

Logarithmic intertwining operators and associative algebras

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Abstract

We establish an isomorphism between the space of logarithmic intertwining operators among suitable generalized modules for a vertex operator algebra and the space of homomorphisms between suitable modules for a generalization of Zhu's algebra given by Dong-Li-Mason.

1 Introduction

In the representation theory of reductive vertex operator algebras (vertex operator algebras for which a suitable category of weak modules is semisimple) and in the construction of rational conformal field theories, intertwining operators introduced in [FHL] are in fact the fundamental mathematical objects from which these theories are developed and constructed. In [FZ], for a reductive vertex operator algebra V , Frenkel and Zhu identified the spaces of intertwining operators among irreducible V -modules with suitable spaces constructed from (right, bi-, left) modules for Zhu's algebra $A(V)$ associated to the irreducible V -modules. See [L] for a generalization and a proof of this result. This result is very useful for the calculation of fusion rules and for the construction of intertwining operators.

To develop the representation theory of vertex operator algebras that are not reductive, it is necessary to consider certain generalized modules that are not completely reducible and the logarithmic intertwining operators among them. The theory of logarithmic intertwining operators corresponds to genus-zero logarithmic conformal field theories in physics. In fact, logarithmic structure in conformal field theory was first introduced by physicists to describe disorder phenomena [G] and logarithmic conformal field theories have been developed rapidly in recent years. See [HLZ1] for an introduction and for references to the study of logarithmic intertwining operators, a logarithmic tensor category theory and their connection with various works of mathematicians and physicists on logarithmic conformal field theories.

In this general setting, we can ask the following natural question: In the case that the generalized modules involved are not necessarily completely reducible, can we identify the spaces of logarithmic intertwining operators among suitable generalized modules with some

spaces constructed from modules for certain associative algebras associated to the vertex operator algebra? We answer this question in the present paper. Our answer needs the generalizations of Zhu's algebra given by Dong, Li and Mason in [DLM]. For a generalized module for the vertex operator algebra, we introduce a bimodule for such an associative algebra. This bimodule generalizes the bimodule for Zhu's algebra given in [FZ]. Our main result establishes an isomorphism between the space of logarithmic intertwining operators among suitable generalized modules and the space of homomorphisms between suitable modules for a generalization of Zhu's algebra given in [DLM]. See Theorem 6.6 for the precise statement of our main result. Our method follows the one used in [H1] and is different from the one used in [L].

Our result will be used in a forthcoming paper on generalized twisted modules associated to a not-necessarily-finite-order isomorphism of a vertex operator algebra (see [H3] for the definition and examples of such generalized twisted modules). In fact, the results on generalized twisted modules in that forthcoming paper is the main motivation for the main theorem that we obtain in this paper.

The present paper is organized as follows: In the next section, we recall basic notions and results on generalized modules for a vertex operator algebra. In Section 3, we recall the generalizations of Zhu's algebra by Dong, Li and Mason in [DLM]. In Section 4, we introduce and study a bimodule structure for such an algebra on a quotient of a lower-bounded generalized module for a vertex operator algebra. In Section 5, we begin our study of the relation between logarithmic intertwining operators and homomorphisms between suitable modules for a generalization of Zhu's algebra. Our main result is stated and proved in Section 6.

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2 Generalized modules for a vertex operator algebra

In this paper, we shall assume that the reader is familiar with the basic notions and results in the theory of vertex operator algebras. In particular, we assume that the reader is familiar with weak modules, \mathbb{N} -gradable weak modules, contragredient modules and related results. Our terminology and conventions follow those in [FLM], [FHL] and [H2]. We shall use \mathbb{Z} , \mathbb{Z}_+ , \mathbb{N} , \mathbb{R} and \mathbb{C} to denote the (sets of) integers, positive integers, nonnegative integers, real numbers and complex numbers, respectively. For $n \in \mathbb{C}$, we use $\Re(n)$ and $\Im(n)$ to denote the real and imaginary parts of n .

In this section, we recall the notion of generalized module for a vertex operator algebra and related notions in [HLZ1] and also some related notions and basic properties in [H2].

We fix a vertex operator algebra V in this paper. (In fact, the results in the present paper are true for a grading-restricted Möbius vertex algebra (see [HLZ1]).) We first recall the definition of generalized V -module and related notions in [HLZ1] (see also [M]):

Definition 2.1. A \mathbb{C} -graded vector space $W = \coprod_{n \in \mathbb{C}} W_{[n]}$ equipped with a linear map

$$\begin{aligned} Y_W : V \otimes W &\rightarrow W((x)) \\ v \otimes w &\mapsto Y_W(v, x)w \end{aligned}$$

is called a *generalized V -module* if it satisfies all the axioms for V -modules except that W does not have to satisfy the two grading-restriction conditions and that instead of requiring $L(0)w = nw$ for $w \in W_{[n]}$, we require that the following weaker version of the $L(0)$ -grading property, still called the *$L(0)$ -grading property*: For $n \in \mathbb{C}$, the homogeneous subspaces $W_{[n]}$ are the generalized eigenspaces of $L(0)$ with eigenvalues n , that is, for $n \in \mathbb{C}$, $w \in W_{[n]}$, there exists $K \in \mathbb{Z}_+$, depending on w , such that $(L(0) - n)^K w = 0$. We define *homomorphisms* (or *module maps*) and *isomorphisms* (or *equivalence*) between generalized V -modules, *generalized V -submodules* and *quotient generalized V -modules* in the obvious way.

Definition 2.2. An *irreducible generalized V -module* is a generalized V -module W such that there is no generalized V -submodule of W that is neither 0 nor W itself. A *lower bounded generalized V -module* is a generalized V -module W such that $W_{[n]} = 0$ when $\Re(n)$ is sufficiently negative. We say that lower-bounded generalized V -module W *has a lowest conformal weight*, or for simplicity, *W has a lowest weight* if there exists $n_0 \in \mathbb{C}$ such that $W_{[n_0]} \neq 0$ but $W_{[n]} = 0$ when $\Re(n) < \Re(n_0)$ or $\Re(n) = \Re(n_0)$ but $\Im(n) \neq \Im(n_0)$. In this case, we call n_0 , $W_{[n_0]}$ and elements of $W_{[n_0]}$ the *lowest conformal weight* or *lowest weight* of W , the *lowest weight space* or *lowest weight space* of W and *lowest conformal weight vectors* or *lowest weight vectors* of W , respectively. A *grading restricted generalized V -module* is a generalized V -module W such that W is lower bounded and $\dim W_{[n]} < \infty$ for $n \in \mathbb{C}$. An *(ordinary) V -module* is a generalized V -module W such that W is grading restricted and $W_{[n]} = W_{(n)}$ for $n \in \mathbb{C}$, where for $n \in \mathbb{C}$, $W_{(n)}$ are the eigenspaces of $L(0)$ with eigenvalues n . An *irreducible V -module* is a V -module such that it is irreducible as a generalized V -module. A *generalized V -module of length l* is a generalized V -module W such that there exist generalized V -submodules $W = W_1 \supset \cdots \supset W_{l+1} = 0$ such that W_i/W_{i+1} for $i = 1, \dots, l$ are irreducible V -modules. A *finite length generalized V -module* is a generalized V -module of length l for some $l \in \mathbb{Z}_+$. *Homomorphisms* and *isomorphisms* between lower-bounded, grading-restricted or finite length generalized V -modules are homomorphisms and isomorphisms between the underlying generalized V -modules.

Remark 2.3. By definition, every V -module is a generalized V -module, but the converse is not true in general. In particular, an irreducible generalized V -module in general might not be a V -module. A generalized V -module W has a lowest weight if W is \mathbb{R} -graded and lower-bounded or if W is lower-bounded and generated by one homogeneous element. In particular, the vertex operator algebra V , any \mathbb{R} -graded V -module, or any irreducible lower-bounded generalized V -module has a lowest weight.

Let $W = \coprod_{n \in \mathbb{C}} W_{[n]}$ equipped with Y_W be a generalized V -module. As in [HLZ1], the *opposite vertex operator on W associated to $v \in V$* is defined by

$$Y_W^o(v, x) = Y_W(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}). \quad (2.1)$$

Let W' be the \mathbb{C} -graded vector subspace of W^* given by

$$W' = \coprod_{n \in \mathbb{C}} (W_{[n]})^*. \quad (2.2)$$

We shall use the notation $\langle \cdot, \cdot \rangle_W$, or $\langle \cdot, \cdot \rangle$ if the underlying space is clear, to denote the canonical pairing between W' and W . As in Section 5.2 of [FHL], we define a vertex operator map Y'_W for W' by

$$\langle Y'_W(v, x)w', w \rangle = \langle w', Y_W^o(v, x)w \rangle \quad (2.3)$$

for $v \in V$, $w' \in W'$ and $w \in W$. The correspondence given by $v \mapsto Y'_W(v, x)$ is a linear map from V to $(\text{End } W')[[x, x^{-1}]]$. Write

$$Y'_W(v, x) = \sum_{n \in \mathbb{Z}} (Y'_W)_n(v) x^{-n-1}$$

$((Y'_W)_n(v) \in \text{End } W')$ and

$$Y_W^o(v, x) = \sum_{n \in \mathbb{Z}} (Y_W^o)_n(v) x^{-n-1}$$

$((Y_W^o)_n(v) \in \text{End } W)$. Then for $v \in V$, $w' \in W'$ and $w \in W$,

$$\langle (Y'_W)_n(v)w', w \rangle = \langle w', (Y_W^o)_n(v)w \rangle. \quad (2.4)$$

We have:

Theorem 2.4. *Let W be a lower-bounded generalized V -module. Then W' equipped with Y'_W is a lower-bounded generalized V -module. Moreover, $Y''_W|_{V \otimes W} = Y_W$. ■*

The proof of this theorem is the same as those of Theorems 5.2.1 and 5.3.1 in [FHL]. Note that in this theorem, W does not have to be grading restricted. The space W' equipped with Y'_W is called the *contragredient* of W .

We also define the operators $L'(n)$ for $n \in \mathbb{Z}$ by

$$Y'_W(\omega, x) = \sum_{n \in \mathbb{Z}} L'(n) x^{-n-2}.$$

As in Section 5.2 of [FHL], by extracting the coefficient of x^{-n-2} in (2.3) with $v = \omega$ and using the fact that $L(1)\omega = 0$, we have

$$\langle L'(n)w', w \rangle = \langle w', L(-n)w \rangle \quad \text{for } n \in \mathbb{Z}. \quad (2.5)$$

The following fact is useful (see [H2]):

Proposition 2.5. *The contragredient of a generalized V -module of length l also has length l . ■*

3 Associative algebras from vertex operator algebras and their modules

In this section, we recall the generalizations of Zhu's algebra $[Z]$ given by Dong, Li and Mason in [DLM] associated to a vertex operator algebra. We prove some elementary but useful results.

Recall that we have fixed our vertex operator algebra V in this paper. For $N \in \mathbb{N}$, define a product $*_N$ on V by

$$u *_N v = \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_x x^{-N-m-1} Y_V((1+x)^{L(0)+N} u, x) v$$

for $u, v \in V$. Let $O_N(V)$ be the subspace of V spanned by elements of the form

$$\text{Res}_x x^{-2N-1-n} Y_V((1+x)^{L(0)+N} u, x) v$$

for $n \in \mathbb{Z}_+$, $u, v \in V$ and of the form $(L(-1) + L(0))u$ for $u \in V$.

Theorem 3.1 ([DLM]). *The subspace $O_N(V)$ is a two-sided ideal of V under the product $*_N$ and the product $*_N$ induces a structure of associative algebra on the quotient $A_N(V) = V/O_N(V)$ with the identity $\mathbf{1} + O_N(V)$ and with $\omega + O_N(V)$ in the center of $A_N(V)$. ■*

Remark 3.2. When $N = 0$, $A_0(V)$ is the associative algebra first introduced and studied by Zhu in [Z].

Let W be a weak V -module and for $N \in \mathbb{N}$, let

$$\Omega_N(W) = \{w \in W \mid (Y_W)_k(u)w = 0 \text{ for homogeneous } u \in V, \text{wt } u - k - 1 < -N\}.$$

Theorem 3.3 ([DLM]). *The map $V \rightarrow \text{End } \Omega_N(W)$ given by $v \mapsto o(v) = (Y_W)_{\text{wt } v - 1}(v)$ for homogeneous $v \in V$ induces a structure of $A_N(V)$ -module on $\Omega_N(W)$. ■*

From the commutator formula for vertex operators, we know that the space \hat{V} of operators on V of the form $(Y_V)_n(u)$ for $u \in V$ and $n \in \mathbb{Z}$, equipped with the Lie bracket for operators, is a Lie algebra. Giving the operators $(Y_V)_n(u)$ for homogeneous u the weights $\text{wt } u - n - 1$, we see that \hat{V} becomes a \mathbb{Z} -graded Lie algebra. We use $\hat{V}_{(n)}$ to denote the homogeneous subspace of weight n . Also, $A_N(V)$ equipped with the Lie bracket induced from the associative algebra structure is a Lie algebra.

Proposition 3.4 ([DLM]). *The map given by $(Y_V)_{\text{wt } v - 1}(v) \mapsto v + O_N(V)$ is a surjective homomorphism of Lie algebras from $\hat{V}_{(0)}$ to the Lie algebra $A_N(V)$. ■*

Let W be a lower-bounded generalized V -module such that $W = \coprod_{n \in h_W + \mathbb{N}} W_{[n]}$ for some $h_W \in \mathbb{C}$ and $W_{[h_W]} \neq 0$. For $N \in \mathbb{N}$, let

$$\Omega_N^0(W) = \prod_{n=0}^N W_{[h_W+n]}.$$

It is clear that $\Omega_N^0(W) \subset \Omega_N(W)$. Since for $u \in V$, $o(u)$ preserves the weights, $\Omega_N^0(W)$ is an $A_N(V)$ -submodule of $\Omega_N(W)$.

Remark 3.5. A generalized V -module W decomposes into generalized submodules corresponding to the congruence classes of its weights modulo \mathbb{Z} . For $\mu \in \mathbb{C}/\mathbb{Z}$, let

$$W^\mu = \coprod_{n \in \mu} W_{[n]}.$$

Then

$$W = \coprod_{\mu \in \mathbb{C}/\mathbb{Z}} W^\mu$$

and each W^μ is a generalized V -submodule of W . In particular, if a generalized module W is indecomposable, then there exists $h \in \mathbb{C}$ such that $W = \coprod_{n \in h + \mathbb{Z}} W_{[n]}$. In the case that W is lower bounded, there exists $h_\mu \in \mathbb{C}$ for $\mu \in \mathbb{C}/\mathbb{Z}$ such that

$$W^\mu = \coprod_{n \in h_\mu + \mathbb{N}} W_{[n]}$$

for $\mu \in \mathbb{C}/\mathbb{Z}$.

For a lower-bounded generalized V -module W , by Remark 3.5, there exists $h_\mu \in \mathbb{C}$ for $\mu \in \mathbb{C}/\mathbb{Z}$ such that

$$W = \coprod_{\mu \in \mathbb{C}/\mathbb{Z}} W^\mu$$

where

$$W^\mu = \coprod_{n \in h_\mu + \mathbb{N}} W_{[n]}$$

for $\mu \in \mathbb{C}/\mathbb{Z}$ are lower-bounded generalized V -submodules of W . Let

$$\Omega_N^0(W) = \sum_{\mu \in \mathbb{C}/\mathbb{Z}} \Omega_N^0(W^\mu) \subset W.$$

Since $\Omega_N^0(W^\mu)$ is an $A_N(V)$ -submodule of $\Omega_N(W^\mu)$ for each $\mu \in \mathbb{C}/\mathbb{Z}$, $\Omega_N^0(W)$ is an $A_N(V)$ -submodule of $\Omega_N(W)$.

Proposition 3.6. *Let W be a lower-bounded generalized V -module generated by $\Omega_N^0(W)$ for some $N \in \mathbb{N}$. Then W is spanned by elements of the form*

$$(Y_W)_{m_1}(u^1) \cdots (Y_W)_{m_k}(u^k)w,$$

where u^1, \dots, u^k are homogeneous elements of V , m_1, \dots, m_k are integers such that $\text{wt } u^i - m_i - 1 > 0$ and $w \in \Omega_N^0(W)$.

Proof. We know that W is spanned by elements of the form

$$(Y_W)_{m_1}(u^1) \cdots (Y_W)_{m_k}(u^k)w,$$

where u^1, \dots, u^k are homogeneous elements of V , m_1, \dots, m_k are integers and $w \in \Omega_N^0(W)$. We have to show that these elements can be written as linear combinations of elements of the same form such that $\text{wt } u^i - m_i - 1 > 0$. If there exists i such that $\text{wt } u^i - m_i - 1 \leq 0$, then we can find an i such that $\text{wt } u^i - m_i - 1 \leq 0$ and $\text{wt } u^j - m_j - 1 > 0$ for $j > i$. The component form of the commutator formula for vertex operators gives

$$\begin{aligned} & (Y_W)_{m_i}(u^i)(Y_W)_{m_{i+1}}(u^{i+1}) - (Y_W)_{m_{i+1}}(u^{i+1})(Y_W)_{m_i}(u^i) \\ &= \sum_{j \in \mathbb{N}} \binom{m_i}{j} (Y_W)_{m_i+m_{i+1}-j}((Y_V)_j(u^i)u^{i+1}). \end{aligned}$$

Thus we have

$$\begin{aligned} & (Y_W)_{m_1}(u^1) \cdots (Y_W)_{m_k}(u^k)w \\ &= (Y_W)_{m_1}(u^1) \cdots (Y_W)_{m_{i-1}}(u^{i-1}) \cdot \\ & \quad \cdot (Y_W)_{m_{i+1}}(u^{i+1})(Y_W)_{m_i}(u^i)(Y_W)_{m_{i+2}}(u^{i+2}) \cdots (Y_W)_{m_k}(u^k)w \\ &+ \sum_{j \in \mathbb{N}} \binom{m_i}{j} (Y_W)_{m_1}(u^1) \cdots (Y_W)_{m_{i-1}}(u^{i-1}) \cdot \\ & \quad \cdot (Y_W)_{m_i+m_{i+1}-j}((Y_V)_j(u^i)u^{i+1})(Y_W)_{m_{i+2}}(u^{i+2}) \cdots (Y_W)_{m_k}(u^k)w. \end{aligned}$$

Using this formula, the fact that $(Y_W)_m(u)\tilde{w} \in \Omega_N^0(W)$ for homogeneous $u \in V$, $m \in \mathbb{Z}$ and $\tilde{w} \in \Omega_N^0(W)$ such that $\text{wt } u - m - 1 \leq 0$, and inductions on k and on the largest number i such that $\text{wt } u^i - m_i - 1 \leq 0$, we see that

$$(Y_W)_{m_1}(u^1) \cdots (Y_W)_{m_k}(u^k)w,$$

can indeed be written as a linear combination of elements of the same form such that $\text{wt } u^i - m_i - 1 > 0$. ■

Using the definition of the opposite vertex operators $Y^o(u, x)$ for $u \in V$, we see that Proposition 3.6 gives:

Corollary 3.7. *Let W be a lower-bounded generalized V -module generated by $\Omega_N^0(W)$ for some $N \in \mathbb{N}$. Then W is spanned by elements of the form*

$$(Y_W^o)_{m_1}(u^1) \cdots (Y_W^o)_{m_k}(u^k)w,$$

where u^1, \dots, u^k are homogeneous elements of V , m_1, \dots, m_k are integers such that $\text{wt } u^i - m_i - 1 < 0$ and $w \in \Omega_N^0(W)$. ■

In the results above, W must be generated by $\Omega_N^0(W)$ for some $N \in \mathbb{N}$. We now show that generalized V -modules of finite length is lower bounded and have this property.

Let W be a generalized V -module of length l and $W = W_1 \supset \cdots \supset W_{l+1} = 0$ a finite composition series of W . Since W_i/W_{i+1} for $i = 1, \dots, l$ are irreducible V -modules and irreducible V -modules as modules are lower bounded, there exist homogeneous elements $w_i \in W_i$ of weights $h_i \in \mathbb{C}$ for $i = 1, \dots, l$ such that $w_i + W_{i+1}$ for $i = 1, \dots, l$ are lowest weight vectors of W_i/W_{i+1} .

Proposition 3.8. *Let W be a generalized V -module of length l , $W = W_1 \supset \cdots \supset W_{l+1} = 0$ a finite composition series of W and $w_i \in W_i$ homogeneous elements of weights $h_i \in \mathbb{C}$ for $i = 1, \dots, l$ such that $w_i + W_{i+1}$ for $i = 1, \dots, l$ are lowest weight vectors of W_i/W_{i+1} . Let N be a positive integer such that $|\Re(h_i) - \Re(h_j)| \leq N$ for $i \neq j$, $i, j \in \{1, \dots, l\}$ and $r = \min_{i \in \{1, \dots, l\}} \Re(h_i)$. Then W is lower bounded, the real number r is the smallest real part of the weights of elements of W and the subset $\{w_1, \dots, w_l\}$ of W is in $\Omega_N^0(W)$ and generates W . In particular, $\Omega_N^0(W)$ generates W .*

Proof. Since $w_i + W_{i+1}$ is a lowest weight vector of the irreducible V -modules W_i/W_{i+1} for $i = 1, \dots, l$, W as a graded vector space is isomorphic to $\coprod_{i=1}^l W_i/W_{i+1}$. Since the lowest weight of W_i/W_{i+1} is h_i for $i = 1, \dots, l$, the real part of the weight of any homogeneous vector of W is larger than or equal to $r = \min_{i \in \{1, \dots, l\}} \Re(h_i)$. So W is lower bounded, r is the smallest real part of the weights of the elements of the graded space $\coprod_{i=1}^l W_i/W_{i+1}$ and thus r is also the smallest real part of the weights of the elements of W .

Since $|\Re(h_i) - \Re(h_j)| \leq N$, $w_1, \dots, w_l \in \coprod_{\Re(n) \leq r+N} W_{[n]}$. By definition, we know that $\coprod_{\Re(n) \leq r+N} W_{[n]} \subset \Omega_N^0(W)$. Thus $w_1, \dots, w_l \in \Omega_N^0(W)$.

Let \widetilde{W} be the generalized V -submodule generated by w_i for $i = 1, \dots, l$. Since W_i/W_{i+1} for $i = 1, \dots, l$ are irreducible, $w_i + W_{i+1}$ for $i = 1, \dots, l$ are generators of W_i/W_{i+1} . We now show that $W = \widetilde{W}$. Since $W_l = W_l/W_{l+1}$ is generated by w_l , we see that $W_l \subset \widetilde{W}$. Now assume that $W_m \subset \widetilde{W}$. Then since W_{m-1}/W_m is generated by $w_{m-1} + W_m$, every element of W_{m-1}/W_m is a linear combination of elements of the form

$$\begin{aligned} & (Y_{W_{m-1}/W_m})_{n_1}(u^1) \cdots (Y_{W_{m-1}/W_m})_{n_k}(u^k)(w_{m-1} + W_m) \\ &= (Y_{W_{m-1}})_{n_1}(u^1) \cdots (Y_{W_{m-1}})_{n_k}(u^k)w_{m-1} + W_m. \end{aligned}$$

Thus elements of W_{m-1} are linear combinations of elements of the form

$$(Y_{W_{m-1}})_{n_1}(u^1) \cdots (Y_{W_{m-1}})_{n_k}(u^k)w_{m-1} + w$$

where $w \in W_m$. Since

$$(Y_{W_{m-1}})_{n_1}(u^1) \cdots (Y_{W_{m-1}})_{n_k}(u^k)w_{m-1} \in \widetilde{W}$$

and $w \in W_m \subset \widetilde{W}$,

$$(Y_{W_{m-1}})_{n_1}(u^1) \cdots (Y_{W_{m-1}})_{n_k}(u^k)w_{m-1} + w \in \widetilde{W}.$$

So $W_{m-1} \subset \widetilde{W}$. By the principle of induction, $W = W_1 \subset \widetilde{W}$. Thus we see that $\{w_1, \dots, w_l\}$ generates W . ■

Remark 3.9. Let W be a generalized V -module of finite length. By Propositions 2.5 and 3.8, the contragredient module W' is generated by $\Omega_N^0(W')$ for some $N \in \mathbb{N}$.

4 $A_N(V)$ -bimodules from generalized V -modules

In this section, for a generalized V -module W and $N \in \mathbb{N}$, we introduce an $A_N(V)$ -bimodule $A_N(W)$. These bimodules should be viewed as generalizations of bimodules for Zhu's algebra introduced in [FZ]. We emphasize that the formulas defining the right actions on the bimodules given in this section are different from but equivalent to the one in [FZ] in the case $N = 0$. Our formulas are more natural and conceptual.

In this section, we fix a generalized V -module W . We need the semisimple part $L(0)_s \in \text{End } W$ of the operator $L(0)$ on W defined by

$$L(0)_s w = n w$$

for $w \in W_{[n]}$, $n \in \mathbb{C}$. Recall from [HLZ1] that we have the commutator formula

$$\begin{aligned} [L(0)_s, Y_W(u, x_0)] &= [L(0), Y_W(u, x_0)] \\ &= Y_W(L(0)u, x_0) + x_0 \frac{d}{dx_0} Y_W(u, x_0) \end{aligned}$$

for $u \in V$. In particular, we have

$$[L(0)_s, L(-1)] = L(-1).$$

Thus we have the $L(0)_s$ -conjugation property

$$y^{L(0)_s} Y_W(u, x) y^{-L(0)_s} = Y_W(y^{L(0)_s} u, xy) \quad (4.6)$$

for $u \in V$ and

$$y^{L(0)_s} e^{xL(-1)} y^{-L(0)_s} = e^{xyL(-1)}. \quad (4.7)$$

These formulas for $L(0)_s$ certainly hold also for $L(0)$.

We also need the map

$$\begin{aligned} Y_{WV}^W : W \otimes V &\longrightarrow W[[x, x^{-1}]] \\ w \otimes u &\longmapsto Y_{WV}^W(w, x)u \end{aligned}$$

defined in [FHL] by

$$Y_{WV}^W(w, x)u = e^{xL(-1)} Y_W(u, -x)w$$

for $u \in V$ and $w \in W$. Proposition 5.1.2 and Remark 5.4.2 in [FHL] give in particular the following:

Proposition 4.1 ([FHL]). *The map Y_{WV}^W is an intertwining operator of type $\binom{W}{WV}$. In particular, the Jacobi identity*

$$\begin{aligned} x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_W(u, x_1) Y_{WV}^W(w, x_2) v - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_{WV}^W(w, x_2) Y_V(u, x_1) v \\ = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y_{WV}^W(Y_W(u, x_0)w, x_2) v \end{aligned} \quad (4.8)$$

holds for $u, v \in V$ and $w \in W$. Moreover, the Jacobi identity

$$\begin{aligned} x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_{WV}^W(w, x_1) Y_V(v, x_2) u - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_W(v, x_2) Y_{WV}^W(w, x_1) u \\ = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y_{WV}^W(Y_{WV}^W(w, x_0) v, x_2) u \end{aligned} \quad (4.9)$$

also holds for $u, v \in V$ and $w \in W$.

We also have:

Proposition 4.2. For $w \in W$,

$$y^{L(0)} Y_{WV}^W(w, x) y^{-L(0)} = Y_{WV}^W(y^{L(0)} w, xy).$$

This formula also holds with $L(0)$ replaced by $L(0)_s$.

Proof. These follow from the definition of Y_{WV}^W and (4.7) for $L(0)$ or $L(0)_s$. ■

For $N \in \mathbb{N}$, $u \in V$ and $w \in W$, we define

$$\begin{aligned} u *_N w &= \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_x x^{-N-m-1} Y_W((1+x)^{L(0)+N} u, x) w, \\ w *_N u &= \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_x x^{-N-m-1} \cdot \\ &\quad \cdot (1+x)^{-(L_W(-1)+L_W(0))} Y_{WV}^W((1+x)^{L(0)+N} w, x) u. \end{aligned}$$

Let $O_N(W)$ be the subspace of W spanned by elements of the form

$$u \circ_N w = \text{Res}_x x^{-2N-2} Y_W((1+x)^{L(0)+N} u, x) w$$

for $u \in V$ and $w \in W$. Let $A_N(W) = W/O_N(W)$.

Remark 4.3. Note that in the case of $N = 0$, our right action is different from the right action in [FZ]. Certainly, these right actions induce the same right action on $A_0(W)$ (see Remark 4.5 below). The advantage of defining the right action using the formula above is that many formulas involving the right action can be proved in the same way as the proofs of the corresponding formulas for the algebra or for the left action, with some of the vertex operator maps replaced by the map Y_{WV}^W . On the other hand, we note that the definition of the right action above makes sense only for generalized V -modules, not for weak V -modules that are not generalized V -modules.

The following lemma generalizes Lemma 2.1 in [DLM]:

Lemma 4.4. Let $u \in V$ and $w \in W$.

1. We have

$$\begin{aligned}
u *_N w &= \sum_{m=0}^N \binom{m+N}{N} (-1)^N \text{Res}_x x^{-N-m-1} \cdot \\
&\quad \cdot (1+x)^{-(L(-1)+L(0))} Y_{WV}^W((1+x)^{L(0)+m-1} w, x) u, \\
w *_N u &= \sum_{m=0}^N \binom{m+N}{N} (-1)^N \text{Res}_x x^{-N-m-1} Y_W((1+x)^{L(0)+m-1} u, x) w.
\end{aligned}$$

2. For $p \geq q \geq 0$, we have

$$\begin{aligned}
\text{Res}_x x^{-2N-2-p} Y_W((1+x)^{L(0)+N+q} u, x) w &\in O_N(W), \\
\text{Res}_x x^{-2N-2-p} (1+x)^{-(L(-1)+L(0))} Y_{WV}^W((1+x)^{L(0)+N+q} w, x) u &\in O_N(W).
\end{aligned}$$

In particular,

$$w \circ_N u = \text{Res}_x x^{-2N-2} (1+x)^{-(L(-1)+L(0))} Y_{WV}^W((1+x)^{L(0)+N} w, x) u \in O_N(W).$$

3. We have

$$\begin{aligned}
u *_N w - w *_N u &= \text{Res}_x Y_W((1+x)^{L(0)-1} u, x) w, \\
w *_N u - u *_N w &= \text{Res}_x (1+x)^{-(L(-1)+L(0))} Y_{WV}^W((1+x)^{L(0)-1} w, x) u.
\end{aligned}$$

Proof. Using the definition of Y_{WV}^W , we obtain

$$\begin{aligned}
Y_W(u, x) w &= (1+x)^{L(-1)+L(0)} Y_{WV}^W \left((1+x)^{-L(0)} w, \frac{-x}{1+x} \right) (1+x)^{-L(0)} u, \\
Y_{WV}^W(w, x) u &= (1+x)^{L(-1)+L(0)} Y_W \left((1+x)^{-L(0)} u, \frac{-x}{1+x} \right) (1+x)^{-L(0)} w.
\end{aligned}$$

Then

$$\begin{aligned}
u *_N w &= \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_y y^{-N-m-1} Y_W((1+y)^{L(0)+N} u, y) w \\
&= \sum_{m=0}^N (-1)^m \binom{m+N}{N} \cdot \text{Res}_y y^{-N-m-1} (1+y)^{L(-1)+L(0)} Y_{WV}^W \left((1+y)^{-L(0)+N} w, \frac{-y}{1+y} \right) u \\
&= \sum_{m=0}^N (-1)^N \binom{m+N}{N} \text{Res}_x x^{-N-m-1} (1+x)^{-(L(-1)+L(0))} Y_{WV}^W((1+x)^{L(0)+m-1} w, x) u
\end{aligned}$$

and

$$\begin{aligned}
w *_N u &= \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_y y^{-N-m-1} (1+y)^{-(L(-1)+L(0))} Y_{WV}^W((1+y)^{L(0)+N} w, y) u \\
&= \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_y y^{-N-m-1} Y_W \left((1+y)^{-L(0)+N} u, \frac{-y}{1+y} \right) w \\
&= \sum_{m=0}^N (-1)^N \binom{m+N}{N} \text{Res}_x x^{-N-m-1} Y_W((1+x)^{L(0)+m-1} u, x) w,
\end{aligned}$$

where in both formulas, the last steps are obtained by changing the variable $x = \frac{-y}{1+y}$. This proves Part 1.

Similarly,

$$\begin{aligned}
w \circ_N u &= \text{Res}_y y^{-2N-2} (1+y)^{-(L(-1)+L(0))} Y_{WV}^W((1+y)^{L(0)+N} w, y) u \\
&= \text{Res}_y y^{-2N-2} Y_W \left((1+y)^{-L(0)+N} u, \frac{-y}{1+y} \right) w \\
&= \text{Res}_x x^{-2N-2} Y_W((1+x)^{L(0)+N} u, x) w \\
&\in O_N(W).
\end{aligned}$$

Now the proof of Part 2 is similar to the proof of Lemma 2.1.2 of [Z].

Using Part 1, we obtain

$$\begin{aligned}
u *_N w - w *_N u &= \text{Res}_x \left(\sum_{m=0}^N \binom{m+N}{N} \frac{(-1)^m (1+x)^{N+1} - (-1)^N (1+x)^m}{x^{N+m+1}} \right) Y_W((1+x)^{L(0)_s-1} u, x) w \\
&= \text{Res}_x Y_W((1+x)^{L(0)_s-1} u, x) w,
\end{aligned}$$

where the last step uses the formula

$$\sum_{m=0}^N \binom{m+N}{N} \frac{(-1)^m (1+x)^{N+1} - (-1)^N (1+x)^m}{x^{N+m+1}} = 1$$

given by Proposition 5.2 in [DLM]. This proves the first property in Part 3. The second is similar. \blacksquare

Remark 4.5. By Part 1 in Lemma 4.4, we see that we can also define the right action $*_N$ of V on W by

$$w *_N u = \sum_{m=0}^N \binom{m+N}{N} (-1)^N \text{Res}_x x^{-N-m-1} Y_W((1+x)^{L(0)+m-1} u, x) w$$

for $u \in V$ and $w \in W$. The advantage of this right action is that W does not have to be a generalized V -module.

Lemma 4.6. *The subspace $O_N(W)$ of W is invariant under the left and right actions of V above.*

Proof. We need to prove

$$(u \circ_N w) *_N v \in O_N(W), \quad (4.10)$$

$$v *_N (u \circ_N w) \in O_N(W) \quad (4.11)$$

for $u, v \in V$ and $w \in W$.

We prove (4.10) first. For homogeneous $u \in V$ and $w \in W$, we have

$$\begin{aligned} u \circ_N w &= \text{Res}_y y^{-2N-2} Y_W((1+y)^{L(0)+N} u, y) w \\ &= \text{Res}_y y^{-2N-2} (1+y)^{\text{wt } u+N} Y_W(u, y) w. \end{aligned}$$

For $L(0)$ acting on W , we write $L(0)_n = L(0) - L(0)_s$. Then for homogeneous $u \in V$, $v \in V$ and homogeneous $w \in W$,

$$\begin{aligned} &(u \circ_N w) *_N v \\ &= \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_y y^{-2N-2} x_2^{-N-m-1} (1+y)^{\text{wt } u+N} \cdot \\ &\quad \cdot (1+x_2)^{-(L(-1)+L(0))} Y_{WV}^W((1+x_2)^{L(0)+N} Y_W(u, y) w, x_2) v \\ &= \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_y y^{-2N-2} x_2^{-N-m-1} (1+y)^{\text{wt } u+N} (1+x_2)^{\text{wt } u+\text{wt } w+N} \cdot \\ &\quad \cdot (1+x_2)^{-(L(-1)+L(0))} Y_{WV}^W(Y_W(u, (1+x_2)y) (1+x_2)^{L(0)_n} w, x_2) v \\ &= \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} x_0^{-2N-2} x_2^{-N-m-1} (1+x_2+x_0)^{\text{wt } u+N} \cdot \\ &\quad \cdot (1+x_2)^{\text{wt } w+2N+1} (1+x_2)^{-(L(-1)+L(0))} Y_{WV}^W(Y_W(u, x_0) (1+x_2)^{L(0)_n} w, x_2) v \\ &= \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} \text{Res}_{x_1} x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) x_0^{-2N-2} x_2^{-N-m-1} \cdot \\ &\quad \cdot (1+x_2)^{\text{wt } w+2N+1} (1+x_2+x_0)^{\text{wt } u+N} (1+x_2)^{-(L(-1)+L(0))} \cdot \\ &\quad \cdot Y_W(u, x_1) Y_{WV}^W((1+x_2)^{L(0)_n} w, x_2) v \\ &\quad - \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} \text{Res}_{x_1} x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) x_0^{-2N-2} x_2^{-N-m-1} \cdot \\ &\quad \cdot (1+x_2+x_0)^{\text{wt } u+N} (1+x_2)^{\text{wt } w+2N+1} (1+x_2)^{-(L(-1)+L(0))} \cdot \\ &\quad \cdot Y_{WV}^W((1+x_2)^{L(0)_n} w, x_2) Y_V(u, x_1) v \\ &= \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} \text{Res}_{x_1} x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) (x_1 - x_2)^{-2N-2} x_2^{-N-m-1} \cdot \\ &\quad \cdot (1+x_1)^{\text{wt } u+N} (1+x_2)^{\text{wt } w+2N+1} (1+x_2)^{-(L(-1)+L(0))} \cdot \end{aligned}$$

$$\begin{aligned}
& \cdot Y_W(u, x_1) Y_{WV}^W((1+x_2)^{L(0)n} w, x_2) v \\
& - \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} \text{Res}_{x_1} x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) (-x_2 + x_1)^{-2N-2} x_2^{-N-m-1} \cdot \\
& \quad \cdot (1+x_1)^{\text{wt } u+N} (1+x_2)^{\text{wt } w+2N+1} (1+x_2)^{-(L(-1)+L(0))} \cdot \\
& \quad \cdot Y_{WV}^W((1+x_2)^{L(0)n} w, x_2) Y_V(u, x_1) v \\
& = \sum_{m=0}^N \sum_{i \in \mathbb{N}} (-1)^{m+i} \binom{m+N}{N} \binom{-2N-2}{i} \text{Res}_{x_2} \text{Res}_{x_1} x_1^{-2N-2-i} x_2^{-N-m-1+i} \cdot \\
& \quad \cdot (1+x_2)^{\text{wt } w+2N+1} (1+x_2)^{-(L(-1)+L(0))} \cdot \\
& \quad \cdot Y_W((1+x_1)^{L(0)+N} u, x_1) Y_{WV}^W((1+x_2)^{L(0)n} w, x_2) v \\
& - \sum_{m=0}^N \sum_{i \in \mathbb{N}} (-1)^{m+i} \binom{m+N}{N} \binom{-2N-2}{i} \text{Res}_{x_2} \text{Res}_{x_1} x_2^{-3N-m-3-i} x_1^i (1+x_1)^{\text{wt } u+N} \cdot \\
& \quad \cdot (1+x_2)^{-(L(-1)+L(0))} Y_{WV}^W((1+x_2)^{L(0)+2N+1} w, x_2) Y_V(u, x_1) v,
\end{aligned}$$

where in the third step, we have changed the variable $y = \frac{x_0}{1+x_2}$ and in the fourth step, we have used the Jacobi identity (4.8). Since all the terms in the first sum of the right-hand side lie in $O_N(W)$ by definition and all the terms in the second sum of the right-hand side lie in $O_N(W)$ by Part 2 of Lemma 4.4, (4.10) holds.

For homogeneous $u, v \in V$ and $w \in W$, using Part 3 in Lemma 4.4 and (4.10), we obtain

$$\begin{aligned}
& v *_N (u \circ_N w) \\
& = (u \circ_N w) *_N v + \text{Res}_{x_1} (1+x_1)^{\text{wt } v-1} Y_W(v, x_1) (u \circ_N w) \\
& \equiv \text{Res}_{x_1} (1+x_1)^{\text{wt } v-1} Y_W(v, x_1) (u \circ_N w) \pmod{O_N(W)} \\
& = \text{Res}_{x_1} \text{Res}_{x_2} (1+x_1)^{\text{wt } v-1} (1+x_2)^{\text{wt } u+N} x_2^{-2N-2} Y_W(v, x_1) Y_W(u, x_2) w \\
& = \text{Res}_{x_1} \text{Res}_{x_2} (1+x_1)^{\text{wt } v-1} (1+x_2)^{\text{wt } u+N} x_2^{-2N-2} Y_W(u, x_2) Y_W(v, x_1) w \\
& \quad + \text{Res}_{x_1} \text{Res}_{x_2} \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \cdot \\
& \quad \cdot (1+x_1)^{\text{wt } v-1} (1+x_2)^{\text{wt } u+N} x_2^{-2N-2} Y_W(Y_V(v, x_0) u, x_2) w \\
& \equiv \text{Res}_{x_2} \text{Res}_{x_0} (1+x_2+x_0)^{\text{wt } v-1} (1+x_2)^{\text{wt } u+N} x_2^{-2N-2} Y_W(Y_V(v, x_0) u, x_2) w \pmod{O_N(W)} \\
& = \sum_{i \in \mathbb{N}} \binom{\text{wt } v-1}{i} \text{Res}_{x_2} (1+x_2)^{\text{wt } u+\text{wt } v+N-1-i} x_2^{-2N-2} Y_W(Y_i(v) u, x_2) w \\
& = \sum_{i \in \mathbb{N}} \binom{\text{wt } v-1}{i} \text{Res}_{x_2} x_2^{-2N-2} Y_W((1+x_2)^{L(0)+N} Y_i(v) u, x_2) w \\
& \in O_N(W),
\end{aligned}$$

proving (4.11). ■

The main result in this section is the following:

Theorem 4.7. *The left and right actions of V on W induce an $A_N(V)$ -bimodule structure on $A_N(W)$.*

Proof. Lemma 4.6 says that the left and right actions of V on W give left and right actions of V on $A_N(W)$. We first need to show that these left and right actions of V on $A_N(W)$ in fact give left and right actions of $A_N(V)$ on $A_N(W)$, that is, we need to prove

$$(L(-1)u + L(0)u) *_N w \in O_N(W), \quad (4.12)$$

$$w *_N (L(-1)u + L(0)u) \in O_N(W), \quad (4.13)$$

$$(u \circ_N v) *_N w \in O_N(W), \quad (4.14)$$

$$w *_N (v \circ_N u) \in O_N(W) \quad (4.15)$$

for $u, v \in V$ and $w \in W$. The proof of these formulas are similar to the proof of Lemma 4.6 and we omit them.

Next we need to prove that these left and right actions indeed give left and right $A_N(V)$ modules, that is, we need to prove

$$u *_N (v *_N w) \equiv (u *_N v) *_N w \text{ mod } O_N(W), \quad (4.16)$$

$$w *_N (v *_N u) \equiv (w *_N v) *_N u \text{ mod } O_N(W) \quad (4.17)$$

$u, v \in V$ and $w \in W$. We prove only (4.17) here; the proof of (4.16) is similar.

For $v \in V$ and homogeneous $w \in W$,

$$\begin{aligned} w *_N v &= \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_y y^{-N-m-1} (1+y)^{-(L(-1)+L(0))} Y_{WV}^W((1+y)^{L(0)+N} w, y) v \\ &= \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_y y^{-N-m-1} (1+y)^{\text{wt } w+N} \cdot \\ &\quad \cdot (1+y)^{-(L(-1)+L(0))} Y_{WV}^W((1+y)^{L(0)+N} w, y) v. \end{aligned}$$

Then for $u \in V$, homogeneous $v \in V$ and homogeneous $w \in W$,

$$\begin{aligned} &(w *_N v) *_N u \\ &= \sum_{m=0}^N \sum_{n=0}^N (-1)^{m+n} \binom{m+N}{N} \binom{n+N}{N} \text{Res}_{x_2} \text{Res}_y y^{-N-m-1} x_2^{-N-n-1} (1+y)^{\text{wt } w+N} \cdot \\ &\quad \cdot (1+x_2)^{-(L(-1)+L(0))} Y_{WV}^W((1+x_2)^{L(0)+N} (1+y)^{-(L(-1)+L(0))} \cdot \\ &\quad \cdot Y_{WV}^W((1+y)^{L(0)+N} w, y) v, x_2) u \\ &= \sum_{m=0}^N \sum_{n=0}^N (-1)^{m+n} \binom{m+N}{N} \binom{n+N}{N} \text{Res}_{x_2} \text{Res}_y y^{-N-m-1} x_2^{-N-n-1} (1+y)^{\text{wt } w+N} \cdot \\ &\quad \cdot (1+x_2)^{-(L(-1)+L(0))} Y_{WV}^W((1+y)^{-((1+x_2)L(-1)+L(0))} (1+x_2)^{L(0)+N} \cdot \\ &\quad \cdot Y_{WV}^W((1+y)^{L(0)+N} w, y) v, x_2) u \end{aligned}$$

$$\begin{aligned}
&= \sum_{m=0}^N \sum_{n=0}^N (-1)^{m+n} \binom{m+N}{N} \binom{n+N}{N} \text{Res}_{x_2} \text{Res}_y y^{-N-m-1} x_2^{-N-n-1} (1+y)^{\text{wt } w+N} \cdot \\
&\quad \cdot (1+x_2)^{-(L(-1)+L(0))} Y_{WV}^W((1+y)^{-(x_2 L(-1)+L(0))} e^{-yL(-1)} (1+x_2)^{L(0)+N} \cdot \\
&\quad \cdot Y_{WV}^W((1+y)^{L(0)_n} w, y) v, x_2) u \\
&= \sum_{m=0}^N \sum_{n=0}^N (-1)^{m+n} \binom{m+N}{N} \binom{n+N}{N} \text{Res}_{x_2} \text{Res}_y y^{-N-m-1} x_2^{-N-n-1} (1+y)^{\text{wt } w+N} \cdot \\
&\quad \cdot (1+x_2)^{-(L(-1)+L(0))} (1+y)^{-(L(-1)+L(0))} Y_{WV}^W((1+x_2)^{L(0)+N} \cdot \\
&\quad \cdot Y_{WV}^W((1+y)^{L(0)_n} w, y) v, x_2) (1+y)^{L(-1)+L(0)} u \\
&= \sum_{m=0}^N \sum_{n=0}^N (-1)^{m+n} \binom{m+N}{N} \binom{n+N}{N} \text{Res}_{x_2} \text{Res}_y y^{-N-m-1} x_2^{-N-n-1} (1+y)^{\text{wt } w+N} \cdot \\
&\quad \cdot (1+x_2)^{\text{wt } w+\text{wt } v+N} (1+x_2)^{-(L(-1)+L(0))} (1+y)^{-(L(-1)+L(0))} \cdot \\
&\quad \cdot Y_{WV}^W(Y_{WV}^W((1+x_2+(1+x_2)y)^{L(0)_n} w, (1+x_2)y) v, x_2) u \\
&= \sum_{m=0}^N \sum_{n=0}^N (-1)^{m+n} \binom{m+N}{N} \binom{n+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} x_0^{-N-m-1} x_2^{-N-n-1} (1+x_2)^{\text{wt } v+N+m} \cdot \\
&\quad \cdot (1+x_2+x_0)^{\text{wt } w+N} (1+x_2+x_0)^{-(L(-1)+L(0))} \cdot \\
&\quad \cdot Y_{WV}^W(Y_{WV}^W((1+x_2+x_0)^{L(0)_n} w, x_0) v, x_2) u \\
&= \sum_{m=0}^N \sum_{n=0}^N (-1)^{m+n} \binom{m+N}{N} \binom{n+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} \text{Res}_{x_1} x_0^{-1} \delta\left(\frac{x_1-x_2}{x_0}\right) \cdot \\
&\quad \cdot x_0^{-N-m-1} x_2^{-N-n-1} (1+x_2+x_0)^{\text{wt } w+N} (1+x_2)^{\text{wt } v+N+m} \cdot \\
&\quad \cdot (1+x_2+x_0)^{-(L(-1)+L(0))} Y_{WV}^W((1+x_2+x_0)^{L(0)_n} w, x_1) Y_V(v, x_2) u \\
&\quad - \sum_{m=0}^N \sum_{n=0}^N (-1)^{m+n} \binom{m+N}{N} \binom{n+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} \text{Res}_{x_1} x_0^{-1} \delta\left(\frac{x_2-x_1}{-x_0}\right) \cdot \\
&\quad \cdot x_0^{-N-m-1} x_2^{-N-n-1} (1+x_2+x_0)^{\text{wt } w+N} (1+x_2)^{\text{wt } v+N+m} \cdot \\
&\quad \cdot (1+x_2+x_0)^{-(L(-1)+L(0))} Y_W(v, x_2) Y_{WV}^W((1+x_2+x_0)^{L(0)_n} w, x_1) u \\
&= \sum_{m=0}^N \sum_{n=0}^N (-1)^{m+n} \binom{m+N}{N} \binom{n+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} \text{Res}_{x_1} x_0^{-1} \delta\left(\frac{x_1-x_2}{x_0}\right) \cdot \\
&\quad \cdot (x_1-x_2)^{-N-m-1} x_2^{-N-n-1} (1+x_1)^{\text{wt } w+N} (1+x_2)^{\text{wt } v+N+m} \cdot \\
&\quad \cdot (1+x_1)^{-(L(-1)+L(0))} Y_{WV}^W((1+x_1)^{L(0)_n} w, x_1) Y_V(v, x_2) u \\
&\quad - \sum_{m=0}^N \sum_{n=0}^N (-1)^{m+n} \binom{m+N}{N} \binom{n+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} \text{Res}_{x_1} x_0^{-1} \delta\left(\frac{x_2-x_1}{-x_0}\right) \cdot \\
&\quad \cdot (-x_2+x_1)^{-N-m-1} x_2^{-N-n-1} (1+x_1)^{\text{wt } w+N} (1+x_2)^{\text{wt } v+N+m} \cdot \\
&\quad \cdot (1+x_1)^{-(L(-1)+L(0))} Y_W(v, x_2) Y_{WV}^W((1+x_1)^{L(0)_n} w, x_1) u
\end{aligned}$$

$$\begin{aligned}
&= \sum_{m=0}^N \sum_{n=0}^N \sum_{i \in \mathbb{N}} (-1)^{m+n+i} \binom{m+N}{N} \binom{n+N}{N} \binom{-N-m-1}{i} \cdot \\
&\quad \cdot \text{Res}_{x_2} \text{Res}_{x_1} x_1^{-N-m-1-i} x_2^{-N-n-1+i} (1+x_2)^m (1+x_1)^{-(L(-1)+L(0))} \cdot \\
&\quad \cdot Y_{WV}^W((1+x_1)^{L(0)+N} w, x_1) Y_V((1+x_2)^{L(0)+N} v, x_2) u \\
&- \sum_{m=0}^N \sum_{n=0}^N \sum_{i \in \mathbb{N}} (-1)^{m+n+i} \binom{m+N}{N} \binom{n+N}{N} \binom{-N-m-1}{i} \cdot \\
&\quad \cdot \text{Res}_{x_2} \text{Res}_{x_1} x_1^i x_2^{-2N-m-n-2-i} (1+x_1)^{-(L(-1)+L(0))} \cdot \\
&\quad \cdot Y_W((1+x_2)^{L(0)+N+m} v, x_2) Y_{WV}^W((1+x_1)^{L(0)+N} w, x_1) u,
\end{aligned}$$

where in the fifth step, we have used (4.13) and Part 3 in Lemma 4.4 to obtain $(L(-1) + L(0))\tilde{w} = \omega *_N \tilde{w} - \tilde{w} *_N \omega$ which is in $O_N(W)$ for $\tilde{w} \in O_N(W)$ by Lemma 4.6, in the sixth step, we have changed the variable $y = \frac{x_0}{1+x_2}$ and in the seventh step, we have used the Jacobi identity (4.9). By Part 2 in Lemma 4.4, we know that every term in the second sum in the right-hand side is of the form $(L(-1) + L(0))^k \tilde{w}$ for some $k \in \mathbb{N}$ and $\tilde{w} \in O_N(W)$ and is thus in $O_N(W)$. Also those terms with $i > N - m$ in the first sum in the right-hand side lie in $O_N(W)$. The sum of those terms with $i \leq N - m$ in the first sum in the right-hand side equals

$$\begin{aligned}
&\sum_{m=0}^N \sum_{n=0}^N \sum_{i=0}^{N-m} (-1)^{m+n+i} \binom{m+N}{N} \binom{n+N}{N} \binom{-N-m-1}{i} \cdot \\
&\quad \cdot \text{Res}_{x_2} \text{Res}_{x_1} x_1^{-N-m-1-i} x_2^{-N-n-1+i} (1+x_2)^m (1+x_1)^{-(L(-1)+L(0))} \cdot \\
&\quad \cdot Y_{WV}^W((1+x_1)^{L(0)+N} w, x_1) Y_V((1+x_2)^{L(0)+N} v, x_2) u \\
&= w *_N (v *_N u) \\
&\quad + \sum_{m=0}^N \sum_{n=0}^N (-1)^{m+n} \binom{m+N}{N} \binom{n+N}{N} \text{Res}_{x_2} \text{Res}_{x_1} x_1^{-N-m-1} x_2^{-N-n-1} \cdot \\
&\quad \cdot \left(\sum_{i=0}^{N-m} \sum_{j \in \mathbb{N}} \binom{-N-m-1}{i} \binom{m}{j} (-1)^i \frac{x_2^{i+j}}{x_1^i} - 1 \right) \cdot \\
&\quad \cdot (1+x_1)^{-(L(-1)+L(0))} Y_{WV}^W((1+x_1)^{L(0)+N} w, x_1) Y_V((1+x_2)^{L(0)+N} v, x_2) u \\
&= w *_N (v *_N u) \\
&\quad + \sum_{n=0}^N (-1)^n \binom{n+N}{N} \text{Res}_{x_2} \text{Res}_{x_1} x_1^{-N-1} x_2^{-N-n-1} \cdot \\
&\quad \cdot \left(\sum_{m=0}^N (-1)^m \binom{m+N}{N} \left(\sum_{i=0}^{N-m} \sum_{j \in \mathbb{N}} \binom{-N-m-1}{i} \binom{m}{j} (-1)^i \frac{x_2^{i+j}}{x_1^{m+i}} - \frac{1}{x_1^m} \right) \right) \cdot \\
&\quad \cdot (1+x_1)^{-(L(-1)+L(0))} Y_{WV}^W((1+x_1)^{L(0)+N} w, x_1) Y_V((1+x_2)^{L(0)+N} v, x_2) u.
\end{aligned}$$

By Proposition 5.3 in [DLM],

$$\sum_{m=0}^N (-1)^m \binom{m+N}{N} \left(\sum_{i=0}^{N-m} \sum_{j \in \mathbb{N}} \binom{-N-m-1}{i} \binom{m}{j} (-1)^i \frac{x_2^{i+j}}{x_1^{m+i}} - \frac{1}{x_1^m} \right) = 0.$$

Thus the calculations above give (4.17).

Finally we also have to show that the left action and the right action of $A_N(V)$ on $A_N(W)$ commute, that is,

$$(u *_N w) *_N v \equiv u *_N (w *_N v) \pmod{O_N(W)}.$$

The proof is similar to the proof of (4.17) and is omitted. ■

5 Logarithmic intertwining operators

We begin our study of the connection between logarithmic intertwining operators and associative algebras $A_N(V)$ in this section. Logarithmic intertwining operators were introduced first in the representation theory of vertex operator algebras by Milas [M]. Here we recall the general definition of logarithmic intertwining operator from [HLZ2].

Definition 5.1. Let W_1 , W_2 and W_3 be generalized V -modules. A *logarithmic intertwining operator of type* $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$ is a linear map

$$\begin{aligned} \mathcal{Y} : W_1 \otimes W_2 &\rightarrow W_3[\log x]\{x\}, \\ w_{(1)} \otimes w_{(2)} &\mapsto \mathcal{Y}(w_{(1)}, x)w_{(2)} = \sum_{n \in \mathbb{C}} \sum_{k \in \mathbb{N}} \mathcal{Y}_{n,k}(w_{(1)})w_{(2)} x^{-n-1} (\log x)^k \in W_3[\log x]\{x\} \end{aligned}$$

satisfying the following conditions:

1. The *lower truncation condition*: For any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $n \in \mathbb{C}$,

$$\mathcal{Y}_{n+m;k}(w_{(1)})w_{(2)} = 0 \quad \text{for } m \in \mathbb{N} \text{ sufficiently large, independent of } k. \quad (5.18)$$

2. The *Jacobi identity*:

$$\begin{aligned} &x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y_{W_3}(v, x_1) \mathcal{Y}(w_{(1)}, x_2) w_{(2)} \\ &\quad - x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right) \mathcal{Y}(w_{(1)}, x_2) Y_{W_2}(v, x_1) w_{(2)} \\ &= x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) \mathcal{Y}(Y_{W_1}(v, x_0) w_{(1)}, x_2) w_{(2)} \end{aligned} \quad (5.19)$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$.

3. The $L(-1)$ -derivative property: for any $w_{(1)} \in W_1$,

$$\mathcal{Y}(L(-1)w_{(1)}, x) = \frac{d}{dx} \mathcal{Y}(w_{(1)}, x). \quad (5.20)$$

Using Proposition 3.6 and Corollary 3.7, we have the following result on logarithmic intertwining operators:

Proposition 5.2. *Let W_1 , W_2 and W_3 be lower-bounded generalized V -modules and \mathcal{Y} a logarithmic intertwining operator of type $\left(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix}\right)$. Let N_2 and N'_3 be positive integers such that W_2 and W'_3 are generated by $\Omega_{N_2}^0(W_2)$ and $\Omega_{N'_3}^0(W'_3)$. For $\tilde{w}_{(1)} \in W_1$, $\tilde{w}_{(2)} \in W_2$, and $\tilde{w}'_{(3)} \in W'_3$, the series $\langle \tilde{w}'_{(3)}, \mathcal{Y}(\tilde{w}_{(1)}, x) \tilde{w}_{(2)} \rangle$ can be expressed as a linear combination of series of the form $\langle w'_{(3)}, \mathcal{Y}(w_{(1)}, x) w_{(2)} \rangle$ for $w_{(1)} \in W_1$, $w_{(2)} \in \Omega_{N_2}^0(W_2)$ and $w'_{(3)} \in \Omega_{N'_3}^0(W'_3)$ with Laurent polynomials of x as coefficients.*

Proof. By the commutator formula for vertex operators and logarithmic intertwining operators, we have

$$(Y_{W_3})_m(u) \mathcal{Y}(w_{(1)}, x) - \mathcal{Y}(w_{(1)}, x) (Y_{W_2})_m(u) = \text{Res}_{x_0} (x_2 + x_0)^m \mathcal{Y}(Y_{W_1}(u, x_0) w_{(1)}, x_2)$$

for $u \in V$ and $w_{(1)} \in W_1$. The conclusion follows from this commutator formula, Proposition 3.6, Corollary 3.7 and induction on the weights of $\tilde{w}_{(2)}$ and $\tilde{w}'_{(3)}$. \blacksquare

Let

$$\mathcal{Y}^0(w_{(1)}, x) = \sum_{n \in \mathbb{C}} \mathcal{Y}_{n,0}(w_{(1)}) x^{-n-1}.$$

Then for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, the Jacobi identity for $\mathcal{Y}^0(w_{(1)}, x)$ holds, that is,

$$\begin{aligned} & x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y_{W_3}(v, x_1) \mathcal{Y}^0(w_{(1)}, x_2) w_{(2)} \\ & - x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right) \mathcal{Y}^0(w_{(1)}, x_2) Y_{W_2}(v, x_1) w_{(2)} \\ & = x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) \mathcal{Y}^0(Y_{W_1}(v, x_0) w_{(1)}, x_2) w_{(2)}. \end{aligned} \quad (5.21)$$

Proposition 5.3 ([M]). *Let W_1 , W_2 , W_3 be lower-bounded generalized V -modules and let \mathcal{Y} be a logarithmic intertwining operator of type $\left(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix}\right)$. Let $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $h_1, h_2 \in \mathbb{C}$, and $k_1, k_2 \in \mathbb{Z}_+$ such that $(L(0) - h_1)^{k_1} w_{(1)} = 0$ and $(L(0) - h_2)^{k_2} w_{(2)} = 0$.*

1. For $w'_{(3)} \in W'_3$, $h_3 \in \mathbb{C}$ and $k_3 \in \mathbb{Z}_+$ such that $(L'(0) - h_3)^{k_3} w'_{(3)} = 0$,

$$\begin{aligned} & \langle w'_{(3)}, \mathcal{Y}(w_{(1)}, x) w_{(2)} \rangle \\ & \in \mathbb{C} x^{h_3 - h_1 - h_2} + \mathbb{C} x^{h_3 - h_1 - h_2} \log x + \dots \oplus \mathbb{C} x^{h_3 - h_1 - h_2} (\log x)^{k_1 + k_2 + k_3 - 3}. \end{aligned}$$

2. Suppose that there exist $h_3 \in \mathbb{C}$ and $k_3 \in \mathbb{Z}_+$ such that for any homogeneous element $w'_{(3)} \in W'_3$, $(L'(0) - h_3)^{k_3} w'_{(3)} = 0$. Then

$$\mathcal{Y}(w_{(1)}, x)w_{(2)} \in x^{h_3-h_1-h_2}W_3[[x, x^{-1}]] + x^{h_3-h_1-h_2}W_3[[x, x^{-1}]] \log x \\ + \dots + x^{h_3-h_1-h_2}W_3[[x, x^{-1}]](\log x)^{k_1+k_2+k_3-3},$$

Let W be a generalized V -module. Recall the operator $x^{\pm L(0)}$ in [HLZ2] defined by

$$x^{\pm L(0)}w = x^{\pm n} \sum_{i \in \mathbb{N}} \frac{(L(0) - n)^i w}{i!} (\pm \log x)^i \in x^{\pm n} W_{[n]}[\log x] \quad (5.22)$$

for $w \in W_{[n]}$. Also recall the $L(0)$ -conjugation property for logarithmic intertwining operators in [HLZ2]: For any logarithmic intertwining operator \mathcal{Y} of type $\binom{W_3}{W_1 W_2}$ and any $w_{(1)} \in W_1$,

$$y^{L(0)} \mathcal{Y}(w_{(1)}, x) y^{-L(0)} = \mathcal{Y}(y^{L(0)} w_{(1)}, xy). \quad (5.23)$$

Let W_1, W_2 and W_3 be lower-bounded generalized V -modules and \mathcal{Y} an logarithmic intertwining operator of type $\binom{W_3}{W_1 W_2}$. Then for $N \in \mathbb{N}$, $A_N(W_1) \otimes_{A_N(V)} \Omega_N^0(W_2)$ and $\Omega_N^0(W_3)$ are both left $A_N(V)$ -modules. Note that $\Omega_N^0(W_3)$ as a subspace of W_3 is also graded and for any $n \in \mathbb{C}$, the image of $\Omega_N^0(W_3)$ under the projection $\pi_n : W_3 \rightarrow (W_3)_{[n]}$ is denoted $(\Omega_N^0(W_3))_{[n]}$.

First, we consider the case that there exists $h_3 \in \mathbb{C}$ such that $W_3 = \coprod_{n \in h_3 + \mathbb{N}} (W_3)_{[n]}$ and $(W_3)_{[h_3]} \neq 0$. As above, let $L(0)_s$ be the semisimple part of the operator $L(0)$ on any module for the Virasoro algebra. Let

$$\rho(\mathcal{Y}) : W_1 \otimes \Omega_N^0(W_2) \rightarrow W_3$$

be defined by

$$\begin{aligned} \rho(\mathcal{Y})(w_{(1)} \otimes w_{(2)}) &= \sum_{n=0}^N \text{Res}_x x^{-h_3-n-1} \mathcal{Y}^0(x^{L(0)_s} w_{(1)}, x) x^{L(0)_s} w_{(2)} \\ &= \sum_{n=0}^N \mathcal{Y}_{\text{wt } w_{(1)} + \text{wt } w_{(2)} - h_3 - n - 1, 0}(w_{(1)}) w_{(2)}, \end{aligned}$$

for homogeneous $w_{(1)} \in W_1$, $w_{(2)} \in \Omega_N^0(W_2)$. In the general case that $W_3 = \coprod_{\mu \in \mathbb{C}/\mathbb{Z}} W_3^\mu$, let $\pi^\mu : W_3 \rightarrow W_3^\mu$ be the projection from W_3 to W_3^μ for $\mu \in \mathbb{C}/\mathbb{Z}$. Then $\pi^\mu \circ \mathcal{Y}$ is a logarithmic intertwining operator of type $\binom{W_3^\mu}{W_1 W_2}$ for each $\mu \in \mathbb{C}/\mathbb{Z}$. We define

$$\rho(\mathcal{Y}) : W_1 \otimes \Omega_N^0(W_2) \rightarrow W_3$$

by

$$\rho(\mathcal{Y})(w_{(1)} \otimes w_{(2)}) = \sum_{\mu \in \mathbb{C}/\mathbb{Z}} \rho(\pi^\mu \circ \mathcal{Y})(w_{(1)} \otimes w_{(2)})$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in \Omega_N^0(W_2)$. We have:

Lemma 5.4. *The image of $W_1 \otimes \Omega_N^0(W_2)$ under $\rho(\mathcal{Y})$ is in $\Omega_N^0(W_3)$. In particular, $\rho(\mathcal{Y})$ is a linear map from $W_1 \otimes \Omega_N^0(W_2)$ to $\Omega_N^0(W_3)$.*

Proof. This follows from the definition by calculating the weights. ■

The following lemma shows that $\rho(\mathcal{Y})$ is in fact a linear map from $A_N(W_1) \otimes \Omega_N^0(W_2)$ to $\Omega_N^0(W_3)$:

Lemma 5.5. *For $w_{(1)} \in O_N(W_1)$ and $w_{(2)} \in \Omega_N^0(W_2)$, $\rho(\mathcal{Y})(w_{(1)} \otimes w_{(2)}) = 0$.*

Proof. We prove the lemma in the case that $W_3 = \coprod_{n \in h_3 + \mathbb{N}} (W_3)_{[n]}$ for some $h_3 \in \mathbb{C}$ and $(W_3)_{[h_3]} \neq 0$,

$$\begin{aligned} w_{(1)} &= u \circ_N w \\ &= \text{Res}_x x^{-2N-2} Y_{W_1}((1+x)^{L(0)+N} u, x) w \\ &= \text{Res}_x x^{-2N-2} (1+x)^{\text{wt } u+N} Y_{W_1}(u, x) w \end{aligned} \tag{5.24}$$

for some homogeneous $u \in V$ and $w \in W_1$, and $w_{(2)} \in (\Omega_N^0(W_2))_{[h_2]}$. The general case follows easily.

In this case, by the definition of $\rho(\mathcal{Y})$, (5.24), the $L(0)_s$ -conjugation property above and the Jacobi identity for \mathcal{Y}^0 , we have

$$\begin{aligned} &\rho(\mathcal{Y})((w_{(1)} + O_N(W_1)) \otimes w_{(2)}) \\ &= \sum_{n=0}^N \text{Res}_{x_2} \text{Res}_{x_0} x_0^{-2N-2} x_2^{h_2-h_3-n-1} (1+x_0)^N \cdot \\ &\quad \cdot \mathcal{Y}^0(x_2^{L(0)s} Y_{W_1}((1+x_0)^{L(0)} u, x_0) w_{(1)}, x_2) w_{(2)} \\ &= \sum_{n=0}^N \text{Res}_{x_2} \text{Res}_{x_0} x_0^{-2N-2} x_2^{h_2-h_3-n-1} (1+x_0)^N \cdot \\ &\quad \cdot \mathcal{Y}^0(Y_{W_1}((x_2+x_0x_2)^{L(0)s} u, x_0x_2) x_2^{L(0)s} w_{(1)}, x_2) w_{(2)} \\ &= \sum_{n=0}^N \text{Res}_{x_2} \text{Res}_{x_0} \text{Res}_{x_1} x_1^{-1} \delta\left(\frac{x_2+x_0x_2}{x_1}\right) x_0^{-2N-2} x_2^{h_2-h_3-n-N-1} (x_2+x_0x_2)^{\text{wt } u+N} \cdot \\ &\quad \cdot \mathcal{Y}^0(Y_{W_1}(u, x_0x_2) x_2^{L(0)s} w_{(1)}, x_2) w_{(2)} \\ &= \sum_{n=0}^N \text{Res}_{x_2} \text{Res}_{x_0} \text{Res}_{x_1} x_1^{-1} \delta\left(\frac{x_2+x_0x_2}{x_1}\right) x_0^{-2N-2} x_2^{h_2-h_3-n-N-1} x_1^{\text{wt } u+N} \cdot \\ &\quad \cdot \mathcal{Y}^0(Y_{W_1}(u, x_0x_2) x_2^{L(0)s} w_{(1)}, x_2) w_{(2)} \\ &= \sum_{n=0}^N \text{Res}_{x_2} \text{Res}_{x_0} \text{Res}_{x_1} (x_0x_2)^{-1} \delta\left(\frac{x_1-x_2}{x_0x_2}\right) x_0^{-2N-2} x_2^{h_2-h_3-n-N-1} \cdot \\ &\quad \cdot x_1^{\text{wt } u+N} Y_{W_3}(u, x_1) \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) w_{(2)} \end{aligned}$$

$$\begin{aligned}
& - \sum_{n=0}^N \text{Res}_{x_2} \text{Res}_{x_0} \text{Res}_{x_1} (x_0 x_2)^{-1} \delta \left(\frac{x_2 - x_1}{-x_0 x_2} \right) x_0^{-2N-2} x_2^{h_2-h_3-n-N-1} \cdot \\
& \quad \cdot x_1^{\text{wt } u+N} \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) Y_{W_2}(u, x_1) w_{(2)} \\
& = \sum_{n=0}^N \text{Res}_{x_2} \text{Res}_{x_1} (x_1 - x_2)^{-2N-2} x_2^{h_2-h_3-n+N} \cdot \\
& \quad \cdot x_1^{\text{wt } u+N} Y_{W_3}(u, x_1) \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) w_{(2)} \\
& \quad - \sum_{n=0}^N \text{Res}_{x_2} \text{Res}_{x_1} (-x_2 + x_1)^{-2N-2} x_2^{h_2-h_3-n+N} \cdot \\
& \quad \cdot x_1^{\text{wt } u+N} \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) Y_{W_2}(u, x_1) w_{(2)} \\
& = \sum_{n=0}^N \sum_{m=0}^N \binom{-2N-2}{l} x_2^{h_2-h_3-n+N+l} \cdot \\
& \quad \cdot \text{Res}_{x_2} (Y_{W_3})_{\text{wt } u-N-2-l}(u) \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) w_{(2)} \\
& \quad - \sum_{n=0}^N \sum_{l \in \mathbb{N}} (-1)^{m-2N-2-l} \binom{-2N-2}{l} x_2^{h_2-h_3-n-N-2-l} \cdot \\
& \quad \cdot \text{Res}_{x_2} \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) (Y_{W_2})_{\text{wt } u+N+l}(u) w_{(2)}. \tag{5.25}
\end{aligned}$$

Since the weight of $(Y_{W_3})_{\text{wt } u-N-2-l}(u)$ is $N+1+l$, the real parts of the weights of the homogeneous components of the coefficients of

$$(Y_{W_3})_{\text{wt } u-N-2-l}(u) \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) w_{(2)}$$

are larger than or equal to $N+1+l+r_3$. Thus the first term in the right-hand side of (5.25) is equal to 0. Since $w_{(2)} \in \Omega_N^0(W_2)$, $(Y_{W_2})_{\text{wt } u+N+l}(u) w_{(2)} = 0$. From (5.25), we see that the second term in the right-hand side of (5.25) is also equal to 0. Thus the left-hand side of (5.25) is 0, proving the lemma. \blacksquare

As we mentioned before, by this lemma, $\rho(\mathcal{Y})$ is in fact a linear map from $A_N(W_1) \otimes \Omega_N^0(W_2)$ to $\Omega_N^0(W_3)$. We now have:

Proposition 5.6. *The map $\rho(\mathcal{Y})$ is in fact an $A_N(V)$ -module homomorphism from $A_N(W_1) \otimes \Omega_N^0(W_2)$ to $\Omega_N^0(W_3)$, that is,*

$$\rho(\mathcal{Y}) \in \text{Hom}_{A_N(V)}(A_N(W_1) \otimes \Omega_N^0(W_2), \Omega_N^0(W_3)).$$

Proof. We need to prove

$$\rho(\mathcal{Y})((u *_N w_{(1)} + O_N(W_1)) \otimes w_{(2)}) = o(u) \rho(\mathcal{Y})((w_{(1)} + O_N(W_1)) \otimes w_{(2)}).$$

for $u \in V$, $w_{(1)} \in W_1$, and $w_{(2)} \in \Omega_N^0(W_2)$. We prove this in the case that $W_3 = \coprod_{n \in h_3 + \mathbb{N}} (W_3)_{[n]}$ for some $h_3 \in \mathbb{C}$ and $(W_3)_{[h_3]} \neq 0$. The general case follows easily.

Let $w_{(2)} \in \Omega_N^0(W_2)$ be homogeneous of weight h_2 . Calculations similar to those in the proof of Lemma 5.5 give

$$\begin{aligned}
& \rho(\mathcal{Y})((u *_N w_{(1)} + O_N(W_1)) \otimes w_{(2)}) \\
&= \sum_{n=0}^N \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} x_0^{-N-m-1} x_2^{h_2-h_3-n-1} (1+x_0)^N \cdot \\
&\quad \cdot \mathcal{Y}^0(x_2^{L(0)s} Y_{W_1}((1+x_0)^{L(0)} u, x_0) w_{(1)}, x_2) w_{(2)} \\
&= \sum_{n=0}^N \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} x_0^{-N-m-1} x_2^{h_2-h_3-n-1} (1+x_0)^N \cdot \\
&\quad \cdot \mathcal{Y}^0(Y_{W_1}((x_2+x_0x_2)^{L(0)} u, x_0x_2) x_2^{L(0)s} w_{(1)}, x_2) w_{(2)} \\
&= \sum_{n=0}^N \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} \text{Res}_{x_1} x_1^{-1} \delta\left(\frac{x_2+x_0x_2}{x_1}\right) x_0^{-N-m-1} \cdot \\
&\quad \cdot x_2^{h_2-h_3-n-N-1} (x_2+x_0x_2)^{\text{wt } u+N} \mathcal{Y}^0(Y_{W_1}(u, x_0x_2) x_2^{L(0)s} w_{(1)}, x_2) w_{(2)} \\
&= \sum_{n=0}^N \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} \text{Res}_{x_1} x_1^{-1} \delta\left(\frac{x_2+x_0x_2}{x_1}\right) x_0^{-N-m-1} \cdot \\
&\quad \cdot x_2^{h_2-h_3-n-N-1} x_1^{\text{wt } u+N} \mathcal{Y}^0(Y_{W_1}(u, x_0x_2) x_2^{L(0)s} w_{(1)}, x_2) w_{(2)} \\
&= \sum_{n=0}^N \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} \text{Res}_{x_1} (x_0x_2)^{-1} \delta\left(\frac{x_1-x_2}{x_0x_2}\right) \cdot \\
&\quad \cdot x_2^{h_2-h_3-n-N-1} x_0^{-N-m-1} x_1^{\text{wt } u+N} Y_{W_3}(u, x_1) \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) w_{(2)} \\
&\quad - \sum_{n=0}^N \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} \text{Res}_{x_1} (x_0x_2)^{-1} \delta\left(\frac{x_2-x_1}{-x_0x_2}\right) \cdot \\
&\quad \cdot x_2^{h_2-h_3-n-N-1} x_0^{-N-m-1} x_1^{\text{wt } u+N} \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) Y_{W_2}(u, x_1) w_{(2)} \\
&= \sum_{n=0}^N \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_1} x_2^{h_2-h_3-n+m-1} (x_1-x_2)^{-N-m-1} x_1^{\text{wt } u+N} \cdot \\
&\quad \cdot Y_{W_3}(u, x_1) \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) w_{(2)} \\
&\quad - \sum_{n=0}^N \sum_{m=0}^N (-1)^m \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_1} x_2^{h_2-h_3-n+m-1} (-x_2+x_1)^{-N-m-1} x_1^{\text{wt } u+N} \cdot \\
&\quad \cdot \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) Y_{W_2}(u, x_1) w_{(2)} \\
&= \sum_{n=0}^N \sum_{m=0}^N \sum_{l \in \mathbb{N}} (-1)^{m+l} \binom{m+N}{N} \binom{-N-m-1}{l} \text{Res}_{x_2} x_2^{h_2-h_3-n+m+l-1} \cdot \\
&\quad \cdot (Y_{W_3})_{\text{wt } u-m-1-l}(u) \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) w_{(2)}
\end{aligned}$$

$$\begin{aligned}
& - \sum_{n=0}^N \sum_{m=0}^N \sum_{l \in \mathbb{N}} (-1)^{m-N-m-1-l} \binom{m+N}{N} \binom{-N-m-1}{l} \text{Res}_{x_2} x_2^{h_2-h_3-n-N-l-2} \cdot \\
& \quad \cdot \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) (Y_{W_2})_{\text{wt } u+N+l}(u) w_{(2)} \\
& = \sum_{n=0}^N \sum_{m=0}^N \sum_{l=0}^{N-m} (-1)^{m+l} \binom{m+N}{N} \binom{-N-m-1}{l} \text{Res}_{x_2} x_2^{h_2-h_3-n+m+l-1} \cdot \\
& \quad \cdot (Y_{W_3})_{\text{wt } u-m-1-l}(u) \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) w_{(2)} \\
& = \sum_{n=0}^N \sum_{m=0}^N \sum_{i=m}^N (-1)^i \binom{m+N}{N} \binom{-N-m-1}{i-m} \text{Res}_{x_2} x_2^{h_2-h_3-n+i-1} \cdot \\
& \quad \cdot (Y_{W_3})_{\text{wt } u-i-1}(u) \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) w_{(2)} \\
& = \sum_{n=0}^N \sum_{i=0}^N \sum_{m=0}^i (-1)^i \binom{m+N}{N} \binom{-N-m-1}{i-m} \text{Res}_{x_2} x_2^{h_2-h_3-n+i-1} \cdot \\
& \quad \cdot (Y_{W_3})_{\text{wt } u-i-1}(u) \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) w_{(2)} \\
& = \sum_{n=0}^N \text{Res}_{x_2} x_2^{h_2-h_3-n-1} (Y_{W_3})_{\text{wt } u-1}(u) \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) w_{(2)} \\
& \quad + \sum_{n=0}^N \sum_{i=1}^N \sum_{m=0}^i (-1)^i \binom{m+N}{N} \binom{-N-m-1}{i-m} x_2^{h_2-h_3-n+i-1} \cdot \\
& \quad \cdot \text{Res}_{x_2} (Y_{W_3})_{\text{wt } u-i-1}(u) \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) w_{(2)}. \tag{5.26}
\end{aligned}$$

Since for $i = 1, \dots, N$,

$$\begin{aligned}
& \sum_{m=0}^i \binom{m+N}{N} \binom{-N-m-1}{i-m} \\
& = \sum_{m=0}^i (-1)^{i-m} \binom{m+N}{N} \binom{N+i}{i-m} \\
& = \sum_{m=0}^i (-1)^{i-m} \binom{N+i}{N} \binom{i}{m} \\
& = \binom{N+i}{N} (1-1)^i \\
& = 0, \tag{5.27}
\end{aligned}$$

the right-hand side of (5.26) is equal to

$$\begin{aligned}
& \sum_{n=0}^N \text{Res}_{x_2} (Y_{W_3})_{\text{wt } u-1}(u) \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) w_{(2)} x_2^{h_2-h_3-n-1} \\
& = o(u) \rho(\mathcal{Y})((w_{(1)} + O_N(W_1)) \otimes w_{(2)}).
\end{aligned}$$

This completes the proof. ■

Proposition 5.7. *The map $\rho(\mathcal{Y})$ is in fact an $A_N(V)$ -module homomorphism from $A_N(W_1) \otimes_{A_N(V)} \Omega_N^0(W_2)$ to $\Omega_N^0(W_3)$, that is,*

$$\rho(\mathcal{Y}) \in \text{Hom}_{A_N(V)}(A_N(W_1) \otimes_{A_N(V)} \Omega_N^0(W_2), \Omega_N^0(W_3)).$$

Proof. We need to prove

$$\rho(\mathcal{Y})((w_{(1)} *_N u + O_N(W_1)) \otimes w_{(2)}) = \rho(\mathcal{Y})((w_{(1)} + O_N(W_1)) \otimes o(u)w_{(2)})$$

for $u \in V$, $w_{(1)} \in W_1$, and $w_{(2)} \in \Omega_N^0(W_2)$. We shall prove this equality only in the case that $W_3 = \coprod_{n \in h_3 + \mathbb{N}} (W_3)_{[n]}$ for some $h_3 \in \mathbb{C}$ and $(W_3)_{[h_3]} \neq 0$. The general case follows easily.

Let $w_{(2)} \in \Omega_N^0(W_2)$ be homogeneous of weight h_2 . By Part 1 in Lemma 4.4,

$$\begin{aligned} & w_{(1)} *_N u + O_N(W_1) \\ &= \sum_{m=0}^N \binom{m+N}{N} (-1)^N \text{Res}_x x^{-N-m-1} Y_W((1+x)^{L(0)+m-1} u, x) w_{(1)} + O_N(W_1). \end{aligned}$$

Then

$$\begin{aligned} & \rho(\mathcal{Y})((w_{(1)} *_N u + O_N(W_1)) \otimes w_{(2)}) \\ &= \sum_{n=0}^N \sum_{m=0}^N (-1)^N \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} x_0^{-N-m-1} x_2^{h_2-h_3-n-1} (1+x_0)^{m-1} \cdot \\ & \quad \cdot \mathcal{Y}^0(x_2^{L(0)s} Y_{W_1}((1+x_0)^{L(0)} u, x_0) w_{(1)}, x_2) w_{(2)} \\ &= \sum_{n=0}^N \sum_{m=0}^N (-1)^N \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} x_0^{-N-m-1} x_2^{h_2-h_3-n-1} (1+x_0)^{m-1} \cdot \\ & \quad \cdot \mathcal{Y}^0(Y_{W_1}((x_2+x_0x_2)^{L(0)} u, x_0x_2) x_2^{L(0)s} w_{(1)}, x_2) w_{(2)} \\ &= \sum_{n=0}^N \sum_{m=0}^N (-1)^N \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} \text{Res}_{x_1} x_1^{-1} \delta\left(\frac{x_2+x_0x_2}{x_1}\right) x_0^{-N-m-1} x_2^{h_2-h_3-n-m} \cdot \\ & \quad \cdot (x_2+x_0x_2)^{\text{wt } u+m-1} \mathcal{Y}^0(Y_{W_1}(u, x_0x_2) x_2^{L(0)s} w_{(1)}, x_2) w_{(2)} \\ &= \sum_{n=0}^N \sum_{m=0}^N (-1)^N \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} \text{Res}_{x_1} x_1^{-1} \delta\left(\frac{x_2+x_0x_2}{x_1}\right) x_0^{-N-m-1} x_2^{h_2-h_3-n-m} \cdot \\ & \quad \cdot x_1^{\text{wt } u+m-1} \mathcal{Y}^0(Y_{W_1}(u, x_0x_2) x_2^{L(0)s} w_{(1)}, x_2) w_{(2)} \\ &= \sum_{n=0}^N \sum_{m=0}^N (-1)^N \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} \text{Res}_{x_1} (x_0x_2)^{-1} \delta\left(\frac{x_1-x_2}{x_0x_2}\right) \cdot \\ & \quad \cdot x_2^{\text{wt } w_{(1)}+h_2-h_3-n-m} x_0^{-N-m-1} x_1^{\text{wt } u+m-1} Y_{W_3}(u, x_1) \mathcal{Y}^0(w_{(1)}, x_2) w_{(2)} \end{aligned}$$

$$\begin{aligned}
& - \sum_{n=0}^N \sum_{m=0}^N (-1)^N \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_0} \text{Res}_{x_1} (x_0 x_2)^{-1} \delta \left(\frac{x_2 - x_1}{-x_0 x_2} \right) \cdot \\
& \quad \cdot x_2^{h_2-h_3-n-m} x_0^{-N-m-1} x_1^{\text{wt } u+m-1} \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) Y_{W_2}(u, x_1) w_{(2)} \\
& = \sum_{n=0}^N \sum_{m=0}^N (-1)^N \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_1} x_2^{\text{wt } w_{(1)}+h_2-h_3-n+N} (x_1 - x_2)^{-N-m-1} \cdot \\
& \quad \cdot x_1^{\text{wt } u+m-1} Y_{W_3}(u, x_1) \mathcal{Y}^0(w_{(1)}, x_2) w_{(2)} \\
& \quad - \sum_{n=0}^N \sum_{m=0}^N (-1)^N \binom{m+N}{N} \text{Res}_{x_2} \text{Res}_{x_1} x_2^{h_2-h_3-n+N} (-x_2 + x_1)^{-N-m-1} \cdot \\
& \quad \cdot x_1^{\text{wt } u+m-1} \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) Y_{W_2}(u, x_1) w_{(2)} \\
& = \sum_{n=0}^N \sum_{m=0}^N \sum_{l \in \mathbb{N}} (-1)^{N+l} \binom{m+N}{N} \binom{-N-m-1}{l} \cdot \\
& \quad \cdot \text{Res}_{x_1} x_1^{\text{wt } u-N-2-l} Y_{W_3}(u, x_1) \mathcal{Y}_{\text{wt } w_{(1)}+h_2-h_3-n+N+l,0}(w_{(1)}) w_{(2)} \\
& \quad - \sum_{n=0}^N \sum_{m=0}^N \sum_{l \in \mathbb{N}} (-1)^{-m-1-l} \binom{m+N}{N} \binom{-N-m-1}{l} \cdot \\
& \quad \cdot \text{Res}_{x_2} x_2^{h_2-h_3-n-m-l-1} \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) (Y_{W_2})_{\text{wt } u+m-1+l}(u) w_{(2)} \\
& = \sum_{n=0}^N \sum_{m=0}^N \sum_{l=0}^{N-m} (-1)^{m+l} \binom{m+N}{N} \binom{-N-m-1}{l} \cdot \\
& \quad \cdot \text{Res}_{x_2} x_2^{h_2-h_3-n-m-l-1} \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) (Y_{W_2})_{\text{wt } u+m-1+l}(u) w_{(2)} \\
& = \sum_{n=0}^N \sum_{m=0}^N \sum_{i=m}^N (-1)^i \binom{m+N}{N} \binom{-N-m-1}{i-m} \cdot \\
& \quad \cdot \text{Res}_{x_2} x_2^{h_2-h_3-n-i-1} \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) (Y_{W_2})_{\text{wt } u+i-1}(u) w_{(2)} \\
& = \sum_{n=0}^N \sum_{i=0}^N \sum_{m=0}^i (-1)^i \binom{m+N}{N} \binom{-N-m-1}{i-m} \cdot \\
& \quad \cdot \text{Res}_{x_2} x_2^{h_2-h_3-n-i-1} \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) (Y_{W_2})_{\text{wt } u+i-1}(u) w_{(2)} \\
& = \sum_{n=0}^N \text{Res}_{x_2} x_2^{h_2-h_3-n-1} \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) (Y_{W_2})_{\text{wt } u-1}(u) w_{(2)} \\
& \quad + \sum_{n=0}^N \sum_{i=1}^N \sum_{m=0}^i (-1)^i \binom{m+N}{N} \binom{-N-m-1}{i-m} \cdot \\
& \quad \cdot \text{Res}_{x_2} x_2^{h_2-h_3-n-i-1} \mathcal{Y}^0(x_2^{L(0)s} w_{(1)}, x_2) (Y_{W_2})_{\text{wt } u+i-1}(u) w_{(2)} \\
& = \rho(\mathcal{Y})((w_{(1)} + O_N(W_1)) \otimes o(u) w_{(2)}), \tag{5.28}
\end{aligned}$$

where in the last step, we used (5.27). This completes the proof. \blacksquare

Let \mathcal{Y} be a logarithmic intertwining operator of type $(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix})$. For homogeneous $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$, let $k_1, k_2, k_3 \in \mathbb{Z}_+$ such that $(L(0) - \text{wt } w_{(1)})^{k_1} w_{(1)} = 0$, $(L(0) - \text{wt } w_{(2)})^{k_2} w_{(2)} = 0$ and $(L'(0) - \text{wt } w'_{(3)})^{k_3} w'_{(3)} = 0$. Then by Proposition 5.3,

$$\begin{aligned} \langle w'_{(3)}, \mathcal{Y}(w_{(1)}, x)w_{(2)} \rangle \\ \in \mathbb{C}x^{\text{wt } w'_{(3)} - \text{wt } w_{(1)} - \text{wt } w_{(2)}} + \mathbb{C}x^{\text{wt } w'_{(3)} - \text{wt } w_{(1)} - \text{wt } w_{(2)}} \log x \\ + \dots + \mathbb{C}x^{\text{wt } w'_{(3)} - \text{wt } w_{(1)} - \text{wt } w_{(2)}} (\log x)^{k_1+k_2+k_3-3}. \end{aligned}$$

For $z \in \mathbb{C}$, let $\log z$ be the value of the logarithm of z such that $0 \leq \arg z < 2\pi$. We substitute $e^{(\text{wt } w'_{(3)} - \text{wt } w_{(1)} - \text{wt } w_{(2)}) \log z}$ for $x^{\text{wt } w'_{(3)} - \text{wt } w_{(1)} - \text{wt } w_{(2)}}$ and $(\log z)^k$ for $(\log x)^k$ for $k = 0, \dots, k_1+k_2+k_3-3$ in $\langle w'_{(3)}, \mathcal{Y}(w_{(1)}, x)w_{(2)} \rangle$ to obtain a complex number $\langle w'_{(3)}, \mathcal{Y}(w_{(1)}, z)w_{(2)} \rangle$. Then we define $\langle w'_{(3)}, \mathcal{Y}(w_{(1)}, z)w_{(2)} \rangle$ for nonhomogeneous $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$ using linearity.

The $L(0)$ -conjugation property (5.23) for logarithmic intertwining operators recalled above gives

$$\langle w'_{(3)}, \mathcal{Y}(w_{(1)}, x)w_{(2)} \rangle = \langle x^{L'(0)} w'_{(3)}, \mathcal{Y}(x^{-L(0)} w_{(1)}, 1) x^{-L(0)} w_{(2)} \rangle \quad (5.29)$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$.

Proposition 5.8. *Assume that W_2 and W'_3 are generated by $\Omega_N^0(W_2)$ and $\Omega_N^0(W'_3)$, respectively. Then the map*

$$\begin{aligned} \rho : \mathcal{V}_{W_1 W_2}^{W_3} &\rightarrow \text{Hom}_{A_N(V)}(A_N(W_1) \otimes_{A_N(V)} \Omega_N^0(W_2), \Omega_N^0(W_3)) \\ \mathcal{Y} &\mapsto \rho(\mathcal{Y}) \end{aligned}$$

is injective.

Proof. Assume that $\rho(\mathcal{Y}) = 0$. We have $W'_3 = \coprod_{\mu \in \mathbb{C}/\mathbb{Z}} (W'_3)^\mu$. For $w_{(2)} \in (\Omega_N^0(W_2))_{[h_2]}$ and $w'_{(3)} \in (\Omega_N^0((W'_3)^\mu))_{[h_3^\mu + n]}$,

$$\begin{aligned} \langle w'_{(3)}, \mathcal{Y}(w_{(1)}, x)w_{(2)} \rangle \\ &= \langle w'_{(3)}, (\pi^\mu \circ \mathcal{Y})(w_{(1)}, x)w_{(2)} \rangle \\ &= \langle x^{L'(0)} w'_{(3)}, (\pi^\mu \circ \mathcal{Y})(x^{-L(0)} w_{(1)}, 1) x^{-L(0)} w_{(2)} \rangle \\ &= \langle x^{L'(0)} w'_{(3)}, \sum_{i \in \mathbb{C}} (\pi^\mu \circ \mathcal{Y})_{i,0}(x^{-L(0)} w_{(1)}) x^{-L(0)} w_{(2)} \rangle \\ &= \langle x^{L'(0)} w'_{(3)}, (\pi^\mu \circ \mathcal{Y})_{\text{wt } w_{(1)} + h_2 - h_3^\mu - n - 1, 0}(x^{-L(0)} w_{(1)}) x^{-L(0)} w_{(2)} \rangle \\ &= \langle x^{L'(0)} w'_{(3)}, \sum_{\mu \in \mathbb{C}/\mathbb{Z}} \sum_{m=0}^N \text{Res}_{x_0} x_0^{h_2 - h_3^\mu - m - 1} (\pi^\mu \circ \mathcal{Y})^0(x_0^{L(0)} x^{-L(0)} w_{(1)}, x_0) x^{-L(0)} w_{(2)} \rangle \\ &= \langle x^{L'(0)} w'_{(3)}, \sum_{\mu \in \mathbb{C}/\mathbb{Z}} \rho(\pi^\mu \circ \mathcal{Y})((x^{-L(0)} w_{(1)} + O_N(W_1)) \otimes x^{-L(0)} w_{(2)}) \rangle \\ &= \langle x^{L'(0)} w'_{(3)}, \rho(\mathcal{Y})((x^{-L(0)} w_{(1)} + O_N(W_1)) \otimes x^{-L(0)} w_{(2)}) \rangle \\ &= 0. \end{aligned}$$

Since h_2 , μ and n are arbitrary, this equality holds for $w_{(2)} \in \Omega_N^0(W_2)$ and $w'_{(3)} \in \Omega_N^0(W'_3)$. Then by Proposition 5.2, we have

$$\langle \tilde{w}'_{(3)}, \mathcal{Y}(\tilde{w}_{(1)}, x)\tilde{w}_{(2)} \rangle = 0$$

for all $\tilde{w}_{(1)} \in W_1$, $\tilde{w}_{(2)} \in W_2$ and $\tilde{w}'_{(3)} \in W'_3$. Thus $\mathcal{Y} = 0$. So ρ is injective. \blacksquare

6 The main theorem

In this section, under conditions stronger than those results in the preceding section, we state and prove our main theorem. The conditions needed in our main theorem is that some lower-bounded generalized V -modules involved should satisfy a certain universal property. Before we state and prove our main theorem, we first give a construction of such lower-bounded generalized V -modules.

Let W be a lower-bounded generalized V -module such that $W = \coprod_{n \in h_W + \mathbb{N}} W_{[n]}$ for some $h_W \in \mathbb{C}$ and $W_{[h_W]} \neq 0$. Then $G_N(W) = W_{[h_W + N]}$ is an $A_N(V)$ -submodule of $\Omega_N^0(W)$. We now would like to construct a generalized V -module from $G_N(W)$ satisfying a certain universal property. We consider the affinization $V[t, t^{-1}] = V \otimes \mathbb{C}[t, t^{-1}]$ of V . For simplicity, we shall use $u(m)$ to denote $u \otimes t^m$ for $u \in V$ and $m \in \mathbb{Z}$. We consider the tensor algebra $T(V[t, t^{-1}])$. For simplicity we shall omit the tensor product symbol when we write elements of $T(V[t, t^{-1}])$.

For any $u \in V, m \in \mathbb{Z}$, $u(m)$ acts from the left on $T(V[t, t^{-1}]) \otimes G_N(W)$. We shall also omit the tensor product symbol when we write elements of $T(V[t, t^{-1}]) \otimes G_N(W)$. The gradings on $T(V[t, t^{-1}])$ and $G_N(W)$ give a grading on $T(V[t, t^{-1}]) \otimes G_N(W)$. Explicitly, for homogeneous $u_i \in V, m_i \in \mathbb{Z}, i = 1, \dots, s$ and homogeneous $w \in G_N(W)$, the weight of

$$u_1(m_1) \cdots u_s(m_s)w$$

is

$$\text{wt } u_1 - m_1 - 1 + \cdots + \text{wt } u_s - m_s - 1 + \text{wt } w.$$

Let

$$Y_{T(V[t, t^{-1}]) \otimes G_N(W)}(u, x) : T(V[t, t^{-1}]) \otimes G_N(W) \rightarrow (T(V[t, t^{-1}]) \otimes G_N(W))[[x, x^{-1}]]$$

be defined by

$$Y_{T(V[t, t^{-1}]) \otimes G_N(W)}(u, x) = \sum_{m \in \mathbb{Z}} u(m)x^{-m-1}$$

for $u \in V$.

Let $I_{V;W}$ be the $T(V[t, t^{-1}])$ -submodule of $T(V[t, t^{-1}]) \otimes G_N(W)$ generated by the elements

$$u(\text{wt } u - 1)w - o(u)w$$

for homogeneous $u \in V$ and $w \in G_N(W)$, the elements

$$u_1(m_1) \cdots u_s(m_s)w$$

for homogeneous $u_i \in V$, $m_i \geq \text{wt } u_i - 1$ satisfying $\sum_{i=1}^s m_i > \sum_{i=1}^s (\text{wt } u_i - 1) + N$, $w \in G_N(W)$ and the coefficients in x_1 and x_2 of

$$\begin{aligned} & Y_{T(V[t, t^{-1}]) \otimes G_N(W)}(u, x_1) Y_{T(V[t, t^{-1}]) \otimes G_N(W)}(v, x_2) w \\ & - Y_{T(V[t, t^{-1}]) \otimes G_N(W)}(v, x_2) Y_{T(V[t, t^{-1}]) \otimes G_N(W)}(u, x_1) w \\ & - \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y_{T(V[t, t^{-1}]) \otimes G_N(W)}(Y_V(u, x_0)v, x_2) w \end{aligned}$$

for $u, v \in V$ and $w \in T(V[t, t^{-1}]) \otimes G_N(W)$.

Let

$$\tilde{S}(V; W) = (T(V[t, t^{-1}]) \otimes G_N(W)) / I_{V; W_2}.$$

We shall use elements of $T(V[t, t^{-1}]) \otimes G_N(W)$ to represent elements of $\tilde{S}(V; W)$. But note that these elements now satisfy the following relations:

$$u(\text{wt } u - 1)w = o(u)w$$

for homogeneous $u \in V$ and $w \in G_N(W)$,

$$u_1(m_1) \cdots u_s(m_s)w = 0$$

for homogeneous $u_i \in V$, $m_i \geq \text{wt } u_i - 1$ satisfying $\sum_{i=1}^s m_i > \sum_{i=1}^s (\text{wt } u_i - 1) + N$, $w \in G_N(W)$ and

$$\begin{aligned} & Y_{T(V[t, t^{-1}]) \otimes G_N(W)}(u, x_1) Y_{T(V[t, t^{-1}]) \otimes G_N(W)}(v, x_2) w \\ & - Y_{T(V[t, t^{-1}]) \otimes G_N(W)}(v, x_2) Y_{T(V[t, t^{-1}]) \otimes G_N(W)}(u, x_1) w \\ & = \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y_{T(V[t, t^{-1}]) \otimes G_N(W)}(Y_V(u, x_0)v, x_2) w \end{aligned}$$

for $u, v \in V$ and $w \in T(V[t, t^{-1}]) \otimes G_N(W)$.

The map $Y_{T(V[t, t^{-1}]) \otimes G_N(W)}$ induces a map for $\tilde{S}(V; W)$ and we shall denote it by $Y_{\tilde{S}(V; W)}$. By the definition of $\tilde{S}(V; W)$, the commutator formula for $Y_{\tilde{S}(V; W)}$ holds. Using this commutator formula and other properties given by the definition of $\tilde{S}(V; W)$, we see that $\tilde{S}(V; W)$ is spanned by elements of the form

$$u_1(m_1) \cdots u_s(m_s)w \tag{6.30}$$

for homogeneous $u_i \in V$, $m_i \in \mathbb{Z}$, $i = 1, \dots, s$ satisfying

$$\begin{aligned} & \text{wt } u_1 - m_1 - 1 \geq \cdots \geq \text{wt } u_s - m_s - 1, \\ & \text{wt } u_1 - m_1 - 1 + \cdots + \text{wt } u_s - m_s - 1 \geq -N, \end{aligned}$$

$w \in G_N(W)$. The grading of $T(V[t, t^{-1}]) \otimes G_N(W)$ induces a grading on $\tilde{S}(V; W)$ such that the weight of the element (6.30) is

$$\text{wt } u_1 - m_1 - 1 + \cdots + \text{wt } u_s - m_s - 1 + \text{wt } w.$$

Thus the real parts of the weights of the elements of $\tilde{S}(V; W)$ are bigger than or equal to $\Re(h_W)$. In particular, for $u \in V$ and $w \in \tilde{S}(V; W)$, $u(m)w = 0$ when m is sufficiently large.

Recall the \mathbb{Z} -graded Lie algebra \hat{V} of operators on V of the form $(Y_V)_n(u)$ for $u \in V$ and $n \in \mathbb{Z}$, equipped with the Lie bracket for operators in Section 3. Since the commutator formula for $Y_{\tilde{S}(V; W)}$ holds, $\tilde{S}(V; W)$ is a graded $U(\hat{V})$ -module. Let $P_N(\hat{V}) = \coprod_{k > N}^{\infty} \hat{V}_{(-k)} \oplus \hat{V}_{(0)}$. Then $P_N(\hat{V})$ is a subalgebra of \hat{V} . We know that $G_N(W)$ is an $A_N(V)$ -module and is therefore a graded module for $\hat{V}_{(0)}$. Let $\hat{V}_{(-k)}$ for $k > N$ act on $G_N(W)$ trivially. Then $G_N(W)$ is a $P_N(\hat{V})$ -module. Let $U(\cdot)$ be the universal enveloping algebra functor from the category of Lie algebras to the category of associative algebras. Then $\tilde{S}(V; W)$ as a graded $U(\hat{V})$ -module is a quotient of the graded $U(\hat{V})$ -module

$$U(\hat{V}) \otimes_{U(P_N(\hat{V}))} G_N(W).$$

Let $J_{V; W}$ be the graded $U(\hat{V})$ -submodule of $\tilde{S}(V; W)$ generated by the coefficients in x

$$Y_{\tilde{S}(V; W)}(L(-1)u, x)w - \frac{d}{dx} Y_{\tilde{S}(V; W)}(u, x)w$$

and the coefficients in x_0, x_2 of

$$\begin{aligned} & Y_{\tilde{S}(V; W)}(Y_V(u, x_0)v, x_2)w - Y_{\tilde{S}(V; W)}(u, x_0 + x_2)Y_{\tilde{S}(V; W)}(v, x_2)w \\ & + \text{Res}_{x_1} x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_{\tilde{S}(V; W)}(v, x_2)Y_{\tilde{S}(V; W)}(u, x_1)w, \end{aligned}$$

for $u, v \in V$ and $w \in \tilde{S}(V; W)$. Let

$$S_N(G_N(W)) = \tilde{S}(V; W) / J_{V; W}.$$

Then $S_N(G_N(W))$ is also a graded $U(\hat{V})$ -module. We shall sometimes still use elements of $T(V[t, t^{-1}]) \otimes \Omega_N(W)$ to represent elements of $S_N(V; W)$. But note that these elements now satisfy more relations than the elements of $\tilde{S}(V; W)$ written in the same form. The vertex operator map $Y_{\tilde{S}(V; W)}$ induces a vertex operator map

$$Y_{S_N(G_N(W))} : V \otimes S_N(G_N(W)) \rightarrow (S_N(G_N(W)))[[x, x^{-1}]].$$

In general, for a lower-bounded generalized V -module W , we know that there exists $h_\mu \in \mathbb{C}$ for $\mu \in \mathbb{C}/\mathbb{Z}$ such that $W = \oplus_{\mu \in \mathbb{C}/\mathbb{Z}} W^\mu$ where W^μ for $\mu \in \mathbb{C}/\mathbb{Z}$ are grading-restricted generalized V -modules such that $W^\mu = \coprod_{n \in h_\mu + \mathbb{N}} (W^\mu)_{[n]}$. For $\mu \in \mathbb{C}/\mathbb{Z}$, we have

defined $G_N(W^\mu)$, $S_N(G_N(W^\mu))$ and $Y_{S_N(G_N(W^\mu))}$ above. Let

$$\begin{aligned} G_N(W) &= \coprod_{\mu \in \mathbb{C}/\mathbb{Z}} G_N(W^\mu), \\ S_N(G_N(W)) &= \coprod_{\mu \in \mathbb{C}/\mathbb{Z}} S_N(G_N(W^\mu)) \end{aligned}$$

and let

$$Y_{S_N(G_N(W))} : V \otimes S_N(G_N(W)) \rightarrow (S_N(G_N(W)))[[x, x^{-1}]]$$

be the map given by

$$Y_{S_N(G_N(W))}(u, x)(w^{\mu_1} + \cdots + w^{\mu_m}) = Y_{S_N(G_N(W^{\mu_1}))}(u, x)w^{\mu_1} + \cdots + Y_{S_N(G_N(W^{\mu_m}))}(u, x)w^{\mu_m}$$

for $u \in V$, $\mu_1, \dots, \mu_m \in \mathbb{C}/\mathbb{Z}$ and $w^{\mu_1} \in W^{\mu_1}, \dots, w^{\mu_m} \in W^{\mu_m}$.

Theorem 6.1. *Let W be a lower-bounded generalized V -module. The graded space $S_N(G_N(W))$ equipped with vertex operator map $Y_{S_N(G_N(W))}$ is a lower-bounded generalized V -module such that $G_N(S_N(G_N(W)))$ is isomorphic to $G_N(W)$. The lower-bounded generalized V -module $S_N(G_N(W))$ satisfies the following universal property: For any lower-bounded generalized V -module \widetilde{W} and any $A_N(V)$ -module map $\phi : G_N(W) \rightarrow G_N(\widetilde{W})$, there is a unique homomorphism $\bar{\phi} : S_N(G_N(W)) \rightarrow \widetilde{W}$ of generalized V -modules such that $\bar{\phi}|_{G_N(W)} = \phi$.*

Proof. We prove the theorem only in the case that $W = \coprod_{n \in h_W + \mathbb{N}} W_{[n]}$; the general case follows. By definition, the commutator formula and the associator formula for the vertex operator map $Y_{S_N(G_N(W))}$ holds. Thus the Jacobi identity holds. The other properties are clearly satisfied.

Since $\tilde{S}(V; W)$ is spanned by elements of form (6.30), we see that the subspace of $\tilde{S}(V; W)$ spanned by homogeneous elements of weight $h_W + N$ is isomorphic to $G_N(W)$. Since the grading on $S_N(G_N(W))$ is induced from the grading on $\tilde{S}(V; W)$, we see that $G_N(S_N(G_N(W)))$ is isomorphic to $G_N(W)$.

The universal property follows from the construction. ■

Remark 6.2. In [DLM], given an $A_N(V)$ -module U , an \mathbb{N} -gradable weak V -module $\bar{M}_N(U)$ is constructed. In the case that $U = G_N(W)$, it can be shown using the universal properties for $\bar{M}_N(U)$ and for $S_N(G_N(W))$ that the generalized V -module $S_N(G_N(W))$ constructed above is isomorphic to $\bar{M}_N(U)$.

We need the following definition:

Definition 6.3. For $k \in \mathbb{Z}_+$, we say that the $L(0)$ -block sizes of a generalized V -module W are less than k if for any homogeneous $w \in W$, $(L(0) - \text{wt } w)^k w = 0$. We say that the $L(0)$ -block sizes of a generalized V -module is bounded if there exists $k \in \mathbb{Z}_+$ such that the $L(0)$ -block sizes of W are less than or equal to k .

Remark 6.4. Let W_1, W_2 and W_3 be lower-bounded generalized V -modules such that $W_1 = \coprod_{n \in h_1 + \mathbb{Z}} (W_1)_{[n]}$, $W_2 = \coprod_{n \in h_2 + \mathbb{Z}} (W_2)_{[n]}$ and $W_3 = \coprod_{n \in h_3 + \mathbb{Z}} (W_3)_{[n]}$ for some $h_1, h_2, h_3 \in \mathbb{C}$ and such that the $L(0)$ -block sizes of W_1, W_2 and W_3 are less than k_1, k_2 and k_3 , respectively. Then by Proposition 5.3, we have

$$\mathcal{Y}(w_{(1)}, x)w_{(2)} \in x^{h_3-h_1-h_2}W_3[[x, x^{-1}]] \oplus x^{h_3-h_1-h_2}W_3[[x, x^{-1}]] \log x \oplus \dots \oplus x^{h_3-h_1-h_2}W_3[[x, x^{-1}]](\log x)^{k_1+k_2+k_3-3}.$$

Let W_1 and W_2 be lower-bounded generalized V -modules. Assume that there exist $h_1, h_2 \in \mathbb{C}$ such that $W_1 = \coprod_{n \in h_1 + \mathbb{Z}} (W_1)_{[n]}$ and $W_2 = \coprod_{n \in h_2 + \mathbb{Z}} (W_2)_{[n]}$ and that there exist $k_1, k_2 \in \mathbb{Z}_+$ such that the $L(0)$ -block sizes of W_1 and W_2 are less than k_1 and k_2 , respectively. Let $h_3 \in \mathbb{C}$ and $k_3 \in \mathbb{Z}_+$. Similarly to the construction of $S_N(G_N(W))$ above, we now construct from W_1 and $G_N(W_2)$ a lower-bounded generalized V -module $S_N(V; W_1, W_2)$ and a logarithmic intertwining operator \mathcal{Y}_t of type $\binom{S_N(V; W_1, W_2)}{W_1 S_N(G_N(W_2))}$ such that $S_N(V; W_1, W_2) = \coprod_{n \in h_3 + \mathbb{Z}} (S_N(V; W_1, W_2))_{[n]}$ and the $L(0)$ -block sizes of $S_N(V; W_1, W_2)$ are less than or equal to k_3 .

Let $h = h_3 - h_1 - h_2$ and $k_0 = k_1 + k_2 + k_3$. From Remark 6.4, if $S_N(V; W_1, W_2)$ and \mathcal{Y}_t are constructed, we must have

$$\mathcal{Y}_t(w_{(1)}, x)w_{(2)} \in x^h S_N(V; W_1, W_2)[[x, x^{-1}]] \oplus x^h S_N(V; W_1, W_2)[[x, x^{-1}]] \log x \oplus \dots \oplus x^h S_N(V; W_1, W_2)[[x, x^{-1}]](\log x)^{k_0-3}.$$

We consider

$$t^{-h}W_1[t, t^{-1}][\log t] = W_1 \otimes t^{-h}\mathbb{C}[t, t^{-1}][\log t].$$

For simplicity, we shall use $u(m)$ and $w_{(1)}(n, k)$ to denote

$$u \otimes t^m \in V[t, t^{-1}]$$

and

$$w_{(1)} \otimes t^n \otimes (\log t)^k \in t^{-h}W_1[t, t^{-1}][\log t],$$

respectively, for $u \in V, w_{(1)} \in W_1, m \in \mathbb{Z}, n \in -h + \mathbb{Z}$ and $k \in \mathbb{N}$. We consider the tensor algebra

$$T(V[t, t^{-1}] \oplus t^{-h}W_1[t, t^{-1}][\log t]).$$

The tensor algebra $T(V[t, t^{-1}])$ is a subalgebra of this tensor algebra and $t^{-h}W_1[t, t^{-1}][\log t]$ is a subspace. Let $T_{V; W_1}$ be the $T(V[t, t^{-1}])$ -sub-bimodule of $T(V[t, t^{-1}] \oplus t^{-h}W_1[t, t^{-1}][\log t])$ generated by $t^{-h}W_1[t, t^{-1}][\log t]$. Then $T_{V; W_1} \otimes G_N(W_2)$ as a $T(V[t, t^{-1}])$ -module is equivalent to

$$T(V[t, t^{-1}]) \otimes t^{-h}W_1[t, t^{-1}][\log t] \otimes T(V[t, t^{-1}]) \otimes G_N(W_2).$$

For simplicity we shall omit the tensor product symbol for elements. In particular, $T_{V; W_1} \otimes G_N(W_2)$ is spanned by elements of the form

$$u_1(m_1) \cdots u_s(m_s)w_{(1)}(n, k)u_{s+1}(m_{s+1}) \cdots u_{s+t}(m_{s+t})w_{(2)},$$

for $u_i \in V$, $m_i \in \mathbb{Z}$, $k \in \mathbb{N}$, $i = 1, \dots, s+t$, $w_{(1)} \in W_1$ and $w_{(2)} \in G_N(W_2)$.

For any $u \in V$, $m \in \mathbb{Z}$, $u(m)$ acts from the left on $T_{V;W_1} \otimes G_N(W_2)$. The natural grading on $t^{-h}W_1[t, t^{-1}]$ gives a grading on $t^{-h}W_1[t, t^{-1}][\log t]$ with the weight of $\log t$ being 0. The gradings on $T(V[t, t^{-1}])$, $t^{-h}W_1[t, t^{-1}][\log t]$ and $G_N(W_2)$ give a grading on $T_{V;W_1} \otimes G_N(W_2)$. Explicitly, for homogeneous $u_i \in V$, $m_i \in \mathbb{Z}$, $i = 1, \dots, s+t$, and homogeneous $w_{(1)} \in W_1$ and $w_{(2)} \in G_N(W_2)$, the weight of

$$u_1(m_1) \cdots u_s(m_s)w_{(1)}(n, k)u_{s+1}(m_{s+1}) \cdots u_{s+t}(m_{s+t})w_{(2)}$$

is

$$\text{wt } u_1 - m_1 - 1 + \cdots + \text{wt } u_{s+t} - m_{s+t} - 1 + \text{wt } w_{(1)} - n - 1 + \text{wt } w_{(2)}.$$

For $u \in V$, recall the map

$$Y_{T(V[t, t^{-1}]) \otimes G_N(W_2)}(u, x) : T(V[t, t^{-1}]) \otimes G_N(W_2) \rightarrow (T(V[t, t^{-1}]) \otimes G_N(W_2))[[x, x^{-1}]].$$

For $u \in V$ and $w_{(1)} \in W_1$, let

$$\begin{aligned} Y_{T_{V;W_1} \otimes G_N(W_2)}(u, x) &: T_{V;W_1} \otimes G_N(W_2) \rightarrow (T_{V;W_1} \otimes G_N(W_2))[[x, x^{-1}]], \\ \mathcal{Y}_t(w_{(1)}, x) &: T(V[t, t^{-1}]) \otimes G_N(W_2) \rightarrow x^h(T_{V;W_1} \otimes G_N(W_2))[[x, x^{-1}]][\log x] \end{aligned}$$

be defined by

$$\begin{aligned} Y_{T_{V;W_1} \otimes G_N(W_2)}(u, x) &= \sum_{m \in \mathbb{Z}} u(m)x^{-m-1}, \\ \mathcal{Y}_t(w_{(1)}, x) &= \sum_{n \in -h + \mathbb{Z}} \sum_{k \in \mathbb{N}} w_{(1)}(n, k)x^{-n-1}(\log x)^k, \end{aligned}$$

respectively.

Let $I_{V;W_1, W_2}$ be the $T(V[t, t^{-1}])$ -submodule of $T_{V;W_1} \otimes G_N(W_2)$ generated by elements of the following forms:

$$au(\text{wt } u - 1)w_{(2)} - ao(u)w_{(2)} \text{ for } a \in T_{V;W_1}, u \in V, \text{ and } w_{(2)} \in G_N(W_2),$$

$$au_1(m_1) \cdots u_s(m_s)w_{(2)} \text{ for } a \in T_{V;W_1}, \text{ homogeneous } u_i \in V, m_i \geq \text{wt } u_i - 1 \text{ satisfying } \sum_{i=1}^s (\text{wt } u_i - m_i - 1) < -N, w_{(2)} \in G_N(W_2),$$

$$u_1(m_1) \cdots u_s(m_s)w_{(1)}(n, k)w_{(2)} \text{ for homogeneous } u_i \in V, m_i \in \mathbb{Z}, w_{(1)} \in W_1, n \in \mathbb{C}, k \in \mathbb{N}, w_{(2)} \in G_N(W_2), \text{ and either } n \notin -h + \mathbb{Z} \text{ or } \sum_{i=1}^s \text{wt } u_i - m_i - 1 + \text{wt } w_{(1)} - n - 1 + \text{wt } w_{(2)} < \Re(h_3) \text{ or } k > k_0 - 3,$$

and the coefficients in x_1 , x_2 and $\log x_2$ of

$$\begin{aligned} &aY_{T(V[t, t^{-1}]) \otimes G_N(W_2)}(u, x_1)Y_{T(V[t, t^{-1}]) \otimes G_N(W_2)}(v, x_2)w \\ &- aY_{T(V[t, t^{-1}]) \otimes G_N(W_2)}(v, x_2)Y_{T(V[t, t^{-1}]) \otimes G_N(W_2)}(u, x_1)w \end{aligned}$$

$$\begin{aligned}
& -\text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) a Y_{T(V[t, t^{-1}]) \otimes G_N(W_2)}(Y_V(u, x_0)v, x_2)w, \\
& Y_{T_{V;W_1} \otimes G_N(W_2)}(u, x_1) Y_{T_{V;W_1} \otimes G_N(W_2)}(v, x_2) \tilde{w} \\
& - Y_{T_{V;W_1} \otimes G_N(W_2)}(v, x_2) Y_{T_{V;W_1} \otimes G_N(W_2)}(u, x_1) \tilde{w} \\
& - \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y_{T_{V;W_1} \otimes G_N(W_2)}(Y_V(u, x_0)v, x_2) \tilde{w}, \\
& Y_{T_{V;W_1} \otimes G_N(W_2)}(u, x_1) \mathcal{Y}_t(w_{(1)}, x_2) w \\
& - \mathcal{Y}_t(w_{(1)}, x_2) Y_{T(V[t, t^{-1}]) \otimes G_N(W_2)}(u, x_1) w \\
& - \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \mathcal{Y}_t(Y_{W_1}(u, x_0)w_{(1)}, x_2) w,
\end{aligned}$$

for $a \in T_{V;W_1}$, $u, v \in V$, $w_{(1)} \in W_1$, $w \in T(V[t, t^{-1}] \otimes G_N(W_2))$ and $\tilde{w} \in T_{V;W_1} \otimes G_N(W_2)$.

Let

$$\tilde{S}(V; W_1, W_2) = (T_{V;W_1} \otimes G_N(W_2)) / I_{V;W_1, W_2}.$$

We shall use elements of $T_{V;W_1} \otimes G_N(W_2)$ to represent elements of $\tilde{S}(V; W_1, W_2)$. But these elements now satisfy relations. Recall the $U(\hat{V})$ -module $\tilde{S}(V; W_2)$ above. The maps $Y_{T_{V;W_1} \otimes G_N(W_2)}(u, x)$ for $u \in V$ and $\mathcal{Y}_t(w_{(1)}, x)$ for $w_{(1)} \in W_1$ induce maps from $\tilde{S}(V; W_1, W_2)$ to $(\tilde{S}(V; W_1, W_2))[[x, x^{-1}]]$ and from $\tilde{S}(V; W_2)$ to $x^h(\tilde{S}(V; W_1, W_2))[[x, x^{-1}]][\log x]$, respectively. We shall use the notation $Y_{\tilde{S}(V; W_1, W_2)}(u, x)$ to denote the first map and the same notation $\mathcal{Y}_t(w_{(1)}, x)$ to denote the second map. These maps for all $u \in V$ and $w_{(1)} \in W_1$ give us maps $Y_{\tilde{S}(V; W_1, W_2)}$ and \mathcal{Y}_t . Recall the map $Y_{\tilde{S}(V; W_2)}(u, x)$ for $u \in V$ and $Y_{\tilde{S}(V; W_2)}$. By the definition of $\tilde{S}(V; W_1, W_2)$, the commutator formulas for $Y_{\tilde{S}(V; W_1, W_2)}$ and for $Y_{\tilde{S}(V; W_2)}$, \mathcal{Y}_t and $Y_{\tilde{S}(V; W_2)}$ hold. Using these commutator formulas and other properties given by the definition of $\tilde{S}(V; W_1, W_2)$, we see that $\tilde{S}(V; W_1, W_2)$ is spanned by elements of the form

$$u_1(m_1) \cdots u_s(m_s) w_{(1)}(n, k) w_{(2)} \quad (6.31)$$

for homogeneous $u_i \in V$, $m_i \in \mathbb{Z}$, $i = 1, \dots, s$, homogeneous $w_{(1)} \in W_1$, $n \in -h + \mathbb{N}$, $0 \leq k \leq k_0 - 3$ and $w_{(2)} \in G_N(W_2)$, satisfying

$$\Re(h_3) \leq \text{wt } w_{(1)} - n - 1 + \text{wt } w_{(2)}, \quad (6.32)$$

$$\Re(h_3) \leq \sum_{i=1}^s (\text{wt } u_i - m_i - 1) + \text{wt } w_{(1)} - n - 1 + \text{wt } w_{(2)}. \quad (6.33)$$

The grading on $T_{V;W_1} \otimes G_N(W_2)$ induces a grading on $\tilde{S}(V; W_1, W_2)$ such that the weight of the element (6.31) is

$$\text{wt } u_1 - m_1 - 1 + \cdots + \text{wt } u_s - m_s - 1 + \text{wt } w_{(1)} - n - 1 + \text{wt } w_{(2)}.$$

Thus the real parts of the weights of the elements of $\tilde{S}(V; W_1, W_2)$ are bigger than or equal to $\Re(h_3)$. In particular, for $u \in V$, $w_{(1)} \in W_1$ and $w \in \tilde{S}(V; W_1, W_2)$, $u(m)w = 0$, $w_{(1)}(n, k)w = 0$ when m , n and k are sufficiently large.

Since the commutator formulas for $Y_{\tilde{S}(V;W_1,W_2)}$ holds, $\tilde{S}(V;W_1,W_2)$ is a $U(\hat{V})$ -module.

Let $J_{V;W_1,W_2}$ be the $U(\hat{V})$ -submodule of $\tilde{S}(V;W_1,W_2)$ generated by the coefficients in x and $\log x$ of

$$\begin{aligned} & aY_{\tilde{S}(V;W_2)}(L(-1)u, x)w - a\frac{d}{dx}Y_{\tilde{S}(V;W_2)}(u, x)w, \\ & Y_{\tilde{S}(V;W_1,W_2)}(L(-1)u, x)\tilde{w} - \frac{d}{dx}Y_{\tilde{S}(V;W_1,W_2)}(u, x)\tilde{w}, \\ & \mathcal{Y}_t(L(-1)w_{(1)}, x)w - \frac{d}{dx}\mathcal{Y}_t(w_{(1)}, x)w, \\ & \mathcal{Y}_t(w_{(1)}, x)w - x^{L(0)}\mathcal{Y}_t(x^{-L(0)}w_{(1)}, 1)x^{-L(0)}w, \end{aligned}$$

and the coefficients in x_0 , x_2 and $\log x_2$ of

$$\begin{aligned} & aY_{\tilde{S}(V;W_2)}(Y_V(u, x_0)v, x_2)w - aY_{\tilde{S}(V;W_2)}(u, x_0 + x_2)Y_{\tilde{S}(V;W_2)}(v, x_2)w \\ & + \text{Res}_{x_1}x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)aY_{\tilde{S}(V;W_2)}(v, x_2)Y_{\tilde{S}(V;W_2)}(u, x_1)w, \\ & Y_{\tilde{S}(V;W_1,W_2)}(Y_V(u, x_0)v, x_2)\tilde{w} - Y_{\tilde{S}(V;W_1,W_2)}(u, x_0 + x_2)Y_{\tilde{S}(V;W_1,W_2)}(v, x_2)\tilde{w} \\ & + \text{Res}_{x_1}x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y_{\tilde{S}(V;W_1,W_2)}(v, x_2)Y_{\tilde{S}(V;W_1,W_2)}(u, x_1)\tilde{w}, \\ & \mathcal{Y}_t(Y_V(u, x_0)w_{(1)}, x_2)w - Y_{\tilde{S}(V;W_1,W_2)}(u, x_0 + x_2)\mathcal{Y}_t(w_{(1)}, x_2)w \\ & + \text{Res}_{x_1}x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\mathcal{Y}_t(w_{(1)}, x_2)Y_{\tilde{S}(V;W_2)}(u, x_1)w, \end{aligned}$$

for $a \in T_{V;W_1}$, $u, v \in V$, $w_{(1)} \in W_1$, $w \in \tilde{S}(V;W_2)$ and $\tilde{w} \in \tilde{S}(V;W_1,W_2)$.

Let

$$S_N(V;W_1,W_2) = \tilde{S}(V;W_1,W_2)/J_{V;W_1,W_2}.$$

Then $S_N(V;W_1,W_2)$ is also a $U(\hat{V})$ -module. We shall use elements of $T_{V;W_1} \otimes G_N(W_2)$ to represent elements of $S_N(V;W_1,W_2)$. But these elements now satisfy more relations than the elements of $\tilde{S}(V;W_1,W_2)$ written in the same form.

The maps $Y_{\tilde{S}(V;W_1,W_2)}(u, x)$ for $u \in V$ and $\mathcal{Y}_t(w_{(1)}, x)$ for $w_{(1)} \in W_1$ induce maps

$$Y_{S_N(V;W_1,W_2)}(u, x) : S_N(V;W_1,W_2) \rightarrow (S_N(V;W_1,W_2))[[x, x^{-1}]]$$

and

$$\mathcal{Y}_t(w_{(1)}, x) : S_N(G_N(W_2)) \rightarrow x^h(S_N(V;W_1,W_2))[[x, x^{-1}]][\log x],$$

respectively. These maps give maps

$$\begin{aligned} Y_{S_N(V;W_1,W_2)} : V \otimes S_N(V;W_1,W_2) & \rightarrow (S_N(V;W_1,W_2))[[x, x^{-1}]] \\ u \otimes w & \mapsto Y_{S_N(V;W_1,W_2)}(u, x)w \end{aligned}$$

and

$$\begin{aligned}\mathcal{Y}_t : W_1 \otimes S_N(G_N(W_2)) &\rightarrow x^h(S_N(V; W_1, W_2))[[x, x^{-1}]][\log x] \\ w_{(1)} \otimes w_{(2)} &\mapsto \mathcal{Y}_t(w_{(1)}, x)w_{(2)}.\end{aligned}$$

By construction, these operators satisfy the lower truncation property, the identity property for $Y_{S_N(V; W_1, W_2)}$, the commutator formula for $Y_{S_N(V; W_1, W_2)}$ and for $Y_{S_N(V; W_1, W_2)}$, \mathcal{Y}_t and $Y_{S_N(G_N(W_2))}$, the associator formula for $Y_{S_N(G_N(W_2))}$ and for $Y_{S_N(V; W_1, W_2)}$, \mathcal{Y}_t and $Y_{S_N(G_N(W_2))}$, the $L(-1)$ -derivative property for $Y_{S_N(V; W_1, W_2)}$ and for \mathcal{Y}_t and the $x^{L(0)}$ -conjugation property for $Y_{S_N(V; W_1, W_2)}$ and for \mathcal{Y}_t .

Theorem 6.5. *Let W_1 and W_2 be lower-bounded generalized V -modules. Assume that there exist $h_1, h_2 \in \mathbb{C}$ such that $W_1 = \coprod_{n \in h_1 + \mathbb{Z}} (W_1)_{[n]}$ and $W_2 = \coprod_{n \in h_2 + \mathbb{Z}} (W_2)_{[n]}$ and that there exist $k_1, k_2 \in \mathbb{Z}_+$ such that the $L(0)$ -block sizes of W_1 and W_2 are less than k_1 and k_2 , respectively. Let $h_3 \in \mathbb{C}$ and $k_3 \in \mathbb{Z}_+$. Then the graded space $S_N(V; W_1, W_2)$ equipped with vertex operator map $Y_{S_N(V; W_1, W_2)}$ is a lower-bounded generalized V -module such that $S_N(V; W_1, W_2) = \coprod_{n \in h_3 + \mathbb{Z}} (S_N(V; W_1, W_2))_{[n]}$ and the $L(0)$ -block sizes of $S_N(V; W_1, W_2)$ are less than or equal to k_3 and \mathcal{Y}_t is a logarithmic intertwining operator of type $\left(\begin{smallmatrix} S_N(V; W_1, W_2) \\ W_1 S_N(G_N(W_2)) \end{smallmatrix} \right)$.*

Proof. Since $Y_{S_N(V; W_1, W_2)}$ satisfies the lower truncation property, the identity property, the commutator formula, the associator formula and the $L(-1)$ -derivative property, $S_N(V; W_1, W_2)$ is a generalized V -module. Since \mathcal{Y}_t satisfies the lower truncation property, the commutator formula, the associator formula and the $L(-1)$ -derivative property, \mathcal{Y}_t is a logarithmic intertwining operator of type $\left(\begin{smallmatrix} S_N(V; W_1, W_2) \\ W_1 S_N(G_N(W_2)) \end{smallmatrix} \right)$. It is clear from the construction and the properties above satisfied by $Y_{S_N(V; W_1, W_2)}$ that $S_N(V; W_1, W_2)$ is lower bounded, $S_N(V; W_1, W_2) = \coprod_{n \in h_3 + \mathbb{Z}} (S_N(V; W_1, W_2))_{[n]}$ and the $L(0)$ -block sizes of $S_N(V; W_1, W_2)$ are less than or equal to k_3 . ■

Now we state and prove our main result:

Theorem 6.6. *Let W_1 , W_2 and W_3 be lower-bounded generalized V -modules whose $L(0)$ -block sizes are bounded. Let N be a nonnegative integer such that W_1 is generated by $\Omega_N^0(W_1)$ and W_2 and W_3' are isomorphic to $S_N(G_N(W_2))$ and $S_N(G_N(W_3'))$, respectively. Then the map*

$$\begin{aligned}\rho : \mathcal{V}_{W_1 W_2}^{W_3} &\rightarrow \text{Hom}_{A_N(V)}(A_N(W_1) \otimes_{A_N(V)} \Omega_N^0(W_2), \Omega_N^0(W_3)) \\ \mathcal{Y} &\mapsto \rho(\mathcal{Y})\end{aligned}$$

is a linear isomorphism.

Proof. Since $S_N(G_N(W_2))$ and $S_N(G_N(W_3'))$ are generated by $G_N(S_N(G_N(W_2))) \subset \Omega_N^0(S_N(G_N(W_2)))$ and $G_N(S_N(G_N(W_3'))) \subset \Omega_N^0(S_N(G_N(W_3')))$, respectively, they are generated by $\Omega_N^0(S_N(G_N(W_2)))$ and $\Omega_N^0(S_N(G_N(W_3')))$, respectively. Since W_2 and W_3' are isomorphic to $S_N(G_N(W_2))$ and $S_N(G_N(W_3'))$, respectively, W_2 and W_3' are generated by

$\Omega_N^0(W_2)$ and $\Omega_N^0(W_3')$, respectively. By Proposition 5.8, ρ is injective. So we need only prove that ρ is surjective. Given any element f of

$$\text{Hom}_{A_N(V)}(A_N(W_1) \otimes_{A_N(V)} \Omega_N^0(W_2), \Omega_N^0(W_3)),$$

we want to construct an element \mathcal{Y}^f of $\mathcal{V}_{W_1 W_2}^{W_3}$ such that $\rho(\mathcal{Y}^f) = f$.

We first construct \mathcal{Y}^f in the case that there exist $h_i \in \mathbb{C}$ for $i = 1, 2, 3$ such that $W_1 = \coprod_{n \in h_1 + \mathbb{Z}} (W_1)_{[n]}$, $W_2 = \coprod_{n \in h_2 + \mathbb{Z}} (W_2)_{[n]}$ and $W_3 = \coprod_{n \in h_3 + \mathbb{Z}} (W_3)_{[n]}$. Since the $L(0)$ -block sizes of W_1 , W_2 and W_3 are bounded, there exists positive integers k_1 , k_2 and k_3 such that the $L(0)$ -block sizes of W_1 , W_2 and W_3 are less than k_1 , k_2 and k_3 , respectively. Let $h = h_3 - h_1 - h_2$ and $k_0 = k_1 + k_2 + k_3$. Then by Proposition 6.4, for the logarithmic intertwining operator \mathcal{Y}^f that we want to construct, we have

$$\mathcal{Y}(w_{(1)}, x)w_{(2)} \in x^h W_3[[x, x^{-1}]] \oplus x^h W_3[[x, x^{-1}]] \log x \oplus \cdots \oplus x^h W_3[[x, x^{-1}]] (\log x)^{k_0-3}.$$

From Theorem 6.5, we have a lower-bounded generalized V -module $S_N(V; W_1, W_2)$ such that $S_N(V; W_1, W_2) = \coprod_{n \in h_3 + \mathbb{Z}} (S_N(V; W_1, W_2))_{[n]}$ and the $L(0)$ -block sizes of $S_N(V; W_1, W_2)$ are less than or equal to k_3 and a logarithmic intertwining operator \mathcal{Y}_t of type $\left(\begin{smallmatrix} S_N(V; W_1, W_2) \\ W_1 S_N(G_N(W_2)) \end{smallmatrix} \right)$. Let $S_N^0(V; W_1, W_2)$ be generalized V -submodule of $S_N(V; W_1, W_2)$ generated by elements of the form $w_{(1)}(n, 0)w_{(2)} \in S_N(V; W_1, W_2)$ where $w_{(1)} \in W_1$ is homogeneous, $n \leq \text{wt } w_{(1)} - 1 + \text{wt } w_{(2)} - h_3$ and $w_{(2)} \in G_N(W_2)$ is homogeneous. Then we have a homomorphism $\mu : S_N^0(V; W_1, W_2) \rightarrow W_3$ of generalized V -modules defined as follows:

Using commutator formulas, we know that $S_N^0(V; W_1, W_2)$ is spanned by elements of the form (6.31) for homogeneous $u_i \in V$, $m_i \in \mathbb{Z}$, $i = 1, \dots, s$, homogeneous $w_{(1)} \in W_1$, $n \in -h + \mathbb{N}$, and $w_{(2)} \in G_N(W_2)$, satisfying (6.32) and (6.33). For $w'_{(3)} \in G_N(W_3')$, we define an element $\mu'(w'_{(3)}) \in (S_N^0(V; W_1, W_2))'$ by

$$\begin{aligned} & \langle \mu'(w'_{(3)}), u_1(m_1) \cdots u_s(m_s) w_{(1)}(n, 0) w_{(2)} \rangle \\ &= \langle w'_{(3)}, (Y_{W_3})_{m_1}(u_1) \cdots (Y_{W_3})_{m_s}(u_s) P_{\text{wt } w_{(1)} + \text{wt } w_{(2)} - n - 1} f((w_{(1)} + O_N(W_1)) \otimes w_{(2)}) \rangle. \end{aligned} \quad (6.34)$$

Note that the only relations among elements of form (6.31) are those given by $I_{V; W_1, W_2}$ and $J_{V; W_1, W_2}$. These relations are also satisfied by elements of W_3 of the form

$$(Y_{W_3})_{m_1}(u_1) \cdots (Y_{W_3})_{m_s}(u_s) P_{\text{wt } w_{(1)} + \text{wt } w_{(2)} - n - 1} f((w_{(1)} + O_N(W_1)) \otimes w_{(2)}).$$

Hence $\mu'(w'_{(3)})$ is well defined. More precisely, we can see that $\mu'(w'_{(3)})$ is well defined as follows: Consider the graded subspace of $T_{V; W_1} \otimes G_N(W_2)$ spanned by elements of the form

$$u_1(m_1) \cdots u_s(m_s) w_{(1)}(n, 0) w_{(2)}$$

for homogeneous $u_i \in V$, $m_i \in \mathbb{Z}$, $i = 1, \dots, s$, homogeneous $w_{(1)} \in W_1$, $n \in -h + \mathbb{N}$, and $w_{(2)} \in G_N(W_2)$, satisfying (6.32) and (6.33). First, we define $\mu'(w'_{(3)})$ using (6.34) to be an element of the graded dual space of this graded subspace of $T_{V; W_1} \otimes G_N(W_2)$. Then from

the definitions of $\mu'(w'_{(3)})$ and $I_{V;W_1,W_2}$, we see that $\mu'(w'_{(3)})$ annihilates the intersection of $I_{V;W_1,W_2}$ and this graded subspace of $T_{V;W_1} \otimes G_N(W_2)$. So $\mu'(w'_{(3)})$ is in fact an element of the graded dual space of the graded subspace of $\tilde{S}(V;W_1,W_2)$ spanned by elements of the form

$$u_1(m_1) \cdots u_s(m_s)w_{(1)}(n,0)w_{(2)}$$

for homogeneous $u_i \in V$, $m_i \in \mathbb{Z}$, $i = 1, \dots, s$, homogeneous $w_{(1)} \in W_1$, $n \in -h + \mathbb{N}$, and $w_{(2)} \in G_N(W_2)$, satisfying (6.32) and (6.33). But from the definitions of $\mu'(w'_{(3)})$ and $J_{V;W_1,W_2}$, we see that $\mu'(w'_{(3)})$ annihilates the intersection of $J_{V;W_1,W_2}$ and this graded subspace of $\tilde{S}(V;W_1,W_2)$. Thus $\mu'(w'_{(3)})$ is in fact an element of $(S_N^0(V;W_1,W_2))'$.

By definition, we see that if $w'_{(3)}$ is homogeneous, then $\mu'(w'_{(3)})$ is also homogeneous and $\text{wt } \mu'(w'_{(3)}) = \text{wt } w'_{(3)}$. Thus $\mu'(w'_{(3)}) \in G_N((S_N^0(V;W_1,W_2))')$ for any $w'_{(3)} \in G_N(W'_3)$ and we obtain a linear map $\mu' : G_N(W'_3) \rightarrow G_N((S_N^0(V;W_1,W_2))')$. The map μ' is in fact a homomorphism of $A_N(V)$ -modules, that is,

$$\mu'((Y'_{W'_3})_{\text{wt } u-1}(u)w'_{(3)}) = (Y_{(S_N^0(V;W_1,W_2))'})_{\text{wt } u-1}(u)\mu'(w'_{(3)}),$$

for homogeneous $u \in V$ and $w'_{(3)} \in G_N(W'_3)$. But this is equivalent to

$$\mu'((Y_{W'_3}^o)_{\text{wt } u-1}(u)w'_{(3)}) = (Y_{(S_N^0(V;W_1,W_2))'}^o)_{\text{wt } u-1}(u)\mu'(w'_{(3)})$$

for homogeneous $u \in V$ and $w'_{(3)} \in G_N(W'_3)$, which follows from the calculation

$$\begin{aligned} & \langle \mu'((Y_{W'_3}^o)_{\text{wt } u-1}(u)w'_{(3)}), u_1(m_1) \cdots u_s(m_s)w_{(1)}(n,0)w_{(2)} \rangle \\ &= \langle (Y_{W'_3}^o)_{\text{wt } u-1}(u)w'_{(3)}, (Y_{W_3})_{m_1}(u_1) \cdots (Y_{W_3})_{m_s}(u_s) \cdot \\ & \quad \cdot P_{\text{wt } w_{(1)} + \text{wt } w_{(2)} - n - 1} f((w_{(1)} + O_N(W_1)) \otimes w_{(2)}) \rangle \\ &= \langle w'_{(3)}, (Y_{W_3})_{\text{wt } u-1}(u)(Y_{W_3})_{m_1}(u_1) \cdots (Y_{W_3})_{m_s}(u_s) \cdot \\ & \quad \cdot P_{\text{wt } w_{(1)} + \text{wt } w_{(2)} - n - 1} f((w_{(1)} + O_N(W_1)) \otimes w_{(2)}) \rangle \\ &= \langle \mu'(w'_{(3)}), u(\text{wt } u - 1)u_1(m_1) \cdots u_s(m_s)w_{(1)}(n,0)w_{(2)} \rangle \\ &= \langle (Y_{(S_N^0(V;W_1,W_2))'}^o)_{\text{wt } u-1}(u)\mu'(w'_{(3)}), u_1(m_1) \cdots u_s(m_s)w_{(1)}(n,0)w_{(2)} \rangle \end{aligned}$$

for homogeneous $u_i \in V$, $m_i \in \mathbb{Z}$, $i = 1, \dots, s$, homogeneous $w_{(1)} \in W_1$, $n \in -h + \mathbb{N}$, and $w_{(2)} \in G_N(W_2)$, satisfying (6.32) and (6.33).

Since W'_3 is isomorphic to $S_N(G_N(W'_3))$, by the universal property of $S_N(G_N(W'_3))$, we obtain a homomorphism, still denoted as μ' , of generalized V -modules from W'_3 to $(S_N^0(V;W_1,W_2))'$, extending $\mu' : G_N(W'_3) \rightarrow G_N((S_N^0(V;W_1,W_2))')$. The adjoint map μ'' of μ' is a homomorphism of generalized V -modules from $(S_N^0(V;W_1,W_2))''$ to W_3 . In particular, the restriction of μ'' to $S_N^0(V;W_1,W_2)$ is a homomorphism μ of generalized V -modules from $S_N^0(V;W_1,W_2)$ to W_3 .

We now define

$$(\mathcal{Y}^f)_{n,0}(w_{(1)})w = \mu(w_{(1)}(n,0)w)$$

for $w_{(1)} \in W_1$, $w \in S_N(G_N(W_2))$ and $n \in -h + \mathbb{N}$. Since W_2 is isomorphic to $S_N(G_N(W_2))$, we obtain maps $(\mathcal{Y}^f)_{n,0}(w_{(1)}) : W_2 \rightarrow W_3$ for $n \in -h + \mathbb{N}$. Let

$$(\mathcal{Y}^f)^0(w_{(1)}, x) = \sum_{n \in -h + \mathbb{N}} (\mathcal{Y}^f)_{n,0}(w_{(1)}) x^{-n-1}.$$

In particular, for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$,

$$\langle w'_{(3)}, (\mathcal{Y}^f)^0(w_{(1)}, 1)w_{(2)} \rangle = \sum_{n \in -h + \mathbb{N}} \langle w'_{(3)}, (\mathcal{Y}^f)_{n,0}(w_{(1)})w_{(2)} \rangle$$

is well defined. Now we construct a logarithmic intertwining operator \mathcal{Y}^f of type $\binom{W_3}{W_1 W_2}$ by

$$\langle w'_{(3)}, \mathcal{Y}^f(w_{(1)}, x)w_{(2)} \rangle = \langle x^{L(0)}w'_{(3)}, (\mathcal{Y}^f)^0(x^{-L(0)}w_{(1)}, 1)x^{-L(0)}w_{(2)} \rangle$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$ (recall (5.29)).

Since \mathcal{Y}_t satisfies the commutator formula, the associative formula and the $L(-1)$ -derivative property, so does \mathcal{Y}^f . Thus \mathcal{Y}^f satisfies the Jacobi identity and the $L(-1)$ -derivative property. So it is a logarithmic intertwining operator of the desired type. It is clear from the construction that $\rho(\mathcal{Y}^f) = f$.

In the general case, by Remark 3.5, W_1 , W_2 and W'_3 can all be decomposed as direct sums of grading generalized modules W such that $G_N(W)$ are spanned by homogeneous elements of weights $h + n$ for $n = 0, \dots, N$. Then the logarithmic intertwining operator \mathcal{Y}^f can be obtained by adding those intertwining operators obtained from the case discussed above. ■

From Proposition 3.8 and Theorem 6.6, we obtain immediately the following result:

Corollary 6.7. *Assume that there exists a positive integer N such that the absolute value of the difference of the lowest weights of any two irreducible V -modules is less than or equal to N . Let W_1 , W_2 and W_3 be grading-restricted generalized V -modules of finite lengths whose $L(0)$ -block sizes are bounded. Assume that W_2 and W'_3 are equivalent to $S_N(G_N(W_2))$ and $S_N(G_N(W'_3))$, respectively. Then the map*

$$\begin{aligned} \rho : \mathcal{V}_{W_1 W_2}^{W_3} &\rightarrow \text{Hom}_{A_N(V)}(A_N(W_1) \otimes_{A_N(V)} \Omega_N^0(W_2), \Omega_N^0(W_3)) \\ \mathcal{Y} &\mapsto \rho(\mathcal{Y}) \end{aligned}$$

is a linear isomorphism. ■

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