Locally convex topological completions of modules for a vertex operator algebra

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Vertex operator algebras are algebraic structures such that conformal field theories can be constructed and studied using modules and intertwining operators for them. See [H2], [H3], [H8] and [H11] for a program to construct conformal field theories using the representation theory of vertex operator algebras.

For a vertex operator algebra satisfying certain natural finiteness and reductivity conditions, the intertwining operators among irreducible modules satisfy commutativity and associativity (operator product expansion) (see [H1] and [H9]). An intertwining operator algebra is roughly speaking the direct sum of the (finitely many) irreducible modules for such a vertex operator algebra equipped with the (finite-dimensional) spaces of intertwining operators satisfying commutativity and associativity (see [H2] and [H6]). The notion of intertwining operator algebra can be viewed as a mathematical definition of chiral genus-zero rational conformal field theory (see [H4]).

Under stronger finiteness and reductivity conditions, intertwining operators have the modular invariance property, that is, the q-traces of products of intertwining operators form modular invariant vector spaces [H10]. These modular invariant vector spaces give genus-one chiral rational conformal field theories.

But to construct chiral weakly conformal field theories in the sense of G. Segal (see [S]), we need to construct locally convex topological completions of modules appearing in the intertwining operator algebras and prove that one can associate continuous and trace-class maps between tensor products of these completions to Riemann surfaces with parametrized boundaries and that Segal's axioms hold. We also expect that such completions should be closely related to the Hilbert spaces in the approach to conformal field theory using conformal nets.

In [H5] and [H7], the author constructed locally convex topological completions of finitely generated modules for a finitely generated vertex operator algebra using the correlation functions obtained from the products of vertex operators for modules and the exponentials of the Virasoro operators. But these completions are not large enough to allow the actions of intertwining operators and operators associated to genus-one elements.

For genus-zero chiral conformal field theories, correlation functions are given by analytic extensions of products of intertwining operators among modules. For genus-one chiral conformal field theories, correlations functions are given by analytic extensions of q-traces of genus-zero correlation functions. To construct locally convex topological completions of modules, we need to use all these correlation functions. But when we construct higher-genus correlation functions, we need to prove that multi-q-traces of genus-zero correlation functions are convergent. This convergence is still a conjecture now. In fact this higher-genus convergence conjecture is the main unsolved problem in the construction of higher-genus rational conformal field theories.

If we assume that this higher-genus convergence conjecture is true, then the idea in the construction in [H5] and [H7] still works. Consider a vertex operator algebra satisfying the finiteness and reductive conditions mentioned above so that the associativity and modular invariance of intertwining operators hold. Assume that the higher-genus convergence conjecture is true. Then using the correlation functions obtained from taking multi-q-traces of the genus-zero correlation functions obtained from the analytic extensions of products of intertwining operators, we can generalize the construction in [H5] and [H7] to obtain locally convex topological completions of modules for the vertex operator algebra. The correlation functions can be extended to maps between tensor products of these completions and we obtain a conformal field theory in the sense of G. Segal.

We now state a conjecture which should be related to the equivalence between the vertex operator algebra approach and the conformal nets approach to unitary rational conformal field theories.

Conjecture. *If the chiral conformal field theory is unitary, then the Hilbert space completions and the locally convex topological completions of modules for the vertex operator algebra are the same.*

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