$. We use induction on n. For $v \in V_{(m)}$ and $w \in W_{[\![n]\!]}$, where $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, $v_{-1}w \in W_{[\![m+n]\!]} \neq W_{[\![0]\!]}$. Hence nonzero elements of $C_1(W)$ cannot be in $W_{[\![0]\!]}$. Therefore $\dim W_{[\![0]\!]} \leq \dim W/C_1(W) < \infty$ since W is C_1-cofinite.

Assume that $\dim W_{[\![k]\!]} < \infty$ for $k = 0, \dots, n$. Then the subspace of $W_{[\![n+1]\!]}$ spanned by elements of the form $v_{-1}w$ for $v \in V_{(n+1-k)}$ and $w \in W_{[\![k]\!]}$ for $k = 0, \dots, n$ is also finite dimensional since $\dim V_{(n+1-k)} < \infty$ and $\dim W_{[\![k]\!]} < \infty$ for $k = 0, \dots, n$. But the quotient of $W_{[\![n+1]\!]}$ by this subspace of $W_{[\![n+1]\!]}$ is by definition a subspace of $W/C_1(W)$. Since $\dim W/C_1(W) < \infty$, this quotient is also finite dimensional. Then we have $\dim W_{[\![n+1]\!]} < \infty$.

Since $\dim W_{[\![n]\!]} < \infty$, we also have $\dim \coprod_{n=0}^{N} W_{[\![n]\!]} < \infty$ for $N \in \mathbb{N}$. But every finite-dimensional subspace of W must be in $\coprod_{n=0}^{N} W_{[\![n]\!]}$ for some $N \in \mathbb{N}$. Take a finite-dimensional subspace M of W such that $W = C_1(W) + M$. Then there exists $N_0 \in \mathbb{N}$ such that $M \subset \coprod_{n=0}^{N_0} W_{[\![n]\!]}$ and thus $\coprod_{n=N_0+\mathbb{Z}_+} W_{[\![n]\!]} \subset C_1(W)$. We can now take M above to be $\coprod_{n=0}^{N_0} W_{[\![n]\!]}$. So from now on, $M = \coprod_{n=0}^{N_0} W_{[\![n]\!]}$.

We are ready to show that W is spanned by elements of the form $v_{-1}^{(1)} \cdots v_{-1}^{(i)} w$ for $i \in \mathbb{N}$, $v^{(1)}, \dots, v^{(i)} \in V$ and $w \in M$. Denote the space spanned by elements of the form above by $\widetilde{W}$. What we want to prove is $W = \widetilde{W}$.

We need to prove that $W_{[\![n]\!]} \subset \widetilde{W}$ for every $n \in \mathbb{N}$. We use induction on n. When $n \leq N_0$, $W_{[\![n]\!]} \subset M \subset \widetilde{W}$. Assume that for $m < n \in N_0 + \mathbb{Z}_+$, $W_{[\![m]\!]} \subset \widetilde{W}$. Then $W_{[\![n]\!]} \subset C_1(W)$ is spanned by elements of the form $v_{-1}w$, where $v \in V_{(l)}$ and $w \in W_{[\![m]\!]}$ for $l \in \mathbb{Z}_+$ and $m \in \mathbb{N}$, such that $l + m = n > N_0$. Since $l > 0$, we have $m < n$. By induction assumption, $w \in \widetilde{W}$ and thus $v_{-1}w \in \widetilde{W}$. So $W_{[\![n]\!]} \subset \widetilde{W}$ and thus Conclusion 2 is true.

Since $W_{[\![n]\!]}$ for fixed $n \in \mathbb{N}$ is invariant under the action of $L_W(0)$ and $\dim W_{[\![n]\!]} < \infty$, $W_{[\![n]\!]}$ can be decomposed as a direct sum of generalized eigenspaces of $L_W(0)$. Hence W can also be decomposed as a direct sum of generalized eigenspaces of $L_W(0)$. In particular, the finite-dimensional invariant subspace $M = \coprod_{n=0}^{N_0} W_{[\![n]\!]}$ of W under $L_W(0)$ is a direct sum of finitely many generalized eigenspaces of $L_W(0)$. So there exists $h_1, \dots, h_k \in \mathbb{C}$ and $\widetilde{N} \in \mathbb{N}$

such that $h_1, \dots, h_k$ are not congruent to each other modulo $\mathbb{Z}$ and $M = \coprod_{i=1}^{k} \coprod_{n=0}^{\widetilde{N}} M_{[h_i+n]}$, where $M_{[h_i+n]}$ for $i = 1, \dots, k$ and $n = 0, \dots, \widetilde{N}$ are finite-dimensional generalized eigenspaces of $L_W(0)$ with eigenvalues $h_i + n$.

On the other hand, since W is spanned by elements of the form $v^{(1)}_{-1} \cdots v^{(i)}_{-1} w$ for $i \in \mathbb{N}$, $v^{(1)}, \dots, v^{(i)} \in V$ and $w \in M$, we see that $W = \coprod_{i=1}^{k} \coprod_{n \in h_i + \mathbb{N}} W_{[n]}$. where $W_{[n]}$ are finite-dimensional generalized eigenspaces of $L_W(0)$ with eigenvalues n. Thus W is a quasi-finite-dimensional generalized V-module. □

Now we give results involving intertwining operators. We refer the reader to [HLZ3] for the precise definition of (logarithmic) intertwining operators among generalized V-modules. As we have mentioned and have been doing in Section 2, in this paper, we omit the word "logarithmic" to call logarithmic intertwining operators simply intertwining operators, unless it is necessary to emphasize that there are logarithms of the variables. Also, we note that (logarithmic) intertwining operators are well defined for weak modules.

We first generalize the results on differential equations of the author in [H3] which is observed in [HLZ8] to hold also for logarithmic intertwining operators. In fact, it is observed in Remark 1.7 in [H3] that if one of the V-modules is not C_1-cofinite but is generated by a lowerest vector, the results on the differential equations in [H3] still hold. In fact, this remark should be corrected by replacing "one of the V-modules" by "the V-module W_0" (the V-module placed at ∞). We now show that the results and proofs on these differential equations in [H3] also work if the generalized V-module placed at ∞ is quasi-finite-dimensional generalized V-modules (not necessarily generated by lowerest vectors).

Theorem 3.2. *Let W_1, W_2, and W_3 be C_1-cofinite grading-restricted generalized V-modules and W_0 quasi-finite-dimensional generalized V-modules. Then given either the singular point $z_1 = \infty$, $z_2 = 0$ or the singular point $z_1 = z_2$, $z_2 = \infty$, for $w_0 \in W_0$, $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$, there exist*

$$a_k(z_1, z_2), b_l(z_1, z_2) \in R = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, (z_1 - z_2)^{-1}]$$

for $k = 1, \dots, m$ and $l = 1, \dots, n$ such that for lower-bounded generalized V-modules W_4, intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of types $\binom{W_0'}{W_1 W_4}$ and $\binom{W_4}{W_2 W_3}$, respectively, the series

$$\langle w_0, \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle \tag{3.1}$$

satisfy the expansions of the system of differential equations

$$\frac{\partial^m \varphi}{\partial z_1^m} + a_1(z_1, z_2)\frac{\partial^{m-1}\varphi}{\partial z_1^{m-1}} + \cdots + a_m(z_1, z_2)\varphi = 0, \tag{3.2}$$

$$\frac{\partial^n \varphi}{\partial z_2^n} + b_1(z_1, z_2)\frac{\partial^{n-1}\varphi}{\partial z_2^{n-1}} + \cdots + b_n(z_1, z_2)\varphi = 0 \tag{3.3}$$

of regular singular point at $z_1 = \infty$, $z_2 = 0$ on the regions $|z_1| > |z_2| > 0$ or at the singular point $z_1 = z_2$, $z_2 = \infty$ on the region $|z_2| > |z_1 - z_2| > 0$.

Proof. In this proof, we show that calculations and proofs used to derive the differential equations in [H3] still work here. So we assume that the reader is familiar with the paper [H3].

In [H3], the reason why we need the C_1-cofiniteness of modules is because we need to show that the R-module T/J is finitely generated, where $R = \mathbb{C}[z_1, z_1^{-1}, z_1, z_1^{-1}, (z_1 - z_2)^{-1}]$, $T = R \otimes W_0 \otimes W_1 \otimes W_2 \otimes W_3$ and J is the R-submodule of T generated by elements $\mathcal{A}(u, w_0, w_1, w_2, w_3)$, $\mathcal{B}(u, w_0, w_1, w_2, w_3)$, $\mathcal{C}(u, w_0, w_1, w_2, w_3)$ and $\mathcal{D}(u, w_0, w_1, w_2, w_3)$ given in [H3] for $u \in V$, $w_0 \in W_0$, $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$. But this strong result that T/J is finitely generated is in fact not needed in [H3]. When the differential equations are derived in [H3] for (3.1) with fixed w_0, w_1, w_2, w_3, only Corollary 1.3 in [H3] is needed. Corollary 1.3 in [H3] states that the R-submodule of T/J generated by elements of the form $[w_0 \otimes L_{W_1}(-1)^i w_1 \otimes w_2 \otimes w_3]$ and $[w_0 \otimes w_1 \otimes L_{W_2}(-1)^j w_2 \otimes w_3]$ (where we use $[X]$ to denote the coset in T/J containing X) for $i, j \in \mathbb{N}$ is finitely generated. So to derive the differential equations, we need only show that for fixed w_0, w_1, w_2, w_3, this R-submodule of T/J is finitely generated.

To prove that this R-submodule of T/J is finitely generated, we first prove that a larger R-module than this R-submodule of T/J is finitely generated. Given $N_0 \in \mathbb{N}$, let

$$T_{N_0} = R \otimes \left(\coprod_{\Re(n) \leq N_0} (W_0)_{[n]} \right) \otimes W_1 \otimes W_2 \otimes W_3.$$

Then T_{N_0} is an R-submodule of T. Note that elements of the form $\mathcal{D}(u, w_0, w_1, w_2, w_3)$ for $u \in V$, $w_0 \in \coprod_{\Re(n) \leq N_0} (W_0)_{[n]}$, $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$ in general might not be in T_{N_0}. But elements of the form $\mathcal{A}(u, w_0, w_1, w_2, w_3)$, $\mathcal{B}(u, w_0, w_1, w_2, w_3)$, and $\mathcal{C}(u, w_0, w_1, w_2, w_3)$ for the same u, w_0, w_1, w_2, w_3 are always in T_{N_0}. Let J_{N_0} be the R-submodule of T_{N_0} spanned by $\mathcal{A}(u, w_0, w_1, w_2, w_3)$, $\mathcal{B}(u, w_0, w_1, w_2, w_3)$, and $\mathcal{C}(u, w_0, w_1, w_2, w_3)$ for $u \in V$, $w_0 \in \coprod_{\Re(n) \leq N_0} (W_0)_{[n]}$, $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$. Since W_0 is quasi-finite dimensional, $\coprod_{\Re(n) \leq N_0} (W_0)_{[n]}$ is finite-dimensional. This fact and the same proofs as those of Proposition 1.1 and Corollary 1.2 in [H3] shows that T_{N_0}/J_{N_0} is finitely generated.

For fixed given w_0, w_1, w_2, w_3, take $N_0 \in \mathbb{N}$ such that $w_0 \in \coprod_{\Re(n) \leq N_0} (W_0)_{[n]}$. Then T_{N_0}/J_{N_0} is finitely generated. Since R is Noetherian, the R-submodule of T_{N_0}/J_{N_0} generated by elements of the form $[w_0 \otimes L_{W_1}(-1)^i w_1 \otimes w_2 \otimes w_3]$ and $[w_0 \otimes w_1 \otimes L_{W_2}(-1)^j w_2 \otimes w_3]$ for $i, j \in \mathbb{N}$ is also finitely generated.

Now the proof of Theorem 1.4 in [H3] gives the differential equations and the proof Theorem 2.3 in [H3] gives the regularity of the singular points. □

The proof of the following result is the same as the proof of Theorem 3.2 above:

Theorem 3.3. *Let W_i for $i = 1, \dots, n$ be C_1-cofinite grading-restricted generalized V-modules and W_0 and W_{n+1} quasi-finite-dimensional generalized V-modules. Then for any $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1, \dots, w_{(n+1)} \in W_{n+1}$, there exist*

$$a_{k,\,l}(z_1, \dots, z_n) \in \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}, (z_1 - z_2)^{-1}, (z_1 - z_3)^{-1}, \dots, (z_{n-1} - z_n)^{-1}],$$

for $k = 1, \dots, m$ and $l = 1, \dots, n$, such that the following holds: For any lower-bounded generalized V-modules $\widetilde{W}_1, \dots, \widetilde{W}_{n-1}$, and any intertwining operators

$$\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_{n-1}, \mathcal{Y}_n$$

of types

$$\binom{W_0}{W_1 \widetilde{W}_1}, \binom{\widetilde{W}_1}{W_2 \widetilde{W}_2}, \dots, \binom{\widetilde{W}_{n-2}}{W_{n-1} \widetilde{W}_{n-1}}, \binom{\widetilde{W}_{n-1}}{W_n W_{n+1}},$$

respectively, the series

$$\langle w'_{(0)}, \mathcal{Y}_1(w_{(1)}, z_1) \cdots \mathcal{Y}_n(w_{(n)}, z_n) w_{(n+1)} \rangle \tag{3.4}$$

satisfies the expansion of the system of differential equations

$$\frac{\partial^m \varphi}{\partial z_l^m} + \sum_{k=1}^{m} a_{k,\,l}(z_1, \dots, z_n) \frac{\partial^{m-k} \varphi}{\partial z_l^{m-k}} = 0, \quad l = 1, \dots, n$$

on the region $|z_1| > \cdots > |z_n| > 0$. Moreover, for any set of possible singular points of the system

$$\frac{\partial^m \varphi}{\partial z_l^m} + \sum_{k=1}^{m} a_{k,\,l}(z_1, \dots, z_n) \frac{\partial^{m-k} \varphi}{\partial z_l^{m-k}} = 0, \quad l = 1, \dots, n \tag{3.5}$$

such that either $z_i = 0$ or $z_i = \infty$ for some i or $z_i = z_j$ for some $i \neq j$, $a_{k,\,l}(z_1, \dots, z_n)$ can be chosen for $k = 1, \dots, m$ and $l = 1, \dots, n$ so that these singular points are regular.

For the next two results, we need the notion of surjectivity of an intertwining operator. Let W_1, W_2 and W_3 be weak V-modules. An intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$ is said to be surjective if W_3 is spanned by the coefficients of $\mathcal{Y}(w_1, x)w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$. If for a weak V-module W_3, there is an surjective intertwining operator of type $\binom{W_3}{W_1 W_2}$, we say that W_3 is a *weak surjective product of W_1 and W_2*. If W_3 is generalized V-module or grading-restricted generalized V-module or other classes of V-modules, we also use the terms *generalized surjective product of W_1 and W_2*, *grading-restricted generalized surjective product of W_1 and W_2* and so on.

Proposition 3.4. *Let W_1 and W_2 be generalized V-modules and W_3 a weak V-module. If W_3 is a weak surjective product of W_1 and W_2, then W_3 is also a generalized V-module.*

Proof. For $w_1 \in W_1$ and $w_2 \in W_2$, we have

$$\mathcal{Y}(w_1, x)w_2 = \sum_{k=0}^{K} \sum_{n \in \mathbb{C}} \mathcal{Y}_{n,k}(w_1) w_2 x^{-n-1} (\log x)^k,$$

where $\mathcal{Y}_{n,k}(w_1) \in \mathrm{Hom}(W_2, W_3)$. We now take w_1 and w_2 to be homogeneous. We have $L_{W_1}(0) = L_{W_1}(0)_S + L_{W_1}(0)_N$ and $L_{W_2}(0) = L_{W_2}(0)_S + L_{W_2}(0)_N$, where $L_{W_1}(0)_S$ and $L_{W_2}(0)_S$ are the semisimple parts of $L_{W_1}(0)$ and $L_{W_2}(0)$, respectively, and $L_{W_1}(0)_N$ and $L_{W_2}(0)_N$ are the nilpotent parts of $L_{W_1}(0)$ and $L_{W_2}(0)$, respectively. Then $L_{W_1}(0)_S w_1 = (\mathrm{wt}\, w_1) w_1$ and $L_{W_2}(0)_S w_2 = (\mathrm{wt}\, w_2) w_2$.

The commutator formula between $L(0)$ and $\mathcal{Y}$ can be written as

$$L_{W_3}(0)\mathcal{Y}(w_1, x)w_2 = x\frac{d}{dx}\mathcal{Y}(w_1, x)w_2 + \mathcal{Y}(L_{W_1}(0)w_1, x)w_2 + \mathcal{Y}(w_1, x)L_{W_2}(0)w_2.$$

Taking the coefficients of $x^{-n-1}(\log x)^k$ on both sides, using the fact that w_1 and w_2 are homogeneous, and then reorganizing the terms, we obtain

$$\begin{aligned}(L_{W_3}(0) &- (\mathrm{wt}\, w_1 - n - 1 + \mathrm{wt}\, w_2))\mathcal{Y}_{n,k}(w_1)w_2 \\ &= \mathcal{Y}_{n,k}(L_{W_1}(0)_N w_1)w_2 + \mathcal{Y}_{n,k}(w_1)L_{W_2}(0)_N w_2 + (k+1)\mathcal{Y}_{n,k+1}(w_1)w_2. \end{aligned} \tag{3.6}$$

Using (3.6) repeatedly, we see that

$$(L_{W_3}(0) - (\mathrm{wt}\, w_1 - n - 1 + \mathrm{wt}\, w_2))^J \mathcal{Y}_{n,k}(w_1)w_2$$

for $J \in \mathbb{Z}_+$ is a linear comnbination of elements of the form

$$\mathcal{Y}_{n,k+l}(L_{W_1}(0)_N^m w_1)L_{W_2}(0)_N^n w_2 \tag{3.7}$$

for $m, n, l \in \mathbb{N}$ such that $m + n + l = J$. When J is sufficiently large, at least one of m, n, l must be sufficiently large. Since $L_{W_1}(0)_N$ and $L_{W_2}(0)_N$ are nilpotent and $\mathcal{Y}_{n,k}(w_1)w_2 = 0$ for $k > K$, (3.7) must be 0 when J is sufficiently large. Hence we obtain

$$(L_{W_3}(0) - (\mathrm{wt}\, w_1 - n - 1 + \mathrm{wt}\, w_2))^J \mathcal{Y}_{n,k}(w_1)w_2 = 0$$

for sufficiently large $J \in \mathbb{Z}_+$. Thus we see that $\mathcal{Y}_{n,k}(w_1)w_2$ is a generalized eigenvector for $L_{W_3}(0)$ with eigenvalue $\mathrm{wt}\, w_1 - n - 1 + \mathrm{wt}\, w_2$. Since $\mathcal{Y}$ is surjective, $\mathcal{Y}_{n,k}(w_1)w_2$ for $w_1 \in W_1$, $w_2 \in W_2$, $n \in \mathbb{C}$ and $0 \leq k \leq K$ span W_3. So W_3 is a generalized V-module. □

The next result is a generalization of the inequality of Nahm in [N]. This generalization is stronger than a key result obtained later by Miyamoto in [Mi]. Here by generalization, we mean that in the theorem below, the inequality holds for any weak surjective product of W_1 and W_2. We do not need the existence of any fusion or tensor product of W_1 and W_2 and the surjective product of W_1 and W_2 can be a weak V-module. In fact, this generalization was essentially formulated by McRae and Sopin in [MS] except that the surjective product of W_1 and W_2 is probably a stronger type of V-modules than a weak V-module. It was observed also in [MS] that the inequality can be obtained using the number of independent solutions of the differential equations in [Mi]. Here we give a full proof of this inequality.

Theorem 3.5. *Let W_1, W_2 be $\mathbb{N}$-gradable weak V-modules. Then for a weak surjective product W_3 of W_1 and W_2,*

$$\dim(W_3/C_1(W_3)) \leq \dim(W_1/C_1(W_1)) \dim(W_2/C_1(W_2)). \tag{3.8}$$

Proof. In the case that at least one of $\dim(W_1/C_1(W_1))$ and $\dim(W_2/C_1(W_2))$ is ∞, (3.8) holds. So we need only prove (3.8) in the case that $\dim W_1/C_1(W_1), \dim W_2/C_1(W_2) < \infty$, that is, in the case that W_1 and W_2 are C_1-cofinite $\mathbb{N}$-gradable weak V-modules. In particular, conclusions of Proposition 3.1 hold for W_1 and W_2.

Let M_1 and M_2 be finite-dimensional graded subspaces of W_1 and W_2, respectively, such that $W_1 = C_1(W_1) \oplus M_1$ and $W_2 = C_1(W_2) \oplus M_2$. Then $\dim(W_1/C_1(W_1)) = \dim M_1$ and $\dim(W_2/C_1(W_2)) = \dim M_2$.

By Proposition 3.4, W_3 is a generalized V-module. In particular, W_3 is $\mathbb{C}$-graded and we have the contragredient W_3' of W_3. Let M_3 be a graded subspace of W_3 such that $W_3 = C_1(W_3) \oplus M_3$. Then $W_3/C_1(W_3)$ is isomorphic to M_3. We need only prove $\dim M_3 \leq \dim M_1 \dim M_2$.

Let

$$C_1(W_3)^\perp = \{w_3' \in W_3' \mid \langle w_3', C_1(W_3)\rangle = 0\} \subset W_3'.$$

and let M_3' be the graded dual of M_3 with respect to the $\mathbb{C}$-grading induced from the one on W_3. We define a linear map $r : C_1(W_3)^\perp \to M_3'$ to be the restriction map sending elements of $C_1(W_3)^\perp \subset W_3'$ to their restrictions to M_3, that is, $(r(w_3'))(w_3) = \langle w_3', w_3\rangle$ for $w_3' \in C_1(W_3)^\perp$ and $w_3 \in M_3$. If $r(w_3') = 0$ for $w_3' \in C_1(W_3)^\perp$, then $\langle w_3', w_3\rangle = 0$ for all $w_3 \in M_3$. But by definition, $\langle w_3', C_1(W_3)\rangle = 0$. Since $W_3 = C_1(W_3) \oplus M_3$, we obtain $\langle w_3', w_3\rangle = 0$ for all $w_3 \in W_3$. Thus $w_3' = 0$, proving that r is injective. Given $w_3' \in M_3'$, we extend it to an element $\bar{w}_3' \in W_3'$ by $\langle \bar{w}_3', \tilde{w}_3 + w_3\rangle = \langle w_3', w_3\rangle$ for $\tilde{w}_3 \in C_1(W_3)$ and $w_3 \in M_3$. By definition, $\bar{w}_3' \in C_1(W_3)^\perp$ and $r(\bar{w}_3') = w_3'$, proving r is surjective. We have proved that r is a linear isomorphism. Therefore we need only prove that $\dim C_1(W_3)^\perp \leq \dim M_1 \dim M_2$.

Let $\mathcal{Y}$ be a surjective intertwining operator of type $\binom{W_3}{W_1 W_2}$. We have proved that for homogeneous $w_1 \in W_1$ and $w_2 \in W_2$,

$$\mathrm{wt}\, \mathcal{Y}_{n,k}(w_1)w_2 = \mathrm{wt}\, w_1 - n - 1 + \mathrm{wt}\, w_2.$$

Then for homogeneous $w_3' \in W_3'$,

$$\langle w_3', \mathcal{Y}(w_1, x)w_2\rangle = \sum_{k=0}^{K} \langle w_3', \mathcal{Y}_{\mathrm{wt}\, w_1 + \mathrm{wt}\, w_2 - \mathrm{wt}\, w_3' - 1, k}(w_1)w_2\rangle x^{-\mathrm{wt}\, w_1 - \mathrm{wt}\, w_2 + \mathrm{wt}\, w_3'} (\log x)^k.$$

Fix $z \in \mathbb{C}^\times$. We use $\log z$ to denote $\log |z| + i \arg z$, where $0 \leq \arg z < 2\pi$. Then

$$\langle w_3', \mathcal{Y}(w_1, z)w_2\rangle = \sum_{k=0}^{K} \langle w_3', \mathcal{Y}_{\mathrm{wt}\, w_1 + \mathrm{wt}\, w_2 - \mathrm{wt}\, w_3' - 1, k}(w_1)w_2\rangle e^{(-\mathrm{wt}\, w_1 - \mathrm{wt}\, w_2 + \mathrm{wt}\, w_3') \log z} (\log z)^k$$

is well defined. We define a linear map $f : C_1(W_3)^\perp \to (M_1 \otimes M_2)^*$ by

$$(f(w_3'))(w_1 \otimes w_2) = \langle w_3', \mathcal{Y}(w_1, z)w_2\rangle$$

for $w_1 \in M_1$, $w_2 \in M_2$ and $w_3' \in C_1(W_3)^\perp$. To prove $\dim C_1(W_3)^\perp \leq \dim M_1 \dim M_2 = \dim(M_1 \otimes M_2)^*$, we need only prove that f is injective.

Assume that $f(w_3') = 0$ for an element $w_3' \in C_1(W_3)^\perp$. Then by the definition of f, for $w_1 \in M_1$, $w_2 \in M_2$,

$$\langle w_3', \mathcal{Y}(w_1, z)w_2\rangle = 0 \tag{3.9}$$

We now prove (3.9) for all $w_1 \in W_1$, $w_2 \in W_2$. By Conclusion 3 of Proposition 3.1, there exist $h_1^{(1)}, \dots, h_p^{(1)}, h_2^{(1)}, \dots, h_q^{(2)} \in \mathbb{C}$ such that $W_1 = \coprod_{i=1}^p \coprod_{n \in h_i^{(1)} + \mathbb{N}} (W_1)_{[n]}$ and $W_2 = \coprod_{j=1}^q \coprod_{n \in h_j^{(2)} + \mathbb{N}} (W_2)_{[n]}$. For $w_1 \in \coprod_{i=1}^p (W_1)_{[h_i^{(1)} + n_1]}$ and $w_2 \in \coprod_{j=1}^q (W_2)_{[h_j^{(2)} + n_2]}$, we use induction on $n_1 + n_2$ to prove (3.9). When $n_1 + n_2 = 0$, $n_1 = n_2 = 0$. We have $w_1 \in \coprod_{i=1}^p (W_1)_{[h_i^{(1)}]}$ and $w_2 \in \coprod_{j=1}^q (W_2)_{[h_j^{(2)}]}$. We can take w_1 and w_2 to be homogeneous. Then there are i and j such that $w_1 \in (W_1)_{[h_i^{(1)}]}$ and $w_2 \in (W_2)_{[h_j^{(2)}]}$. Since $W_1 = C_1(W_1) \oplus M_1$ and $W_2 = C_1(W_1) \oplus M_2$, there exist homogeneous $u^{(k)}, v^{(l)} \in V$, $\tilde{w}_1^{(k)} \in W_1$, $\tilde{w}_2^{(l)} \in W_2$, for $k = 1, \dots, s$, $l = 1, \dots, t$, $\tilde{w}_1 \in M_1$, and $\tilde{w}_2 \in M_2$ such that $w_1 = \sum_{k=1}^s u_{-1}^{(k)} \tilde{w}_1^{(k)} + \tilde{w}_1$ and $w_2 = \sum_{l=1}^t v_{-1}^{(l)} \tilde{w}_2^{(l)} + \tilde{w}_2$. But for $k = 1, \dots, s$,

$$\Re(\mathrm{wt}\, \tilde{w}_1^{(k)}) < \mathrm{wt}\, u^{(k)} + \Re(\mathrm{wt}\, \tilde{w}_1^{(k)}) = \Re(\mathrm{wt}\, u_{-1}^{(k)} \tilde{w}_1^{(k)}) = \Re(\mathrm{wt}\, w_1) = \Re(h_i^{(1)}).$$

So $u_{-1}^{(k)} \tilde{w}_1^{(k)} = 0$ and $w_1 = \tilde{w}_1 \in M_1$. Similarly, $w_2 \in M_2$. In this case, (3.9) is true.

Assume that when $n_1 + n_2 < m$, (3.9) is true. In the case $n_1 + n_2 = m$, we still take $w_1 \in (W_1)_{[h_i^{(1)} + n_1]}$ and still have $w_1 = \sum_{k=1}^s u_{-1}^{(k)} \tilde{w}_1^{(k)} + \tilde{w}_1$ for homogeneous $u^{(k)} \in V$, $\tilde{w}_1^{(k)} \in W_1$ for $k = 1, \dots, s$, and $\tilde{w}_1 \in M_1$. In this case,

$$\Re(\mathrm{wt}\, \tilde{w}_1^{(k)}) < \mathrm{wt}\, u^{(k)} + \Re(\mathrm{wt}\, \tilde{w}_1^{(k)}) = \Re(\mathrm{wt}\, u_{-1}^{(k)} \tilde{w}_1^{(k)}) = \Re(\mathrm{wt}\, w_1) = \Re(h_i + n_1).$$

So $\tilde{w}_1^{(k)} \in \coprod_{m_1 < n_1} (W_1)_{[h_i^{(1)} + m_1]}$ Similarly, we still take $w_2 \in (W_2)_{[h_j^{(2)} + n_2]}$ and still have $w_2 = \sum_{l=1}^t v_{-1}^{(l)} \tilde{w}_2^{(l)} + \tilde{w}_2$ for homogeneous $v^{(l)} \in V$, $\tilde{w}_2^{(l)} \in W_1$ for $l = 1, \dots, t$, and $\tilde{w}_2 \in M_2$. Then $\tilde{w}_2^{(l)} \in \coprod_{m_2 < n_2} (W_2)_{[h_j^{(2)} + m_2]}$. Using the Jacobi identity and the fact that $\langle w_3', C_1(W_3)\rangle = 0$, we see that $\langle w_3', \mathcal{Y}(u_{-1}^{(k)} \tilde{w}_1^{(k)}, x)w_2\rangle$ is a linear combination of formal series in x of the form $\langle w_3', \mathcal{Y}(\tilde{w}_1^{(k)}, x)\hat{w}_2\rangle$, where $\hat{w}_2 \in \coprod_{\hat{n}_2 \le n_2} (W_2)_{[h_j^{(2)} + \hat{n}_2]}$ with Laurent polynomial in x as coefficients. Since $\tilde{w}_1^{(k)} \in \coprod_{m_1 < n_1} (W_1)_{[h_i^{(1)} + m_1]}$ and $\hat{w}_2 \in \coprod_{\hat{n}_2 \le n_2} (W_2)_{[h_j^{(2)} + \hat{n}_2]}$ and $m_1 + \hat{n}_2 < n_1 + n_2$, by induction assumption, $\langle w_3', \mathcal{Y}(\tilde{w}_1^{(k)}, z)\hat{w}_2\rangle = 0$. So $\langle w_3', \mathcal{Y}(u_{-1}^{(k)} \tilde{w}_1^{(k)}, z)w_2\rangle = 0$. Similarly, $\langle w_3', \mathcal{Y}(\tilde{w}_1, x)v_{-1}^{(l)} \tilde{w}_2^{(l)}\rangle$ is a linear combination of formal series in x of the form $\langle w_3', \mathcal{Y}(\hat{w}_1, x)\tilde{w}_2^{(l)}\rangle$, where $\hat{w}_1 \in \coprod_{\hat{n}_1 \le n_1} (W_1)_{[h_i^{(1)} + \hat{n}_1]}$ with Laurent polynomials in x as coefficients. Since $\hat{w}_1 \in \coprod_{\hat{n}_1 \le n_1} (W_1)_{[h_i^{(1)} + \hat{n}_1]}$ and $\tilde{w}_2^{(l)} \in \coprod_{m_2 < n_2} (W_2)_{[h_j^{(2)} + m_2]}$ and $\hat{n}_1 + m_2 < n_1 + n_2$, by induction assumption, $\langle w_3', \mathcal{Y}(\hat{w}_1, z)\tilde{w}_2^{(l)}\rangle = 0$. So $\langle w_3', \mathcal{Y}(\tilde{w}_1, x)v_{-1}^{(l)} \tilde{w}_2^{(l)}\rangle = 0$. Thus

$$\begin{aligned}
\langle w_3', \mathcal{Y}(w_1, z)w_2\rangle &= \sum_{k=1}^s \langle w_3', \mathcal{Y}(u_{-1}^{(k)} \tilde{w}_1^{(k)}, z)w_2\rangle + \langle w_3', \mathcal{Y}(\tilde{w}_1, z)w_2\rangle \\
&= \langle w_3', \mathcal{Y}(\tilde{w}_1, z)w_2\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=1}^{t} \langle w_3', \mathcal{Y}(\tilde{w}_1, z) v_{-1}^{(l)} \tilde{w}_2^{(l)} \rangle + \langle w_3', \mathcal{Y}(\tilde{w}_1, z)\tilde{w}_2 \rangle \\
&= \langle w_3', \mathcal{Y}(\tilde{w}_1, z)\tilde{w}_2 \rangle \\
&= 0,
\end{aligned}
$$

proving (3.9) for all $w_1 \in W_1$ and $w_2 \in W_2$.

Using the $L(-1)$-derivative property for $\mathcal{Y}$, we have

$$\langle w_3', \mathcal{Y}(w_1, \xi)w_2 \rangle = \langle w_3', \mathcal{Y}(e^{(\xi - z)L_{W_1}(-1)}w_1, z)w_2 \rangle = 0$$

on the region $|\xi - z| < |z|$. But $\langle w_3', \mathcal{Y}(w_1, \xi)w_2 \rangle$ is an analytic function of ξ. This analytic function is equal to 0 on a region means that it is equal to 0 on its domain. So we obtain $\langle w_3', \mathcal{Y}(w_1, \xi)w_2 \rangle = 0$ on the region $\xi \neq 0$. Thus $\langle w_3', \mathcal{Y}_{n,k}(w_1)w_2 \rangle = 0$ for $w_1 \in W_1$, $w_2 \in W_2$, $n \in \mathbb{C}$, $k = 1, \dots, K$. Since $\mathcal{Y}$ is surjective, we must have $w_3' = 0$, proving the injectivity of f. □

Theorem 3.5 has the following immediate consequence:

Corollary 3.6. *Let W_1 and W_2 be C_1-cofinite grading-restricted generalized V-modules (or equivalenetly, C_1-cofinite $\mathbb{N}$-gradable weak V-modules). Then a weak surjective product of W_1 and W_2 is a C_1-cofinite generalized V-module. In particular, a lower-bounded generalized surjective product of W_1 and W_2 is a C_1-cofinite grading-restricted generalized V-module.*

Proof. Since W_1 and W_2 are generalized V-modules, by Proposition 3.4, a weak surjective product W_3 of W_1 and W_2 is a generalized V-module. Since W_1 and W_2 are C_1-cofinite, by (3.8) in Theorem 3.5, $\dim(W_3/C_3(W_3)) \leq \dim(W_1/C_1(W_1))\dim(W_2/C_1(W_2)) < \infty$. So W_3 is a C_1-cofinite generalized V-module.

In the case that W_3 is lower-bounded generalized surjective product of W_1 and W_2, since W_3 is also C_1-cofinite, it is grading-restricted (in fact quasi-finite-dimensional) by Conclusion 3 of Proposition 3.1. □

Remark 3.7. Miyamoto proved the following result in [Mi]: Let W_1 and W_2 be C_1-cofinite $\mathbb{N}$-gradable weak V-modules. Then there exists $d_{W_1,W_2} \in \mathbb{N}$ such that for any $\mathbb{N}$-gradable weak surjective product W_3 of W_1 and W_2, $\dim(W_3/C_1(W_3)) \leq d_{W_1,W_2}$. We can obtain this result from Theorem 3.5 by taking $d_{W_1,W_2} = \dim(W_1/C_1(W_1))\dim(W_2/C_1(W_2))$. Also in this result in [Mi], W_3 is an $\mathbb{N}$-gradable weak V-nodule while in Theorem 3.5, W_3 is a weak V-module. In fact, Creutzig, McRae and Yang generalized this result in [Mi] to the case that W_3 is a generalized V-module in Corollary 2.14 in [CMY2]. Combining Proposition 3.4 above and Corollary 2.14 in [CMY2], this result in [Mi] is immediately generalized to the case that W_3 is a weak V-module. Also, the proof of this result in [Mi] uses differential equations, which are equivalent to the special case of the differential equations derived in [H3] for one intertwining operator. The proof of Corollary 2.14 in [CMY2] uses this result in [Mi] and, in particular, also uses differential equations implicitly. The proof of Theorem 3.5 does not use differential equations at all. On the other hand, as mentioned above, McRae and Sopin observed in [MS] that the inequality of Nahm can be obtained using the number of independent solutions of the differential equations in [Mi].

4 Proof of the main theorem

In this section, we take $\mathcal{C}$ to be the category of C_1-cofinite grading-restricted generalized V-modules. We verify in this section that $\mathcal{C}$ satisfies Assumptions 1, 4, 6, 8, 9, 11, 12 given in Section 2. Then by Theorem 2.8, we obtain Theorem 1.2.

For this category $\mathcal{C}$, Assumption 1 is by definition satisfied. Since a grading-restricted Möbius vertex algebra is C_1-cofinite, Assumption 4 is also satisfied. We still need to verify Assumptions 6, 8, 9, 11, and 12.

We first verify Assumptions 6 and 8.

Proposition 4.1. *The convergence and extension property for products of intertwining operators in $\mathcal{C}$ holds. Moreover, the products of more than two intertwining operators are absolutely convergent in the corresponding region and can be analytically extended to multi-valued analytic functions defined on the region where the complex variables in the intertwining operators are not equal* 0 *and each other.*

Proof. The first conclusion (Assumption 8) follows from Theorem 3.2. The proof is the same as the proof of Theorem 3.5 in [H3] in the case that the intertwining operators do not contain logarithms of the variables and the observation in the proof of Theorem 11.8 in [HLZ8] that the proof of Theorem 3.5 in [H3] still works in the case of logarithmic intertwining operators.

The second conclusion (Assumption 6) follows immediately from Theorem 3.3 by the theory of differential equations of regular singular points. □

Remark 4.2. In fact, the main difficult part of the proof of Theorem 3.5 in [H3] is the proof of the result that in the semisimple case, there is no logarithm after the products of intertwining operators are analytically extended to the region for the iterates of intertwining operators and expanded as series in powers and logarithms of the variables. So the proof in our logarithmic case is in fact simpler.

We now verify Assumption 9 using Proposition 3.1.

Proposition 4.3. *For any object W in $\mathcal{C}$, the weights of homogeneous elements of W are congruent to finitely many complex numbers modulo $\mathbb{Z}$ and there exists $K \in \mathbb{Z}_+$ such that $L_W(0)_N^K = 0$ where $L_W(0)_N$ is the nilpotent part of $L_W(0)$.*

Proof. The first part of this result follows immediately from concludion 3 of Proposition 3.1. So we need to prove only the second part.

By conclusion 2 of Proposition 3.1, there is a finite-dimensional subspace M of W such that W is spanned by elements of the form $v_{-1}^{(1)} \cdots v_{-1}^{(i)} w$ for $i \in \mathbb{N}$, $v^{(1)}, \dots, v^{(i)} \in V$ and $w \in M$. Let $w_1, \dots, w_k$ be a basis of M. Then there is $K_i \in \mathbb{Z}_+$ for $i = 1, \dots, k$ such that $L_W(0)_N^{K_i} w_i = 0$. Let $K = \max(K_1, \dots, K_k)$. We now show that $L_W(0)_N^K = 0$.

For any element $w \in M$, by the definition of K, we have $L_W(0)_N^K w = 0$. To prove $L_W(0)_N^K w = 0$ for general $w \in W$, we first notice that $L_W(0)_N$ and the vertex operator $Y_W(v, x)$ for $v \in V$ commute. In fact, this is the special case of (4.2) in [H6] when the twisted module is an untwisted V-module (twisted by the identity on V). In the untwisted case, the vertex operator $Y_W(v,x)$ does not contain $\log x$ and thus the commutator of $L_W(0)_N$ and $Y_W(v,x)$ is 0. In particular, $L_W(0)_N$ and v_{-1} commute. Thus for an element $v^{(1)}_{-1}\cdots v^{(i)}_{-1}w$ for $i\in\mathbb{N}$, $v^{(1)},\dots,v^{(i)}\in V$ and $w\in M$, we have

$$L_W(0)_N^K(v^{(1)}_{-1}\cdots v^{(i)}_{-1}w)=v^{(1)}_{-1}\cdots v^{(i)}_{-1}L_W(0)_N^Kw=0.$$

Since W is spanned by such elements, we obtain $L_W(0)_N^K=0$. □

For Assumption 11, we need only verify that $\mathcal{C}$ is closed under $P(z)$-tensor products for some $z\in\mathbb{C}^\times$. To verify this part of Assumption 11 and Assumption 12, we prove a general result on grading-restricted generalized surjective product of W_1 and W_2 and lower-bounded generalized V-modules contained in $(W_1\otimes W_2)^*$. Then Assumptions 11 and 12 are immediate consequences. This result can also be proved by modifying the relevant results and their proofs in [CJORY], [CY], [Mc] and [CMOY].

Theorem 4.4. *Let W_1 and W_2 be C_1-cofinite grading-restricted generalized V-modules.*

1. *There are $h_1,\dots,h_k\in\mathbb{C}$, depending only on W_1 and W_2, such that for any grading-restricted generalized surjective product W_0 of W_1 and W_2, we have*

$$W_0=\coprod_{i=1}^k\coprod_{n\in\mathbb{N}}(W_0)_{[h_i+n]}.$$

2. *Let W be a lower-bounded generalized V-module contained in $(W_1\otimes W_2)^*$ (equipped with the action $Y'_{P(z)}(\cdot,x)$ of V and the corresponding actions of $L(-1)$, $L(0)$ and $L(1)$ on $(W_1\otimes W_2)^*$). Then W' is grading-restricted generalized surjective product of W_1 and W_2. In particular, $W\subset W_1\boxdot_{P(z)}W_1$,*

$$W=\coprod_{i=1}^k\coprod_{n\in\mathbb{N}}W_{[h_i+n]},\tag{4.10}$$

where $h_1,\dots,h_k$ are complex numbers given above, and W' is C_1-cofinite,

Proof. We consider the set of $\dim(W_0/C_1(W_0))$ for all grading-restricted generalized surjective product W_0 of W_1 and W_2. Then by Theorem 3.5, this set of nonnegative integers is bounded from above by $\dim(W_1/C_1(W_1))\dim(W_2/C_1(W_2))$. Hence there must be a maximum of this set. Let $W_0^{\max}$ be a grading-restricted generalized surjective product of W_1 and W_2 such that $\dim W_0/C_1(W_0)$ is the maximum of the set. Then by Proposition 3.1, there are $h_1,\dots,h_k\in\mathbb{C}$ such that

$$W_0^{\max}=\coprod_{i=1}^k\coprod_{n\in\mathbb{N}}(W_0^{\max})_{[h_i+n]}.$$

Given any grading-restricted generalized surjective product W_0 of W_1 and W_2, we have a surjective intertwining operator $\mathcal{Y}$ of type $\binom{W_0}{W_1W_2}$. For any fixed $z \in \mathbb{C}^\times$, we have a $P(z)$-intertwining map $I_{\mathcal{Y},0} = \mathcal{Y}(\cdot, z)\cdot$ of type $\binom{W_0}{W_1W_2}$. The $P(z)$-intertwining map $I_{\mathcal{Y},0}$ gives a V-module map $I'_{\mathcal{Y},0}$ from W_0' to the generalized V-module $W_1 \boxbackslash_{P(z)} W_2$ (see Definition 5.3.1 in [HLZ5]). Since the intertwining operator $\mathcal{Y}$ is surjective, this V-module map is injective. The image of W_0' under this V-module map is a grading-restricted generalized V-submdoule of $W_1 \boxbackslash_{P(z)} W_2$. This is also true for $W_0^{\max}$. Let W be the sum of the image of W_0' under the V-module map from W_0' to $W_1 \boxbackslash_{P(z)} W_2$ and the image of $(W_0^{\max})'$ under the V-module map from $(W_0^{\max})'$ to $W_1 \boxbackslash_{P(z)} W_2$. Then W is also a grading-restricted generalized V-submodule of $W_1 \boxbackslash_{P(z)} W_2$. Let $J : W \to W_1 \boxbackslash_{P(z)} W_2$ be the inclusion map. Then $J' : W_1 \otimes W_2 \to \overline{W'}$ is a $P(z)$-intertwining map (see Section 5 of [HLZ5], especially Notation 5.25). The $P(z)$-intertwining map J' gives a surjective intertwining operator $\mathcal{Y}_{J',0}$ (see (4.18) in [HLZ4]) of type $\binom{W'}{W_1W_2}$. Since W is grading-restricted, W' is also grading-restricted. By Corollary 3.6, W' is a C_1-cofinite grading-restricted generalized V-module and $\dim W'/C_1(W') \leq \dim W_0^{\max}/C_1(W_0^{\max})$.

We now prove

$$W' = \coprod_{i=1}^{k} \coprod_{n \in \mathbb{N}} (W')_{[h_i+n]}.$$

By definition, we have an injective V-module map from $(W^{\max})'$ to W. Its adjoint is a surjective V-module map $f : W' \to W_0^{\max}$. Since f is a V-module map, $f(C_1(W')) \subset C_1(W_0^{\max})$. Then the map f induces a surjective linear map $\bar{f} : W'/C_1(W') \to W_0^{\max}/C_1(W_0^{\max})$. In particular, we have $\dim(W'/C_1(W')) \geq \dim(W_0^{\max}/C_1(W_0^{\max}))$. But we already have $\dim(W'/C_1(W')) \leq \dim(W_0^{\max}/C_1(W_0^{\max}))$. So we obtain

$$\dim(W'/C_1(W')) = \dim(W_0^{\max}/C_1(W_0^{\max})).$$

Thus $\bar{f}$ is injective and $\ker \bar{f} = 0$. Since $\bar{f}$ is induced from f, we obtain $\ker f \subset C_1(W')$.

Since W' is a C_1-cofinite grading-restricted generalized V-module, by Proposition 3.1, there are $\tilde{h}_1, \dots, \tilde{h}_l \in \mathbb{C}$ such that

$$W' = \coprod_{i=1}^{l} \coprod_{n \in \mathbb{N}} (W')_{[\tilde{h}_i+n]},$$

where $(W')_{[\tilde{h}_i+n]}$ for $n \in \mathbb{N}$ are finite-dimensional. Moreover, we can always find such $\tilde{h}_1, \dots, \tilde{h}_l \in \mathbb{C}$ such that $(W')_{[\tilde{h}_i]} \neq 0$ for $i = 1, \dots, l$. Note that $(W')_{[\tilde{h}_i]}$ for $i = 1, \dots, l$ cannot be contained in $C_1(W')$ and thus cannot be in $\ker f$. So for each i, nonzero elements of $(W')_{[\tilde{h}_i]}$ are mapped to nonzero homogeneous elements of $W_0^{\max}$. Hence there exist $1 \leq j \leq k$ and $n \in \mathbb{N}$ such that $\tilde{h}_i = h_j + n$. Hence we obtain

$$W' = \coprod_{i=1}^{k} \coprod_{n \in \mathbb{N}} (W')_{[h_i+n]}.$$

As a consequence, the contragredient W of W' can also be written as

$$W = \coprod_{i=1}^{k} \coprod_{n \in \mathbb{N}} W_{[h_i+n]}.$$

Then as a V-submodule of W, W_0' can be written as

$$W_0' = \coprod_{i=1}^{k} \coprod_{n \in \mathbb{N}} (W_0')_{[h_i+n]}.$$

Thus W_0 can be written as

$$W_0 = \coprod_{i=1}^{k} \coprod_{n \in \mathbb{N}} (W_0)_{[h_i+n]}.$$

We now prove the second part of the theorem. Let $i : W \to (W_1 \otimes W_2)^*$ be the inclusion map. Then we have a $P(z)$-intertwining map i' of type $\binom{W'}{W_1 W_2}$ (as in the proof of Theorem 4.4, see Section 5 of [HLZ5]). If W is grading-restricted, then W' is also grading-restricted and, as in the proof of the first part above (see (4.18) in [HLZ4]), the $P(z)$-intertwining map i' gives a surjective intertwining operator $\mathcal{Y}_{i',0}$ of type $\binom{W'}{W_1 W_2}$. Then W' is a grading-restricted generalized surjective product of W_1 and W_2. So to prove the second part of the theorem, we need only prove that W is grading-restricted.

Let W_0 be the image of i' (meaning the generalized V-submodule of W' spanned by homogeneous components of elements of the form $i'(w_1 \otimes w_2)$ for $w_1 \in W_1$ and $w_2 \in W_2$). Then we obtain an $P(z)$-intertwining map I_0 of type $\binom{W_0}{W_1 W_2}$. This $P(z)$-intertwining map I_0 in turn gives a surjective intertwining operator $\mathcal{Y}_{I_0,0}$ of type $\binom{W_0}{W_1 W_2}$ (as in the proof of the first part above, see (4.18) in [HLZ4]). Since W_0 is a generalized V-submodule of the lower-bounded generalized V-module W', it is also lower-bounded. By Corollary 3.6 and Proposition 3.1, W_0 is a C_1-cofinite grading-restricted generalized V-module. Then the image $I_0'(W_0')$ of $I_0' : W_0' \to (W_1 \otimes W_2)^*$ is in $W_1 \boxdot_{P(z)} W_2$. If $I'(w_0') = 0$ for $w_0' \in W_0'$, we have

$$\langle w_0', I_0(w_1 \otimes w_2) \rangle = (I'(w_0'))(w_1 \otimes w_2) = 0$$

for all $w_1 \in W_1$ and $w_2 \in W_2$. But W_0 is the image of I_0. So we obtain $w_0' = 0$, showing that I_0' is injective. Then $I_0'(W_0')$ is equivalent to W_0' as a generalized V-module. In particular, $I_0'(W_0')$ is a grading-restricted generalized V-submodule of $W_1 \boxdot_{P(z)} W_2$ whose contragredient is C_1-cofinite. Let $\chi : W_0 \to W'$ be the inclusion map. Then by definition, we have $i' = \chi \circ I_0$. So we obtain $i'' = I_0' \circ \chi'$, where $i'' : W'' \to (W_1 \otimes W_2)^*$ is the adjoint of i'. Since W can be embedded into W'', we can identify W with a generalized V-submodule of W''. Hence we have $i = I_0' \circ \chi'|_W$, where $\chi'|_W$ is the restriction of χ' to W. Thus $W = i(W) = I_0'(\chi'(W)) \subset I_0'(W_0')$. Since $I_0'(W_0')$ is grading restricted, W is also grading-restricted. □

It is now easy to verify that $\mathcal{C}$ is closed under $P(z)$-tensor products for any $z \in \mathbb{C}^\times$ and thus Assumption 11 is satisfied.

Corollary 4.5. *Let W_1 and W_2 be C_1-cofinite grading-restricted generalized V-modules. Then the contragredient $W_1 \boxtimes_{P(z)} W_2$ of $W_1 \boxbackslash_{P(z)} W_2$ is a C_1-cofinite grading-restricted generalized V-module. In particular, $\mathcal{C}$ is closed under the $P(z)$-tensor product.*

Proof. By Proposition 2.5, $W_1 \boxbackslash_{P(z)} W_2$ is the sum of grading-restricted generalized V-modules contained in $(W_1 \otimes W_2)^*$ such that their contragredient are objects of $\mathcal{C}$. By the second part of Theorem 4.4, these grading-restricted generalized V-modules are of the form

$$W = \coprod_{i=1}^{k} \coprod_{n \in \mathbb{N}} (W)_{[h_i+n]}.$$

Thus as a sum of such generalized V-modules, $W_1 \boxbackslash_{P(z)} W_2$ is lower-bounded. Then by the second part of Theorem 4.4 again, $W_1 \boxbackslash_{P(z)} W_2$ itself is also grading restricted and its contragredient $W_1 \boxtimes_{P(z)} W_2$ is C_1-cofinite. □

We see that Assumption 12 is also satisfied.

Corollary 4.6. *Let W_1 and W_2 be C_1-cofinite grading-restricted generalized V-modules. Let $\lambda \in (W_1 \otimes W_2)^*$ be a generalized eigenvector of $L_{P(z)}(0)$ satisfying the compatibility condition. Assuming that the generalized V-module W_λ generated by λ is lower bounded. Then W_λ is grading-restricted and W'_λ is C_1-cofinite.*

Proof. This result is the special case of the second part of Theorem 4.4 with $W = W_\lambda$ □

Proof of Theorem 1.2. We have verified Assumptions 1, 4, 6, 8, 9, 11, 12 for the category $\mathcal{C}$ of C_1-cofinite grading-restricted generalized V-modules. By Theorem 2.8, we obtain the conclusion of Theorem 1.2. □

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