# Modular invariance in conformal field theory

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## Contents



## 1 Tori and its moduli space

We shall call a closed connected genus-one Riemann surface a torus. Then every torus can be obtained from a parallelogram in the complex plane $\mathbb{C}$ by identifying its opposite boundary intervals. Using translations, rotations and dilations, every parallelogram in the complex plane is conformally equivalent to a parallellogram whose four vertices are given by $0, 1, \tau, \tau+1$, where $\tau \in \mathbb{H}$, where $\mathbb{H}$ is the (open) upper half plane. In particular, for every conformal equivalence class of tori, we obtain an lement $\tau \in \mathbb{H}$.

On the other hand, two different numbers in $\mathbb{H}$ might give conformal equivalent tori. The first problem is to determine what are the relations between two numbers in $\mathbb{H}$ that give conformally equivalent tori. The answer is given by the following basic result:

**Proposition 1.1.** *Let $\tau, \tau' \in \mathbb{H}$. The tori corresponding to $\tau$ and $\tau'$ are conformally equivalent if and only if there exists*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{Z}),$$

*where $SL(2, \mathbb{Z})$ is the group of $2 \times 2$ matrices with entries in $\mathbb{Z}$ and determinants equal to $1$, such that*

$$\tau' = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}.$$

The proof of this proposition is given by the correspondence between tori and $\tau \in \mathbb{H}$ together with Proposition 2 in Section 2, Chapter VII in [Ser].

The space of all $\tau$ is in fact the Teichmüller space of genus-one Riemann surfaces. The space of all conformally equivalence classes of genus-one Riemann surfaces is the moduli space of such Riemann surfaces. From Proposition 1.1, we see that this moduli space can be studied using the quotient of the Teichüller space $\mathbb{H}$ by the action of the modular group (the mapping class group in the genus-one case) $SL(2, \mathbb{Z})$.

## 2 Graded dimensions and modular invariance

Let $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ be a $\mathbb{C}$-graded vector space. Assume that $\dim W_{[n]} < \infty$. The graded dimension of $W$ is defined to be the formal series

$$\dim_* W = \sum_{n\in\mathbb{C}} \dim W_{[n]} q^n.$$

Let $W_1 = \coprod_{n\in\mathbb{C}} (W_1)_{[n]}$ and $W_2 = \coprod_{n\in\mathbb{C}} (W_2)_{[n]}$ be $\mathbb{C}$-graded vector spaces such that

$$\dim \coprod_{n_1+n_2=m} (W_1)_{[n_1]} \otimes (W_2)_{[n_2]} < \infty$$

for $m \in \mathbb{C}$. Then we have

$$\dim_*(W_1 \otimes W_2) = (\dim_* W_1)(\dim_* W_2).$$

In general, for $k \in \mathbb{N}$, we also have

$$\dim_*(W_1 \otimes \cdots \otimes W_k) = (\dim_* W_1)\cdots(\dim_* W_k).$$

Moreover, this can be generalized to the graded dimension of an infinite tensor product, that is,

$$\dim_* \left( \bigotimes_{n\in\mathbb{N}} W_n \right) = \prod_{n\in\mathbb{N}} \dim_* W_n.$$

Let $W = \coprod_{n\in\mathbb{N}} W_{[n]}$ such that $\dim W_{[n]} < \infty$ for $n \in \mathbb{N}$. Consider the symmetric algebra

$$S(W) = \coprod_{k\in\mathbb{N}} S^k(W),$$

which is also $\mathbb{N}$-graded with finite-dimensional homogeneous subspaces. The space $S(W)$ is in fact isomorphic as a graded space to the graded space $\bigotimes_{n=0}^{\infty} S(W_{[n]})$ of infinite tensor product of $S(W_{[n]})$ as follows: $S(W)$ is spanned by elements of the form

$$w_{i_1}^{n_1} \cdots w_{i_{p_1}}^{n_1} \cdots w_{i_l}^{n_l} \cdots w_{i_{p_l}}^{n_l}, \tag{2.1}$$

where $n_1 < \cdots < n_l$. We define a map from $S(W)$ to $\bigotimes_{n\in\mathbb{N}} S(W_{[n]})$ by sending the element (2.1) to

$$1 \otimes \cdots \otimes (w_{i_1}^{n_1} \cdots w_{i_{p_1}}^{n_{p_1}}) \otimes \cdots \otimes (w_{i_l}^{n_l} \cdots w_{i_{p_l}}^{n_l}) \otimes 1 \otimes \cdots .$$

Then this is a graded isomorphism. Thus we have

$$\dim_* S(W) = \dim_* \left( \bigotimes_{n\in\mathbb{N}} S(W_{[n]}) \right) = \prod_{n\in\mathbb{N}} \dim_* S(W_{[n]}).$$

It is known that $S(W_{[n]})$ is in fact graded isomorphic to the polynomial algebra with $\dim W_{[n]}$ variables of degree $n$. Using induction, we can show that the graded dimension of such a polynomial ring is $\sum_{m\in\mathbb{N}} \binom{\dim W_{[n]}+m-1}{m} q^{mn}$. So we obtain

$$\dim_* S(W_{[n]}) = \sum_{m\in\mathbb{N}} \binom{\dim W_{[n]}+m-1}{m} q^{mn} = \sum_{m\in\mathbb{N}} \binom{-\dim W_{[n]}}{m} (-q^n)^m = (1-q^n)^{-\dim W_{[n]}}.$$

Then we have

$$\dim_* S(W) = \prod_{n\in\mathbb{N}} (1-q^n)^{-\dim W_{[n]}}.$$

For the Fock space $S(\hat{\mathfrak{h}}^-)$, we have

$$\dim_* S(\hat{\mathfrak{h}}^-) = q^{\frac{\dim \mathfrak{h}}{24}} \eta(\tau)^{-\dim \mathfrak{h}},$$

where

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n\in\mathbb{N}} (1-q^n)$$

for $q = e^{2\pi i\tau}$ is the Dedekind $\eta$ function. It has the following modular transformation property:

$$\eta\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = \epsilon(\alpha,\beta,\gamma,\delta)(\gamma\tau+\delta)^{\frac{1}{2}}\eta(\tau).$$

In particular, we have

$$\eta(\tau+1) = e^{\frac{\pi i}{12}}\eta(\tau),$$
$$\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\eta(\tau).$$

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