Meromorphic open-string vertex algebras

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Abstract

A notion of meromorphic open-string vertex algebra is introduced. A meromorphic open-string vertex algebra is an open-string vertex algebra in the sense of Kong and the author satisfying additional rationality (or meromorphicity) conditions for vertex operators. The vertex operator map for a meromorphic open-string vertex algebra satisfies rationality and associativity but in general does not satisfy the Jacobi identity, commutativity, the commutator formula, the skew-symmetry or even the associator formula. Given a vector space $\mathfrak{h}$, we construct a meromorphic open-string vertex algebra structure on the tensor algebra of the negative part of the affinization of $\mathfrak{h}$ such that the vertex algebra struture on the symmetric algebra of the negative part of the Heisenberg algebra associated to $\mathfrak{h}$ is a quotient of this meromorphic open-string vertex algebra. We also introduce the notion of left module for a meromorphic open-string vertex algebra and construct left modules for the meromorphic open-string vertex algebra above.

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1 Introduction

Vertex (operator) algebras arose naturally in the study of two-dimensional conformal field theories in physics (see the first systematic study using the method of operator product expansion in [BPZ] by Belavin, Polyakov and Zamolodchikov) and in the vertex operator construction of representations of affine Lie algebras and in the construction and study of the "moonshine module" for the Monster finite simple group in mathematics (see the announcement [B] by Borcherds and the monograph [FLM] by Frenkel, Lepowsky and Meurman).

Vertex (operator) algebras can be viewed as the "closed-string-theoretic" analogues of both Lie algebras and commutative associative algebras. A vertex (operator) algebra is defined in terms of either the Jacobi identity or the duality property or parts of these axioms. The Jacobi identity contains the commutator formula for vertex operators and the duality property includes in particular commutativity. The commutator formula and commutativity are fundamental to vertex (operator) algebras. Many of the results on vertex (operator) algebras and their representations depend heavily on the commutator formula and commutativity. The commutator formula and commutativity also play an important role in the construction of examples of vertex (operator) algebras, especially in the construction of vertex operator algebras associated to affine Lie algebras and the Virasoro algebra. It was proved in [FHL] that associativity and thus other properties, including in particular the Jacobi identity, follows from commutativity and other minor axioms. Geometrically, it was shown in [H1] and [H3] by the author that commutativity is equivalent to a meromorphicity property on an algebra over the partial operad of the moduli space of spheres with punctures and standard local coordinates at the punctures.

Beyond topological field theories, two-dimensional conformal field theories are the only mathematically successful quantum field theories. Many people attribute this success to the existence of the infinite-dimensional conformal symmetry. But from the experience in the study of two-dimensional conformal field theories in terms of the representation theory of vertex operator algebras, at least in the genus-zero case, commutativity or other equivalent properties is the main reason why two-dimensional conformal field theories are mathematically much better understood than other non-topological quantum field theories. In fact, it is the commutator formula that allows one to apply the Lie-theoretic method to the study of vertex operator algebras and their representations.

Two-dimensional conformal field theories are deep mathematical theories that play important roles in both mathematics and physics. But there are also other non-topological quantum field theories of fundamental importance in both mathematics and physics. Nonlinear sigma models whose target spaces are not Calabi-Yau manifolds are such examples. The most important example is Yang-Mills theory. In physics, Yang-Mills theory is known to describe fundamental interactions and in mathematics, one of the major unsolved problem is the existence of quantum Yang-Mills theory and the mass gap conjecture. Unfortunately, for these theories, we do not expect that they have a commutativity property as strong as the commutativity property for two-dimensional conformal field theories. This is actually the reason why it is very difficult to generalize the veretx-algebraic approach in the study of two-dimensional conformal field theories to other non-topological quantum field theories.

On the other hand, we have another fundamental property of vertex (operator) algebras: Associativity. Associativity for vertex (operator) algebras is a strong form of operator product expansion for meromorphic fields. While we do not expect that commutativity holds for general non-topological quantum field theories, we do believe that operator product expansion holds for these theories. Therefore, to study non-topological quantum field theories that are not two-dimensional conformal field theories, one approach is to find algebraic structures satisfying certain associativity property but not necessarily any commutativity property.

In dimension 2, in physicis, Cardy initiated the study of boundary conformal field theory in [C] and, in mathematics, Kong and the author later introduced and constructed open-string vertex algebras in [HK]. An open-string vertex algebra satisfies associativity but not commutativity. However, the examples given in [HK] are constructed using modules and intertwining operators for a vertex operator algebra belonging to the meromorphic center of the open-string vertex algebra. Since intertwining operators for the meromorphic center satisfy the commutativity property for intertwining operators (a generalization of commutativity for vertex (operator) algebras formulated and proved in [H2], [H4] and [H5]), these examples of open-string vertex algebras still satisfy a certain generalized version of commutativity. In fact, these open-string vertex algebra are still part of an open-closed two-dimensional conformal field theory describing the interaction of boundary states or open strings. To go beyond conformal field theories in dimension 2, we need to find examples of open-string vertex algebras that are not constructed from modules and intertwining operators for vertex (operator) algebras.

In the present paper, we construct a class of such examples. In fact, the examples that we construct in the present paper satisfy stronger conditions than those open-string vertex algebras constructed in [HK]. Like vertex (operator) algebras, the products and iterates of vertex operators of these open-string vertex algebras are expansions of rational functions. We call an open-string vertex algebra satisfying such a rationality property a meromorphic open-string vertex algebra. The vertex operator map for a meromorphic open-string vertex algebra satisfies rationality and associativity but in general does not satisfy the Jacobi identity, commutativity, the commutator formula, the skew-symmetry or even the associator formula.

Given a vector space $\mathfrak{h}$, we have the Heisenberg algebra $\hat{\mathfrak{h}} = \hat{\mathfrak{h}}_+ \oplus \hat{\mathfrak{h}}_- \oplus \hat{\mathfrak{h}}_0$, where $\hat{\mathfrak{h}}_+ = \mathfrak{h} \otimes t\mathbb{C}[t]$, $\hat{\mathfrak{h}}_- = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]$, $\hat{\mathfrak{h}}_0 = \mathfrak{h} \oplus \mathbb{C}\mathbf{k}$ and $\mathbb{C}\mathbf{k}$ is the center of $\hat{\mathfrak{h}}$. Instead of the universal enveloping algebra $U(\hat{\mathfrak{h}})$ of $\hat{\mathfrak{h}}$, we consider the quotient $N(\hat{\mathfrak{h}})$ of the tensor algebra $T(\hat{\mathfrak{h}})$ of $\hat{\mathfrak{h}}$ by only the commutator relations between $\hat{\mathfrak{h}}_+$ and $\hat{\mathfrak{h}}_-$, between $\hat{\mathfrak{h}}_+$ and $\hat{\mathfrak{h}}_0$, between $\hat{\mathfrak{h}}_-$ and $\hat{\mathfrak{h}}_0$ and between $\mathfrak{h}$ and $\mathbb{C}\mathbf{k}$, but not the commutator relations between $\hat{\mathfrak{h}}_+$ and itself, between $\hat{\mathfrak{h}}_-$ and itself and between $\mathfrak{h}$ and itself. In other words, we do not assume that $\mathfrak{h}$ and consequently $\hat{\mathfrak{h}}_+$ and $\hat{\mathfrak{h}}_-$ are abelian Lie algebras. Actually, since we work with the tensor algebras of $\hat{\mathfrak{h}}_+$ and $\hat{\mathfrak{h}}_-$ and $\mathfrak{h}$, we do not assume any relations among linearly independent elements of these vector spaces. From a left module M for the tensor algebra $T(\mathfrak{h})$ of $\mathfrak{h}$, we construct an induced left module for $N(\hat{\mathfrak{h}})$ and prove that this induced left module is linearly isomorphic to $T(\hat{\mathfrak{h}}_-) \otimes M$ where $T(\hat{\mathfrak{h}}_-)$ is the tensor algebra of $\hat{\mathfrak{h}}_-$. In the case that M is the trivial left module $\mathbb{C}$ for $T(\mathfrak{h})$, we construct a meromorphic open-string vertex algebra structure on $T(\hat{\mathfrak{h}}_-) \simeq T(\hat{\mathfrak{h}}_-) \otimes \mathbb{C}$. We know that the symmetric algebra $S(\hat{\mathfrak{h}}_-)$ of $\hat{\mathfrak{h}}_-$ has a natural grading-restricted vertex algebra structure. In particular, $S(\hat{\mathfrak{h}}_-)$ is also a meromorphic open-string vertex algebra. Thus $S(\hat{\mathfrak{h}}_-)$ is in fact a quotient of $T(\hat{\mathfrak{h}}_-)$ as meromorphic open-string vertex algebras. We also introduce the notion of left module for a meromorphic open-string vertex algebra and construct a structure of left module for $T(\hat{\mathfrak{h}}_-)$ on $T(\hat{\mathfrak{h}}_-)\otimes M$ for a left $T(\mathfrak{h})$-module M. Comparing to the construction of the vertex operator algebra associated to the universal enveloping algebra $U(\hat{\mathfrak{h}})$ of the Heisenberg algebra $\hat{\mathfrak{h}}$, the construction in the present paper involves much more complicated calculations because of the noncommutativity of the tensor algebras $T(\hat{\mathfrak{h}}_+)$, $T(\hat{\mathfrak{h}}_-)$ and $T(\mathfrak{h})$ of $\hat{\mathfrak{h}}_+$, $\hat{\mathfrak{h}}_-$ and $\mathfrak{h}$, respectively.

The present paper grew out of the author's study of nonlinear sigma models using the representation theory of vertex operator algebras. Though there is indeed a vertex operator algebra associated to a Riemannian manifold and representations of this algebra can be constructed from smooth functions on the manifold, the author has noticed that the more fundamental structure associated to a Riemannian manifold is a meromorphic open-string vertex algebra. It was conjectured by physicists that in general quantum nonlinear sigma models are not conformal field theories. These theories are believed to be "gapped" or massive theories. Therefore it is reasonable to expect that the fundamental algebraic structure of a nonlinear sigma model satisfies the operator product expansion condition but in general might not have a commutativity property. The construction of a meromorphic open-string vertex algebra and left modules from a Riemannian manifold is given in [H7].

We also have notions of right module and bimodule for a meromorphic open-string vertex algebra. We also have a similar construction of right $T(\mathfrak{h})$-modules and we can also construct $T(\mathfrak{h})$-bimodules. These notions, constructions and a study of these modules and left modules will be given in a paper on the representation theory of meromorphic open-string vertex algebras.

When the homogeneous subspaces of a meromorphic open-string vertex algebra is finite dimensional, we say that it is grading-restricted. The notion of grading-restricted meromorphic open-string vertex algebra should be viewed as a noncommutative generalization of the notion of grading-restricted vertex algebra (which really should be called grading-restricted closed-string vertex algebra). In [H6], the author introduced cohomologies of grading-restricted vertex algebras by constructing certain complexes analogous to the Hochschild complex for associative algebras and then consider subcomplexes analogous to the Harrison complex for commutative associative algebras. The complexes in [H6] that are analogous to the Hochschild complex are in fact also defined for meromorphic open-string vertex algebras and give cohomologies for such algebras. We shall discuss this cohomology theory in a future publication.

The construction in the present paper can be generalized to higher dimensions. There have been efforts by mathematicians to generalize vertex (operator) algebras to higher dimensions. But these efforts are not very successful mainly because the examples constructed are mostly free field theories or theories obtained by tensoring two-dimensional conformal field theories in a suitable sense. The main difficulty is that for those higher-dimensional quantum field theories of fundamental importance in mathematics and physics, there might not be commutativity or equivalent properties. On the other hand, we do want operator product expansion to hold. Our generalizations of meromorphic open-string vertex algebras in higher dimensions satisfy associativity but not necessarily commutativity and the examples obtained by generalizing the construction in the present paper are not from free field theories. We shall give these generalizations and constructions in another future publication.

The present paper is organized as follows: In Section 2, we introduce the notion of meromorphic open-string vertex algebra and explain that they are indeed open-string vertex algebra defined in [HK]. For a vector space $\mathfrak{h}$, we introduce a quotient algebra $N(\hat{\mathfrak{h}})$ of the tensor algebra $T(\hat{\mathfrak{h}})$ mentioned above and construct induced left modules for $N(\hat{\mathfrak{h}})$ in Section 3. As a preparation for our construction of examples of meromorphic open-string vertex algebras and left modules, we define and study normal ordering and vertex operators in Section 4. This is the main technical section of the present paper. In Section 5, we construct the class of meromorphic open-string vertex algebras mentioned above. In Section 6, we introduce the notion of left module for a meromorphic open-string vertex algebra and construct left modules for the meromorphic open-string vertex algebras constructed in Section 5.

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2 Definition of meromorphic open-string vertex algebra

In this section, we give the definition of meromorphic open-string vertex algebra. We also recall the notion of open-string vertex algebra introduced by Kong and the author in [HK] and explain that a meromorphic open-string vertex algebra is indeed an open-string vertex algebra.

Since the applications we have in mind are always over the field of complex numbers, for convenience, we shall assume that all the vector spaces in the present paper are over the complex numbers. But every definition, except for the recalled notion of open-string vertex algebra, construction or result in the present paper can be formulated, carried out or obtained over a field of characteristic 0 without additional efforts. We shall denote the set of nonnegative integers and the set of positive integers by $\mathbb{N}$ and $\mathbb{Z}_+$, respectively. We use both formal variables and complex variables. We shall use $x, x_1, \dots, y, y_1, \dots$ to denote commuting formal variables and $z, z_1, \dots$ to denote complex variables or complex numbers. When we write down the expression such as $(x_1 - x_2)^n$ for $n \in \mathbb{Z}$ and commuting formal variables x_1 and x_2, we always mean the expansion in nonnegative powers of x_2, the second formal variable. But when we write down the expression $(z_1 - z_2)^n$ for $n \in \mathbb{Z}$ and complex variables or numbers z_1 and z_2, we mean the usual analytic function or the complex number. In the region $|z_1| > |z_2|$, this analytic function or complex number is in fact equal to the sum of the series obtained by substituting z_1 and z_2 for x_1 and x_2 in the formal series $(x_1 - x_2)^n$.

For a $\mathbb{C}$-graded vector space $V = \coprod_{n\in\mathbb{C}} V_{(n)}$, we use V' to denote the graded dual $\coprod_{n\in\mathbb{C}} V^*_{(n)}$ of V. We first introduce meromorphic open-string vertex algebras:

Definition 2.1. A *meromorphic open-string vertex algebra* is a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ (graded by weights) equipped with a linear map

$$\begin{array}{rcl} Y_V : V & \to & (\mathrm{End}\, V)[[x, x^{-1}]] \\ u & \mapsto & Y_V(u, x), \end{array}$$

or equivalently, a linear map

$$\begin{array}{rcl} Y_V : V \otimes V & \to & V[[x, x^{-1}]] \\ u \otimes v & \mapsto & Y_V(u, x)v \end{array}$$

called *vertex operator map* a *vacuum* $\mathbf{1} \in V$, satisfying the following conditions:

1. *Lower bound condition*: When n is sufficiently negative, $V_{(n)} = 0$.

2. Properties for the vacuum: $Y_V(\mathbf{1}, x) = 1_V$ (the *identity property*) and for $u \in V$, $Y_V(u, x)\mathbf{1} \in V[[x]]$ and $\lim_{x\to 0} Y_V(u, x)\mathbf{1} = u$ (the *creation property*).

3. *Rationality*: For $u_1, \ldots, u_n, v \in V$ and $v' \in V'$, the series
$$\begin{aligned}&\langle v', Y_V(u_1, z_1) \cdots Y_V(u_n, z_n)v\rangle \\ &\quad = \langle v', Y_V(u_1, x_1) \cdots Y_V(u_n, x_n)v\rangle\Big|_{x_1=z_1,\ldots,x_n=z_n}\end{aligned} \tag{2.1}$$
converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function in $z_1, \ldots, z_n$ with the only possible poles at $z_i = 0$ for $i = 1, \ldots, n$ and $z_i = z_j$ for $i \neq j$. For $u_1, u_2, v \in V$ and $v' \in V'$, the series
$$\begin{aligned}&\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle \\ &\quad = \langle v', Y_V(Y_V(u_1, x_0)u_2, x_2)v\rangle\Big|_{x_0=z_1-z_2,\ x_2=z_2}\end{aligned} \tag{2.2}$$
converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to a rational function with the only possible poles at $z_1 = 0$, $z_2 = 0$ and $z_1 = z_2$.

4. *Associativity*: For $u_1, u_2, v \in V$, $v' \in V'$, we have
$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle = \langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle \tag{2.3}$$
when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

5. **d**-*bracket property*: Let $\mathbf{d}_V$ be the grading operator on V, that is, $\mathbf{d}_V u = mu$ for $m \in \mathbb{Z}$ and $u \in V_{(m)}$. For $u \in V$,
$$[\mathbf{d}_V, Y_V(u, x)] = Y_V(\mathbf{d}_V u, x) + x\frac{d}{dx}Y_V(u, x). \tag{2.4}$$

6. The *D-derivative property* and the *D-commutator formula*: Let $D_V : V \to V$ be defined by
$$D_V(u) = \lim_{x\to 0} \frac{d}{dx}Y_V(u, x)\mathbf{1}$$

for $u \in V$. Then for $u \in V$,

$$\begin{aligned} \frac{d}{dx} Y_V(u, x) &= Y_V(D_V u, x) \\ &= [D_V, Y_V(u, x)]. \end{aligned} \tag{2.5}$$

A meromorphic open-string vertex algebra is said to be *grading restricted* if $\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$. *Homomorphisms*, *isomorphisms*, *subalgebras* of meromorphic open-string vertex algebras are defined in the obvious way.

We shall denote the meromorphic open-string vertex algebra defined above by $(V, Y_V, \mathbf{1})$ or simply by V. For $u \in V$, we call the map $Y_V(u, x) : V \to V[[x, x^{-1}]]$ the *vertex operator associated to u*.

Remark 2.2. Note that from the definition, a meromorphic open-string vertex algebra in general might not be a vertex algebra. But a $\mathbb{Z}$-graded vertex algebra such that the $\mathbb{Z}$-grading is lower bounded is a meromorphic open-string vertex algebra. In particular, a grading-restricted vertex algebra in the sense of [H6] or a vertex operator algebra in the sense of [FLM] and [FHL] is a grading-restricted meromorphic open-string vertex algebra. (In [H6] and in the present paper, we use the term open-string vertex algebra because such an algebra can be interpreted as describing the interaction of open stings at a "vertex." See the discussion below and the discussion in [HK]. In fact, a (grading-restricted) vertex algebra should have been called a (meromorphic) closed-string vertex algebra.)

Remark 2.3. In the definition above, we require that a meromorphic open-string vertex algebra satisfy some strong conditions, for example, the lower bound condition. We can define weaker versions of meromorphic open-string vertex algebra but here we put these stronger conditions since the examples we construct in this paper satisfy these stronger conditions. These conditions are also important for the development and applications of the theory of meromorphic open-string vertex algebras.

For a $\mathbb{C}$-graded vector space $V = \coprod_{n \in \mathbb{C}} V_{(n)}$, we use $\overline{V}$ to denote the algebraic completion $\prod_{n \in \mathbb{C}} V_{(n)}$ of V. We now recall the notion of open-string vertex algebra from [HK]:

Definition 2.4. An *open-string vertex algebra* is an $\mathbb{R}$-graded vector space $V = \coprod_{n \in \mathbb{R}} V_{(n)}$ (graded by *weights*) equipped with a *vertex map*

$$\begin{aligned} Y^O : V \times \mathbb{R}_+ &\to \mathrm{Hom}(V, \overline{V}) \\ (u, r) &\mapsto Y^O(u, r) \end{aligned}$$

such that for $r \in \mathbb{R}_+$ the map given by $u \mapsto Y^O(u, r)$ is linear, or equivalently,

$$\begin{aligned} Y^O : (V \otimes V) \times \mathbb{R}_+ &\to \overline{V} \\ (u \otimes v, r) &\mapsto Y^O(u, r)v \end{aligned}$$

such that for $r \in \mathbb{R}_+$ the map given by $u\otimes \mapsto Y^O(u, r)v$ is linear, a *vacuum* $\mathbf{1} \in V$ and an operator $D \in \mathrm{End}\ V$ of weight 1, satisfying the following conditions:

1. *Vertex map weight property*: For $n_1, n_2 \in \mathbb{R}$, there exist a finite subset $N(n_1, n_2) \subset \mathbb{R}$ such that the image of $\left(\coprod_{n \in n_1+\mathbb{Z}} V_{(n)} \otimes \coprod_{n \in n_2+\mathbb{Z}} V_{(n)}\right) \times \mathbb{R}_+$ under Y^O is in $\prod_{n \in N(n_1,n_2)+\mathbb{Z}} V_{(n)}$.

2. Properties for the vacuum: For any $r \in \mathbb{R}_+$, $Y^O(\mathbf{1}, r) = 1_V$ (the *identity property*) and $\lim_{r\to 0} Y^O(u, r)\mathbf{1}$ exists and is equal to u (the *creation property*).

3. *Local-truncation property for D'*: Let $D' : V' \to V'$ be the adjoint of D. Then for any $v' \in V'$, there exists a positive integer k such that $(D')^k v' = 0$.

4. *Convergence properties*: For $v_1, \dots, v_n, v \in V$ and $v' \in V'$, the series

$$\langle v', Y^O(v_1, r_1) \cdots Y^O(v_n, r_n)v\rangle$$

converges absolutely when $r_1 > \cdots > r_n > 0$. For $v_1, v_2, v \in V$ and $v' \in V'$, the series

$$\langle v', Y^O(Y^O(v_1, r_0)v_2, r_2)v\rangle$$

converges absolutely when $r_2 > r_0 > 0$.

5. *Associativity*: For $v_1, v_2, v \in V$ and $v' \in V'$,

$$\langle v', Y^O(v_1, r_1)Y^O(v_2, r_2)v\rangle = \langle v', Y^O(Y^O(v_1, r_1 - r_2)v_2, r_2)v\rangle$$

for $r_1, r_2 \in \mathbb{R}$ satisfying $r_1 > r_2 > r_1 - r_2 > 0$.

6. $\mathbf{d}$-*bracket property*: Let $\mathbf{d}$ be the grading operator on V, that is, $\mathbf{d}u = mu$ for $m \in \mathbb{R}$ and $u \in V_{(m)}$. For $u \in V$, $Y^O(u, r)$ as a function of $r \in \mathbb{R}_+$ valued in $\mathrm{Hom}(V, \overline{V})$ is differentiable (that is, for $v \in V$, $v' \in V'$, $\langle v', Y^O(u, r)v\rangle$ as a function of r is differentiable) and

$$[\mathbf{d}, Y^O(u, r)] = Y^O(\mathbf{d}u, r) + r\frac{d}{dr}Y^O(u, r). \tag{2.6}$$

7. *D-derivative property*: We still use D to denote the natural extension of D to $\mathrm{Hom}(\overline{V}, \overline{V})$. For $u \in V$,

$$\frac{d}{dr} Y^O(u, r) = [D, Y^O(u, r)] = Y^O(Du, r). \tag{2.7}$$

The open-string vertex algebra defined above is denoted $(V, Y^O, \mathbf{1}, D)$ or simply V.

In [HK], a formal-variable vertex operator map

$$\begin{aligned} \mathcal{Y}^f : V &\to (\mathrm{End}\, V)\{x\} = \left\{ \sum_{n\in\mathbb{C}} a_n x^n \mid a_n \in \mathrm{End}\, V \right\} \\ u &\mapsto \mathcal{Y}^f(u, x) \end{aligned}$$

is constructed such that $Y^O(u, r) = \mathcal{Y}^f(u, r)$ for $u \in V$ and $r \in \mathbb{R}_+$. In particular, the open-string vertex algebra can be studied in terms of $\mathcal{Y}^f$.

Given a meromorphic open-string vertex algebra $(V, Y_V, \mathbf{1})$, let

$$\begin{aligned} Y_V^O : V \times \mathbb{R}_+ &\to \mathrm{Hom}(V, \overline{V}) \\ (u, r) &\mapsto Y_V^O(u, r) \end{aligned}$$

be defined by $Y_V^O(u, r) = Y_V(u, r)$. Then we have:

Proposition 2.5. *The quadruple $(V, Y_V^O, \mathbf{1}, D_V)$ is an open-string vertex algebra.*

Proof. The vertex map weight property, the identity property, the creation property, the convergence properties, associativity, the **d**-bracket property and the D-derivative property hold obviously. The local-truncation property for D' holds because the meromorphic open-string vertex algebra satisfies the lower bound condition. ∎

In the applications of meromorphic open-string vertex algebras, we also need direct products of such algebras.

Definition 2.6. Let $(V_\alpha, Y_{V_\alpha}, \mathbf{1}_\alpha)$ for $\alpha \in \mathcal{A}$ be meromorphic open-string vertex algebras. Assume that the weights of V_α for $\alpha \in \mathcal{A}$ are bounded from below by a common number. Let $V = \prod_{\alpha\in\mathcal{A}} V_\alpha$. Then V together with the direct products of the $\mathbb{Z}$-gradings, vertex operators and the vacuums of V_α

for $\alpha \in \mathcal{A}$ is a meromorphic open-string vertex algebra. For example, for $u = \sum_{\alpha\in\mathcal{A}} u_\alpha, v = \sum_{\alpha\in\mathcal{A}} v_\alpha \in V$,

$$Y_V(u,x)v = \sum_{\alpha\in\mathcal{A}} Y_{V_\alpha}(u_\alpha, x)v_\alpha$$

and

$$d_V u = \sum_{\alpha\in\mathcal{A}} d_{V_\alpha} u_\alpha.$$

Thus we obtain the d_V-bracket formula

$$\begin{aligned}
[d_V, Y_V(u,x)]v &= \sum_{\alpha\in\mathcal{A}} [d_{V_\alpha}, Y_{V_\alpha}(u_\alpha, x)]v_\alpha \\
&= \sum_{\alpha\in\mathcal{A}} \left(Y_{V_\alpha}(d_{V_\alpha}u_\alpha, x) + x\frac{d}{dx}Y_{V_\alpha}(u_\alpha, x) \right) v_\alpha \\
&= Y_V(d_V u, x)v + x\frac{d}{dx}Y_V(u,x)v.
\end{aligned}$$

The other axioms can be proved similarly. This meromorphic open-string vertex algebra is called the *direct product meromorphic open-string vertex algebra* of $(V_\alpha, Y_{V_\alpha}, \mathbf{1}_\alpha)$, $\alpha \in \mathcal{A}$.

3 A quotient algebra of the tensor algebra of the affinization $\hat{\mathfrak{h}}$ of a vector space $\mathfrak{h}$

Examples of open-string vertex algebras were constructed in [HK] using modules and intertwining operators for vertex operator algebras. In this section, we study a quotient algebra of the tensor algebra of the affinization of a vector space and its modules. We shall use these structures in later sections to construct directly a class of meromorphic open-string vertex operator algebras and left modules and thus new examples of open-string vertex algebras and left modules, without using the theory of vertex operator algebras.

Let $\mathfrak{h}$ be a vector space over $\mathbb{C}$ equipped with a nondegenerate bilinear form $(\cdot,\cdot)$. The Heisenberg algebra $\hat{\mathfrak{h}}$ associated with $\mathfrak{h}$ and $(\cdot,\cdot)$ is the vector space $\mathfrak{h} \otimes [t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$ equipped with the bracket operation defined by

$$\begin{aligned}
[a\otimes t^m, b\otimes t^n] &= m(a,b)\delta_{m+n,0}\mathbf{k}, \\
[a\otimes t^m, \mathbf{k}] &= 0,
\end{aligned}$$

for $a, b \in \mathfrak{h}$ and $m, n \in \mathbb{Z}$. It is a $\mathbb{Z}$-graded Lie algebra. In particular, we have the universal enveloping algebra $U(\mathfrak{h})$ of $\mathfrak{h}$. The universal enveloping algebra $U(\mathfrak{h})$ is constructed as a quotient of the tensor algebra $T(\mathfrak{h})$ of the vector space $\mathfrak{h}$. We have a triangle decomposition

$$\hat{\mathfrak{h}} = \hat{\mathfrak{h}}_- \oplus \hat{\mathfrak{h}}_0 \oplus \hat{\mathfrak{h}}_+,$$

where

$$\begin{aligned}
\hat{\mathfrak{h}}_- &= \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}],\\
\hat{\mathfrak{h}}_+ &= \mathfrak{h} \otimes t\mathbb{C}[t],\\
\hat{\mathfrak{h}}_0 &= \mathfrak{h} \otimes \mathbb{C} \oplus \mathbb{C}\mathbf{k}\\
&\simeq \mathfrak{h} \oplus \mathbb{C}\mathbf{k},\\
\mathfrak{h} &\simeq \mathfrak{h} \otimes \mathbb{C}
\end{aligned}$$

are subalgebras of $\hat{\mathfrak{h}}$.

The meromorphic open-string vertex algebras and left modules in the present paper are constructed from left modules for a quotient algebra $N(\hat{\mathfrak{h}})$ of the tensor algebra $T(\hat{\mathfrak{h}})$ such that $U(\hat{\mathfrak{h}})$ is a quotient of $N(\hat{\mathfrak{h}})$. Let I be the two-sided ideal of $T(\hat{\mathfrak{h}})$ generated by elements of the form

$$\begin{gathered}
(a \otimes t^m) \otimes (b \otimes t^n) - (b \otimes t^n) \otimes (a \otimes t^m) - m(a, b)\delta_{m+n,0}\mathbf{k},\\
(a \otimes t^k) \otimes (b \otimes t^0) - (b \otimes t^0) \otimes (a \otimes t^k),\\
(a \otimes t^l) \otimes \mathbf{k} - \mathbf{k} \otimes (a \otimes t^l)
\end{gathered}$$

for $m \in \mathbb{Z}_+$, $n \in -\mathbb{Z}_+$, $k \in \mathbb{Z} \setminus \{0\}$ and $l \in \mathbb{Z}$. Let $N(\hat{\mathfrak{h}}) = T(\hat{\mathfrak{h}})/I$. By definition, we see that $U(\hat{\mathfrak{h}})$ is a quotient algebra of $N(\hat{\mathfrak{h}})$.

We have the following Poincaré-Birkhof-Witt type result for $N(\hat{\mathfrak{h}})$;

Proposition 3.1. *As a vector space, $N(\hat{\mathfrak{h}})$ is linearly isomorphic to*

$$T(\hat{\mathfrak{h}}_-) \otimes T(\hat{\mathfrak{h}}_+) \otimes T(\mathfrak{h}) \otimes T(\mathbb{C}\mathbf{k}) \tag{3.1}$$

where $T(\hat{\mathfrak{h}}_-)$, $T(\hat{\mathfrak{h}}_+)$, $T(\mathfrak{h})$ and $T(\mathbb{C}\mathbf{k})$ are the tensor algebras of the vector spaces $\hat{\mathfrak{h}}_-$, $\hat{\mathfrak{h}}_+$, $\mathfrak{h}$ and $\mathbb{C}\mathbf{k}$, respectively.

Proof. We first show that for any $k \in \mathbb{N}$, any element of $T(\hat{\mathfrak{h}})$ of the form $u_1 \otimes \cdots \otimes u_k$ for $u_1, \ldots, u_k$ of either the form $a \otimes t^m$ or $\mathbf{k}$ is a sum of an element of (3.1) and an element of I. We use induction on the number of elements that are not $\mathbf{k}$ in the set $\{u_1, \ldots, u_k\}$. When this number is 0, the element we are considering is $\mathbf{k} \otimes \cdots \otimes \mathbf{k}$ and is in (3.1). Assume that when there are less than n elements that are not $\mathbf{k}$ in the set $\{u_1, \ldots, u_k\}$, this statement is true. Modulo elements of I, we can move all factors of the form $\mathbf{k}$ in the element $u_1 \otimes \cdots \otimes u_k$ to the right and then move those $u_i \in \mathfrak{h}_0$ to the immediate left of the tensor powers of $\mathbf{k}$ but keep the order of these elements. Thus we can assume that $u_1 \otimes \cdots \otimes u_k$ is of the form

$$(a_1 \otimes t^{m_1}) \otimes \cdots \otimes (a_l \otimes t^{m_l}) \otimes a_{l+1} \otimes \cdots \otimes a_n \otimes \mathbf{k} \otimes \cdots \otimes \mathbf{k}$$

where $a_1, \ldots, a_n \in \mathfrak{h}$ and $m_1, \ldots, m_l \in \mathbb{Z} \setminus \{0\}$. If there is an integer j satisfying $1 \leq j \leq l$ such that $m_1, \ldots, m_j < 0$ and $m_{j+1}, \ldots, m_l > 0$, then this element is in (3.1). Otherwise, modulo elements of I and elements of the form $u_1 \otimes \cdots \otimes u_k$ with less than n factors not equal to $\mathbf{k}$, we can move factors of the form $a_i \otimes t^{m_i}$ with positive m_i to the right of the factors of the form $a_i \otimes t^{m_i}$ with negative m_i and keep the order of such factors with positive m_i and the order of such factors with negative m_i. The resulting element is in (3.1). By induction assumption, elements of the form $u_1 \otimes \cdots \otimes u_k$ with less than n factors not equal to $\mathbf{k}$ are sums of elements of (3.1) and I. Thus in the case that there are n elements that are not $\mathbf{k}$ in the set $\{u_1, \ldots, u_k\}$, the statement is true.

We have proved that $T(\hat{\mathfrak{h}})$ is the sum of (3.1) and I. Now we show that the intersection of (3.1) and I is 0. The proof is in fact the same as Jacobson's proof of the linear independence part of the Poincaré-Birkhof-Witt theorem for the universal envelop algebra of a Lie algebra.

Given an element of $T(\hat{\mathfrak{h}})$ of the form $u_1 \otimes \cdots \otimes u_k$ for $u_1, \ldots, u_k$ of either the form $a \otimes t^m$ or $\mathbf{k}$, a pair u_i and u_j is said to be in *wrong order* if (i) $u_i \in \hat{\mathfrak{h}}_+$ and $u_j \in \hat{\mathfrak{h}}_-$, (ii) $u_i \in \mathfrak{h}$ and $u_j \in \hat{\mathfrak{h}}_-$, (iii) $u_i \in \mathfrak{h}$ and $u_j \in \hat{\mathfrak{h}}_+$, (iv) $u_i = \mathbf{k}$ and $u_j \in \hat{\mathfrak{h}}_-$, (v) $u_i = \mathbf{k}$ and $u_j \in \hat{\mathfrak{h}}_+$ or (vi) $u_i = \mathbf{k}$ and $u_j \in \mathfrak{h}$. We now define a linear map $f : T(\hat{\mathfrak{h}}) \to T(\hat{\mathfrak{h}})$ such that

$$f(u_1 \otimes \cdots \otimes u_k) = u_1 \otimes \cdots \otimes u_k \tag{3.2}$$

if there is no pair in wrong order in $u_1, \ldots, u_k$ and

$$\begin{aligned} f(u_1 \otimes \cdots \otimes u_k) &= f(u_1 \otimes \cdots \otimes u_{i-1} \otimes u_{i+1} \otimes u_i \otimes u_{i+2} \otimes \cdots\cdots \otimes u_k) \\ &\quad + f(u_1 \otimes \cdots \otimes u_{i-1} \otimes [u_{i+1}, u_i] \otimes u_{i+2} \otimes \cdots\cdots \otimes u_k) \end{aligned} \tag{3.3}$$

if u_i and u_{i+1} is a pair in wrong order. If f exists, then $f(I) = 0$ and f restricted to (3.1) is the identity map. Thus the intersection of (3.1) and I is 0.

We define f inductively on the tensor power of $\hat{\mathfrak{h}}$ and the number of pairs in wrong order in $u_1, \dots, u_k$. Let $T^k(\hat{\mathfrak{h}})$ be the k-th tensor power of $\hat{\mathfrak{h}}$. Then $T(\hat{\mathfrak{h}}) = \coprod_{k\in\mathbb{N}} T^k(\hat{\mathfrak{h}})$. We define f on $\coprod_{k\leq 1} T^k(\hat{\mathfrak{h}})$ to be the identity map. Assume that when $k < m$, $f(u_1 \otimes \cdots \otimes u_m)$ is defined. We define $f(u_1 \otimes \cdots \otimes u_m)$ using (3.2) if there is no pair in wrong order in $u_1, \dots, u_m$. Assume that when the number of pairs in wrong order in $u_1, \dots, u_m$ is less than $n \in \mathbb{Z}_+$, $f(u_1 \otimes \cdots \otimes u_m)$ is defined. When the number of pairs in the wrong order in $u_1, \dots, u_m$ is equal to n, there exists i such that the pair u_i and u_{i+1} is in wrong order. In this case, we define $f(u_1 \otimes \cdots \otimes u_m)$ using (3.3). We still have to check that f is well defined, that is, is independent of the choice of the pair u_i and u_{i+1}. Let u_i and u_{i+1} and $u_{i'}$ and $u_{i'+1}$ be two such pairs. We can assume that $i < i'$. If $i + 1 < i'$, then it is straitforward to check that $f(u_1 \otimes \cdots \otimes u_m)$ defined using u_i and u_{i+1} and using u_i and u_{i+1} are equal. We now check the case $i + 1 = i'$. Note that in this case, u_i and $u_{i'+1} = u_{i+2}$ is also a pair in wrong order. In this case, using the pair u_i and u_{i+1} first, then the pair u_i and u_{i+2} and finally the pair u_{i+1} and u_{i+2}, we obtain

$$\begin{aligned}
&f(u_1 \otimes \cdots \otimes u_m)\\
&\quad = f(u_1 \otimes \cdots \otimes u_{i+2} \otimes u_{i+1} \otimes u_i \otimes \cdots \otimes u_m)\\
&\qquad + f(u_1 \otimes \cdots \otimes [u_{i+1}, u_{i+2}] \otimes u_i \otimes \cdots \otimes u_m)\\
&\qquad + f(u_1 \otimes \cdots \otimes u_{i+1} \otimes \otimes [u_i, u_{i+2}] \otimes \cdots \otimes u_m)\\
&\qquad + f(u_1 \otimes \cdots \otimes [u_i, u_{i+1}] \otimes u_{i+2} \otimes \cdots \otimes u_m).
\end{aligned} \tag{3.4}$$

On the other hand, using the pair u_{i+1} and u_{i+2} first, then the pair u_i and u_{i+2} and finally the pair u_i and u_{i+1}, we obtain

$$\begin{aligned}
&f(u_1 \otimes \cdots \otimes u_m)\\
&\quad = f(u_1 \otimes \cdots \otimes u_{i+2} \otimes u_{i+1} \otimes u_i \otimes \cdots \otimes u_m)\\
&\qquad + f(u_1 \otimes \cdots \otimes u_{i+2} \otimes [u_i, u_{i+1}] \otimes \cdots \otimes u_m)\\
&\qquad + f(u_1 \otimes \cdots \otimes [u_i, u_{i+2}] \otimes u_{i+1} \otimes \cdots \otimes u_m)\\
&\qquad + f(u_1 \otimes \cdots \otimes u_i \otimes [u_{i+1}, u_{i+2}] \otimes \cdots \otimes u_m).
\end{aligned} \tag{3.5}$$

Note that $[u_i, u_{i+1}]$, $[u_i \otimes u_{i+2}]$ and $[u_{i+1}, u_{i+2}]$ are either 0 or belong to $\mathbb{C}\mathbf{k}$. Also by the induction assumption, f is well defined on $\coprod_{k<m} T^k(\hat{\mathfrak{h}})$. In

particular we have

$$
\begin{aligned}
&f(u_1 \otimes \cdots \otimes [u_{i+1}, u_{i+2}] \otimes u_i \otimes \cdots \otimes u_m)\\
&\qquad = f(u_1 \otimes \cdots \otimes u_i \otimes [u_{i+1}, u_{i+2}] \otimes \cdots \otimes u_m), \qquad (3.6)\\
&f(u_1 \otimes \cdots \otimes u_{i+1} \otimes [u_i, u_{i+2}] \otimes \cdots \otimes u_m)\\
&\qquad = f(u_1 \otimes \cdots \otimes [u_i, u_{i+2}] \otimes u_{i+1} \otimes \cdots \otimes u_m), \qquad (3.7)\\
&f(u_1 \otimes \cdots \otimes u_{i+2} \otimes [u_i, u_{i+1}] \otimes \cdots \otimes u_m)\\
&\qquad = f(u_1 \otimes \cdots \otimes [u_i, u_{i+1}] \otimes u_{i+2} \otimes \cdots \otimes u_m). \qquad (3.8)
\end{aligned}
$$

Using (3.6), (3.7) and (3.8), we see that the right-hand sides of (3.4) and (3.5) are equal, proving that the definition of $f(u_1 \otimes \cdots \otimes u_m)$ is independent of the choice of the pair u_i and u_{i+1}.

Since the intersection of (3.1) and I is 0, $T(\hat{\mathfrak{h}})$ is the direct sum of (3.1) and I. Thus $N(\hat{\mathfrak{h}}) = T(\hat{\mathfrak{h}})/I$ is linearly isomorphic to (3.1). ∎

Now we construct left modules for $N(\hat{\mathfrak{h}})$. Let M be a left $T(\mathfrak{h})$-module. We define the action of $\mathbf{k}$ on M to be 1 and the actions of elements of $\hat{\mathfrak{h}}_+$ on M to be 0. Then M is also a left module for the subalgebra $N(\hat{\mathfrak{h}}_+ \oplus \hat{\mathfrak{h}}_0)$ of $N(\hat{\mathfrak{h}})$ generated by elements of $\hat{\mathfrak{h}}_+$ and $\hat{\mathfrak{h}}_0$. We consider the induced left module $N(\hat{\mathfrak{h}}) \otimes_{N(\hat{\mathfrak{h}}_+ \oplus \hat{\mathfrak{h}}_0)} M$. By Proposition 3.1, we see that $N(\hat{\mathfrak{h}}) \otimes_{N(\hat{\mathfrak{h}}_+ \oplus \hat{\mathfrak{h}}_0)} M$ is linearly isomorphic to $T(\hat{\mathfrak{h}}_-) \otimes M$. We shall identify $N(\hat{\mathfrak{h}}) \otimes_{N(\hat{\mathfrak{h}}_+ \oplus \hat{\mathfrak{h}}_0)} M$ with $T(\hat{\mathfrak{h}}_-) \otimes M$. The left $N(\hat{\mathfrak{h}})$-module structure on $T(\hat{\mathfrak{h}}_-) \otimes M$ can be obtained explicitly by using the commutator relations defining the algebra $N(\hat{\mathfrak{h}})$ and the left $N(\hat{\mathfrak{h}}_+ \oplus \hat{\mathfrak{h}}_0)$-module structure on M.

For a left $N(\hat{\mathfrak{h}})$-module, we denote the representation images of $a \otimes t^n \in \hat{\mathfrak{h}}$ for $a \in \mathfrak{h}$ and $n \in \mathbb{Z}$ acting on the left module by $a(n)$. Then a left $N(\hat{\mathfrak{h}})$-module $T(\hat{\mathfrak{h}}_-) \otimes M$ constructed from a left $T(\mathfrak{h})$-module M is spanned by elements of the form $a_1(-n_1) \cdots a_k(-n_k)w$, where $a_1, \ldots, a_k \in \hat{\mathfrak{h}}$, $n_1, \ldots, n_k \in \mathbb{Z}_+$ and $w \in M$.

4 Normal ordering and vertex operators

In this section, we define the normal ordering for certain operators on a left $N(\hat{\mathfrak{h}})$-module of the form $T(\hat{\mathfrak{h}}_-) \otimes M$ and vertex operators acting on such a left $N(\hat{\mathfrak{h}})$-module. We then prove a number of technical formulas for products of normal ordered products of operators and products of vertex operators.

This section contains the main technical material of the present paper. Many of the calculations are much more complicated than the Heisenberg algebra case because of the noncommutativity of the operators.

Given a left $N(\hat{\mathfrak{h}})$-module, we define a *normal ordering* map $\mathop{\circ}\limits_{\circ}\cdot\mathop{\circ}\limits_{\circ}$ from the space of operators on the left module spanned by operators of the form $a_1(n_1)\cdots a_k(n_k)$ to itself by rearranging the order of $a_1(n_1),\dots,a_k(n_k)$ in $a_1(n_1)\cdots a_k(n_k)$ by moving $a_i(n_i)$ with $n_i<0$ to the left, $a_i(n_i)$ with $n_i>0$ to the middle and $a_i(n_i)$ with $n_i=0$ to the right but keeping the orders of $a_i(n_i)$ with $n_i<0$, with $n_i<0$ or with $n_i=0$. For example,

$$\begin{aligned}&\mathop{\circ}\limits_{\circ}a_1(-1)a_2(0)a_3(4)a_4(0)a_5(-3)a_6(-1)a_7(10)\mathop{\circ}\limits_{\circ}\\&\quad=a_1(-1)a_5(-3)a_6(-1)a_3(4)a_7(10)a_2(0)a_4(0).\end{aligned}$$

More explicitly,

$$\mathop{\circ}\limits_{\circ}a_1(n_1)\cdots a_k(n_k)\mathop{\circ}\limits_{\circ}=a_{\sigma(1)}(n_{\sigma(1)})\cdots a_{\sigma(k)}(n_{\sigma(k)}),$$

where $\sigma\in S_k$ is the unique permutation such that

$$\begin{gathered}\sigma(1)<\cdots<\sigma(\alpha),\\ \sigma(\alpha+1)<\cdots<\sigma(\beta),\\ \sigma(\beta+1)<\cdots<\sigma(k),\\ n_{\sigma(1)},\dots,n_{\sigma(\alpha)}<0,\\ n_{\sigma(\alpha+1)},\dots,n_{\sigma(\beta)}>0,\\ n_{\sigma(\beta+1)},\dots,n_{\sigma(k)}=0,\end{gathered}$$

for some integers α and β satisfying $0\le\alpha\le\beta\le k$.

Given an induced left $N(\hat{\mathfrak{h}})$-module $W=T(\hat{\mathfrak{h}}_-)\otimes M$, $a_1,\dots,a_k\in\mathfrak{h}$ and $m_1,\dots,m_k\in\mathbb{Z}_+$, we define the vertex operator $Y_W(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1},x)$ associated to $a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}\in T(\hat{\mathfrak{h}}_-)$ by

$$\begin{aligned}&Y_W(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1},x)\\&\quad=\mathop{\circ}\limits_{\circ}\frac{1}{(n_1-1)!}\left(\frac{d^{m_1-1}}{dx^{m_1-1}}a_1(x)\right)\cdots\frac{1}{(m_k-1)!}\left(\frac{d^{m_k-1}}{dx^{m_k-1}}a_k(x)\right)\mathop{\circ}\limits_{\circ},\end{aligned}\tag{4.1}$$

where

$$a_i(x)=\sum_{n\in\mathbb{Z}}a_i(n)x^{-n-1}$$

for $i=1,\dots,k$ and $a_i(n)$ for $i=1,\dots,k$ and $n\in\mathbb{Z}$ are the representation images of $a_i\otimes t^n$ on W.

We need the following commutator formula:

Lemma 4.1. *For $a, b \in \mathfrak{h}$,*

$$\left[\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial x_1^{m-1}}a^+(x_1), \frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial x_2^{n-1}}b^-(x_2)\right] = n(a,b)\binom{-n-1}{m-1}(x_1-x_2)^{-m-n}, \tag{4.2}$$

where for $a \in \mathfrak{h}$,

$$a^{\pm}(x) = \sum_{n\in\pm\mathbb{Z}_+} a(n)x^{-n-1}$$

and a negative power of $x_1 - x_2$, as in the formal calculus in the theory of vertex operator algebras, is understood as the binomial expansion in the nonnegative powers of the formal variable x_2.

Proof. The proof is a straightforward calculation. ∎

We also need an explicit expression of a vertex operator. For $k \in \mathbb{Z}_+$ and $\alpha, \beta \in \mathbb{N}$ satisfying $0 \le \alpha \le \beta \le k$, let $J(k;\alpha,\beta)$ be the set of elements of S_k which preserve the orders of the first α numbers, the next $\beta-\alpha$ numbers, and the last $k-\beta$ numbers, that is,

$$J_{k;\alpha,\beta} = \{\sigma \in S_k \mid \sigma(1) < \cdots < \sigma(\alpha), \sigma(\alpha+1) < \cdots < \sigma(\beta),\ \sigma(\beta+1) < \cdots < \sigma(k)\}.$$

Lemma 4.2. *For $a_1, \ldots, a_k \in \hat{\mathfrak{h}}$ and $n_1, \ldots, n_k \in \mathbb{Z}_+$,*

$$\begin{aligned}
&Y_W(a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}, x)\\
&= \sum_{0\le\alpha\le\beta\le k}\sum_{\sigma\in J(k;\alpha,\beta)} \left(\frac{1}{(n_{\sigma(1)}-1)!}\frac{\partial^{n_{\sigma(1)}-1}}{\partial x^{n_{\sigma(1)}-1}}a^-_{\sigma(1)}(x)\right)\cdot\\
&\qquad \cdots\left(\frac{1}{(n_{\sigma(\alpha)}-1)!}\frac{\partial^{n_{\sigma(\alpha)}-1}}{\partial x^{n_{\sigma(\alpha)}-1}}a^-_{\sigma(\alpha)}(x)\right)\cdot\\
&\qquad \cdot\left(\frac{1}{(n_{\sigma(\alpha+1)}-1)!}\frac{\partial^{n_{\sigma(\alpha+1)}-1}}{\partial x^{n_{\sigma(\alpha+1)}-1}}a^+_{\sigma(\alpha+1)}(x)\right)\cdot\\
&\qquad \cdots\left(\frac{1}{(n_{\sigma(\beta)}-1)!}\frac{\partial^{n_{\sigma(\beta)}-1}}{\partial x^{n_{\sigma(\beta)}-1}}a^+_{\sigma(\beta)}(x)\right)\cdot\\
&\qquad \cdot\left(\frac{1}{(n_{\sigma(\alpha+1)}-1)!}\frac{\partial^{n_{\sigma(\beta+1)}-1}}{\partial x^{n_{\sigma(\beta+1)}-1}}a_{\sigma(\beta+1)}(0)x^{-1}\right)\cdot\\
&\qquad \cdots\left(\frac{1}{(n_{\sigma(k)}-1)!}\frac{\partial^{n_{\sigma(k)}-1}}{\partial x^{n_{\sigma(k)}-1}}(a_{\sigma(k)}(0)x^{-1})\right).
\end{aligned} \tag{4.3}$$

Proof. The expression follows immediately from the definition of the vertex operator and the definition of the normal ordering. ∎

We need the following:

Proposition 4.3. *For* $a_0, \ldots, a_k \in \mathfrak{h}$ *and* $m_0, \ldots, m_k \in \mathbb{Z}_+$,

$$\begin{aligned}
&\left(\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}}{\partial x_0^{m_0-1}}a_0(x_0)\right)\cdot \\
&\qquad \cdot {}^{\circ}_{\circ}\left(\frac{1}{(m_1-1)!}\frac{\partial^{m_1-1}}{\partial x_1^{m_1-1}}a_1(x_1)\right)\cdots\left(\frac{1}{(m_k-1)!}\frac{\partial^{m_k-1}}{\partial x_k^{m_k-1}}a_k(x_k)\right){}^{\circ}_{\circ} \\
&\quad = {}^{\circ}_{\circ}\left(\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}}{\partial x_0^{m_0-1}}a_0(x_0)\right)\cdot \\
&\qquad \cdot\left(\frac{1}{(m_1-1)!}\frac{\partial^{m_1-1}}{\partial x_1^{m_1-1}}a_1(x_1)\right)\cdots\left(\frac{1}{(m_k-1)!}\frac{\partial^{m_k-1}}{\partial x_k^{m_k-1}}a_k(x_k)\right){}^{\circ}_{\circ} \\
&\qquad + \sum_{p=1}^{k} m_p(a_0, a_p)\binom{-m_p-1}{m_0-1}(x_0-x_p)^{-m_0-m_p}\cdot \\
&\qquad \cdot {}^{\circ}_{\circ}\left(\frac{1}{(m_1-1)!}\frac{\partial^{m_1-1}}{\partial x_1^{m_1-1}}a_1(x_1)\right)\cdot \\
&\qquad\quad \cdots\left(\frac{1}{(m_{p-1}-1)!}\frac{\partial^{m_{p-1}-1}}{\partial x_{p-1}^{m_{p-1}-1}}a_{p-1}(x_{p-1})\right)\cdot \\
&\qquad\quad \cdot\left(\frac{1}{(m_{p+1}-1)!}\frac{\partial^{m_{p+1}-1}}{\partial x_{p+1}^{m_{p+1}-1}}a_{p+1}(x_{p+1})\right)\cdot \\
&\qquad\quad \cdots\left(\frac{1}{(m_k-1)!}\frac{\partial^{m_k-1}}{\partial x_k^{m_k-1}}a_k(x_k)\right){}^{\circ}_{\circ}.
\end{aligned} \tag{4.4}$$

Proof. The proof is a tedious but straightforward calculation. By (4.3) and (4.2), the left-hand side of (4.4) is equal to

$$\begin{aligned}
&\left(\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}}{\partial x_0^{m_0-1}}(a_0^+(x_0) + a_0(0)x_0^{-1} + a_0^-(x_0))\right)\cdot \\
&\qquad \cdot\left(\sum_{0\le\alpha\le\beta\le k}\ \sum_{\sigma\in J(k;\alpha,\beta)}\left(\frac{1}{(m_{\sigma(1)}-1)!}\frac{\partial^{m_{\sigma(1)}-1}}{\partial x_{\sigma(1)}^{m_{\sigma(1)}-1}}a^-_{\sigma(1)}(x_{\sigma(1)})\right)\cdot\right.
\end{aligned}$$

$$\cdots\left(\frac{1}{(m_{\sigma(\alpha)}-1)!}\frac{\partial^{m_{\sigma(\alpha)}-1}}{\partial x_{\sigma(\alpha)}^{m_{\sigma(\alpha)}-1}}a_{\sigma(\alpha)}^{-}(x_{\sigma(\alpha)})\right)\cdot$$

$$\cdot\left(\frac{1}{(m_{\sigma(\alpha+1)}-1)!}\frac{\partial^{m_{\sigma(\alpha+1)}-1}}{\partial x_{\sigma(\alpha+1)}^{m_{\sigma(\alpha+1)}-1}}a_{\sigma(\alpha+1)}^{+}(x_{\sigma(\alpha+1)})\right)\cdot$$

$$\cdots\left(\frac{1}{(m_{\sigma(\beta)}-1)!}\frac{\partial^{m_{\sigma(\beta)}-1}}{\partial x_{\sigma(\beta)}^{m_{\sigma(\beta)}-1}}a_{\sigma(\beta)}^{+}(x_{\sigma(\beta)})\right)\cdot$$

$$\cdot\left(\frac{1}{(m_{\sigma(\alpha+1)}-1)!}\frac{\partial^{m_{\sigma(\beta+1)}-1}}{\partial x_{\sigma(\alpha+1)}^{m_{\sigma(\beta+1)}-1}}a_{\sigma(\beta+1)}(0)x_{\sigma(\alpha+1)}^{-1}\right)\cdot$$

$$\cdots\left(\frac{1}{(m_{\sigma(l)}-1)!}\frac{\partial^{m_{\sigma(k)}-1}}{\partial x_{\sigma(k)}^{m_{\sigma(k)}-1}}(a_{\sigma(k)}(0)x_{\sigma(k)}^{-1})\right)\Bigg)$$

$$=\,{}^{\circ}_{\circ}\left(\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}}{\partial x_0^{m_0-1}}a_0(x_0)\right)\cdot$$

$$\cdot\left(\frac{1}{(m_1-1)!}\frac{\partial^{m_1-1}}{\partial x_1^{m_1-1}}a_1(x_1)\right)\cdots\left(\frac{1}{(m_k-1)!}\frac{\partial^{m_k-1}}{\partial x_k^{m_k-1}}a_k(x_k)\right){}^{\circ}_{\circ}$$

$$+\sum_{0\le\alpha\le\beta\le k}\;\sum_{\sigma\in J(k;\alpha,\beta)}\;\sum_{p=1}^{k}$$

$$\Bigg(\left(\frac{1}{(m_{\sigma(1)}-1)!}\frac{\partial^{m_{\sigma(1)}-1}}{\partial x_{\sigma(1)}^{m_{\sigma(1)}-1}}a_{\sigma(1)}^{-}(x_{\sigma(1)})\right)\cdot$$

$$\cdots\left(\frac{1}{(m_{\sigma(p-1)}-1)!}\frac{\partial^{m_{\sigma(p-1)}-1}}{\partial x_{\sigma(p-1)}^{m_{\sigma(p-1)}-1}}a_{\sigma(p-1)}^{-}(x_{\sigma(p-1)})\right)\cdot$$

$$\cdot\left[\left(\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}}{\partial x_0^{m_0-1}}a_0^{+}(x_0)\right),\left(\frac{1}{(m_{\sigma(p)}-1)!}\frac{\partial^{m_{\sigma(p)}-1}}{\partial x_{\sigma(p)}^{m_{\sigma(p)}-1}}a_{\sigma(p)}^{-}(x_{\sigma(p)})\right)\right]\cdot$$

$$\cdot\left(\frac{1}{(m_{\sigma(p+1)}-1)!}\frac{\partial^{m_{\sigma(p+1)}-1}}{\partial x_{\sigma(p+1)}^{m_{\sigma(p+1)}-1}}a_{\sigma(p+1)}^{-}(x_{\sigma(p+1)})\right)\cdot$$

$$
\begin{aligned}
&\cdots\left(\frac{1}{(m_{\sigma(\alpha)}-1)!}\frac{\partial^{m_{\sigma(\alpha)}-1}}{\partial x_{\sigma(\alpha)}^{m_{\sigma(\alpha)}-1}}a^-_{\sigma(\alpha)}(x_{\sigma(\alpha)})\right)\cdot\\
&\cdot\left(\frac{1}{(m_{\sigma(\alpha+1)}-1)!}\frac{\partial^{m_{\sigma(\alpha+1)}-1}}{\partial x_{\sigma(\alpha+1)}^{m_{\sigma(\alpha+1)}-1}}a^+_{\sigma(\alpha+1)}(x_{\sigma(\alpha+1)})\right)\cdot\\
&\cdots\left(\frac{1}{(m_{\sigma(\beta)}-1)!}\frac{\partial^{m_{\sigma(\beta)}-1}}{\partial x_{\sigma(\beta)}^{m_{\sigma(\beta)}-1}}a^+_{\sigma(\beta)}(x_{\sigma(\beta)})\right)\cdot\\
&\cdot\left(\frac{1}{(m_{\sigma(\alpha+1)}-1)!}\frac{\partial^{m_{\sigma(\beta+1)}-1}}{\partial x_{\sigma(\alpha+1)}^{m_{\sigma(\beta+1)}-1}}a_{\sigma(\beta+1)}(0)x^{-1}_{\sigma(\alpha+1)}\right)\cdot\\
&\cdots\left(\frac{1}{(m_{\sigma(k)}-1)!}\frac{\partial^{m_{\sigma(k)}-1}}{\partial x_{\sigma(k)}^{m_{\sigma(k)}-1}}a_{\sigma(k)}(0)x^{-1}_{\sigma(k)})\right)\Bigg)\\
=&\,{}^\circ_\circ\left(\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}}{\partial x_0^{m_0-1}}a_0(x_0)\right)\cdot\\
&\cdot\left(\frac{1}{(m_1-1)!}\frac{\partial^{m_1-1}}{\partial x_1^{m_1-1}}a_1(x_1)\right)\cdots\left(\frac{1}{(m_k-1)!}\frac{\partial^{m_k-1}}{\partial x_k^{m_k-1}}a_k(x_k)\right){}^\circ_\circ\\
&+\sum_{0\le\alpha\le\beta\le k}\sum_{\sigma\in J(k;\alpha,\beta)}\sum_{p=1}^{k}m_{\sigma(p)}(a_0,a_{\sigma(p)})\binom{-m_{\sigma(p)}-1}{m_0-1}(x_0-x_{\sigma(p)})^{-m_0-m_{\sigma(p)}}\cdot\\
&\cdot\Bigg(\left(\frac{1}{(m_{\sigma(1)}-1)!}\frac{\partial^{m_{\sigma(1)}-1}}{\partial x_{\sigma(1)}^{m_{\sigma(1)}-1}}a^-_{\sigma(1)}(x_{\sigma(1)})\right)\cdot\\
&\cdots\left(\frac{1}{(m_{\sigma(p-1)}-1)!}\frac{\partial^{m_{\sigma(p-1)}-1}}{\partial x_{\sigma(p-1)}^{m_{\sigma(p-1)}-1}}a^-_{\sigma(p-1)}(x_{\sigma(p-1)})\right)\cdot\\
&\cdot\left(\frac{1}{(m_{\sigma(p+1)}-1)!}\frac{\partial^{m_{\sigma(p+1)}-1}}{\partial x_{\sigma(p+1)}^{m_{\sigma(p+1)}-1}}a^-_{\sigma(p+1)}(x_{\sigma(p+1)})\right)\cdot\\
&\cdots\left(\frac{1}{(m_{\sigma(\alpha)}-1)!}\frac{\partial^{m_{\sigma(\alpha)}-1}}{\partial x_{\sigma(\alpha)}^{m_{\sigma(\alpha)}-1}}a^-_{\sigma(\alpha)}(x_{\sigma(\alpha)})\right)\cdot
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\left(\frac{1}{(m_{\sigma(\alpha+1)}-1)!}\frac{\partial^{m_{\sigma(\alpha+1)}-1}}{\partial x_{\sigma(\alpha+1)}^{m_{\sigma(\alpha+1)}-1}}a^{+}_{\sigma(\alpha+1)}(x_{\sigma(\alpha+1)})\right)\cdot\\
&\cdots\left(\frac{1}{(m_{\sigma(\beta)}-1)!}\frac{\partial^{m_{\sigma(\beta)}-1}}{\partial x_{\sigma(\beta)}^{m_{\sigma(\beta)}-1}}a^{+}_{\sigma(\beta)}(x_{\sigma(\beta)})\right)\cdot\\
&\cdot\left(\frac{1}{(m_{\sigma(\alpha+1)}-1)!}\frac{\partial^{m_{\sigma(\beta+1)}-1}}{\partial x_{\sigma(\beta+1)}^{m_{\sigma(\beta+1)}-1}}a_{\sigma(\beta+1)}(0)x^{-1}_{\sigma(\beta+1)}\right)\cdot\\
&\cdots\left(\frac{1}{(m_{\sigma(k)}-1)!}\frac{\partial^{m_{\sigma(k)}-1}}{\partial x_{\sigma(k)}^{m_{\sigma(k)}-1}}(a_{\sigma(k)}(0)x^{-1}_{\sigma(k)})\right)\Bigg)\\
=&\,{\circ\atop\circ}\left(\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}}{\partial x_0^{m_0-1}}a_0(x_0)\right)\cdot\\
&\cdot\left(\frac{1}{(m_1-1)!}\frac{\partial^{m_1-1}}{\partial x_1^{m_1-1}}a_1(x_1)\right)\cdots\left(\frac{1}{(m_k-1)!}\frac{\partial^{m_k-1}}{\partial x_k^{m_k-1}}a_k(x_k)\right){\circ\atop\circ}\\
&+\sum_{0\le\alpha\le\beta\le k}\sum_{\sigma\in J(k;\alpha,\beta)}\sum_{p=1}^{k}m_p(a_0,a_p)\binom{-m_p-1}{m_0-1}(x_0-x_p)^{-m_0-m_p}\cdot\\
&\cdot\Bigg(\left(\frac{1}{(m_{\sigma(1)}-1)!}\frac{\partial^{m_{\sigma(1)}-1}}{\partial x_1^{m_{\sigma(1)}-1}}a^{-}_{\sigma(1)}(x_1)\right)\cdot\\
&\cdots\left(\frac{1}{(m_{\sigma(p-1)}-1)!}\frac{\partial^{m_{\sigma(p-1)}-1}}{\partial x_{\sigma(p-1)}^{m_{\sigma(p-1)}-1}}a^{-}_{\sigma(p-1)}(x_{\sigma(p-1)})\right)\cdot\\
&\cdot\left(\frac{1}{(m_{\sigma(p+1)}-1)!}\frac{\partial^{m_{\sigma(p+1)}-1}}{\partial x_{\sigma(p+1)}^{m_{\sigma(p+1)}-1}}a^{-}_{\sigma(p+1)}(x_{\sigma(p+1)})\right)\cdot\\
&\cdots\left(\frac{1}{(m_{\sigma(\alpha)}-1)!}\frac{\partial^{m_{\sigma(\alpha)}-1}}{\partial x_{\sigma(\alpha)}^{m_{\sigma(\alpha)}-1}}a^{-}_{\sigma(\alpha)}(x_{\sigma(\alpha)})\right)\cdot\\
&\cdot\left(\frac{1}{(m_{\sigma(\alpha+1)}-1)!}\frac{\partial^{m_{\sigma(\alpha+1)}-1}}{\partial x_{\sigma(\alpha+1)}^{m_{\sigma(\alpha+1)}-1}}a^{+}_{\sigma(\alpha+1)}(x_{\sigma(\alpha+1)})\right)\cdot
\end{aligned}
$$

$$
\cdots\left(\frac{1}{(m_{\sigma(\beta)}-1)!}\frac{\partial^{m_{\sigma(\beta)}-1}}{\partial x_{\sigma(\beta)}^{m_{\sigma(\beta)}-1}}a^+_{\sigma(\beta)}(x_{\sigma(\beta)})\right)\cdot
$$
$$
\cdot\left(\frac{1}{(m_{\sigma(\alpha+1)}-1)!}\frac{\partial^{m_{\sigma(\beta+1)}-1}}{\partial x_{\sigma(\beta+1)}^{m_{\sigma(\beta+1)}-1}}a_{\sigma(\beta+1)}(0)x^{-1}_{\sigma(\beta+1)}\right)\cdot
$$
$$
\cdots\left(\frac{1}{(m_{\sigma(k)}-1)!}\frac{\partial^{m_{\sigma(k)}-1}}{\partial x_{\sigma(k)}^{m_{\sigma(k)}-1}}(a_{\sigma(k)}(0)x^{-1}_{\sigma(k)})\right)\Bigg)
$$
$$
= {}^{\circ}_{\circ}\left(\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}}{\partial x_0^{m_0-1}}a_0(x_0)\right)\cdot
$$
$$
\cdot\left(\frac{1}{(m_1-1)!}\frac{\partial^{m_1-1}}{\partial x_1^{m_1-1}}a_1(x_1)\right)\cdots\left(\frac{1}{(m_k-1)!}\frac{\partial^{m_k-1}}{\partial x_k^{m_k-1}}a_k(x_k)\right){}^{\circ}_{\circ}
$$
$$
+\sum_{p=1}^{k}m_p(a_0,a_p)\binom{-m_p-1}{m_0-1}(x_0-x_p)^{-m_0-m_p}\cdot
$$
$$
\cdot{}^{\circ}_{\circ}\left(\frac{1}{(m_1-1)!}\frac{\partial^{m_1-1}}{\partial x_1^{m_1-1}}a_1(x_1)\right)\cdot
$$
$$
\cdots\left(\frac{1}{(m_{p-1}-1)!}\frac{\partial^{m_{p-1}-1}}{\partial x_{p-1}^{m_{p-1}-1}}a_{p-1}(x_{p-1})\right)\cdot
$$
$$
\cdot\left(\frac{1}{(m_{p+1}-1)!}\frac{\partial^{m_{p+1}-1}}{\partial x_{p+1}^{m_{p+1}-1}}a_{p+1}(x_{p+1})\right)\cdot
$$
$$
\cdots\left(\frac{1}{(m_k-1)!}\frac{\partial^{m_k-1}}{\partial x_k^{m_k-1}}a_k(x_k)\right){}^{\circ}_{\circ}.
$$

∎

Remark 4.4. In (4.4), the formal variables $x_1,\ldots,x_k$ can be taken to be the same but cannot be equal to x_0.

From (4.4), we obtain immediately the following result:

Corollary 4.5. *For $a_0,\ldots,a_k\in\mathfrak{h}$ and $m_0,\ldots,m_k\in\mathbb{Z}_+$,*

$$
Y_W(a_0(-m_0)\mathbf{1},x_1)Y_W(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1},x_2)
$$

$$
\begin{aligned}
&= {\circ\atop\circ}\left(\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}}{\partial x_1^{m_0-1}}a_0(x_1)\right)\cdot\\
&\quad\cdot\left(\frac{1}{(m_1-1)!}\frac{\partial^{m_1-1}}{\partial x_2^{m_1-1}}a_1(x_2)\right)\cdots\left(\frac{1}{(m_k-1)!}\frac{\partial^{m_k-1}}{\partial x_2^{m_k-1}}a_k(x_2)\right){\circ\atop\circ}\\
&\quad+\sum_{p=1}^{l}m_p(a_0,a_p)\binom{-m_p-1}{m_0-1}(x_1-x_2)^{-m_0-m_p}\cdot\\
&\quad\quad\cdot Y_W(a_1(-m_1)\cdots\widehat{a_1(-m_p)}\cdots a_k(-m_k)\mathbf{1},x_2),
\end{aligned}
\tag{4.5}
$$

where for $p=1,\dots,k$, we use $\widehat{a_p(-m_p)}$ to denote that $a_p(-m_p)$ is missing from a product.

Proof. Taking x_0 to be x_1 and $x_1,\dots,x_k$ to be x_2 in (4.4) (note Remark 4.4), we obtain (4.5). ∎

Corollary 4.6. *For $a_0,\dots,a_k\in\mathfrak{h}$ and $m_0,\dots,m_k\in\mathbb{Z}_+$,*

$$
\begin{aligned}
&Y_W(a_0(-m_0)\cdots a_k(-m_k)\mathbf{1},x_1)\\
&\quad=\lim_{x_2\to x_1}\Bigg(Y_W(a_0(-m_0)\mathbf{1},x_1)Y_W(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1},x_2)\\
&\quad\quad-\sum_{p=1}^{k}m_p(a_0,a_p)\binom{-m_p-1}{m_0-1}(x_1-x_2)^{-m_0-m_p}\cdot\\
&\quad\quad\quad\cdot Y_W(a_1(-m_1)\cdots\widehat{a_p(-m_p)}\cdots a_k(-m_k)\mathbf{1},x_2)\Bigg).
\end{aligned}
\tag{4.6}
$$

Proof. Note that we can let $x_1=x_2$ in the first term of the right-hand side of (4.5) and the resulting formal series is

$$Y_W(a_0(-m_0)\cdots a_k(-m_k)\mathbf{1},x_2).$$

Hence if we move the second term in the right-hand side of (4.5) to the left-hand side, we can also let $x_1=x_2$ in the left-hand side of the resulting equality. Thus we obtain (4.6). ∎

Remark 4.7. Note that in the right-hand side of (4.6), we might not be able to take the limit (that is, let x_2 be equal to x_1) of individual terms since we do not know whether the limits or substitutions exist algebraically. But the limit or substitution of the sum indeed exists algebraically, as is shown in the proof of the corollary above.

We now prove the following formula for the product of two normal ordered products:

Proposition 4.8. *For $a_1, \dots, a_k, b_1, \dots, b_l \in \hat{\mathfrak{h}}$ and $m_1, \dots, m_k, n_1, \dots, n_l \in \mathbb{Z}_+$,*

$$\begin{aligned}
&{}^{\circ}_{\circ}\left(\frac{1}{(m_1-1)!}\frac{\partial^{m_1-1}}{\partial x_1^{m_1-1}}a_1(x_1)\right)\cdots\left(\frac{1}{(m_k-1)!}\frac{\partial^{m_k-1}}{\partial x_k^{m_k-1}}a_k(x_k)\right){}^{\circ}_{\circ}\cdot\\
&\qquad\cdot{}^{\circ}_{\circ}\left(\frac{1}{(n_1-1)!}\frac{\partial^{n_1-1}}{\partial y_1^{n_1-1}}b_1(y_1)\right)\cdots\left(\frac{1}{(n_l-1)!}\frac{\partial^{n_l-1}}{\partial y_l^{n_l-1}}b_l(y_l)\right){}^{\circ}_{\circ}\\
&= \sum_{i=0}^{\min(k,l)}\sum_{\substack{k\geq p_1>\cdots>p_i\geq 1\\ 1\leq \text{distinct } q_1,\dots,q_i\leq l}} n_{q_1}\cdots n_{q_i}(a_{p_1},b_{q_1})\cdots(a_{p_i},b_{q_i})\cdot\\
&\qquad\cdot\binom{-n_{q_1}-1}{m_{p_1}-1}\cdots\binom{-n_{q_i}-1}{m_{p_i}-1}\cdot\\
&\qquad\cdot(x_{p_1}-y_{q_1})^{-m_{p_1}-n_{q_1}}\cdots(x_{p_i}-y_{q_i})^{-m_{p_i}-n_{q_i}}\cdot\\
&\qquad\cdot{}^{\circ}_{\circ}\left(\prod_{p\neq p_1,\dots,p_i}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial x_p^{m_p-1}}a_p(x_p)\right)\cdot\\
&\qquad\qquad\cdot\left(\prod_{q\neq q_1,\dots,q_i}\frac{1}{(n_q-1)!}\frac{\partial^{n_q-1}}{\partial y_q^{n_q-1}}b_q(y_q)\right){}^{\circ}_{\circ}. \qquad (4.7)
\end{aligned}$$

Proof. We prove (4.7) using induction on k. When $k = 0$, (4.7) holds.

Now assume that (4.7) holds for $k \leq K$. We prove (4.7) in the case $k = K + 1$. For notational convenience, instead of (4.7) in the case of $k = K + 1$, we prove (4.7) with $a_1, \dots, a_{K+1}$ and $m_1, \dots, m_{K+1}$ replaced by $a_0, \dots, a_K$ and $m_0, \dots, m_K$. Since (4.7) holds for $k = K$, we have

$$\left(\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}}{\partial x_0^{m_0-1}}a_0(x_0)\right)$$

$$
\begin{aligned}
&\cdot\overset{\circ}{\circ}\left(\frac{1}{(m_1-1)!}\frac{\partial^{m_1-1}}{\partial x_1^{m_1-1}}a_1(x_1)\right)\cdots\left(\frac{1}{(m_K-1)!}\frac{\partial^{m_K-1}}{\partial x_K^{m_K-1}}a_k(x_k)\right)\overset{\circ}{\circ}\cdot\\
&\cdot\overset{\circ}{\circ}\left(\frac{1}{(n_1-1)!}\frac{\partial^{n_1-1}}{\partial y_1^{n_1-1}}b_1(y_1)\right)\cdots\left(\frac{1}{(n_l-1)!}\frac{\partial^{n_l-1}}{\partial y_l^{n_l-1}}b_l(y_l)\right)\overset{\circ}{\circ}\\
= &\sum_{i=0}^{\min(K,l)}\sum_{\substack{K\geq p_1>\cdots>p_i\geq 1\\ 1\leq \text{distinct } q_1,\ldots,q_i\leq l}} n_{q_1}\cdots n_{q_i}(a_{p_1},b_{q_1})\cdots(a_{p_i},b_{q_i})\cdot\\
&\cdot\binom{-n_{q_1}-1}{m_{p_1}-1}\cdots\binom{-n_{q_i}-1}{m_{p_i}-1}\cdot\\
&\cdot(x_{p_1}-y_{q_1})^{-m_{p_1}-n_{q_1}}\cdots(x_{p_i}-y_{q_i})^{-m_{p_i}-n_{q_i}}\cdot\\
&\cdot\left(\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}}{\partial x_0^{m_0-1}}a_0(x_0)\right)\cdot\\
&\cdot\overset{\circ}{\circ}\left(\prod_{p\neq 0,p_1,\ldots,p_i}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial x_p^{m_p-1}}a_p(x_p)\right)\cdot\\
&\cdot\left(\prod_{q\neq q_1,\ldots,q_i}\frac{1}{(n_q-1)!}\frac{\partial^{n_q-1}}{\partial y_q^{n_q-1}}b_q(y_q)\right)\overset{\circ}{\circ}.
\end{aligned}
\tag{4.8}
$$

Using (4.4), the right-hand side of (4.8) is equal to

$$
\begin{aligned}
&\sum_{i=0}^{\min(K,l)}\sum_{\substack{K\geq p_1>\cdots>p_i\geq 1\\ 1\leq \text{distinct } q_1,\ldots,q_i\leq l}} n_{q_1}\cdots n_{q_i}(a_{p_1},b_{q_1})\cdots(a_{p_i},b_{q_i})\cdot\\
&\cdot\binom{-n_{q_1}-1}{m_{p_1}-1}\cdots\binom{-n_{q_i}-1}{m_{p_i}-1}\cdot\\
&\cdot(x_{p_1}-y_{q_1})^{-m_{p_1}-n_{q_1}}\cdots(x_{p_i}-y_{q_i})^{-m_{p_i}-n_{q_i}}\cdot\\
&\cdot\overset{\circ}{\circ}\left(\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}}{\partial x_0^{m_0-1}}a_0(x_0)\right)\cdot\\
&\cdot\left(\prod_{p\neq 0,p_1,\ldots,p_i}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial x_p^{m_p-1}}a_p(x_p)\right)\cdot\\
&\cdot\left(\prod_{q\neq q_1,\ldots,q_i}\frac{1}{(n_q-1)!}\frac{\partial^{n_q-1}}{\partial y_q^{n_q-1}}b_q(y_q)\right)\overset{\circ}{\circ}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=0}^{\min(K,l)} \sum_{\substack{K\geq p_1>\cdots>p_i\geq 1\\ 1\leq \text{distinct } q_1,\ldots,q_i\leq l}} \sum_{s\neq 0,p_1,\ldots,p_i}\\
&\qquad m_s n_{q_1}\cdots n_{q_i}(a_0,a_s)(a_{p_1},b_{q_1})\cdots(a_{p_i},b_{q_i})\cdot\\
&\qquad \cdot\binom{-m_s-1}{m_0-1}\binom{-n_{q_1}-1}{m_{p_1}-1}\cdots\binom{-n_{q_i}-1}{m_{p_i}-1}\cdot\\
&\qquad \cdot(x_0-x_s)^{-m_0-m_s}(x_{p_1}-y_{q_1})^{-m_{p_1}-n_{q_1}}\cdots(x_{p_i}-y_{q_i})^{-m_{p_i}-n_{q_i}}\cdot\\
&\qquad \cdot{}^{\circ}_{\circ}\left(\prod_{p\neq 0,s,p_1,\ldots,p_i}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial x_p^{m_p-1}}a_p(x_p)\right)\cdot\\
&\qquad\qquad \cdot\left(\prod_{q\neq q_1,\ldots,q_i}\frac{1}{(n_q-1)!}\frac{\partial^{n_q-1}}{\partial y_q^{n_q-1}}b_q(y_q)\right){}^{\circ}_{\circ}\\
&+\sum_{i=0}^{\min(K,l)} \sum_{\substack{K\geq p_1>\cdots>p_i\geq 1\\ 1\leq \text{distinct } q_1,\ldots,q_i\leq l}} \sum_{t\neq q_1,\ldots,q_i}\\
&\qquad n_t n_{q_1}\cdots n_{q_i}(a_0,b_t)(a_{p_1},b_{q_1})\cdots(a_{p_i},b_{q_i})\cdot\\
&\qquad \cdot\binom{-n_t-1}{m_0-1}\binom{-n_{q_1}-1}{m_{p_1}-1}\cdots\binom{-n_{q_i}-1}{m_{p_i}-1}\cdot\\
&\qquad \cdot(x_0-y_q)^{-m_0-n_t}(x_{p_1}-y_{q_1})^{-m_{p_1}-n_{q_1}}\cdots(x_{p_i}-y_{q_i})^{-m_{p_i}-n_{q_i}}\cdot\\
&\qquad \cdot{}^{\circ}_{\circ}\left(\prod_{p\neq 0,p_1,\ldots,p_i}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial x_p^{m_p-1}}a_p(x_p)\right)\cdot\\
&\qquad\qquad \cdot\left(\prod_{q\neq t,q_1,\ldots,q_i}\frac{1}{(n_q-1)!}\frac{\partial^{n_q-1}}{\partial y_q^{n_q-1}}b_q(y_q)\right){}^{\circ}_{\circ}. \qquad (4.9)
\end{aligned}
$$

Since (4.7) holds for $k\leq K$, we also have

$$
\begin{aligned}
&{}^{\circ}_{\circ}\left(\prod_{1\leq p\leq K,\ p\neq s}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial x_p^{m_p-1}}a_p(x_p)\right){}^{\circ}_{\circ}\,{}^{\circ}_{\circ}\left(\prod_{q=1}^{l}\frac{1}{(n_q-1)!}\frac{\partial^{n_q-1}}{\partial y_q^{n_q-1}}b_q(y_q)\right){}^{\circ}_{\circ}\\
&\quad=\sum_{i=0}^{\min(K,l)} \sum_{\substack{K\geq p_1>\cdots>s>\cdots>p_i\geq 1\\ 1\leq \text{distinct } q_1,\ldots,q_i\leq l}} n_{q_1}\cdots n_{q_i}(a_{p_1},b_{q_1})\cdots(a_{p_i},b_{q_i})\cdot
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\binom{-n_{q_1}-1}{m_{p_1}-1}\cdots\binom{-n_{q_i}-1}{m_{p_i}-1}\cdot\\
&\cdot(x_{p_1}-y_{q_1})^{-m_{p_1}-n_{q_1}}\cdots(x_{p_i}-y_{q_i})^{-m_{p_i}-n_{q_i}}\cdot\\
&\cdot{:}\left(\prod_{p\neq 0,s,p_1,\ldots,p_i}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial x_p^{m_p-1}}a_p(x_p)\right)\cdot\\
&\quad\cdot\left(\prod_{q\neq q_1,\ldots,q_i}\frac{1}{(n_q-1)!}\frac{\partial^{n_q-1}}{\partial y_q^{n_q-1}}b_q(y_q)\right){:}
\end{aligned}
\tag{4.10}
$$

for $s=1,\ldots,K$.

From the calculations given by (4.8), (4.9) and (4.10) we obtain

$$
\begin{aligned}
&\left(\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}}{\partial x_0^{m_0-1}}a_0(x_0)\right)\\
&\qquad\cdot{:}\left(\frac{1}{(m_1-1)!}\frac{\partial^{m_1-1}}{\partial x_1^{m_1-1}}a_1(x_1)\right)\cdots\left(\frac{1}{(m_k-1)!}\frac{\partial^{m_k-1}}{\partial x_k^{m_k-1}}a_k(x_k)\right){:}\cdot\\
&\qquad\cdot{:}\left(\frac{1}{(n_1-1)!}\frac{\partial^{n_1-1}}{\partial y_1^{n_1-1}}b_1(y_1)\right)\cdots\left(\frac{1}{(n_l-1)!}\frac{\partial^{n_l-1}}{\partial y_l^{n_l-1}}b_l(y_l)\right){:}\\
&\quad-\sum_{s=1}^{K}m_s(a_0,a_s)\binom{-m_s-1}{m_0-1}(x-x_1)^{-m_0-m_s}\cdot\\
&\qquad\cdot{:}\left(\prod_{1\leq p\leq K,\ p\neq s}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial x_p^{m_p-1}}a_p(x_p)\right){:}\cdot\\
&\qquad\cdot{:}\left(\prod_{q=1}^{l}\frac{1}{(n_q-1)!}\frac{\partial^{n_q-1}}{\partial y_q^{n_q-1}}b_q(y_q)\right){:}\\
&=\sum_{i=0}^{\min(K,l)}\sum_{\substack{K\geq p_1>\cdots>p_i\geq 1\\ 1\leq \text{distinct } q_1,\ldots,q_i\leq l}} n_{q_1}\cdots n_{q_i}(a_{p_1},b_{q_1})\cdots(a_{p_i},b_{q_i})\cdot\\
&\qquad\cdot\binom{-n_{q_1}-1}{m_{p_1}-1}\cdots\binom{-n_{q_i}-1}{m_{p_i}-1}\cdot\\
&\qquad\cdot(x_{p_1}-y_{q_1})^{-m_{p_1}-n_{q_1}}\cdots(x_{p_i}-y_{q_i})^{-m_{p_i}-n_{q_i}}\cdot\\
&\qquad\cdot{:}\left(\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}}{\partial x_0^{m_0-1}}a_0(x_0)\right)\cdot
\end{aligned}
$$

$$\cdot\left(\prod_{p\neq 0,p_1,\ldots,p_i}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial x_p^{m_p-1}}a_p(x_p)\right)\cdot$$

$$\cdot\left(\prod_{q\neq q_1,\ldots,q_i}\frac{1}{(n_q-1)!}\frac{\partial^{n_q-1}}{\partial y_q^{n_q-1}}b_q(y_q)\right)\mathop{\circ}\limits_{\circ}$$

$$+\sum_{i=0}^{\min(K,l)}\sum_{\substack{K\geq p_1>\cdots>p_i\geq 1\\ 1\leq \text{distinct } q_1,\ldots,q_i\leq l}}\sum_{t\neq q_1,\ldots,q_i}$$

$$n_t n_{q_1}\cdots n_{q_i}(a_0,b_t)(a_{p_1},b_{q_1})\cdots(a_{p_i},b_{q_i})\cdot$$

$$\cdot\binom{-n_t-1}{m_0-1}\binom{-n_{q_1}-1}{m_{p_1}-1}\cdots\binom{-n_{q_i}-1}{m_{p_i}-1}\cdot$$

$$\cdot(x_0-x_t)^{-m_0-n_t}(x_{p_1}-y_{q_1})^{-m_{p_1}-n_{q_1}}\cdots(x_{p_i}-y_{q_i})^{-m_{p_i}-n_{q_i}}\cdot$$

$$\cdot\mathop{\circ}\limits_{\circ}\left(\prod_{p\neq 0,p_1,\ldots,p_i}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial x_p^{m_p-1}}a_p(x_p)\right)\cdot$$

$$\cdot\left(\prod_{q\neq t,q_1,\ldots,q_i}\frac{1}{(n_q-1)!}\frac{\partial^{n_q-1}}{\partial y_q^{n_q-1}}b_q(y_q)\right)\mathop{\circ}\limits_{\circ}$$

$$=\sum_{i=0}^{\min(K+1,l)}\sum_{\substack{K\geq p_1>\cdots>p_i\geq 0\\ 1\leq \text{distinct } q_1,\ldots,q_i\leq l}} n_{q_1}\cdots n_{q_i}(a_{p_1},b_{q_1})\cdots(a_{p_i},b_{q_i})\cdot$$

$$\cdot\binom{-n_{q_1}-1}{m_{p_1}-1}\cdots\binom{-n_{q_i}-1}{m_{p_i}-1}$$

$$\cdot(x_{p_1}-y_{q_1})^{-m_{p_1}-n_{q_1}}\cdots(x_{p_i}-y_{q_i})^{-m_{p_i}-n_{q_i}}\cdot$$

$$\cdot\mathop{\circ}\limits_{\circ}\left(\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}}{\partial x_0^{m_0-1}}a_0(x_0)\right)\cdot$$

$$\cdot\left(\prod_{p\neq 0,p_1,\ldots,p_i}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial x_p^{m_p-1}}a_p(x_p)\right)\cdot$$

$$\cdot\left(\prod_{q\neq q_1,\ldots,q_i}\frac{1}{(n_q-1)!}\frac{\partial^{n_q-1}}{\partial x_q^{n_q-1}}b_q(x_q)\right)\mathop{\circ}\limits_{\circ}. \tag{4.11}$$

By (4.4), the left-hand side of (4.11) is equal to

$$
\begin{aligned}
&{\circ\atop\circ}\left(\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}}{\partial x_0^{m_0-1}}a_0(x_0)\right)\cdot\\
&\qquad\cdot\left(\frac{1}{(m_1-1)!}\frac{\partial^{m_1-1}}{\partial x_1^{m_1-1}}a_1(x_1)\right)\cdots\left(\frac{1}{(m_k-1)!}\frac{\partial^{m_k-1}}{\partial x_k^{m_k-1}}a_k(x_k)\right){\circ\atop\circ}\cdot\\
&\qquad\cdot{\circ\atop\circ}\left(\frac{1}{(n_1-1)!}\frac{\partial^{n_1-1}}{\partial y_1^{n_1-1}}b_1(y_1)\right)\cdots\left(\frac{1}{(n_l-1)!}\frac{\partial^{n_l-1}}{\partial y_l^{n_l-1}}b_l(y_l)\right){\circ\atop\circ},
\end{aligned}
$$

proving (4.7) in the case $k = K+1$. ∎

Corollary 4.9. *For $a_1,\dots,a_k,b_1,\dots,b_l \in \hat{\mathfrak{h}}$ and $m_1,\dots,m_k,n_1,\dots,n_l \in \mathbb{Z}_+$,*

$$
\begin{aligned}
&Y_W(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1},x_1)Y_W(b_1(-n_1)\cdots b_k(-n_l)\mathbf{1},x_2)\\
&\quad=\sum_{i=0}^{\min(k,l)}\sum_{\substack{k\geq p_1>\cdots>p_i\geq 1\\ 1\leq \text{distinct } q_1,\dots,q_i\leq l}} n_{q_1}\cdots n_{q_i}(a_{p_1},b_{q_1})\cdots(a_{p_i},b_{q_i})\cdot\\
&\qquad\cdot\binom{-n_{q_1}-1}{m_{p_1}-1}\cdots\binom{-n_{q_i}-1}{m_{p_i}-1}(x_1-x_2)^{-m_{p_1}-n_{q_1}-\cdots-m_{p_i}-n_{q_i}}\cdot\\
&\qquad\cdot{\circ\atop\circ}\left(\prod_{p\neq p_1,\dots,p_i}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial x_1^{m_p-1}}a_p(x_1)\right)\cdot\\
&\qquad\quad\cdot\left(\prod_{q\neq q_1,\dots,q_i}\frac{1}{(n_q-1)!}\frac{\partial^{n_q-1}}{\partial x_2^{n_q-1}}b_q(x_2)\right){\circ\atop\circ}.
\end{aligned}
\tag{4.12}
$$

Proof. When $x_2,\dots,x_k$ are taken to be equal to x_1 and $y_1,\dots,y_l$ are taken to be equal to x_2, the left-hand and right-hand sides of (4.7) exist and are equal to the left-hand and right-hand sides of (4.12). Thus (4.12) holds. ∎

The formula (4.7) can be generalized to the product of n normal ordered products. But we do not need the explicit formula of the coefficients. What we need is the following:

Corollary 4.10. *For $a_j \in \hat{\mathfrak{h}}$ and $m_j \in \mathbb{Z}_+$ for $j = 1, \dots, n$ and $k_0 = 0, k_1, \dots, k_{l-1}, k_l = n \in \mathbb{Z}$ satisfying $k_0 = 0 < k_1 < \cdots < k_{l-1} < k_l$,*

$$
{}^\circ_\circ \prod_{j=1}^{k_1} \left(\frac{1}{(m_j-1)!} \frac{\partial^{m_j-1}}{\partial x_j^{m_j-1}} a_j(x_j) \right) {}^\circ_\circ \cdots {}^\circ_\circ \prod_{j=k_{l-1}+1}^{n} \left(\frac{1}{(m_j-1)!} \frac{\partial^{m_j-1}}{\partial x_j^{m_j-1}} a_j(x_j) \right) {}^\circ_\circ \tag{4.13}
$$

is a linear combination of formal series of the form

$$
\prod_{(j_1,j_2)\in A} (x_{j_1} - x_{j_2})^{-m_{j_1}-m_{j_2}} {}^\circ_\circ \prod_{j\notin B} \left(\frac{1}{(m_j-1)!} \frac{\partial^{m_j-1}}{\partial x_j^{m_j-1}} a_j(x_j) \right) {}^\circ_\circ, \tag{4.14}
$$

where A is a subset of

$$
\{(j_1, j_2) \mid \exists\, p, q \in \mathbb{Z} \text{ such that } 0 \le p < q < l, \quad k_p + 1 \le j_1 \le k_{p+1},\ k_q + 1 \le j_2 \le k_{q+1}\}
$$

and B is the subset of $\{1, \dots, n\}$ consisting those $j \in \{1, \dots, n\}$ such that either $(j, j') \in A$ for some j' or $(j', j) \in A$ for some j'.

Proof. We use induction on l. When $l = 1$, the conclusion is certainly true. When $l = 2$, (4.7) gives the linear combination explicitly. Assume that the conclusion is true when $l = L$. Then using (4.7), we see that the conclusion is also true when $l = L + 1$. ∎

Taking x_j to be x_{p+1} when $k_p + 1 \le j \le k_{p+1}$ in Corollary 4.10, we obtain:

Corollary 4.11. *For $a_j \in \hat{\mathfrak{h}}$ and $m_j \in \mathbb{Z}_+$ for $j = 1, \dots, n$ and $k_0 = 0, k_1, \dots, k_{l-1}, k_l = n \in \mathbb{Z}$ satisfying $k_0 = 0 < k_1 < \cdots < k_{l-1} < k_l$,*

$$
Y_W(a_1(-m_1) \cdots a_{k_1}(-m_{k_1})\mathbf{1}, x_1) \cdot \cdots Y_W(a_{k_{l-1}+1}(-m_{k_{l-1}+1}) \cdots a_n(-m_n)\mathbf{1}, x_l) \tag{4.15}
$$

is a linear combination of formal series of the form

$$
\prod_{(j_1,j_2)\in A} (y_{j_1} - y_{j_2})^{-m_{j_1}-m_{j_2}} {}^\circ_\circ \prod_{j\notin B} \left(\frac{1}{(m_j-1)!} \frac{\partial^{m_j-1}}{\partial y_j^{m_j-1}} a_j(y_j) \right) {}^\circ_\circ, \tag{4.16}
$$

where in the right-hand side, $y_j = x_{p+1}$ when $k_p + 1 \le j \le k_{p+1}$ and A and B are the same as in Corollary 4.10. ∎

5 A meromorphic open-string vertex algebra structure on $T(\hat{\mathfrak{h}}_-)$

In this section, we construct a meromorphic open-string vertex algebra structure on the left $N(\hat{\mathfrak{h}})$-module $T(\hat{\mathfrak{h}}_-)\otimes\mathbb{C}$ where we view $\mathbb{C}$ as a trivial left $T(\mathfrak{h})$-module.

The left $N(\hat{\mathfrak{h}})$-module $T(\hat{\mathfrak{h}}_-)\otimes\mathbb{C}$ is canonically linearly isomorphic to $T(\hat{\mathfrak{h}}_-)$. Let $\mathbf{1}=1\in T(\hat{\mathfrak{h}}_-)$. By definition, $T(\hat{\mathfrak{h}}_-)$ is spanned by elements of the form $a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}$, where $a_1,\dots,a_k\in\hat{\mathfrak{h}}$ and $m_1,\dots,m_k\in\mathbb{Z}_+$.

We define the weight of $a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}$ for $a_1,\dots,a_k\in\mathfrak{h}$ and $m_1,\dots,m_k\in\mathbb{Z}_+$ to be $m_1+\cdots+m_k$. Then $T(\hat{\mathfrak{h}}_-)$ becomes a $\mathbb{Z}$-graded vector space. If we denote the homogeneous subspace of $T(\hat{\mathfrak{h}}_-)$ of weight m by $(T(\hat{\mathfrak{h}}_-))_{(m)}$, then

$$T(\hat{\mathfrak{h}}_-)=\coprod_{m\in\mathbb{Z}}(T(\hat{\mathfrak{h}}_-))_{(m)}.$$

The element $\mathbf{1}\in T(\hat{\mathfrak{h}}_-)$ is the vacuum of $T(\hat{\mathfrak{h}}_-)$.

We have a vertex operator map

$$\begin{aligned} Y_{T(\hat{\mathfrak{h}}_-)}: T(\hat{\mathfrak{h}}_-) &\to (\mathrm{End}\, T(\hat{\mathfrak{h}}_-))[[x,x^{-1}]] \\ v &\mapsto Y_{T(\hat{\mathfrak{h}}_-)}(u,x)=\sum_{n\in\mathbb{Z}}u_nx^{-n-1} \end{aligned}$$

where $Y_{T(\hat{\mathfrak{h}}_-)}(u,x)$ is defined in (4.1) with $W=T(\hat{\mathfrak{h}}_-)$.

We have the following main result of the present paper:

Theorem 5.1. *The triple $(T(\hat{\mathfrak{h}}_-),Y_{T(\hat{\mathfrak{h}}_-)},\mathbf{1})$ defined above is a meromorphic open-string vertex algebra. In the case that $\mathfrak{h}$ is finite dimensional, $(T(\hat{\mathfrak{h}}_-),Y_{T(\hat{\mathfrak{h}}_-)},\mathbf{1})$ is a grading-restricted meromorphic open-string vertex algebra.*

Proof. The lower bound condition, the identity property and the creation property are easy to verify. We omit the proofs. It is also clear that when $\mathfrak{h}$ is finite dimensional, the homogeneous subspaces of $T(\hat{\mathfrak{h}}_-)$ are finite dimensional.

We first prove that the product (2.1) is absolutely convergent in the region $|z_1|>|z_2|>0$ to a rational function. By Corollary 4.11,

$$\begin{aligned} &\langle v',Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots a_{k_1}(-m_{k_1})\mathbf{1},z_1)\cdot \\ &\qquad\cdots Y_{T(\hat{\mathfrak{h}}_-)}(a_{k_{l-1}+1}(-m_{k_{l-1}+1})\cdots a_n(-m_n)\mathbf{1},z_l)v\rangle \qquad (5.1) \end{aligned}$$

is a linear combination of series of the form

$$\prod_{(j_1,j_2)\in A}(y_{j_1}-y_{j_2})^{-m_{j_1}-m_{j_2}}\big|_{y_j=z_{p+1}\text{ when }k_p+1\le j\le k_{p+1}}\cdot$$
$$\cdot\left\langle v', {}^{\circ}_{\circ}\prod_{j\notin B}\left(\frac{1}{(m_j-1)!}\frac{\partial^{m_j-1}}{\partial y_j^{m_j-1}}a_j(y_j)\right){}^{\circ}_{\circ}v\right\rangle\Bigg|_{y_j=z_{p+1}\text{ when }k_p+1\le j\le k_{p+1}}, \tag{5.2}$$

where A and B are the same as in Corollary 4.10. Since

$$\left\langle v', {}^{\circ}_{\circ}\prod_{j\notin B}\left(\frac{1}{(m_j-1)!}\frac{\partial^{m_j-1}}{\partial y_j^{m_j-1}}a_j(y_j)\right){}^{\circ}_{\circ}v\right\rangle\Bigg|_{y_j=z_{p+1}\text{ when }k_p+1\le j\le k_{p+1}}$$

is a Laurent polynomial in $z_1,\dots,z_l$, (5.2) is the expansion in the region $|z_1|>\cdots>|z_l|>0$ of a rational function in $z_1,\dots,z_l$ with the only possible poles at $z_i=0$ for $i=1,\dots,l$ and $z_i=z_j$ for $i\neq j$. Since (5.1) is a linear combination of series of the form (5.2), it is also the expansion in the region $|z_1|>\cdots>|z_l|>0$ of a rational function in $z_1,\dots,z_l$ with the only possible poles at $z_i=0$ for $i=1,\dots,l$ and $z_i=z_j$ for $i\neq j$. Thus the rationality for products of vertex operators holds.

In particular, we have rationality for products of two vertex operators. But in order to prove the associativity, we need an explicit expression of the products of two vertex operators. By (4.12), we have

$$\begin{aligned}
&\langle v', Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1},z_1)Y_{T(\hat{\mathfrak{h}}_-)}(b_1(-n_1)\cdots b_l(-n_l)\mathbf{1},z_2)v\rangle\\
&\quad=\sum_{i=0}^{\min(k,l)}\sum_{\substack{k\ge p_1>\cdots>p_i\ge 1\\ 1\le\text{ distinct }q_1,\dots,q_i\le l}} n_{q_1}\cdots n_{q_i}(a_{p_1},b_{q_1})\cdots(a_{p_i},b_{q_i})\cdot\\
&\qquad\cdot\binom{-n_{q_1}-1}{m_{p_1}-1}\cdots\binom{-n_{q_i}-1}{m_{p_i}-1}(z_1-z_2)^{-m_{p_1}-n_{q_1}-\cdots-m_{p_i}-n_{q_i}}\cdot\\
&\qquad\cdot\left\langle v', {}^{\circ}_{\circ}\left(\prod_{p\neq p_1,\dots,p_i}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial z_1^{m_p-1}}a_p(z_1)\right)\cdot\right.\\
&\qquad\qquad\left.\cdot\left(\prod_{q\neq q_1,\dots,q_i}\frac{1}{(n_q-1)!}\frac{\partial^{n_q-1}}{\partial z_2^{n_q-1}}b_q(z_2)\right){}^{\circ}_{\circ}v\right\rangle.
\end{aligned}\tag{5.3}$$

Next we discuss iterates of two vertex operators. As in the theory of vertex operator algebras, for a formal Laurent series $b(y)$ in y, we use $\mathrm{Res}_y b(y)$ to denote the coefficient of y^{-1} in $b(y)$. Using this notation, we have

$$b(-n) = \mathrm{Res}_y y^{-n} b(y)$$

for $n \in \mathbb{Z}$ and $b(y) = \sum_{m\in\mathbb{Z}} b(m) y^{-m-1}$. From (4.7), we have

$$\begin{aligned}
&Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}, x_0) b_1(-n_1)\cdots b_l(-n_l)\mathbf{1}\\
&\quad= \mathrm{Res}_{y_1}\cdots \mathrm{Res}_{y_l} y_1^{-n_1}\cdots y_l^{-n_l}\cdot\\
&\qquad\qquad \cdot {}^{\circ}_{\circ}\left(\frac{1}{(m_1-1)!}\frac{\partial^{m_1-1}}{\partial x_0^{m_1-1}}a_1(x_0)\right)\cdots\left(\frac{1}{(m_k-1)!}\frac{\partial^{m_k-1}}{\partial x_0^{m_k-1}}a_k(x_0)\right){}^{\circ}_{\circ}\cdot\\
&\qquad\qquad \cdot {}^{\circ}_{\circ} b_1(y_1)\cdots b_l(y_l){}^{\circ}_{\circ}\mathbf{1}\\
&\quad= \mathrm{Res}_{y_1}\cdots \mathrm{Res}_{y_l} y_1^{-n_1}\cdots y_l^{-n_l}\cdot\\
&\qquad\qquad \cdot \sum_{i=0}^{\min(k,l)} \sum_{\substack{k\geq p_1>\cdots>p_i\geq 1\\ 1\leq \text{distinct } q_1,\dots,q_i\leq l}} (a_{p_1}, b_{q_1})\cdots(a_{p_i}, b_{q_i})\cdot\\
&\qquad\qquad\quad \cdot\binom{-2}{m_{p_1}-1}\cdots\binom{-2}{m_{p_i}-1}\cdot\\
&\qquad\qquad\quad \cdot(x_0-y_{q_1})^{-m_{p_1}-1}\cdots(x_0-y_{q_i})^{-m_{p_i}-1}\cdot\\
&\qquad\qquad\quad \cdot {}^{\circ}_{\circ}\left(\prod_{p\neq p_1,\dots,p_i}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial x_0^{m_p-1}}a_p(x_0)\right)\left(\prod_{q\neq q_1,\dots,q_i} b_q(y_q)\right){}^{\circ}_{\circ}\mathbf{1}\\
&\quad= \sum_{i=0}^{\min(k,l)} \sum_{\substack{k\geq p_1>\cdots>p_i\geq 1\\ 1\leq \text{distinct } q_1,\dots,q_i\leq l}} n_{q_1}\cdots n_{q_l}(a_{p_1}, b_{q_1})\cdots(a_{p_i}, b_{q_i})\cdot\\
&\qquad\qquad\quad \cdot\binom{-n_{q_1}-1}{m_{p_1}-1}\cdots\binom{-n_{q_1}-1}{m_{p_i}-1} x_0^{-m_{p_1}-n_{q_1}-\cdots-m_{p_i}-n_{q_i}}\cdot\\
&\qquad\qquad\quad \cdot {}^{\circ}_{\circ}\left(\prod_{p\neq p_1,\dots,p_i}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial x_0^{m_p-1}}a_p^-(x_0)\right)\left(\prod_{q\neq q_1,\dots,q_i} b_q(-n_q)\right){}^{\circ}_{\circ}\mathbf{1}\\
&\quad= \sum_{i=0}^{\min(k,l)} \sum_{\substack{k\geq p_1>\cdots>p_i\geq 1\\ 1\leq \text{distinct } q_1,\dots,q_i\leq l}} n_{q_1}\cdots n_{q_l}(a_{p_1}, b_{q_1})\cdots(a_{p_i}, b_{q_i})\cdot
\end{aligned}$$

$$
\cdot\binom{-n_{q_1}-1}{m_{p_1}-1}\cdots\binom{-n_{q_1}-1}{m_{p_i}-1}x_0^{-m_{p_1}-n_{q_1}-\cdots-m_{p_i}-n_{q_i}}\cdot
$$
$$
\cdot\left(\prod_{p\neq p_1,\ldots,p_i}\sum_{s_p\in\mathbb{Z}_+}\binom{s_p-1}{m_p-1}a_p(-s_p)x_0^{s_p-m_p}\right)\cdot
$$
$$
\cdot\left(\prod_{q\neq q_1,\ldots,q_i}b_q(-n_q)\right)\mathbf{1}. \tag{5.4}
$$

Thus

$$
\begin{aligned}
&Y_{T(\hat{\mathfrak{h}}_-)}(Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1},x_0)b_1(-n_1)\cdots b_l(-n_l)\mathbf{1},x_2)\\
&=\sum_{i=0}^{\min(k,l)}\sum_{\substack{k\geq p_1>\cdots>p_i\geq 1\\ 1\leq \text{distinct } q_1,\ldots,q_i\leq l}} n_{q_1}\cdots n_{q_l}(a_{p_1},b_{q_1})\cdots(a_{p_i},b_{q_i})\cdot\\
&\qquad\cdot\binom{-n_{q_1}-1}{m_{p_1}-1}\cdots\binom{-n_{q_1}-1}{m_{p_i}-1}x_0^{-m_{p_1}-n_{q_1}-\cdots-m_{p_i}-n_{q_i}}\cdot\\
&\qquad\cdot{}^{\circ}_{\circ}\left(\prod_{p\neq p_1,\ldots,p_i}\sum_{s_p\in\mathbb{Z}_+}\binom{s_p-1}{m_p-1}\cdot\right.\\
&\qquad\qquad\left.\cdot\left(\frac{1}{(s_p-1)!}\frac{\partial^{s_p-1}}{\partial x_2^{s_p-1}}a_p(x_2)\right)x_0^{s_p-m_p}\right)\cdot\\
&\qquad\cdot\left(\prod_{q\neq q_1,\ldots,q_i}\left(\frac{1}{(n_q-1)!}\frac{\partial^{n_q-1}}{\partial x_2^{n_q-1}}b_q(x_2)\right)\right){}^{\circ}_{\circ}\\
&=\sum_{i=0}^{\min(k,l)}\sum_{\substack{k\geq p_1>\cdots>p_i\geq 1\\ 1\leq \text{distinct } q_1,\ldots,q_i\leq l}} n_{q_1}\cdots n_{q_l}(a_{p_1},b_{q_1})\cdots(a_{p_i},b_{q_i})\cdot\\
&\qquad\cdot\binom{-n_{q_1}-1}{m_{p_1}-1}\cdots\binom{-n_{q_1}-1}{m_{p_i}-1}x_0^{-m_{p_1}-n_{q_1}-\cdots-m_{p_i}-n_{q_i}}\cdot\\
&\qquad\cdot{}^{\circ}_{\circ}\left(\prod_{p\neq p_1,\ldots,p_i}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial x_0^{m_p-1}}\cdot\right.
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad \cdot \sum_{s_p\in\mathbb{Z}_+}\left(\left(\frac{1}{(s_p-1)!}\frac{\partial^{s_p-1}}{\partial x_2^{s_p-1}}a_p(x_2)\right)x_0^{s_p-1}\right)\cdot\\
&\qquad \cdot\left(\prod_{q\neq q_1,\dots,q_i}\left(\frac{1}{(n_q-1)!}\frac{\partial^{n_q-1}}{\partial x_2^{n_q-1}}b_q(x_2)\right)\right)\,{}^{\circ}_{\circ}\\
&= \sum_{i=0}^{\min(k,l)}\sum_{\substack{k\geq p_1>\cdots>p_i\geq 1\\ 1\leq \text{distinct } q_1,\dots,q_i\leq l}} n_{q_1}\cdots n_{q_l}(a_{p_1},b_{q_1})\cdots(a_{p_i},b_{q_i})\cdot\\
&\qquad \cdot\binom{-n_{q_1}-1}{m_{p_1}-1}\cdots\binom{-n_{q_1}-1}{m_{p_i}-1}x_0^{-m_{p_1}-n_{q_1}-\cdots-m_{p_i}-n_{q_i}}\cdot\\
&\qquad \cdot{}^{\circ}_{\circ}\left(\prod_{p\neq p_1,\dots,p_i}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial x_0^{m_p-1}}a_p(x_2+x_0)\right)\cdot\\
&\qquad \cdot\left(\prod_{q\neq q_1,\dots,q_i}\left(\frac{1}{(n_q-1)!}\frac{\partial^{n_q-1}}{\partial x_2^{n_q-1}}b_q(x_2)\right)\right){}^{\circ}_{\circ}.
\end{aligned}
\tag{5.5}
$$

For $v\in T(\hat{\mathfrak{h}}_-)$, $v'\in T(\hat{\mathfrak{h}}_-)'$, from (5.5) we see that when $|z_2|>|z_1-z_2|>0$, the series

$$
\begin{aligned}
&\langle v', Y_{T(\hat{\mathfrak{h}}_-)}(Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1},z_1-z_2)b_1(-n_1)\cdots b_l(-n_l)\mathbf{1},z_2)v\rangle\\
&= \sum_{i=0}^{\min(k,l)}\sum_{\substack{k\geq p_1>\cdots>p_i\geq 1\\ 1\leq \text{distinct } q_1,\dots,q_i\leq l}} n_{q_1}\cdots n_{q_l}(a_{p_1},b_{q_1})\cdots(a_{p_i},b_{q_i})\cdot\\
&\qquad \cdot\binom{-n_{q_1}-1}{m_{p_1}-1}\cdots\binom{-n_{q_1}-1}{m_{p_i}-1}(z_1-z_2)^{-m_{p_1}-n_{q_1}-\cdots-m_{p_i}-n_{q_i}}\cdot\\
&\qquad \cdot\left\langle v', {}^{\circ}_{\circ}\left(\prod_{p\neq p_1,\dots,p_i}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial x_0^{m_p-1}}a_p(x_2+x_0)\right)\cdot\right.\\
&\qquad \left.\left.\cdot\left(\prod_{q\neq q_1,\dots,q_i}\left(\frac{1}{(n_q-1)!}\frac{\partial^{n_q-1}}{\partial x_2^{n_q-1}}b_q(x_2)\right)\right){}^{\circ}_{\circ}v\right\rangle\right|_{x_0=z_1-z_2,\ x_2=z_2}
\end{aligned}
\tag{5.6}
$$

is absolutely convergent to the same rational function to which (5.3) converges to. Thus rationality for iterates of two vertex operators and associativity hold.

We now prove the $\mathbf{d}$-bracket property. From the definitions of the $\mathbb{Z}$-grading on $T(\hat{\mathfrak{h}}_-)$, the operator $\mathbf{d}_{T(\hat{\mathfrak{h}}_-)}$ and $a(n)$ for $a \in \mathfrak{h}$ and $n \in \mathbb{Z}$, we have

$$[\mathbf{d}_{T(\hat{\mathfrak{h}}_-)}, a(n)] = -na(n).$$

Then for $a \in \mathfrak{h}$,

$$[\mathbf{d}_{T(\hat{\mathfrak{h}}_-)}, a^{\pm}(x)] = a^{\pm}(x) + x\frac{d}{dx}a^{\pm}(x). \tag{5.7}$$

For $m \in \mathbb{Z}_+$, taking $m-1$-st derivatives with respect to x in both sides of (5.7), we obtain

$$\begin{aligned}&\left[\mathbf{d}_{T(\hat{\mathfrak{h}}_-)}, \frac{1}{(m-1)!}\frac{d^{m-1}}{dx^{m-1}}a^{\pm}(x)\right]\\ &\quad = m\frac{1}{(m-1)!}\frac{d^{m-1}}{dx^{m-1}}a^{\pm}(x) + x\frac{d}{dx}\left(\frac{1}{(m-1)!}\frac{d^{m-1}}{dx^{m-1}}a^{\pm}(x)\right). \end{aligned}\tag{5.8}$$

We also have

$$\begin{aligned}&\left[\mathbf{d}_{T(\hat{\mathfrak{h}}_-)}, \frac{1}{(m-1)!}\frac{d^{m-1}}{dx^{m-1}}(a(0)x^{-1})\right]\\ &\quad = \frac{1}{(m-1)!}\frac{d^{m-1}}{dx^{m-1}}(a(0)x^{-1}) + x\frac{d}{dx}\left(\frac{1}{(m-1)!}\frac{d^{m-1}}{dx^{m-1}}(a(0)x^{-1})\right). \end{aligned}\tag{5.9}$$

Using (5.8), (5.9) and (4.3), we obtain the $\mathbf{d}_{T(\hat{\mathfrak{h}}_-)}$-bracket property.

We still need to prove the D-derivative property and the D-commutator formula. By definition,

$$\begin{aligned}&\frac{d}{dx}Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}, x)\\ &\quad = \frac{d}{dx}\left({\circ\atop\circ}\left(\frac{1}{(m_1-1)!}\frac{\partial^{m_1-1}}{\partial x^{m_1-1}}a_1(x)\right)\cdots\left(\frac{1}{(m_k-1)!}\frac{\partial^{m_k-1}}{\partial x^{m_k-1}}a_k(x)\right){\circ\atop\circ}\right)\\ &\quad = \sum_{p=1}^{k} m_p {\circ\atop\circ}\left(\frac{1}{(m_1-1)!}\frac{\partial^{m_1-1}}{\partial x^{m_1-1}}a_1(x)\right)\cdot\\ &\qquad\qquad \cdots\left(\frac{1}{(m_{p-1}-1)!}\frac{\partial^{m_{p-1}-1}}{\partial x^{m_{p-1}-1}}a_{p-1}(x)\right)\left(\frac{1}{m_p!}\frac{\partial^{m_p}}{\partial x^{m_p}}a_p(x)\right)\cdot\end{aligned}$$

$$\cdot \left(\frac{1}{(m_{p+1}-1)!} \frac{\partial^{m_{p+1}-1}}{\partial x^{m_{p+1}-1}} a_{p+1}(x) \right) \cdot \cdots \left(\frac{1}{(m_k-1)!} \frac{\partial^{m_k-1}}{\partial x^{m_k-1}} a_k(x) \right) \circ\atop\circ$$

$$= \sum_{p=1}^{k} m_p Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots a_{p-1}(-m_{p-1}) \cdot \cdot a_p(-(m_p+1))a_{p+1}(-m_{p+1})\cdots a_k(-m_k)\mathbf{1}, x). \quad (5.10)$$

From (5.10), we obtain

$$\begin{aligned} &D_{T(\hat{\mathfrak{h}}_-)} a_1(-m_1)\cdots a_k(-m_k)\mathbf{1} \\ &\quad = \lim_{x\to 0} \frac{d}{dx} Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}, x)\mathbf{1} \\ &\quad = \sum_{p=1}^{k} m_p \lim_{x\to 0} Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots a_{p-1}(-m_{p-1}) \cdot \\ &\qquad\qquad \cdot a_p(-(m_p+1))a_{p+1}(-m_{p+1})\cdots a_k(-m_k)\mathbf{1}, x) \\ &\quad = \sum_{p=1}^{k} m_p a_1(-m_1)\cdots a_{p-1}(-m_{p-1}) \cdot \\ &\qquad\qquad \cdot a_p(-(m_p+1))a_{p+1}(-m_{p+1})\cdots a_k(-m_k)\mathbf{1}. \end{aligned} \quad (5.11)$$

From (5.10) and (5.11), we obtain

$$\begin{aligned} &\frac{d}{dx} Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}, x) \\ &\quad = Y_{T(\hat{\mathfrak{h}}_-)}\left(\sum_{p=1}^{k} m_p a_1(-m_1)\cdots a_{p-1}(-m_{p-1}) \cdot \right. \\ &\qquad\qquad \left. \cdot a_p(-(m_p+1))a_{p+1}(-m_{p+1})\cdots a_k(-m_k)\mathbf{1}, x \right) \\ &\quad = Y_{T(\hat{\mathfrak{h}}_-)}(D a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}, x). \end{aligned} \quad (5.12)$$

To prove

$$\begin{aligned} &\frac{d}{dx} Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}, x) \\ &\quad = [D_{T(\hat{\mathfrak{h}}_-)}, Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}, x)] \end{aligned} \quad (5.13)$$

for $a_1, \ldots, a_k \in \mathfrak{h}$ and $m_1, \ldots, m_k \in \mathbb{Z}_+$, we use induction on k. When $k = 0$, (5.13) holds. We also need to prove (5.13) in the case $k = 1$. From (5.11), we have

$$[D_{T(\hat{\mathfrak{h}}_-)}, a_1(-m_1)] = ma_1(-m-1) \tag{5.14}$$

for $a_1 \in \mathfrak{h}$ and $m \in \mathbb{Z}_+$. For $b_1, \ldots, b_l \in \mathfrak{h}$ and $n_1, \ldots, n_l \in \mathbb{Z}_+$, we have

$$\begin{aligned}
&[D_{T(\hat{\mathfrak{h}}_-)}, a_1(m)]b_1(-n_1)\cdots b_l(-n_l)\mathbf{1}\\
&\quad = D_{T(\hat{\mathfrak{h}}_-)}a_1(m)b_1(-n_1)\cdots b_l(-n_l)\mathbf{1} - a_1(m)D_{T(\hat{\mathfrak{h}}_-)}b_1(-n_1)\cdots b_l(-n_l)\mathbf{1}\\
&\quad = \sum_{p=1}^{l} m(a_1, b_p)\delta_{m-n_p,0}Db_1(-n_1)\cdots \widehat{b_p(-n_p)}\cdots b_l(-n_l)\mathbf{1}\\
&\qquad - \sum_{p=1}^{l} n_p a_1(m)b_1(-n_1)\cdots b_{p-1}(-n_{p-1})\cdot\\
&\qquad\qquad\qquad \cdot b_p(-(n_p+1))b_{p+1}(-n_{p+1})\cdots b_l(-n_l)\mathbf{1}\\
&\quad = \sum_{p=1}^{l}\sum_{q\neq p} mn_q(a_1, b_p)\delta_{m-n_p,0}\cdot\\
&\qquad\qquad\qquad \cdot b_1(-n_1)\cdots \widehat{b_p(-n_p)}\cdots b_q(-(n_q+1)\cdots b_l(-n_l)\mathbf{1}\\
&\qquad - \sum_{p=1}^{l}\sum_{q\neq p} mn_p(a_1, b_q)\delta_{m-n_q,0}b_1(-n_1)\cdots \widehat{b_q(-n_q)}\cdots b_{p-1}(-n_{p-1})\cdot\\
&\qquad\qquad\qquad \cdot b_p(-(n_p+1))b_{p+1}(-n_{p+1})\cdots b_l(-n_l)\mathbf{1}\\
&\qquad - \sum_{p=1}^{l} mn_p(a_1, b_p)\delta_{m-n_p-1,0}b_1(-n_1)\cdots b_{p-1}(-n_{p-1})\cdot\\
&\qquad\qquad\qquad \cdot b_{p+1}(-n_{p+1})\cdots b_l(-n_l)\mathbf{1}\\
&\quad = -m\sum_{p=1}^{l}(m-1)(a_1, b_p)\delta_{m-n_p-1,0}b_1(-n_1)\cdots b_{p-1}(-n_{p-1})\cdot\\
&\qquad\qquad\qquad \cdot b_{p+1}(-n_{p+1})\cdots b_l(-n_l)\mathbf{1}\\
&\quad = -ma_1(m-1)b_1(-n_1)\cdots b_l(-n_l)\mathbf{1}.
\end{aligned} \tag{5.15}$$

Thus we obtain

$$[D_{T(\hat{\mathfrak{h}}_-)}, a_1(m)] = -ma_1(m-1) \tag{5.16}$$

for $a_1 \in \mathfrak{h}$ and $m \in \mathbb{Z}_+$. The commutator formula (5.14) says that (5.16) holds when $m \in -\mathbb{Z}_+$. Clearly (5.16) also holds when $m = 0$. From (5.16) for $m \in \mathbb{Z}$, we obtain

$$\begin{aligned}
&[D_{T(\hat{\mathfrak{h}}_-)}, Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\mathbf{1}, x)] \\
&\quad = \left[D_{T(\hat{\mathfrak{h}}_-)}, \frac{1}{(m_1-1)!}\frac{d^{m_1-1}}{dx^{m_1-1}}a_1(x)\right] = \frac{1}{(m_1-1)!}\frac{d^{m_1}}{dx^{m_1}}a_1(x) \\
&\quad = \frac{d}{dx}Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\mathbf{1}, x) \qquad (5.17)
\end{aligned}$$

for $a_1 \in \mathfrak{h}$ and $m_1 \in \mathbb{Z}_+$, proving (5.13) in the case $k = 1$.

Now assume that (5.13) holds when $k = K$. For $a_0, a_1, \ldots, a_k \in \mathfrak{h}$ and $m_0, m_1, \ldots, m_k \in \mathbb{Z}_+$, from (4.6) and (5.13) in the case $k = 1$ and $k = K$, we obtain

$$\begin{aligned}
&\frac{d}{dx}Y_{T(\hat{\mathfrak{h}}_-)}(a_0(-m_0)a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}, x) \\
&\quad = \lim_{x_2\to x}\left(\frac{d}{dx}+\frac{d}{dx_2}\right)\cdot \\
&\qquad\qquad \cdot\Bigg(Y_{T(\hat{\mathfrak{h}}_-)}(a_0(-m_0)\mathbf{1}, x)Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}, x_2) \\
&\qquad\qquad\qquad -\sum_{p=1}^{k} m_p(a_0, a_p)\binom{-m_p-1}{m_0-1}(x-x_2)^{-m_0-m_p}\cdot \\
&\qquad\qquad\qquad\qquad \cdot Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots \widehat{a_p(-m_p)}\cdots a_k(-m_k)\mathbf{1}, x_2)\Bigg) \\
&\quad = \lim_{x_2\to x}\Bigg(\frac{d}{dx}Y_{T(\hat{\mathfrak{h}}_-)}(a_0(-m_0)\mathbf{1}, x)Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}, x_2) \\
&\qquad\qquad\qquad +Y_{T(\hat{\mathfrak{h}}_-)}(a_0(-m_0)\mathbf{1}, x)\frac{d}{dx_2}Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}, x_2) \\
&\qquad\qquad\qquad -\sum_{p=1}^{k} m_p(a_0, a_p)\binom{-m_p-1}{m_0-1}(x-x_2)^{-m_0-m_p}\cdot \\
&\qquad\qquad\qquad\qquad \cdot\frac{d}{dx_2}Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots \widehat{a_p(-m_p)}\cdots a_k(-m_k)\mathbf{1}, x_2)\Bigg) \\
&\quad = \lim_{x_2\to x}\Bigg([D_{T(\hat{\mathfrak{h}}_-)}, Y_{T(\hat{\mathfrak{h}}_-)}(a_0(-m_0)\mathbf{1}, x)]Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}, x_2) \\
&\qquad\qquad +Y_{T(\hat{\mathfrak{h}}_-)}(a_0(-m_0)\mathbf{1}, x)[D_{T(\hat{\mathfrak{h}}_-)}, Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}, x_2)]
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{p=1}^{k} m_p(a_0,a_p)\binom{-m_p-1}{m_0-1}(x-x_2)^{-m_0-m_p}\cdot\\
&\qquad\cdot[D_{T(\hat{\mathfrak{h}}_-)}, Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots \widehat{a_p(-m_p)}\cdots a_k(-m_k)\mathbf{1}, x_2)]\Big)\\
&=\Big[D_{T(\hat{\mathfrak{h}}_-)}, \lim_{x_2\to x}\Big(Y_{T(\hat{\mathfrak{h}}_-)}(a_0(-m_0)\mathbf{1}, x)Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}, x_2)\\
&\qquad -\sum_{p=1}^{k} m_p(a_0,a_p)\binom{-m_p-1}{m_0-1}(x-x_2)^{-m_0-m_p}\cdot\\
&\qquad\qquad\cdot Y_{T(\hat{\mathfrak{h}}_-)}(a_1(-m_1)\cdots \widehat{a_p(-m_p)}\cdots a_k(-m_k)\mathbf{1}, x_2)\Big)\Big]\\
&=[D_{T(\hat{\mathfrak{h}}_-)}, Y_{T(\hat{\mathfrak{h}}_-)}(a_0(-m_0)a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}, x)], \qquad (5.18)
\end{aligned}
$$

proving (5.13) in the case $k=K$. ∎

Remark 5.2. The symmetric algebra $S(\hat{\mathfrak{h}}_-)$ has a natural structure of vertex operator algebra (see [B] and [FLM], where $S(\mathfrak{h}_-)$ is constructed as a subalgebra of the vertex operator algebra associated to a even positive definite lattice). In particular, by Remark 2.2, it is a grading-restricted meromorphic open-string vertex algebra. Let $\pi : T(\hat{\mathfrak{h}}_-) \to S(\hat{\mathfrak{h}}_-)$ be the canonical projection. Then π is a homomorphism of meromorphic open-string vertex algebras from $T(\hat{\mathfrak{h}}_-)$ to $S(\hat{\mathfrak{h}}_-)$. It is clear that the kernel of a homomorphism of meromorphic open-string vertex algebras is a subalgebra of the first meromorphic open-string vertex algebra and the quotient of a meromorphic open-string vertex algebra by a subalgebra is a meromorphic open-string vertex algebra. Thus we see that $S(\hat{\mathfrak{h}}_-)$ as a meromorphic open-string vertex algebra is isomorphic to a quotient of the meromorphic open-string vertex algebra $T(\hat{\mathfrak{h}}_-)$.

We end this section by calculating explicitly certain "four-point functions" and to show by these explicit examples that the commutativity does not hold.

Example 5.3. We first calculate the "four-point function" (with the four points $\infty, z_1, z_2, 0$)

$$
\langle v', Y_{T(\hat{\mathfrak{h}}_-)}(a(-1)\mathbf{1}, z_1)Y_{T(\hat{\mathfrak{h}}_-)}(b(-1)\mathbf{1}, z_2)c(-k)\mathbf{1}\rangle \qquad (5.19)
$$

for $a, b, c, d \in \mathfrak{h}$, $k \in \mathbb{Z}_+$, $v' \in T(\hat{\mathfrak{h}}_-)^*_{(k)} \oplus T(\hat{\mathfrak{h}}_-)^*_{(k+3)} \subset T(\hat{\mathfrak{h}}_-)'$ satisfying

$$\langle v', a(-2)b(-1)c(-k)\mathbf{1}\rangle = \langle v', a(-k)\mathbf{1}\rangle = \langle v', b(-k)\mathbf{1}\rangle = \langle v', c(-k)\mathbf{1}\rangle = 1$$

and

$$\langle v', a(-1)b(-2)c(-k)\mathbf{1}\rangle = \langle v', a(-k-3)\mathbf{1}\rangle = \langle v', b(-k-3)\mathbf{1}\rangle = 0$$

As a special case of (5.3), we see that (5.19) is absolutely convergent in the region $|z_1| > |z_2| > 0$ and is equal to

$$\begin{aligned}
&\langle v', {}^{\circ}_{\circ}a(z_1)b(z_2){}^{\circ}_{\circ}c(-k)\mathbf{1}\rangle \\
&\quad +(a, b)(z_1 - z_2)^{-2}\langle v', c(-k)\mathbf{1}\rangle \\
&= \sum_{m,n\in\mathbb{Z}} \langle v', {}^{\circ}_{\circ}a(m)b(n){}^{\circ}_{\circ}c(-k)\mathbf{1}\rangle z_1^{-m-1}z_2^{-n-1} \\
&\quad +(a, b)(z_1 - z_2)^{-2} \\
&= \sum_{m,n\in\mathbb{Z}_+} \langle v', a(m)b(n)c(-k)\mathbf{1}\rangle z_1^{-m-1}z_2^{-n-1} \\
&\quad + \sum_{m,n\in-\mathbb{Z}_+} \langle v', a(m)b(n)c(-k)\mathbf{1}\rangle z_1^{-m-1}z_2^{-n-1} \\
&\quad + \sum_{m\in Z_+, n\in-\mathbb{Z}_+} \langle v', b(n)a(m)c(-k)\mathbf{1}\rangle z_1^{-m-1}z_2^{-n-1} \\
&\quad + \sum_{m\in-Z_+, n\in\mathbb{Z}_+} \langle v', a(m)b(n)c(-k)\mathbf{1}\rangle z_1^{-m-1}z_2^{-n-1} \\
&\quad + \sum_{m\in Z\setminus\{0\}} \langle v', a(m)b(0)c(-k)\mathbf{1}\rangle z_1^{-m-1}z_2^{-1} \\
&\quad + \sum_{n\in Z\setminus\{0\}} \langle v', b(n)a(0)c(-k)\mathbf{1}\rangle z_1^{-1}z_2^{-n-1} \\
&\quad +\langle v', a(0)b(0)c(-k)\mathbf{1}\rangle z_1^{-1}z_2^{-1} \\
&\quad +(a, b)(z_1 - z_2)^{-2} \\
&= z_1 + k(a, c)z_1^{-k-1}z_2^{k-1} + k(b, c)z_1^{k-1}z_2^{-k-1} + (a, b)(z_1 - z_2)^{-2}.
\end{aligned} \tag{5.20}$$

If we require in addition that v' satisfies

$$\langle v', b(-2)a(-1)c(-k)\mathbf{1}\rangle 1$$

and

$$\langle v', b(-1)a(-2)c(-k)\mathbf{1}\rangle = 0$$

(note that it is possible to find such a, b, c, k and v' when $\dim \mathfrak{h} \geq 2$), then the same calculation gives

$$\begin{aligned} &\langle v', Y_{T(\hat{\mathfrak{h}}_-)}(b(-1)\mathbf{1}, z_2) Y_{T(\hat{\mathfrak{h}}_-)}(a(-1)\mathbf{1}, z_1) c(-k)\mathbf{1}\rangle \\ &\quad = z_2 + k(a,c) z_1^{-k-1} z_2^{k-1} + k(b,c) z_1^{k-1} z_2^{-k-1} + (a,b)(z_1 - z_2)^{-2} \end{aligned} \tag{5.21}$$

in the region $|z_2| > |z_1| > 0$. Note that (5.20) and the right-hand side of (5.21) are different. This is an explicit example showing that the commutativity does not hold for the meromorphic open-string vertex algebra $T(\hat{\mathfrak{h}}_-)$.

6 Left modules for the meromorphic open-string vertex operator algebra $T(\hat{\mathfrak{h}}_-)$

In this section, we introduce the notion of left module for a meromorphic open-string vertex operator algebra. Then we construct a structure of a left module for the meromorphic open-string vertex algebra $T(\hat{\mathfrak{h}}_-)$ on the left $N(\hat{\mathfrak{h}})$-module $T(\hat{\mathfrak{h}}_-) \otimes M$ for a left $T(\mathfrak{h})$-module M.

Definition 6.1. Let $(V, Y_V, \mathbf{1})$ be a meromorphic open-string vertex algebra. A *module for V* or a *V-module* is a $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{(n)}$ (graded by *weights*), equipped with a *vertex operator map*

$$\begin{aligned} Y_W : V &\to (\mathrm{End}\, W)[[x, x^{-1}]] \\ u &\mapsto Y_W(u, x), \end{aligned}$$

or equivalently,

$$\begin{aligned} Y_W : V \otimes W &\to W[[x, x^{-1}]] \\ u \otimes w &\mapsto Y_W(u, x)w, \end{aligned}$$

an operator D_W of weight 1, satisfying the following conditions:

1. *Lower bound condition*: When $\Re(n)$ is sufficiently negative, $W_{(n)} = 0$
2. The *identity property*: $Y_W(\mathbf{1}, x) = 1_W$.

3. *Rationality*: For $u_1, \ldots, u_n, w \in W$ and $w' \in W'$, the series

$$\langle w', Y_W(u_1, z_1) \cdots Y_W(u_n, z_n)w\rangle \tag{6.1}$$

converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function in $z_1, \ldots, z_n$ with the only possible poles at $z_i = 0$ for $i = 1, \ldots, n$ and $z_i = z_j$ for $i \neq j$. For $u_1, u_2, w \in W$ and $w' \in W'$, the series

$$\langle w', Y_W(Y_V(u_1, z_1 - z_2)u_2, z_2)w\rangle \tag{6.2}$$

converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to a rational function with the only possible poles at $z_1 = 0$, $z_2 = 0$ and $z_1 = z_2$.

4. *Associativity*: For $u_1, u_2, w \in W$, $w' \in W'$,

$$\langle w', Y_W(u_1, z_1)Y_W(u_2, z_2)w\rangle = \langle w', Y_W(Y_V(u_1, z_1 - z_2)u_2, z_2)w\rangle \tag{6.3}$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

5. $\mathbf{d}$-*bracket property*: Let $\mathbf{d}_W$ be the grading operator on W, that is, $\mathbf{d}_W w = mw$ for $m \in \mathbb{R}$ and $w \in W_{(m)}$. For $u \in V$,

$$[\mathbf{d}_W, Y_W(u, x)] = Y_W(\mathbf{d}_V u, x) + x\frac{d}{dx}Y_W(u, x). \tag{6.4}$$

6. The D-*derivative property* and the D-*commutator formula*: For $u \in V$,

$$\begin{aligned} \frac{d}{dx}Y_W(u, x) &= Y_W(D_V u, x) \\ &= [D_W, Y_W(u, x)]. \end{aligned} \tag{6.5}$$

A left V-module is said to be *grading restricted* if $\dim W_{(n)} < \infty$ for $n \in \mathbb{C}$.

We denote the left V-module just defined by (W, Y_W, D_W).

Remark 6.2. Let V be a $\mathbb{Z}$-graded vertex algebra such that the $\mathbb{Z}$-grading is lower bounded. By Remark 2.2, V is a meromorphic open-string vertex algebra. Then a V-module is a left module for the meromorphic open-string vertex algebra structure.

Definition 6.3. Let $(V, Y_V, \mathbf{1})$ be a meromorphic open-string vertex algebra and (W_1, Y_{W_1}, D_{W_1}) and (W_1, Y_{W_1}, D_{W_1}) left V-modules. A *homomorphism* or *module map* from (W_1, Y_{W_1}, D_{W_1}) to (W_1, Y_{W_1}, D_{W_1}) is a linear map $f : W_1 \to W_2$ such that

$$\begin{aligned} f(Y_{W_1}(u,x)w) &= Y_{W_2}(u,x)f(w), \\ f(D_{W_1}w) &= D_{W_2}f(w) \end{aligned}$$

for $u \in V$ and $w \in W_1$. A *grading-preserving homomorphism* of left V-modules is a homomorphism preserving the gradings. *Isomorphisms* or *equivalences* (*grading-preserving isomorphisms* or *grading-preserving equivalence*, respectively) are invertible homomorphisms (grading-preserving homomorphisms, respectively). *Left submodules* (*grading-preserving left submodules*, respectively) of a left V-module are left V-modules whose underlying vector spaces are subspaces of the left V-module such that the embedding maps are homomorphisms (grading-preserving homomorphisms, respectively).

Remark 6.4. We also have notions of right module and bimodule for a meromorphic open-string vertex algebra. These notions and a study of these modules and left modules will be given in another paper on the representation theory of meromorphic open-string vertex algebras.

Let M be a left $T(\mathfrak{h})$-module. Then we have the left $N(\hat{\mathfrak{h}})$-module $W = T(\hat{\mathfrak{h}}_-) \otimes M$. We have a vertex operator map

$$\begin{aligned} Y_W : T(\hat{\mathfrak{h}}_-) &\to (\mathrm{End}\, W)[[x, x^{-1}]] \\ v &\mapsto Y_W(u,x) = \sum_{n\in\mathbb{Z}} u_n x^{-n-1}, \end{aligned}$$

where $Y_W(u,x)$ is defined in (4.1). Assume that M is $\mathbb{C}$-graded (graded by weights) such that elements of $T(\mathfrak{h})$ preserve the grading and the $\mathbb{C}$-grading is lower bounded. For example, we can just define M to be homogeneous with an arbitrary complex number as the weight. Then this grading on M together with the grading on $T(\hat{\mathfrak{h}}_-)$ gives a grading on $W = T(\hat{\mathfrak{h}}_-) \otimes M$. Let D_M be an operator on M such that D_M is of weight 1 with respect to the grading on M and commutes with the action of the elements of $T(\mathfrak{h})$. For example, we can take D_M to be 0. We define an operator D_W on W by

$$D_W(u \otimes w) = D_{T(\hat{\mathfrak{h}}_-)}u \otimes w + u \otimes D_M w$$

for $u \in T(\hat{\mathfrak{h}}_-)$ and $w \in M$. Then we have:

Theorem 6.5. *The triple (W, Y_W, D_W) given above is a left module for the meromorphic open-string vertex algebra $T(\hat{\mathfrak{h}}_-)$.*

Proof. The proof is in fact completely analogous to the proof of Theorem 5.1. We omit the proof here. ∎

Remark 6.6. On M, there are actually infinitely many lower bounded $\mathbb{C}$-gradings. For example, for any complex number, we can let the weight of every element of M be this number. Let W_1 and W_2 be the left $T(\hat{\mathfrak{h}}_-)$-modules obtained from the same left $T(\mathfrak{h})$-module M as above and the same $D_M = 0$ but with different lower bounded $\mathbb{C}$-gradings. Then W_1 and W_2 are isomorphic but in general are not grading preserving.

Remark 6.7. If M is an $S(\mathfrak{h})$-module with a lower bounded $\mathbb{C}$-grading (graded by weights) such that elements of $S(\mathfrak{h})$ preserve the weights, then the canonical projection $\pi : T(\mathfrak{h}) \to S(\mathfrak{h})$ gives M a left $T(\mathfrak{h})$-module structure with a lower bounded $\mathbb{C}$-grading such that elements of $T(\mathfrak{h})$ preserve the weights. Let D_M be an operator on M such that D_M is of weight 1 with respect to the grading on M and commutes with the actions of elements of $S(\mathfrak{h})$. Then $S(\hat{\mathfrak{h}}_-) \otimes M$ is a module for the underlying grading-restricted vertex algebra of the vertex operator algebra $S(\hat{\mathfrak{h}}_-)$. The homomorphism $\pi : T(\hat{\mathfrak{h}}_-) \to S(\hat{\mathfrak{h}}_-)$ of meromorphic open-string vertex algebras gives $S(\hat{\mathfrak{h}}_-) \otimes M$ a left $T(\hat{\mathfrak{h}}_-)$-module structure. On the other hand, by Theorem 6.5, $T(\hat{\mathfrak{h}}_-) \otimes M$ is a left $T(\hat{\mathfrak{h}}_-)$-module. Let 1_M be the identity operator on M. Then the map $\pi \otimes 1_M : T(\hat{\mathfrak{h}}_-) \otimes M \to S(\hat{\mathfrak{h}}_-) \otimes M$ is a homomorphism of left $T(\hat{\mathfrak{h}}_-)$-modules.

Remark 6.8. We also have a construction of a right $T(\hat{\mathfrak{h}}_-)$-module from a right $T(\mathfrak{h})$-module using the construction of left $T(\hat{\mathfrak{h}}_-)$-modules above. We can also construct $T(\hat{\mathfrak{h}}_-)$-bimodules. These constructions will be given together with the notions of right module and bimodule and a study of these modules and left modules in a paper on the representation theory of meromorphic open-string vertex algebras mentioned above.

As in Section 5, we also end this section by calculating explicitly certain "four-point functions" with some particular elements in $T(\hat{\mathfrak{h}}_-)$ inserted at the points z_1 and z_2, a particular element of a left module for $T(\hat{\mathfrak{h}}_-)$ inserted at the point 0 and a particular element of the graded dual of the left module at the point ∞.

Example 6.9. Let M be a $T(\mathfrak{h})$-module. Give M a $\mathbb{C}$-grading by defining M to be homogeneous of weight 0. Then we have the left $T(\hat{\mathfrak{h}}_-)$-module $T(\hat{\mathfrak{h}}_-)\otimes M$. We shall identify M with the subspace $\mathbb{C}\otimes M$ of $T(\hat{\mathfrak{h}}_-)\otimes M$. Let $a, b\in \mathfrak{h}$. We choose M such that there exist $w\in M\subset T(\hat{\mathfrak{h}}_-)\otimes M$ and $w'\in (T(\hat{\mathfrak{h}}_-)\otimes M)'_{(0)}$ such that

$$\langle w', w\rangle = \langle w', a(0)b(0)w\rangle = 1$$

and

$$\langle w', b(0)a(0)w\rangle = 0.$$

Note that when $\dim \mathfrak{h}\geq 2$, such M exists. Then we have the left $T(\hat{\mathfrak{h}}_-)$-module $T(\hat{\mathfrak{h}}_-)\otimes M$. We first calculate

$$\langle w', Y_{T(\hat{\mathfrak{h}}_-)\otimes M}(a(-1)\mathbf{1}, z_1)Y_{T(\hat{\mathfrak{h}}_-)\otimes M}(b(-1)\mathbf{1}, z_2)w\rangle \tag{6.6}$$

which we know is absolutely convergent in the region $|z_1| > |z_2| > 0$. The formula (5.3) still hold when v and v' are replaced by elements of $T(\hat{\mathfrak{h}}_-)\otimes M$ and $(T(\hat{\mathfrak{h}}_-)\otimes M)'$, respectively, and $Y_{T(\hat{\mathfrak{h}}_-)}$ is replaced by $Y_{T(\hat{\mathfrak{h}}_-)\otimes M}$. Then (6.6) is equal to

$$\begin{aligned}
&\langle w', {}^{\circ}_{\circ}a(z_1)b(z_2){}^{\circ}_{\circ}w\rangle + (a,b)(z_1-z_2)^{-2}\langle w', w\rangle\\
&\quad = \sum_{m,n\in\mathbb{Z}}\langle w', {}^{\circ}_{\circ}a(m)b(n){}^{\circ}_{\circ}w\rangle z_1^{-m-1}z_2^{-n-1} + (a,b)(z_1-z_2)^{-2}\\
&\quad = \sum_{m,n\in\mathbb{Z}_+}\langle w', a(m)b(n)w\rangle z_1^{-m-1}z_2^{-n-1}\\
&\qquad + \sum_{m,n\in-\mathbb{Z}_+}\langle w', a(m)b(n)w\rangle z_1^{-m-1}z_2^{-n-1}\\
&\qquad + \sum_{m\in Z_+, n\in-\mathbb{Z}_+}\langle w', b(n)a(m)w\rangle z_1^{-m-1}z_2^{-n-1}\\
&\qquad + \sum_{m\in -Z_+, n\in\mathbb{Z}_+}\langle w', a(m)b(n)w\rangle z_1^{-m-1}z_2^{-n-1}\\
&\qquad + \sum_{m\in Z\setminus\{0\}}\langle w', a(m)b(0)w\rangle z_1^{-m-1}z_2^{-1}\\
&\qquad + \sum_{n\in Z\setminus\{0\}}\langle w', b(n)a(0)w\rangle z_1^{-1}z_2^{-n-1}\\
&\qquad + \langle v', a(0)b(0)w\rangle z_1^{-1}z_2^{-1}
\end{aligned}$$

$$
\begin{aligned}
&+(a,b)(z_1-z_2)^{-2}\\
&= z_1^{-1}z_2^{-1}+(a,b)(z_1-z_2)^{-2}.
\end{aligned} \tag{6.7}
$$

The same calculation also gives

$$
\langle w', Y_{T(\hat{\mathfrak{h}}_-)\otimes M}(b(-1)\mathbf{1},z_2)Y_{T(\hat{\mathfrak{h}}_-)\otimes M}(a(-1)\mathbf{1},z_1)w\rangle = (a,b)(z_1-z_2)^{-2} \tag{6.8}
$$

in the region $|z_2|>|z_1|>0$. Note that the right-hand side of (6.7) and the right-hand side of (6.8) are different, showing that the commutativity does not hold.

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