

Tensor categories and the mathematics of rational and logarithmic conformal field theory

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Abstract

We review the construction of braided tensor categories and modular tensor categories from representations of vertex operator algebras, which correspond to chiral algebras in physics. The extensive and general theory underlying this construction also establishes the operator product expansion for intertwining operators, which correspond to chiral vertex operators, and more generally, it establishes the logarithmic operator product expansion for logarithmic intertwining operators. We review the main ideas in the construction of the tensor product bifunctors and the associativity isomorphisms. For rational and logarithmic conformal field theories, we review the precise results that yield braided tensor categories, and in the rational case, modular tensor categories as well. In the case of rational conformal field theory, we also briefly discuss the construction of the modular tensor categories for the Wess-Zumino-Novikov-Witten models and, especially, a recent discovery concerning the proof of the fundamental rigidity property of the modular tensor categories for this important special case. In the case of logarithmic conformal field theory, we mention suitable categories of modules for the triplet \mathcal{W} -algebras as an example of the applications of our general construction of the braided tensor category structure.

1 Introduction

Vertex (operator) algebras, often called chiral algebras in the physics literature, are a fundamental class of algebraic structures whose extensive theory has been developed and used in recent years to provide the means to illuminate and to solve many problems in a wide variety of areas of mathematics and theoretical physics. In 1984, Belavin, Polyakov and Zamolodchikov [BPZ] formalized the relation between the operator product expansion, chiral correlation functions and representation theory, especially for the Virasoro algebra, and Knizhnik and Zamolodchikov [KZ] established fundamental relations between conformal field theory and the representation theory of affine Lie algebras. The mathematical notions of

vertex algebra and of vertex operator algebra were formulated in 1986 by Borchers in [B] and in a variant form in 1988 by Frenkel-Lepowsky-Meurman in [FLM2]. The representation theory of vertex (operator) algebras plays deep roles in both mathematics and physics, including in particular in the representation theory of infinite-dimensional Lie algebras, the study of sporadic finite simple groups, notably including the Monster, the construction of knot invariants and 3-manifold invariants, the theory of q -series identities and fermionic formulas, and the study of certain structures in algebraic geometry, as well as in conformal field theory, string theory and quantum computing.

Since the introduction of string theory and conformal field theory in physics and of the areas such as those mentioned above in mathematics, many discoveries have been made by physicists and many others have been made by mathematicians (and many jointly), and these have often influenced what mathematicians study, and, respectively, what physicists study, in remarkable and unexpected ways. Benefiting greatly from such interactions, a vast and (relatively) new interdisciplinary area that cannot be considered to be purely mathematics or purely physics has been blossoming for several decades by now. In a moment we will discuss more instances—the subjects of the present paper—of this phenomenon. Independently of its applications to physics, the established mathematics of this new area—mathematics created by both physicists and mathematicians—is deep and is permanent, and it continues to enjoy rapid development. Especially with the boundary between mathematics and physics sometimes indistinct, it is a good idea to keep in mind that what counts as established mathematics consists of theorems supported by mathematical proofs (whether supplied by mathematicians or by physicists), just as established physical theories are those that have experimental confirmation (although this new interdisciplinary area is clearly so deep, rich and broad that the time horizon for experimental verification of the many aspects of the physics should be flexible!). Mathematicians' and physicists' conjectures and unproved assumptions, which should of course be recognized as such until they have proofs, have played extremely influential roles in the development of the theory, and we will be discussing some particularly interesting ones.

We turn to the theme of this paper—a review of braided and modular tensor categories in rational and in logarithmic conformal field theory. Our theme indeed involves valuable interactions between physicists' and mathematicians' ideas and work.

Tensor product operations for modules play central roles in the representation theory of many important classical algebraic structures, such as Lie algebras, groups (or group algebras), commutative associative algebras, Hopf algebras, and in particular, quantum groups. They give new modules from known ones, but more importantly, they provide powerful tools for studying modules. Most importantly, suitable categories of modules for such algebras, equipped with tensor product bifunctors, contragredient functors, appropriate natural isomorphisms, and related data, become rigid symmetric or rigid braided tensor categories. These tensor category structures play such a fundamental role that many results in the representation theory of such an algebraic structure and its applications depend heavily on such tensor category structure. On the other hand, a large part of these tensor category structures is so easy to construct that these ubiquitous tensor category structures are often

not even explicitly mentioned. (Rigidity for a tensor category, which we will be mentioning often, is an abstraction of the classical compatibility properties, for triple tensor products involving a finite-dimensional vector space X and its dual space X^* , relating the natural maps $\mathbb{C} \rightarrow X^* \otimes X$ and $X^* \otimes X \rightarrow \mathbb{C}$.)

In 1988, motivated partly by Verlinde’s conjecture [V] on fusion rules and modular transformations for rational conformal field theories, Moore and Seiberg ([MS1], [MS2]) obtained a set of polynomial equations for fusing, braiding and modular transformations in rational conformal field theories, based on a number of explicit, very strong, unproved assumptions, including in particular the existence of a suitable operator product expansion for “chiral vertex operators” (which correspond to intertwining operators, or more precisely, to intertwining maps, in vertex operator algebra theory) and the modular invariance of suitable traces of compositions of these chiral vertex operators.

It is important to note that Moore and Seiberg mentioned a number of issues, which turned out to be very substantial in the later mathematical constructions reviewed below, that would arise if one were to try to *prove* these strong assumptions using representations of chiral algebras.

They observed an analogy between certain of these polynomial equations and the coherence properties of tensor categories. Later, Turaev formulated a precise notion of modular tensor category in [T1] and [T2] and gave examples of such tensor categories from representations of quantum groups at roots of unity, based on results obtained by many people on quantum groups and their representations, especially those in the pioneering work [ReT1] and [ReT2] of Reshetikhin and Turaev on the construction of knot and 3-manifold invariants from representations of quantum groups.

On the other hand, on the rational conformal field theory side as opposed to the quantum group side, (rigid and) modular tensor categories for the Wess-Zumino-Novikov-Witten models, and more generally, for rational conformal field theories, were then believed to exist by both physicists and mathematicians, but no one had constructed them at that time. It was a deep unsolved problem to construct such modular tensor categories.

This problem has now been solved. The solution took many years and a great deal of effort for mathematicians to eventually obtain a complete construction of the desired modular tensor categories.

Among many mathematical works on the Wess-Zumino-Novikov-Witten models, the works of Kazhdan-Lusztig [KL1]–[KL5] (which handled negative-level analogues), Finkelberg [Fi1]–[Fi2], Huang-Lepowsky [HL6] and Bakalov-Kirillov [BK] were early explicit contributions toward the construction of the desired modular tensor categories for this particular important class of models.

However, even for the Wess-Zumino-Novikov-Witten models, the construction was not accomplished until 2005, when the first author completed a general construction [H6], for all (suitable) rational conformal field theories, of the desired (rigid and) modular tensor categories that had been conjectured to exist; the papers carrying out this work are discussed below.

See the end of Section 3 for a more detailed discussion of the construction for the Wess-

Zumino-Novikov-Witten models.

For rational conformal field theories in general, braided tensor category structure on categories of modules for vertex operator algebras satisfying suitable finiteness and reductivity conditions was constructed by the authors in the papers [HL1], [HL3]–[HL5], [HL2] and [HL7] together with the papers [H1] and [H4] by the first author. The modularity of these braided tensor categories—that is, the properties of rigidity and of nondegeneracy—were proved by the first author in [H8] by the use of the Moore-Seiberg equations, which the first author had proved for suitable classes of representations of vertex operator algebras in [H7].

Recall that in [MS1], [MS2], these equations had been obtained from the very strong unproved assumptions referred to above, and these strong assumptions were even harder to prove than the Moore-Seiberg equations.

In particular, modular tensor category structure on these module categories was constructed in the papers mentioned above. Certain of the works entering into this construction also established the operator product expansion for intertwining maps, or chiral vertex operators (see [H1] and [H4]) and the modular invariance of the space of q -traces of compositions of an arbitrary number of these intertwining maps (see [H5]).

These constructions and results were established in particular in the important special cases of the Wess-Zumino-Novikov-Witten models and the minimal models. But it is important to note that the theory itself is general vertex operator algebra representation theory, including substantial analytic reasoning; the representation theory of affine Lie algebras and the representation theory of the Virasoro algebra play virtually no role at all in the general theory, until one verifies, after the general theory has been constructed, that the hypotheses for applying the general theory hold for affine Lie algebras at positive integral level and for the minimal models.

For *nonrational* conformal field theories, braided tensor category structure on suitable categories of generalized modules for vertex algebras satisfying suitable conditions was constructed in a series of papers by the authors jointly with Zhang, [HLZ0]–[HLZ8], together with a paper by the first author [H9]. These categories include those for the conformal field theories associated to, for example, Lorentzian lattices (which do not involve logarithms), as well as those for the much deeper logarithmic conformal field theories.

In particular, suitable categories of modules for the triplet \mathcal{W} -algebras, which are examples of vertex operator algebras corresponding to important logarithmic conformal field theories, are braided tensor categories, by an application of our general constructions and theory to this case.

In this paper, we review these constructions, along with the underlying deep theory. In the next section, we discuss the main ideas in the constructions of the tensor product bifunctors and of the associativity isomorphisms. In Section 3, we review the precise results that give modular tensor categories for rational conformal field theories. In this section we also briefly discuss the construction of the modular tensor categories for the Wess-Zumino-Novikov-Witten models and, especially, a recent discovery concerning the proof of the rigidity property in this special case. In Section 4, we review the precise results that produce braided tensor categories for logarithmic conformal field theories, and in this section we also discuss

suitable categories of modules for the triplet \mathcal{W} -algebras as an example illustrating the application of our general theory.

We refer the reader to [HLZ1] for a much more detailed description of the mathematical theory and for a much more extensive discussion of the relevant mathematics and physics literature.

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2 Ideas of the construction of the tensor product bifunctors and the associativity isomorphisms

In the tensor category theory for vertex operator algebras, the tensor product bifunctors are not built on the classical tensor product bifunctor for vector spaces. Correspondingly, the construction of the natural associativity isomorphisms is highly nontrivial. It plays a deep role in the construction of braided and modular tensor category structure. In this section, we present the main ideas of the construction of the tensor product bifunctors and the associativity isomorphisms for suitable categories of modules or generalized modules for vertex operator algebras. Suitable classes of vertex algebras more general than vertex operator algebras are also handled in [HLZ1]–[HLZ8]. Here we purposely suppress a large number of important technical difficulties that had to be (and were) addressed and resolved (see the authors' cited works, along with the basic treatments [FLM2], [FHL] and [LL] for background), so that the reader can see the flow of these ideas without getting into details. (In particular, many statements that follow are intentionally oversimplified. For instance, in the correct mathematical theory, formal variables as well as complex variables are crucially needed.) In the next two sections, we shall more specifically discuss the cases of the material presented here corresponding to rational conformal field theories and to logarithmic conformal field theories, respectively.

At the end of this section we shall suggest a guide for further reading.

The central concept underlying the constructions is the notion of $P(z)$ -intertwining map, where z is a nonzero complex number and $P(z)$ the Riemann sphere $\hat{\mathbb{C}}$ with one negatively oriented puncture at ∞ and two positively oriented punctures at z and 0 , with local coordinates $1/w$, $w - z$ and w , respectively, at these three punctures.

Let $V = \bigoplus_{n \in \mathbb{Z}} V_{(n)}$ be a vertex operator algebra, $\mathbf{1} \in V_{(0)}$ its vacuum vector and $Y(\cdot, z)$ its vertex operator map, which defines the algebra structure. In language more familiar to physicists, the vertex operators $Y(v, z)$ for $v \in V$ form a chiral algebra and the vector space underlying this chiral algebra is isomorphic to V under the map given by $Y(v, z) \mapsto v = \lim_{z \rightarrow 0} Y(v, z)\mathbf{1}$. In the present paper we will not need to say much about the ever-present mode-expansion of vertex operators such as $Y(v, z)$ in terms of powers of z , and we will not need to specifically discuss the conformal vector (in $V_{(2)}$), whose vertex operator gives the Virasoro algebra. For a module W for V , let W' be the restricted (graded) dual module and

let $\langle \cdot, \cdot \rangle$ be the natural pairing between W' and W . (In physics notation, the elements of W' and W are written as $\langle \phi |$ and $|\psi \rangle$, respectively and the pairing between $\langle \phi |$ and $|\psi \rangle$ is written as $\langle \phi | \psi \rangle$.)

Let W_1 , W_2 and W_3 be modules for V , and let $Y_1(\cdot, z)$, $Y_2(\cdot, z)$ and $Y_3(\cdot, z)$ be the corresponding vertex operator maps. Given $v \in V$, the operators $Y_1(v, z)$, $Y_2(v, z)$ and $Y_3(v, z)$ are the actions of the element $Y(v, z)$ of the chiral algebra on W_1 , W_2 and W_3 , respectively. Let z be a fixed nonzero complex number. A $P(z)$ -intertwining map of type $\binom{W_3}{W_1 W_2}$ is a linear map

$$I : W_1 \otimes W_2 \rightarrow \overline{W}_3, \quad (2.1)$$

where \overline{W}_3 is a natural algebraic completion of W_3 related to its \mathbb{C} -grading (typically, the full dual space of the restricted dual of W_3), such that for $w'_{(3)} \in W'_3$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $v \in V$, the series

$$\begin{aligned} & \langle w'_{(3)}, Y_3(v, z_1) I(w_{(1)} \otimes w_{(2)}) \rangle, \\ & \langle w'_{(3)}, I(Y_1(v, z_1 - z) w_{(1)} \otimes w_{(2)}) \rangle, \\ & \langle w'_{(3)}, I(w_{(1)} \otimes Y_2(v, z_1) w_{(2)}) \rangle \end{aligned}$$

are absolutely convergent in the regions $|z_1| > |z| > 0$, $|z| > |z_1 - z| > 0$ and $|z| > |z_1| > 0$, respectively, and for a rational function $f(z_1, z)$ whose only possible poles are at $z_1, z = 0$ and $z_1 = z$ and a loop C_1 in the complex plane enclosing z and 0, we have

$$\begin{aligned} & \int_{C_1} f(z_1, z) \langle w'_{(3)}, Y_3(v, z_1) I(w_{(1)} \otimes w_{(2)}) \rangle dz_1 \\ &= \int_{C_2} f(z_1, z) \langle w'_{(3)}, I(Y_1(v, z_1 - z) w_{(1)} \otimes w_{(2)}) \rangle dz_1 \\ &+ \int_{C_3} f(z_1, z) \langle w'_{(3)}, I(w_{(1)} \otimes Y_2(v, z_1) w_{(2)}) \rangle dz_1, \end{aligned}$$

where C_2 is a loop in the complex plane enclosing z but not 0 and C_3 is a loop enclosing 0 but not z .

It was proved in [HL3], [HL5] and [HLZ3] that for a nonzero complex number z , these $P(z)$ -intertwining *maps* are in fact the evaluations of intertwining *operators* (or logarithmic intertwining operators, in the logarithmic theory) at z , that is, given a $P(z)$ -intertwining map I and a choice of branch of $\log z$, there exists an intertwining operator or logarithmic intertwining operator \mathcal{Y} of the same type such that

$$I(w_{(1)} \otimes w_{(2)}) = \mathcal{Y}(w_{(1)}, z) w_{(2)}, \quad (2.2)$$

where the right-hand side is evaluated using the given branch of $\log z$. (An intertwining operator involves a *formal* variable, while an intertwining map is based on a nonzero complex number z , as above. Intertwining maps correspond more closely to chiral operators in conformal field theory, but both notions are essential in the theory.)

There is a natural linear injection

$$\mathrm{Hom}(W_1 \otimes W_2, \overline{W}_3) \rightarrow \mathrm{Hom}(W'_3, (W_1 \otimes W_2)^*). \quad (2.3)$$

Under this injection, a map $I \in \mathrm{Hom}(W_1 \otimes W_2, \overline{W}_3)$ amounts to a map $I' : W'_3 \rightarrow (W_1 \otimes W_2)^*$:

$$w'_{(3)} \mapsto \langle w'_{(3)}, I(\cdot \otimes \cdot) \rangle. \quad (2.4)$$

For a $P(z)$ -intertwining map I , the map (2.4) intertwines two natural V -actions, on W'_3 and on $(W_1 \otimes W_2)^*$. The space $(W_1 \otimes W_2)^*$ is typically not a V -module, not even in any weak sense. The images of all the elements $w'_{(3)} \in W'_3$ under this map satisfy certain very subtle conditions, called the “ $P(z)$ -compatibility condition” and the “ $P(z)$ -local grading restriction condition,” as formulated in [HL5], after Proposition 13.7, and in [HLZ4], before Theorem 5.44.

Given a suitable category of generalized V -modules (as precisely formulated in the cited works) and generalized modules W_1 and W_2 in this category, the $P(z)$ -tensor product of W_1 and W_2 is then defined to be a pair (W_0, I_0) , where W_0 is a generalized module in the category and I_0 is a $P(z)$ -intertwining map of type $\binom{W_0}{W_1 W_2}$, such that for any pair (W, I) with W a generalized module in the category and I a $P(z)$ -intertwining map of type $\binom{W}{W_1 W_2}$, there is a unique morphism $\eta : W_0 \rightarrow W$ such that $I = \bar{\eta} \circ I_0$, where $\bar{\eta} : \overline{W}_0 \rightarrow \overline{W}$ is the linear map naturally extending η to the completion. This universal property characterizes (W_0, I_0) up to canonical isomorphism, *if it exists*. The $P(z)$ -tensor product of W_1 and W_2 , if it exists, is denoted by $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$, and the image of $w_{(1)} \otimes w_{(2)}$ under $\boxtimes_{P(z)}$, an element of $\overline{W}_1 \boxtimes_{P(z)} \overline{W}_2$, *not* of $W_1 \boxtimes_{P(z)} W_2$, is denoted by $w_{(1)} \boxtimes_{P(z)} w_{(2)}$.

It is crucial to note that the tensor product operation depends on an arbitrary nonzero complex number, and that we must allow this complex number to vary, as we will explain. Correspondingly, the resulting tensor category structure will be much more than a braided tensor category; it will be what we call a “vertex tensor category,” as formalized in [HL2].

From the definition and the natural map (2.3), one finds that if the $P(z)$ -tensor product of W_1 and W_2 exists, then its contragredient module can be realized as the union of the ranges of all the maps of the form (2.4) as W'_3 and I vary. Even if the $P(z)$ -tensor product of W_1 and W_2 does not exist, we denote this union (which is always a subspace stable under a natural action of V) by $W_1 \boxdot_{P(z)} W_2$. If the tensor product does exist, then

$$W_1 \boxtimes_{P(z)} W_2 = (W_1 \boxdot_{P(z)} W_2)', \quad (2.5)$$

$$W_1 \boxdot_{P(z)} W_2 = (W_1 \boxtimes_{P(z)} W_2)'; \quad (2.6)$$

examining (2.5) will show the reader why the notation \boxdot was chosen in the papers [HL1]–[HL5] ($\boxdot = \boxtimes'$!).

By Theorem 13.10 in [HL5] and Theorem 5.50 in [HLZ4], $W_1 \boxdot_{P(z)} W_2$ is equal to the subspace of $(W_1 \otimes W_2)^*$ consisting of the elements satisfying the $P(z)$ -compatibility condition and the $P(z)$ -local grading restriction condition mentioned above. With such a characterization of $W_1 \boxdot_{P(z)} W_2$, algorithms for calculating $W_1 \boxdot_{P(z)} W_2$ and $W_1 \boxtimes_{P(z)} W_2$ (if the tensor product $W_1 \boxtimes_{P(z)} W_2$ of W_1 and W_2 exists) can be developed. It will be interesting to see

how such algorithms are related to the Nahm-Gaberdiel-Kausch algorithm ([N], [GK]) used by physicists. On the other hand, we would like to emphasize that what our theory does is to establish the basic tensor category structure that is assumed to exist in many works, including those developing algorithms for calculating the equivalence class of the tensor product of two objects in the category, and it is necessary to use the formulations and results in our papers (for which algorithms can also be developed) in order to establish the required tensor category structure and associated properties (such as rigidity and modularity, for instance). This is a far different matter from considering only the tensor product modules (or fusion products, or fusion rules, or properties of fusion rules).

Even if the tensor product does not exist, the contragredient $(W_1 \boxtimes_{P(z)} W_2)'$ is still a generalized V -module and might still be useful, but this generalized V -module might not be in the category of generalized V -modules that we start with. What is important is that our goal has been to construct a tensor category, not just a tensor product operation on the set of equivalence classes of generalized V -modules in this category. In particular, besides tensor products of objects, we also need to construct tensor products of morphisms as well as an array of natural isomorphisms relating more than two modules. The universal property and our $\boxtimes_{P(z)}$ constructions are crucial for the constructions of tensor products of morphisms and, more deeply, of these required natural isomorphisms.

Here is a rough analogy from classical algebra: Before the advent of category-theoretic understanding in the twentieth century, tensor products of modules for a group or Lie algebra, etc., were often described in terms of tensor products of basis elements, which would form a basis of the desired tensor product module, whose structure would then need to be shown to be basis-independent, or were often described in terms of multiplicities of irreducible modules occurring in a tensor product, corresponding to fusion rules—among other ways of describing tensor products classically. But the natural associativity maps and coherence properties amounting to tensor category structure were most smoothly formulated only later. In the representation theory of vertex operator algebras, the particular formulations of tensor product that we work with are the formulations needed for the constructions and proofs entering into the necessary tensor category structure and rigidity and modularity, and so on.

In particular, in order to construct tensor category structure, we need to construct appropriate natural associativity isomorphisms. Assuming the existence of the relevant tensor products, we in fact need to construct an appropriate natural isomorphism from $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$ to $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ for complex numbers z_1, z_2 satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$. Note that we are using two distinct nonzero complex numbers, and that certain inequalities hold. This situation corresponds to the fact that a Riemann sphere with one negatively oriented puncture and three positively oriented punctures can be seen in two different ways as the “product” of two Riemann spheres, each of them with one negatively oriented puncture and two positively oriented punctures.

To construct this natural isomorphism, we first consider compositions of certain intertwining maps. As we have mentioned, a $P(z)$ -intertwining map I of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$ maps into \overline{W}_3 rather than into W_3 . Thus the existence of compositions of suitable intertwining maps always entails the (absolute) convergence of certain series. In particular, the existence of

the composition $w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})$ when $|z_1| > |z_2| > 0$ and the existence of the composition $(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}$ when $|z_2| > |z_1 - z_2| > 0$, for general elements $w_{(i)}$ of W_i , $i = 1, 2, 3$, require the proof of certain convergence conditions.

Let us now assume these convergence conditions and let z_1, z_2 satisfy $|z_1| > |z_2| > |z_1 - z_2| > 0$. To construct the desired associativity isomorphism from $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$ to $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$, it is equivalent (by duality) to give a suitable natural isomorphism from $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ to $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$. Instead of constructing this isomorphism directly, we embed both of these spaces, separately, into the single space $(W_1 \otimes W_2 \otimes W_3)^*$.

The space $(W_1 \otimes W_2 \otimes W_3)^*$ carries a natural V -action analogous to the contragredient of the diagonal action in Lie algebra theory (as was also true for the action of V on $(W_1 \otimes W_2)^*$ mentioned above). Also, for nonzero and distinct complex numbers z_1 and z_2 and four generalized V -modules W_1, W_2, W_3 and W_4 , we have a canonical notion of “ $P(z_1, z_2)$ -intertwining map from $W_1 \otimes W_2 \otimes W_3$ to \overline{W}_4 ”. The relation between these two concepts comes from the natural linear injection

$$\begin{aligned} \text{Hom}(W_1 \otimes W_2 \otimes W_3, \overline{W}_4) &\rightarrow \text{Hom}(W'_4, (W_1 \otimes W_2 \otimes W_3)^*) \\ F &\mapsto F', \end{aligned} \quad (2.7)$$

where $F' : W'_4 \rightarrow (W_1 \otimes W_2 \otimes W_3)^*$ is given by

$$\nu \mapsto \nu \circ F, \quad (2.8)$$

which is indeed well defined. Under this natural map, the $P(z_1, z_2)$ -intertwining maps correspond precisely to the maps from W'_4 to $(W_1 \otimes W_2 \otimes W_3)^*$ that intertwine the two natural V -actions on W'_4 and on $(W_1 \otimes W_2 \otimes W_3)^*$.

Now for generalized modules W_1, W_2, W_3, W_4, M_1 , and a $P(z_1)$ -intertwining map I_1 and a $P(z_2)$ -intertwining map I_2 of types $\binom{W_4}{W_1 M_1}$ and $\binom{M_1}{W_2 W_3}$, respectively, it turns out that the composition $I_1 \circ (1_{W_1} \otimes I_2)$ exists and is a $P(z_1, z_2)$ -intertwining map when $|z_1| > |z_2| > 0$. Analogously, for a $P(z_2)$ -intertwining map I^1 and a $P(z_1 - z_2)$ -intertwining map I^2 of types $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively, where M_2 is also a generalized module, the composition $I^1 \circ (I^2 \otimes 1_{W_3})$ is a $P(z_1, z_2)$ -intertwining map when $|z_2| > |z_1 - z_2| > 0$. Hence we have two maps intertwining the V -actions:

$$\begin{aligned} W'_4 &\rightarrow (W_1 \otimes W_2 \otimes W_3)^* \\ \nu &\mapsto \nu \circ F_1, \end{aligned} \quad (2.9)$$

where F_1 is the intertwining map $I_1 \circ (1_{W_1} \otimes I_2)$, and

$$\begin{aligned} W'_4 &\rightarrow (W_1 \otimes W_2 \otimes W_3)^* \\ \nu &\mapsto \nu \circ F_2, \end{aligned} \quad (2.10)$$

where F_2 is the intertwining map $I^1 \circ (I^2 \otimes 1_{W_3})$.

It is important to note that we can express these compositions $I_1 \circ (1_{W_1} \otimes I_2)$ and $I^1 \circ (I^2 \otimes 1_{W_3})$, which involve intertwining *maps*, in terms of (logarithmic) intertwining *operators*. Let

$\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}^1$ and \mathcal{Y}^2 be the intertwining operators (or logarithmic intertwining operators, in the logarithmic case) corresponding to I_1, I_2, I^1 and I^2 , respectively. Then the compositions $I_1 \circ (1_{W_1} \otimes I_2)$ and $I^1 \circ (I^2 \otimes 1_{W_3})$ correspond to the “product” $\mathcal{Y}_1(\cdot, z_1)\mathcal{Y}_2(\cdot, z_2)\cdot$ and the “iterate” $\mathcal{Y}^1(\mathcal{Y}^2(\cdot, z_1 - z_2)\cdot, z_2)\cdot$ of (logarithmic) intertwining operators, respectively. (These products and iterates, which are obtained by specializing formal variables to the indicated complex variables, involve a branch of the log function and also certain convergence.)

The special cases in which the generalized modules W_4 are two iterated tensor product modules, and the “intermediate” modules M_1 and M_2 are two suitable tensor product modules, are particularly interesting: When $W_4 = W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ and $M_1 = W_2 \boxtimes_{P(z_2)} W_3$, and I_1 and I_2 are the corresponding canonical intertwining maps, (2.9) gives the natural V -homomorphism

$$\begin{aligned} W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) &\rightarrow (W_1 \otimes W_2 \otimes W_3)^* \\ \nu &\mapsto (w_{(1)} \otimes w_{(2)} \otimes w_{(3)} \mapsto \\ &\quad \nu(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}))), \end{aligned} \quad (2.11)$$

and when $W_4 = (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$ and $M_2 = W_1 \boxtimes_{P(z_1-z_2)} W_2$, and I^1 and I^2 are the corresponding canonical intertwining maps, (2.10) gives the natural V -homomorphism

$$\begin{aligned} (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 &\rightarrow (W_1 \otimes W_2 \otimes W_3)^* \\ \nu &\mapsto (w_{(1)} \otimes w_{(2)} \otimes w_{(3)} \mapsto \\ &\quad \nu((w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)})). \end{aligned} \quad (2.12)$$

It turns out that both of these maps are injections, so that we are embedding both of the spaces $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ and $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$ into the space $(W_1 \otimes W_2 \otimes W_3)^*$. By giving a precise description of the ranges of these two maps, it was proved in [H1] and [HLZ6] that under suitable conditions, the two ranges are the same; this provided the desired construction of the natural associativity isomorphisms.

More precisely, for any $P(z_1, z_2)$ -intertwining map F , the image of any $\nu \in W'_4$ under F' (recall (2.8)) satisfies certain conditions, which we call the “ $P(z_1, z_2)$ -compatibility condition” and the “ $P(z_1, z_2)$ -local grading restriction condition” (see [H1], (14.26)–(14.29), and [HLZ5], before Proposition 8.17). Hence, as special cases, the elements of $(W_1 \otimes W_2 \otimes W_3)^*$ in the ranges of either of the maps (2.9) or (2.10), and in particular, of (2.11) or (2.12), satisfy these conditions.

In addition, any $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ induces two “evaluation” maps $\mu_\lambda^{(1)} : W_1 \rightarrow (W_2 \otimes W_3)^*$ and $\mu_\lambda^{(2)} : W_3 \rightarrow (W_1 \otimes W_2)^*$, defined by $(\mu_\lambda^{(1)}(w_{(1)}))(w_{(2)} \otimes w_{(3)}) = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})$ and $(\mu_\lambda^{(2)}(w_{(3)}))(w_{(1)} \otimes w_{(2)}) = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})$, respectively. Any element λ of the range of (2.9), and in particular, of (2.11), must satisfy the condition that the elements $\mu_\lambda^{(1)}(w_{(1)})$ all lie, roughly speaking, in a suitable completion of the subspace $W_2 \boxtimes_{P(z_2)} W_3$ of $(W_2 \otimes W_3)^*$, and any element λ of the range of (2.10), and in particular, of (2.12), must satisfy the condition that the elements $\mu_\lambda^{(2)}(w_{(3)})$ all lie, again roughly speaking, in a suitable completion of the subspace $W_1 \boxtimes_{P(z_1-z_2)} W_2$ of $(W_1 \otimes W_2)^*$. These conditions are called the “ $P^{(1)}(z)$ -local grading

restriction condition” and the “ $P^{(2)}(z)$ -local grading restriction condition,” respectively (see [H1], (14.30)–(14.35), and [HLZ6], before Remark 9.7).

It turns out that the construction of the desired natural associativity isomorphism follows from showing that the ranges of both of (2.11) and (2.12) satisfy both of these conditions. This amounts to a certain condition that we call the “expansion condition” on our module category. When a suitable convergence condition and this expansion condition are satisfied, we show that the desired associativity isomorphisms do exist, and that in addition, the “associativity of intertwining maps” holds. That is, let z_1 and z_2 be complex numbers satisfying the inequalities $|z_1| > |z_2| > |z_1 - z_2| > 0$. Then for any $P(z_1)$ -intertwining map I_1 and $P(z_2)$ -intertwining map I_2 of types $\begin{pmatrix} W_4 \\ W_1 M_1 \end{pmatrix}$ and $\begin{pmatrix} M_1 \\ W_2 W_3 \end{pmatrix}$, respectively, there is a suitable module M_2 , and a $P(z_2)$ -intertwining map I^1 and a $P(z_1 - z_2)$ -intertwining map I^2 of types $\begin{pmatrix} W_4 \\ M_2 W_3 \end{pmatrix}$ and $\begin{pmatrix} M_2 \\ W_1 W_2 \end{pmatrix}$, respectively, such that

$$\langle w'_{(4)}, I_1(w_{(1)} \otimes I_2(w_{(2)} \otimes w_{(3)})) \rangle = \langle w'_{(4)}, I^1(I^2(w_{(1)} \otimes w_{(2)}) \otimes w_{(3)}) \rangle \quad (2.13)$$

for $w_{(1)} \in W_1, w_{(2)} \in W_2, w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$; and conversely, given I^1 and I^2 as indicated, there exist a suitable module M_1 and maps I_1 and I_2 with the indicated properties. In terms of (logarithmic) intertwining operators (recall the comments above), the equality (2.13) reads

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, z_1) \mathcal{Y}_2(w_{(2)}, z_2) w_{(3)} \rangle = \langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, z_1 - z_2) w_{(2)}, z_2) w_{(3)} \rangle, \quad (2.14)$$

where $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}^1$ and \mathcal{Y}^2 are the (logarithmic) intertwining operators corresponding to I_1, I_2, I^1 and I^2 , respectively. (As we have been mentioning, the two sides of (2.14) involve a branch of the log function and also certain convergence.) In this sense, the associativity asserts that the product of two suitable intertwining maps can be written as the iterate of two suitable intertwining maps, and vice versa.

From this construction of the natural associativity isomorphisms, $(w_{(1)} \boxtimes_{P(z_1 - z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}$ is mapped naturally to $w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})$ under the natural extension of the corresponding associativity isomorphism (these elements in general lying in the algebraic completions of the corresponding tensor product modules). In fact, this property

$$(w_{(1)} \boxtimes_{P(z_1 - z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)} \mapsto w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \quad (2.15)$$

for $w_{(1)} \in W_1, w_{(2)} \in W_2$ and $w_{(3)} \in W_3$ characterizes the associativity isomorphism

$$(W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3 \rightarrow W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3). \quad (2.16)$$

The coherence property of the associativity isomorphisms, giving rise to the desired tensor category structure, follows from this fact.

This fact also highlights why the complex numbers parametrizing our tensor product bifunctors must be allowed to vary; the indicated complex numbers must satisfy the inequalities mentioned above. Formula (2.15) is at the core of our notion of “vertex tensor category.” When the various complex numbers are systematically specialized to $z = 1$, by a

nontrivial procedure, we obtain an actual braided tensor category. But it is crucial to realize that this specialization procedure loses information. Our vertex tensor category structure is much richer than braided tensor category structure, and it provides the only route to the modular tensor category structure, including the rigidity property, discussed in the next section. Braided tensor category structure carries only “topological” information, while this vertex tensor category structure carries the full, and necessary, conformal-geometric and analytic information. We are doing “conformal field theory,” and it was natural that the (necessarily) mathematical construction and formulation of the desired braided tensor category structures would have to be inherently conformal-geometric, for both rational and logarithmic conformal field theories.

Note that (2.14) can be written as

$$\mathcal{Y}_1(w_{(1)}, z_1)\mathcal{Y}_2(w_{(2)}, z_2) = \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, z_1 - z_2)w_{(2)}, z_2), \quad (2.17)$$

with the “generic” vectors $w_{(3)}$ and $w'_{(4)}$ being implicit. This (rigorous) equation amounts to the (logarithmic) operator product expansion in the physics literature on (logarithmic) conformal field theory; indeed, in our language, if we expand the right-hand side of (2.17) in powers of $z_1 - z_2$ (as well as in nonnegative integral powers of $\log(z_1 - z_2)$ in the logarithmic case), we find that a product of intertwining maps is expressed as a convergent expansion in powers of $z_1 - z_2$ (and nonnegative integral powers of $\log(z_1 - z_2)$), with coefficients that are again intertwining maps, of the form $\mathcal{Y}^1(w, z_2)$. When all three modules are the vertex operator algebra itself, and all the intertwining operators are the canonical vertex operator $Y(\cdot, x)$ itself, this operator product expansion follows easily from the Jacobi identity axiom (see [FLM2]) in the definition of vertex operator algebra. But for intertwining operators (or logarithmic intertwining operators) in general, it is a deep matter to construct the (logarithmic) operator product expansion, that is, to prove the assertions involving (2.13) and (2.14) above. This was accomplished in [H1] in the finitely reductive (“rational”) setting and was considerably generalized in [HLZ6] to the logarithmic setting.

The construction of the operator product expansion, then, is intimately related to the fact that in this theory, it is vertex tensor categories rather than merely braided tensor categories that form the central notion. Braided tensor category structure alone is not enough, but it indeed follows from our vertex tensor category structure.

Readers who are interested in the full details of the general construction of braided tensor category structure, in both the rational and logarithmic cases, can consult [HLZ0]–[HLZ8] and the references there. As we mentioned in the introduction, the paper [HLZ1] gives a detailed description of the whole theory; this includes both the part of the theory presented in the series [HLZ0]–[HLZ8] and the part beyond the braided tensor category structure. For the modular tensor category structure (including the rigidity) in the rational case, see [H6] and [H8].

The papers [HLZ1]–[HLZ8] are written in an essentially self-contained way, because what we call “logarithmic tensor category theory” is new and the theory had to be constructed from the foundations; the only prerequisites for the reader are the basics of vertex operator algebra theory in the form presented in the relevant parts of [FLM2], [FHL] and [LL].

The theory as presented in this series includes, as a special case, a self-contained treatment of the earlier, rational (non-logarithmic) tensor category theory of [HL3]–[HL5] and [H1], and it is in fact easier to understand that theory in the context of the more general treatment in [HLZ1]–[HLZ8] than from the original papers because of certain new results (discussed in the introductory material in [HLZ1]) in this recent series. In particular, the results on the rational theory in [HLZ1]–[HLZ8] are stronger than the results in the original papers.

The papers [HLZ1]–[HLZ8] do not specifically address examples in detail, but references to the literature are given, and whenever a logarithmic or non-logarithmic conformal-theoretic tensor product (or fusion product) operation satisfies the universal property of the present theory, and the hypotheses for applying the present theory can be verified, then this theory provides braided tensor category structure and, more strongly, vertex tensor category structure and convergent operator product expansions, and, under the appropriate further conditions, (rigid and) modular tensor category structure.

3 Modular tensor categories and rational conformal field theories

Though the ideas described in the preceding section are natural, it was highly nontrivial to carry them out completely. Also, these ideas do not work for general vertex operator algebras; certain subtle and deep conditions are necessary. Moreover, the proofs of the rigidity and modularity in the case corresponding to rational conformal field theories required further results, proved by the first author, beyond the vertex-algebraic tensor category theory developed by the authors. In this section, we expand on some comments above by discussing the background and these results, for rational conformal field theories.

The vertex-algebraic study of tensor category structure on module categories for suitable vertex operator algebras was stimulated by the work of Moore and Seiberg, [MS1] and [MS2], in which, in the study of rational conformal field theory, they obtained a set of polynomial equations based on the deep and explicit assumption of the existence of a suitable operator product expansion for “chiral vertex operators,” which, as we have mentioned, correspond to intertwining maps in vertex operator algebra theory, and they observed an analogy between the theory of this set of polynomial equations and the theory of tensor categories. Earlier, in [BPZ], Belavin, Polyakov and Zamolodchikov had already formalized the relation between the operator product expansion, chiral correlation functions and representation theory, for the Virasoro algebra in particular, and Knizhnik and Zamolodchikov [KZ] had established fundamental relations between conformal field theory and the representation theory of affine Lie algebras. As we have discussed in the introductory material in [HL2], [HL3] and [HL6], such study of conformal field theory is deeply connected with the vertex-algebraic construction and study of tensor categories, and also with other mathematical approaches to the construction of tensor categories in the spirit of conformal field theory. Concerning the latter approaches, we would like to mention in particular the works of Tsuchiya-Ueno-Yamada [TUY], Beilinson-Feigin-Mazur [BFM], Kazhdan-Lusztig [KL1]–[KL5], Finkelberg [Fi1]–[Fi2]

and Bakalov-Kirillov [BK].

The operator product expansion and resulting braided tensor category structure constructed by the theory in [HL3], [HL4], [HL5], [H1] were originally structures whose existence was only conjectured: It was in their important study of conformal field theory that Moore and Seiberg ([MS1], [MS2]) first discovered a set of polynomial equations from a suitable axiom system for a “rational conformal field theory.” Inspired by a comment of Witten, they observed an analogy between the theory of these polynomial equations and the theory of tensor categories. The structures given by these Moore-Seiberg equations were called “modular tensor categories” by I. Frenkel. However, in the work of Moore and Seiberg, as they commented, neither tensor product structure nor other related structures were either formulated or constructed mathematically. Later, Turaev formulated a precise notion of modular tensor category in [T1] and [T2] and gave examples of such tensor categories from representations of quantum groups at roots of unity, based on results obtained by many people on quantum groups and their representations, especially those in the pioneering work [ReT1] and [ReT2] of Reshetikhin and Turaev on the construction of knot and 3-manifold invariants from representations of quantum groups. On the other hand, on the rational conformal field theory side, this mathematical formulation of the notion of modular tensor category led to a precise conjecture that the category generated by the integrable highest weight modules of a fixed integral positive integral level for an affine Lie algebra, and much more generally, certain module categories for chiral algebras associated with rational conformal field theories, could be endowed with modular tensor category structure. This conjecture was believed and even assumed, but prematurely so, to be true by physicists and even mathematicians, and its proof in fact took mathematicians many years and a great deal of effort.

The general vertex-algebraic tensor category theory developed by the authors in [HL1], [HL3]–[HL5], [HL2] and [HL7] and by the first author in [H1] and [H4] gave a construction of braided tensor category structure and, more importantly, as we have mentioned, vertex tensor category structure, on the category of modules for a vertex operator algebra satisfying suitable finiteness and reductivity conditions. In [H5], the first author proved the modular invariance for compositions of an arbitrary number of intertwining maps for a vertex operator algebra satisfying stronger finiteness and reductivity conditions. (Interestingly, Zhu’s methods in his pioneering work [Zhu] on modular invariance unfortunately could not be used or adapted to handle the necessary general case of compositions of two or more intertwining maps, essentially because intertwining operators do not satisfy a commutator formula, and so new, analytic, ideas had to be introduced for the solution of this problem in [H5].) Using this modular invariance (more precisely, the modular invariance for compositions of two intertwining maps), the first author proved the Moore-Seiberg equations for suitable representations of vertex operator algebras in [H7], at the same time providing in particular a much stronger version of the Verlinde formula relating the fusion rules, modular transformations, and braiding and fusing matrices than had been previously considered. Using these constructions and results, the first author proved the following rigidity and modularity result in [H8] (see also [H6]; cf. [L]):

Theorem 3.1 *Let V be a simple vertex operator algebra satisfying the conditions:*

1. V is of positive energy ($V_{(0)} = \mathbb{C}\mathbf{1}$ and $V_{(n)} = 0$ for $n < 0$) and the contragredient V' , as a V -module, is equivalent to V .
2. Every \mathbb{N} -gradable weak V -module is a direct sum of irreducible V -modules. (In fact, the results proved in [H9] imply that this condition can be weakened to the condition that every grading-restricted generalized V -module is a direct sum of irreducible V -modules.)
3. V is C_2 -cofinite (the quotient space $V/C_2(V)$ is finite dimensional, where $C_2(V)$ is the subspace of V spanned by the elements of the form $\text{Res}_z z^{-2}Y(u, z)v$ for $u, v \in V$).

Then the category of V -modules has a natural structure of rigid and in fact modular tensor category.

The following families of vertex operator algebras satisfy the three conditions above and thus by Theorem 3.1, the category of modules for each such vertex operator algebra has a natural structure of (rigid and) modular tensor category:

1. The vertex operator algebras V_L associated with positive definite even lattices L ; see [B] and [FLM2] for these vertex operator algebras and see [D1], [DL] and Section 12 of [DLM] for the conditions needed for invoking Theorem 3.1 above.
2. The vertex operator algebras $L(k, 0)$ associated with affine Lie algebras and positive integral levels k ; see [FZ] for these vertex operator algebras and [FZ], [HL6] and Section 12 of [DLM] for the conditions. These structures correspond to the Wess-Zumino-Novikov-Witten models.
3. The “minimal series” of vertex operator algebras associated with the Virasoro algebra; see [FZ] for these vertex operator algebras and [W], [H2] and Section 12 of [DLM] for the conditions.
4. Frenkel, Lepowsky and Meurman’s moonshine module V^\natural ; see [FLM1], [B] and [FLM2] for this vertex operator algebra and [D2] and Section 12 of [DLM] for the conditions.
5. The fixed-point vertex operator subalgebra of V^\natural under the standard involution; see [FLM1] and [FLM2] for this vertex operator algebra and [D2], [H3] and Section 12 of [DLM] for the conditions.

In addition, the following family of vertex operator superalgebras satisfies the conditions needed to apply the tensor category theory developed by the authors in the series of papers [HL1], [HL3]–[HL5], [HL2] and [HL7], and thus the category of modules for such a vertex operator superalgebra has a natural structure of braided tensor category:

6. The “minimal series” of vertex operator superalgebras (suitably generalized vertex operator algebras) associated with the Neveu-Schwarz superalgebra and also the “unitary series” of vertex operator superalgebras associated with the $N = 2$ superconformal algebra; see [KW] and [Ad2] for the corresponding $N = 1$ and $N = 2$ vertex operator superalgebras, respectively, and [Ad1], [Ad3], [HM1] and [HM2] for the conditions.

It is also expected that they satisfy the three conditions in the theorem above and thus it is expected that the category of modules for such a vertex operator superalgebra has a natural structure of modular tensor category.

In the special case of the second family of vertex operator algebras listed above, those corresponding to the Wess-Zumino-Novikov-Witten models, many mathematicians have believed for a long time (at least twenty years) that these (rigid and) modular tensor categories must have been constructed either by using the works of Tsuchiya-Ueno-Yamada [TUY], and/or Beilinson-Feigin-Mazur [BFM] and Bakalov-Kirillov [BK], or by using the works of Kazhdan-Lusztig [KL1]–[KL5] and Finkelberg [Fi1]–[Fi2]. In particular, the Verlinde formula conjectured by Verlinde in [V] would have been an easy consequence of such a construction, had it indeed been achieved.

But unfortunately, it turns out that this belief has recently been shown to be wrong. First, it has now been known, and acknowledged, for a while that, despite a statement in the book [BK] of Bakalov-Kirillov, the algebro-geometric methods in the works of Tsuchiya-Ueno-Yamada [TUY], Beilinson-Feigin-Mazur [BFM] and Bakalov-Kirillov [BK] cannot in fact be used to prove the rigidity of such a tensor category or to identify the S -matrix for such a tensor category with the modular transformation associated to $\tau \mapsto -1/\tau$ on the space spanned by the “characters” of the irreducible modules for such a vertex operator algebra; we shall explain these interesting issues below.

Second, most recently, it has been discovered by the first author, and graciously acknowledged by Finkelberg in [Fi3], that the works of Kazhdan-Lusztig [KL1]–[KL5] and Finkelberg [Fi1]–[Fi2] alone did not prove the rigidity of these tensor categories and thus also did not identify these S -matrices; for such a proof and for such an identification, one in fact needs results proved using different methods, as we discuss. In the course of his argument in [Fi2] that the categories based on modules for an affine Lie algebra at positive integral levels could be embedded as subquotients of Kazhdan-Lusztig’s rigid braided tensor categories at negative levels, Finkelberg proved that the elements of a certain space are proportional to elements of a certain other space. But it was not proved in [Fi2], and it is not possible to use the methods in [Fi1] or [Fi2] to prove, that these proportionality constants are nonzero, so that one could not in fact conclude that these two spaces are isomorphic, which had been the key step in [Fi2]. This subtle issue reminded the first author that his proof of rigidity in [H8] also amounted to a proof that certain proportionality constants are nonzero; the proof in [H8] needed the strong version of the Verlinde formula proved in [H7] involving fusion rules, modular transformations and braiding and fusing matrices.

After the first author pointed out and simultaneously corrected the error by invoking either (i) his general theorem in [H7] that had proved the Verlinde formula or, as an alternative, (ii) his general theorems in [H8] that had established rigidity and identified the S -matrices, Finkelberg gave, in [Fi3], still another alternative correction, using the Verlinde formula proved by Faltings [Fa] (for many but not all classes of simple Lie algebras) and by Teleman [Te] (for all classes of simple Lie algebras).

However, in the cases (i) E_6 level 1 (that is, $k = 1$), (ii) E_7 level 1, and (iii) E_8 levels 1 and 2, even the Verlinde formula proved by Faltings and Teleman does not help because, as

has always been known, the works of Kazhdan-Lusztig and Finkelberg simply do not apply to these excluded cases (and were never claimed to apply to these cases). In these cases, especially in the deep case E_8 level 2, the only proof of the rigidity and the only identification of the S -matrices mentioned above were given by the first author in [H8], using, as we have mentioned, (i) the general vertex-algebraic tensor category theory constructed by the authors in the papers [HL1], [HL3]–[HL5], [HL2] and [HL7] together with [H1] and [H4] by the first author, and (ii) the general vertex-algebraic theorems on modular invariance for compositions of intertwining maps in [H5] and on the Verlinde conjecture in [H7], by the first author. Note that our theory applies, in particular, to all the five classes of vertex operator algebras mentioned above. Most significantly, the theory is vertex-algebraically conceptual and general (although, necessarily, very elaborate) and does not exclude any individual cases (such as for instance E_8 level 2 among the Wess-Zumino-Novikov-Witten models). In [KL1]–[KL5], some of the deep properties of the constructed rigid braided tensor categories needed certain representation theory for affine Lie algebras, including the Knizhnik-Zamolodchikov equations, and followed from corresponding properties of categories of quantum group modules for the rigidity in particular, while the present theory is intrinsically vertex-algebraic; as we mentioned above, the only role that affine Lie algebras play is in the verification of the hypotheses for applying the general theory.

From these discussions, we can see why the works of Tsuchiya-Ueno-Yamada [TUY], Beilinson-Feigin-Mazur [BFM] and Bakalov-Kirillov [BK] cannot in fact be used to prove the rigidity or to identify the S -matrix. We discuss only the rigidity here; the discussion for the identification of the S -matrix is similar. Substantial works using different ideas, methods and results were needed in the proofs of the rigidity in both the work of Finkelberg [Fi1]–[Fi3] and the work of the first author [H8]. The work [Fi1]–[Fi3] was based on the whole theory of Kazhdan-Lusztig [KL1]–[KL5] on the equivalence of the braided tensor categories associated to quantum groups and to affine Lie algebras at negative levels. The work [H8] needed the formula (4.29) in [H7], which in turn needed the modular invariance for compositions of two intertwining maps proved in [H5]. Neither the equivalence of Kazhdan-Lusztig nor the modular invariance of the first author for compositions of intertwining maps can be proved using only the works of Tsuchiya-Ueno-Yamada [TUY], Beilinson-Feigin-Mazur [BFM] and Bakalov-Kirillov [BK]. The work [HK] by Kong and the first author provided another piece of evidence. From [H8] and [HK], it is easy to see that the rigidity is in fact equivalent to the nondegeneracy of suitable bilinear forms on spaces of intertwining operators among the objects of the category, or equivalently, on suitable spaces of morphisms between objects of the category. The proof of this nondegeneracy in [HK] needed a formula ((4.9) in [H7]) obtained in the process of proving the Verlinde formula in [H7]. In particular, the modular invariance proved in [H5] was needed. On the other hand, from the works of Tsuchiya-Ueno-Yamada [TUY], Beilinson-Feigin-Mazur [BFM] and Bakalov-Kirillov [BK], bilinear forms on suitable spaces of morphisms can still be constructed but the nondegeneracy of these forms cannot be proved without additional input. In fact, it has been known that these works gave only the weak rigidity of the braided tensor category.

In [H10], it was pointed out that while the *statement* of rigidity in fact involves only

genus-*zero* conformal field theory, the *proof* of rigidity in [H8] needs genus-*one* conformal field theory (the modular invariance for compositions of intertwining maps in [H8]), and that correspondingly, there must be something deep going on here. The recent discovery that the works of Kazhdan-Lusztig and Finkelberg also require knowledge of the Verlinde formula in order to prove the rigidity in the case of affine Lie algebras enhances this observation in [H10].

4 Braided tensor categories and logarithmic conformal field theories

The semisimplicity of the module categories needed in the preceding section is related to another property of these modules, namely, that each module is a direct sum of its “weight spaces,” which are the eigenspaces of the familiar operator $L(0)$ coming from the Virasoro algebra action on the module. But there are important situations in which module categories are not semisimple and in which modules are not direct sums of their weight spaces. The tensor categories in this case are intimately related to logarithmic conformal field theories in physics. In this section, we discuss the background and the results in this case corresponding to logarithmic conformal field theories.

For the vertex operator algebras $L(k, 0)$ associated with affine Lie algebras, when the sum of k and the dual Coxeter number of the corresponding simple Lie algebra is not a non-negative rational number, the vertex operator algebra $L(k, 0)$ is not finitely reductive, and, working with Lie algebra theory rather than with vertex operator algebra theory, Kazhdan and Lusztig constructed a natural braided tensor category structure on a certain category of modules of level k for the affine Lie algebra in [KL1]–[KL5]. This work of Kazhdan-Lusztig in fact motivated the authors to develop an analogous theory for vertex operator algebras rather than for affine Lie algebras, as was explained in detail in the introductory material in [HL1], [HL2], [HL3], [HL4] and [HL6]. However, this general theory, in its original form, did not apply to Kazhdan-Lusztig’s context, because the vertex operator algebra modules considered in [HL1], [HL2], [HL3], [HL4], [HL5], [H1], [HL7], [H4] are assumed to be the direct sums of their weight spaces (with respect to $L(0)$), and the non-semisimple modules considered by Kazhdan-Lusztig are not in general the direct sums of their weight spaces. Although their setup, based on Lie theory, and ours, based on vertex operator algebra theory, are very different, we expected to be able to recover (and further extend) their results through our vertex operator algebraic approach, which is very general, as we discussed above. This motivated us, jointly with Zhang, in the work [HLZ0]–[HLZ8], to generalize the earlier work of the authors by considering modules with *generalized* weight spaces, and especially, intertwining operators associated with such generalized kinds of modules. As we discuss below, this required us to use *logarithmic* intertwining operators, and we have been able to construct braided tensor category structure, and even vertex tensor category structure, on important module categories that are not semisimple. Using this theory, Zhang ([Zha1], [Zha2]) has indeed recovered the braided tensor category structure of Kazhdan-Lusztig, and

has also extended it to vertex tensor category structure. While in our theory, logarithmic structure plays a fundamental role, logarithmic structure did not show up explicitly in the work of Kazhdan-Lusztig. As we mentioned above, the Kazhdan-Lusztig work used properties of categories of quantum group modules for the rigidity. The work [HLZ0]–[HLZ8] and [Zha1], [Zha2] does not prove rigidity.

In [Mil1], Milas introduced and studied what he called “logarithmic modules” and “logarithmic intertwining operators”; see also [Mil2]. Roughly speaking, logarithmic modules are weak modules for a vertex operator algebra that are direct sums of generalized eigenspaces for the operator $L(0)$. Such weak modules are called “generalized modules” in [HLZ0]–[HLZ8]. Logarithmic intertwining operators are operators that depend not only on (general) powers of a variable z , but also on its logarithm $\log z$.

From the viewpoint of the general representation theory of vertex operator algebras, it would be unnatural to study only semisimple modules or only $L(0)$ -semisimple modules; focusing, artificially, on only such modules would be analogous to focusing only on semisimple modules for general (not necessarily semisimple) finite-dimensional Lie algebras. And as we have pointed out, working in this generality leads to logarithmic structure; thus the general representation theory of vertex operator algebras requires logarithmic structure.

Logarithmic structure in conformal field theory was in fact first introduced by physicists to describe Wess-Zumino-Novikov-Witten models on supergroups ([RoS], [SS]) and disorder phenomena [Gu]. A great deal of progress has been made on this subject. Our paper [HLZ1] includes a discussion of the literature. Here we would like to mention, in particular, [FGST1] for the modular group representations relevant to logarithmic triplet models, [FHST], [GR] and [Ra2] for Verlinde-like formulas, and [GT] for analogues of pseudo-trace functions. One particularly interesting class of logarithmic conformal field theories is the class associated to the triplet \mathcal{W} -algebras $\mathcal{W}_{1,p}$ of central charge $1 - 6\frac{(p-1)^2}{p}$, $p = 2, 3, \dots$, which we shall discuss.

Here is how such logarithmic structure also arises naturally in the representation theory of vertex operator algebras: In the construction of intertwining operator algebras, the first author proved (see [H4]) that if modules for the vertex operator algebra satisfy a certain cofiniteness condition, then products of the usual intertwining operators satisfy certain systems of differential equations with regular singular points. In addition, it was proved in [H4] that if the vertex operator algebra satisfies certain finite reductivity conditions, then the analytic extensions of products of the usual intertwining operators have no logarithmic terms. In the case when the vertex operator algebra satisfies only the cofiniteness condition but not the finite reductivity conditions, the products of intertwining operators still satisfy systems of differential equations with regular singular points, but in this case, the analytic extensions of such products of intertwining operators might indeed have logarithmic terms. This means that if we want to generalize the results in [HL1], [HL2]–[HL5], [H1] and [H4] to the case in which the finite reductivity properties are not always satisfied, we have to consider intertwining operators involving logarithmic terms.

Logarithmic structure also appears naturally in modular invariance results for vertex operator algebras and in the genus-one parts of conformal field theories. In [H5], for vertex operator algebras satisfying the conditions in Theorem 3.1, by deriving certain differential

equations and using the duality properties for intertwining operators, the first author was able to prove the modular invariance for q -traces of products and iterates of intertwining maps. (As we mentioned above, the methods here for the hard part of the argument had to be different from those in [Zhu].) If the vertex operator algebra is of positive energy and is C_2 -cofinite (see Conditions 1 and 3 in Theorem 3.1) but does not necessarily satisfy Condition 2 in Theorem 3.1, suitable “pseudo q -traces” (as introduced by Miyamoto in [Miy]) of products and iterates of intertwining operators still satisfy the same differential equations, but now they involve the logarithm of q . To generalize the Verlinde conjecture proved in [H7] and the modular tensor category structure obtained in [H8] on the category of V -modules, one will need the general logarithmic modular invariance of such pseudo q -traces of products and iterates of intertwining maps.

The work [HLZ0]–[HLZ8] constructed a braided tensor category structure on a suitable category of generalized modules for a vertex operator algebra (or more generally a conformal vertex algebra or a Möbius vertex algebra) under a number of natural assumptions. In [H9], by verifying the assumptions in the papers [HLZ0]–[HLZ8], the first author proved the following result:

Theorem 4.1 *Let V be a C_2 -cofinite vertex operator algebra of positive energy. Then the category of grading-restricted generalized V -modules has a natural structure of braided tensor category.*

The main work in [H9] was in proving that the category is closed under the tensor product operation and that every finitely-generated lower-bounded generalized V -module is grading restricted. The following conjecture on rigidity was made in [H10]:

Conjecture 4.2 *Let V be a simple vertex operator algebra satisfying Conditions 1 and 3 in Theorem 3.1. Then the braided tensor category given in Theorem 4.1 is rigid.*

Triplet \mathcal{W} -algebras $\mathcal{W}_{1,p}$, mentioned above, are a class of vertex operator algebras of central charge $1 - 6\frac{(p-1)^2}{p}$, $p = 2, 3, \dots$ which have attracted a lot of attention from physicists and mathematicians. These algebras were introduced by Kausch [K] and have been studied extensively by physicists and mathematicians. See the introduction of [HLZ1] for more references on the representation theory of these algebras. Such a triplet \mathcal{W} -algebra satisfies the positive energy condition and the C_2 -cofiniteness condition: The C_2 -cofiniteness condition was proved by Abe [Ab] in the simplest case $p = 2$ and by Carqueville-Flohr [CF] and Adamović-Milas [AM1] in the general case. Thus, as a corollary of Theorem 4.1, we have:

Theorem 4.3 *The category of grading-restricted generalized V -modules for a triplet \mathcal{W} -algebra $\mathcal{W}_{1,p}$ has a natural structure of braided tensor category.*

See Tsuchiya-Wood [TW] for the rigidity of the braided tensor category in Theorem 4.3 (which was constructed, along with the logarithmic operator product expansion, in [HLZ0]–[HLZ8] together with [H9], [Ab] (for $p = 2$), [CF] and [AM1]).

In general, the braided tensor category in Theorem 4.1 might not be rigid. An example of such a nonrigid braided tensor category is given by the triplet \mathcal{W} -algebra $\mathcal{W}_{2,3}$ of central charge 0 (see [FGST2], [Ra1], [GRW] and [AM2]).

In addition to these logarithmic issues, another way in which the present work generalizes our earlier tensor category theory for module categories for a vertex operator algebra is that we now allow the algebras to be somewhat more general than vertex operator algebras, in order, for example, to accommodate module categories for the vertex algebras V_L where L is a nondegenerate even lattice that is not necessarily positive definite (cf. [B], [DL]); see [Zha1].

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