On the concepts of intertwining operator and tensor product module in vertex operator algebra theory

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Abstract

We produce counterexamples to show that in the definition of the notion of intertwining operator for modules for a vertex operator algebra, the commutator formula cannot in general be used as a replacement axiom for the Jacobi identity. We further give a sufficient condition for the commutator formula to imply the Jacobi identity in this definition. Using these results we illuminate the crucial role of the condition called the "compatibility condition" in the construction of the tensor product module in vertex operator algebra theory, as carried out in work of Huang and Lepowsky. In particular, we prove by means of suitable counterexamples that the compatibility condition was indeed needed in this theory.

1 Introduction

With motivation from both mathematics and physics, a tensor product theory of modules for a vertex operator algebra was developed in [HL1]–[HL3] and [H2]. This theory has had a number of applications and has been generalized to additional important settings. The main purpose of this paper is to study and elucidate certain subtle aspects of this theory and of the fundamental notion of intertwining operator in vertex operator algebra theory. In particular, we answer a number of questions that have arisen in the theory.

A central theme in vertex (operator) algebra theory is that the theory cannot be reduced to Lie algebra theory, even though the theory is, and always has been, intimately related to Lie algebra theory. Before we discuss the notions of intertwining operator and tensor product module in vertex operator algebra theory, we first recall that for the three basic notions of vertex (operator) algebra, of module and of intertwining operator, there is a uniform main axiom: the Jacobi identity; see [FLM] and [FHL]. For the notion of vertex (operator) algebra itself, the "commutator formula" or "commutativity" or "weak commutativity" ("locality") can alternatively be taken to be the main axiom; see [FLM], [FHL], [DL], [Li3]; cf. [LL]. (Borcherds's original definition of the notion of vertex algebra [B] used "skew-symmetry" and the "associator formula"; cf. [LL].) By contrast, for the notion of module for a vertex (operator) algebra, the "associator formula" or "associativity" or "weak associativity" can be used in place of the Jacobi identity as the main axiom; see [B], [FLM], [FHL], [DL], [Li3]; cf. [LL]. And furthermore, in the definition of this notion of module, the commutator formula (or commutativity or weak commutativity or locality) *cannot* be taken as the main replacement axiom. This is actually an easy observation, and in an Appendix of the present paper we give examples to verify and illustrate this fact. This already illustrates how vertex operator algebra theory cannot be reduced to Lie algebra theory.

In the main text of the present paper, we discuss and analyze the extent to which the Jacobi identity can be replaced by the "commutator formula" in the definition of the notion of intertwining operator among modules for a vertex (operator) algebra. This will in particular explain the crucial nature of the "compatibility condition" in [HL1]–[HL3]. (We shall recall in Section 2 below the precise meaning of the term "commutator formula" in the context of intertwining operators.)

In this paper we assume the reader is familiar with the basic concepts in the theory of vertex operator algebras, including modules and intertwining operators; we shall use the theory as developed in [FLM], [FHL] and [LL], and the terminology and notation of these works.

In the tensor product theory of modules for a vertex operator algebra (see [HL1]–[HL3]), the tensor product functor depends on an element of a certain moduli space of three-punctured spheres with local coordinates at the punctures. In this paper we shall focus on the important moduli space element denoted $P(z)$ in [H1] and [HL1], where z is a nonzero complex number. The corresponding tensor product functor is denoted $W_1 \boxtimes_{P(z)} W_2$ for modules W_1 and W_2 for a suitable vertex operator algebra V. This tensor product module $W_1 \boxtimes_{P(z)} W_2$ can be constructed by means of its contragredient module, which in turn can be realized as a certain subspace $W_1 \boxbackslash_{P(z)} W_2$ of $(W_1 \otimes W_2)^*$ (the dual vector space of the vector space tensor product $W_1 \otimes W_2$ of W_1 and W_2). The elements of $W_1 \boxbackslash_{P(z)} W_2$ satisfy the "lower truncation condition" and the "$P(z)$-compatibility condition" defined and discussed in [HL1]–[HL3]. It was proved in [HL1]–[HL3] that these two conditions together imply the Jacobi identity, and hence that any element of $(W_1 \otimes W_2)^*$ satisfying these two conditions generates a weak module for the vertex operator algebra V. We show here that the converse of this statement is not true in general; specifically, an element of $(W_1 \otimes W_2)^*$ generating a weak V-module does not need to satisfy the compatibility condition. It follows in particular that the largest weak V-module in $(W_1 \otimes W_2)^*$, which we shall write as $W_1 \boxed{\mathbf{z}}\, W_2$ and shall read as "W_1 warning W_2," can indeed be (strictly) larger than the desired space, $W_1 \boxbackslash_{P(z)} W_2$. In particular, for each of the examples, or really counterexamples, that we give, we will see that when the modules W_1 and W_2 are taken to be V itself, neglecting the compatibility condition results in a V-module $V \boxed{\mathbf{z}}\, V$ whose contragredient module is indeed (strictly) larger (in the sense of homogeneous subspaces) than the correct tensor product $V \boxtimes_{P(z)} V$, which is naturally isomorphic to V itself. This of course shows that the compatibility condition cannot in general be omitted.

In Section 2 of this paper we will show that the compatibility condition for elements of $(W_1 \otimes W_2)^*$ reflects in a precise way the Jacobi identity for intertwining operators and intertwining maps, while on the other hand, the *Jacobi identity* (which, as we have been discussing, is implied by the compatibility condition) for elements of $(W_1 \otimes W_2)^*$ reflects in a precise way the *commutator formula* for intertwining operators and intertwining maps.

Thus we have the natural question (which we already mentioned above): In the notion of intertwining operator, can the commutator formula be used as a replacement axiom for the Jacobi identity? In other words, does the Jacobi identity imply the compatibility condition in $(W_1 \otimes W_2)^*$? As one should expect, the answer is no. We shall correspondingly call a "quasi-intertwining operator" an operator satisfying all the conditions for an intertwining operator except that the Jacobi identity in the definition is replaced by the commutator formula. We shall exhibit a straightforward counterexample (a quasi-intertwining operator that is not an intertwining operator) when the vertex algebra is constructed from a commutative associative algebra with identity. However, when the vertex algebra has a conformal vector and nonzero central charge (for instance, when the vertex algebra is a vertex operator algebra with nonzero central charge), we will see that the answer is instead yes—the commutator formula indeed implies the Jacobi identity in this case. We establish this and related results and construct relevant counterexamples in Sections 3 and 4.

As we also show, all these results actually hold in the presence of logarithmic variables, when the modules involved are only direct sums of generalized $L(0)$-eigenspaces, instead of $L(0)$-eigenspaces (see [Mi], [HLZ1], [HLZ2] for these notions).

In the Appendix we show that unless a vertex (operator) algebra is one-dimensional, there exists a non-module that satisfies all the axioms for a module except that the Jacobi identity is replaced by the commutator formula.

We would like to add a few more words concerning why one should expect that for the notion of intertwining operator, the commutator formula does not imply the Jacobi identity (and consequently, the Jacobi identity does not imply the compatibility condition). Consider the elementary situation in which a vertex algebra V is based on a commutative associative algebra. A V-module is exactly the same as a module for the underlying commutative associative algebra, since the notion of V-module can be described via associativity. In particular, if $\dim V > 1$, a V-module is not in general the same as a module for V viewed as a commutative Lie algebra (since this would amount to a vector space of commuting operators acting on the module); commutativity cannot be used as a replacement axiom in the definition of the notion of module. Thus in this situation, the notion of V-module is essentially ring-theoretic and not Lie-algebra-theoretic, while the notion of quasi-intertwining operator, based as it is on the commutator formula, is essentially Lie-algebra-theoretic and not ring-theoretic. These considerations motivated our (straightforward) construction of examples showing that the commutator formula does not imply the Jacobi identity in the definition of the notion of intertwining operator.

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2 Quasi-intertwining operators and the compatibility condition

Throughout this section we let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra (in the precise sense of [FHL], [FLM], [LL] or [HL1]–[HL3]). (Recall that $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ is the underlying $\mathbb{Z}$-graded vector space, Y is the vertex operator map, $\mathbf{1}$ is the vacuum vector, ω is the conformal vector, and $Y(\omega, x) = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}$.) Let z be a fixed nonzero complex number. In this section we first define the notion of quasi-intertwining operator and quasi-($P(z)$-)intertwining map, generalizing (and weakening) the notions of intertwining operator (see [FHL]) and of ($P(z)$-)intertwining map (see Section 4 of [HL1]). We establish the correspondence between these two notions, similar to the correspondence between intertwining operators and intertwining maps. We then show that for V-modules W_1 and W_2, a quasi-$P(z)$-intertwining map of type $\binom{W_3}{W_1\,W_2}$ gives a weak V-module inside $(W_1 \otimes W_2)^*$. The notion of logarithmic quasi-intertwining operator is also defined in this section.

The notion of quasi-intertwining operator is defined in the same way as the notion of intertwining operator except that the Jacobi identity is replaced by the "commutator formula." We now in fact give the definition in the greater generality of weak V-modules; a *weak module* for our vertex operator algebra V is a module for V viewed as a vertex algebra, in the sense of Definition 4.1.1 in [LL].

Definition 2.1 Let (W_1, Y_1), (W_2, Y_2) and (W_3, Y_3) be weak V-modules. A *quasi-intertwining operator of type* $\binom{W_3}{W_1\,W_2}$ is a linear map $\mathcal{Y} : W_1 \otimes W_2 \to W_3\{x\}$ (the space of formal series in complex powers of x with coefficients in W_3), or equivalently,

$$\begin{array}{rcl} W_1 & \to & (\mathrm{Hom}\,(W_2, W_3))\{x\} \\ w & \mapsto & \mathcal{Y}(w, x) = \displaystyle\sum_{n\in\mathbb{C}} w_n x^{-n-1} \quad (\text{where } \ w_n \in \mathrm{Hom}\,(W_2, W_3)) \end{array}$$

such that for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, we have the lower truncation condition

$$(w_{(1)})_n w_{(2)} = 0 \ \text{ for } \ n \ \text{ whose real part is sufficiently large;}$$

the "commutator formula"

$$\begin{aligned} &Y_3(v, x_1)\mathcal{Y}(w_{(1)}, x_2)w_{(2)} - \mathcal{Y}(w_{(1)}, x_2)Y_2(v, x_1)w_{(2)} \\ &\qquad = \mathrm{Res}_{x_0} x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\mathcal{Y}(Y_1(v, x_0)w_{(1)}, x_2)w_{(2)}; \end{aligned} \tag{2.1}$$

and the $L(-1)$-derivative property

$$\frac{d}{dx}\mathcal{Y}(w_{(1)}, x) = \mathcal{Y}(L(-1)w_{(1)}, x), \tag{2.2}$$

where $L(-1)$ is the operator acting on W_1.

Remark 2.2 For the notions of vertex (operator) algebra and module for a vertex (operator) algebra, the term "commutator formula" has the intuitive meaning—it is a formula for the commutator of two operators (acting on the same space). In the context of intertwining operators, even though the similar formula, (2.1), does not involve a commutator of two operators acting on the same space, we still call it the "commutator formula."

Clearly, a quasi-intertwining operator $\mathcal{Y}$ is an intertwining operator if and only if it further satisfies the Jacobi identity

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) & Y_3(v,x_1)\mathcal{Y}(w_{(1)},x_2)w_{(2)} \\ & -x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\mathcal{Y}(w_{(1)},x_2)Y_2(v,x_1)w_{(2)} \\ & = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}(Y_1(v,x_0)w_{(1)},x_2)w_{(2)} \end{aligned} \tag{2.3}$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$; (2.1) of course follows from (2.3) by taking Res_{x_0}. It is clear that the quasi-intertwining operators of the same type form a vector space containing the space of intertwining operators as a subspace.

Recall from [Mi] (see also [HLZ1], [HLZ2]) the notion of logarithmic intertwining operator:

Definition 2.3 Let W_1, W_2, W_3 be weak modules for a vertex operator algebra V. A *logarithmic intertwining operator of type* $\binom{W_3}{W_1\, W_2}$ is a linear map

$$\mathcal{Y}(\cdot,x)\cdot : W_1 \otimes W_2 \to W_3\{x\}[\log x],$$

or equivalently,

$$w_{(1)} \otimes w_{(2)} \mapsto \mathcal{Y}(w_{(1)},x)w_{(2)} = \sum_{n\in\mathbb{C}}\sum_{k\in\mathbb{N}} {w_{(1)}}_{n;\,k}^{\mathcal{Y}} w_{(2)} x^{-n-1}(\log x)^k \in W_3\{x\}[\log x]$$

for all $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, such that the following conditions are satisfied: the lower truncation condition: for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $k \in \mathbb{N}$,

$${w_{(1)}}_{n;\,k}^{\mathcal{Y}} w_{(2)} = 0 \quad \text{for} \;\; n \;\; \text{whose real part is sufficiently large;}$$

the Jacobi identity (2.3) for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$; and the $L(-1)$-derivative property (2.2) for any $w_{(1)} \in W_1$.

Remark 2.4 The notion of logarithmic intertwining operator defined in [HLZ1] and [HLZ2] is slightly more general than this one. In this paper, for brevity we adopt the original definition from [Mi] instead.

By analogy with the notion of quasi-intertwining operator, we can define the notion of logarithmic quasi-intertwining operator, as follows:

Definition 2.5 Let W_1, W_2, W_3 be weak modules for a vertex operator algebra V. A *logarithmic quasi-intertwining operator of type* $\binom{W_3}{W_1\, W_2}$ is a linear map

$$\mathcal{Y}(\cdot, x)\cdot : W_1 \otimes W_2 \to W_3\{x\}[\log x]$$

that satisfies all the axioms in the definition of logarithmic intertwining operator in Definition 2.3 except that the Jacobi identity is replaced by the commutator formula (2.1).

From now on, unless otherwise stated, (W_1, Y_1), (W_2, Y_2) and (W_3, Y_3) are assumed to be *generalized* V*-modules* in the sense of [HLZ1] and [HLZ2], that is, weak V-modules satisfying all the axioms in the definition of the notion of V-module (see [FHL], [FLM], [LL] or [HL1]) except that the underlying vector spaces are allowed to be direct sums of generalized eigenspaces, not just eigenspaces, of the operator $L(0)$; in particular, the $L(0)$-generalized eigenspaces are finite dimensional. We refer the reader to [HLZ1] and [HLZ2] for basic notions related to generalized V-modules. In particular, we have the notions of algebraic completion and of contragredient module for a generalized V-module.

In parallel to the notion of quasi-intertwining operator, we have:

Definition 2.6 A *quasi-*$P(z)$*-intertwining map of type* $\binom{W_3}{W_1 W_2}$ is a linear map $F : W_1 \otimes W_2 \to \overline{W}_3$ (the algebraic completion of W_3 with respect to the grading by weights) satisfying the condition

$$\begin{aligned} Y_3(v, x_1)F(w_{(1)} \otimes w_{(2)}) - F(w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \\ = \mathrm{Res}_{x_0} z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) F(Y_1(v, x_0)w_{(1)} \otimes w_{(2)}) \end{aligned} \tag{2.4}$$

for $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$.

Note that the left-hand side of (2.4) is well defined, by the same argument as was used for the left-hand side of formula (4.2) in [HL1]; that argument indeed remains valid for generalized modules.

Clearly, a quasi-$P(z)$-intertwining map $\mathcal{Y}$ of type $\binom{W_3}{W_1\, W_2}$ is a $P(z)$-intertwining map if and only if it further satisfies the Jacobi identity

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - z}{x_0}\right) Y_3(v, x_1)F(w_{(1)} \otimes w_{(2)}) = \\ = z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) F(Y_1(v, x_0)w_{(1)} \otimes w_{(2)}) \\ + x_0^{-1}\delta\left(\frac{z - x_1}{-x_0}\right) F(w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \end{aligned} \tag{2.5}$$

for $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$; (2.4) follows from (2.5) by taking Res_{x_0}. (It is important to keep in mind that the left-hand side of (2.5) is well defined, by the considerations at the beginning of Section 4 of [HL1]. Clearly, the quasi-$P(z)$-intertwining maps of the same type form a vector space containing the space of $P(z)$-intertwining maps as a subspace.

In case W_1, W_2 and W_3 are ordinary V-modules, given a fixed integer p, by analogy with the maps defined in (12.3) and (12.4) in [HL3], we have the following maps between the spaces of quasi-intertwining operators and of quasi-$P(z)$-intertwining maps of the same type: For a quasi-intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1\,W_2}$, define $F_{\mathcal{Y},p} : W_1 \otimes W_2 \to \overline{W}_3$ by

$$F_{\mathcal{Y},p}(w_{(1)} \otimes w_{(2)}) = \mathcal{Y}(w_{(1)}, e^{l_p(z)})w_{(2)}$$

for all $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, where we follow the notation

$$\begin{aligned} \log z &= \log |z| + i \arg z \ \text{ for } 0 \le \arg z < 2\pi, \\ l_p(z) &= \log z + 2\pi i p,\ p \in \mathbb{Z} \end{aligned}$$

in [HL1] for branches of the log function. On the other hand, let F be a quasi-$P(z)$-intertwining map of type $\binom{W_3}{W_1\,W_2}$. For homogeneous elements $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ and $n \in \mathbb{C}$, define $(w_{(1)})_n w_{(2)}$ to be the projection of $F(w_{(1)} \otimes w_{(2)})$ to the homogeneous subspace of W_3 of weight $\mathrm{wt}\, w_{(1)} - n - 1 + \mathrm{wt}\, w_{(2)}$ multiplied by $e^{(n+1)l_p(z)}$, and define

$$\mathcal{Y}_{F,p}(w_{(1)}, x)w_{(2)} = \sum_{n \in \mathbb{C}} (w_{(1)})_n w_{(2)} x^{-n-1};$$

then extend by linearity to define $\mathcal{Y}_{F,p} : W_1 \otimes W_2 \to W_3\{x\}$.

It was shown in Proposition 12.2 of [HL3] (see also Proposition 4.7 in [HL1]) that these two maps give linear isomorphisms between the space of intertwining operators and the space of $P(z)$-intertwining maps of the same type. By replacing all Jacobi identities by the corresponding commutator formulas in the proof, we see that these two maps also give linear isomorphisms between the space of quasi-intertwining operators and the space of quasi-$P(z)$-intertwining maps of the same type. (The straightforward argument is carried out in [HLZ2].) That is, we have:

Proposition 2.7 *Assume that W_1, W_2 and W_3 are ordinary V-modules. For $p \in \mathbb{Z}$, the correspondence $\mathcal{Y} \mapsto F_{\mathcal{Y},p}$ is a linear isomorphism from the space of quasi-intertwining operators of type $\binom{W_3}{W_1\,W_2}$ to the space of quasi-$P(z)$-intertwining maps of the same type. Its inverse map is given by $F \mapsto \mathcal{Y}_{F,p}$.* $\square$

More generally, if W_1, W_2 and W_3 are generalized (rather than ordinary) V-modules in the sense of [HLZ1] and [HLZ2], then following the argument in [HLZ2] we have a result similar to Proposition 2.7 giving the correspondence between the quasi-$P(z)$-intertwining maps and the logarithmic quasi-intertwining operators.

Here is an easy example of a quasi-$P(z)$-intertwining map that is not a $P(z)$-intertwining map:

Example 2.8 Take V to be the vertex operator algebra constructed from a finite-dimensional commutative associative algebra with identity, with the vertex operator map defined by $Y(a, x)b = ab$ for $a, b \in V$, with the vacuum vector $\mathbf{1}$ taken to be 1 and with $\omega = 0$. Then since the notion of module for a vertex algebra can be characterized in terms of an associativity property, the modules for V as a vertex operator algebra are precisely the finite-dimensional modules for V as an associative algebra (see [B]; cf. [LL]). For two V-modules W_1 and W_2, the vector space $W_1 \otimes W_2$ is a V-module under the action given by

$$Y(v, x)(w_{(1)} \otimes w_{(2)}) \ (= v \cdot (w_{(1)} \otimes w_{(2)})) = w_{(1)} \otimes (v \cdot w_{(2)}) \tag{2.6}$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. The identity map on $W_1 \otimes W_2$ is a quasi-$P(z)$-intertwining map of type $\binom{W_1 \otimes W_2}{W_1\, W_2}$ (formula (2.4) being just (2.6) itself). However it is not a $P(z)$-intertwining map because the Jacobi identity demands that

$$(v \cdot w_{(1)}) \otimes w_{(2)} = w_{(1)} \otimes (v \cdot w_{(2)})$$

for any $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, which of course is not true in general. See Remark 2.20 below for a further discussion of this (counter)example.

Remark 2.9 By the above, this example of course immediately gives a quasi-intertwining operator that is not an intertwining operator.

In the following we will sometimes use results from [HLZ1] and [HLZ2] for our generalized V-modules W_1, W_2 and W_3, but the reader may simply take W_1, W_2 and W_3 to be ordinary V-modules.

Just as in [HL1], formulas (3.4) and (3.5), set

$$Y_t(v, x) = v \otimes x^{-1}\delta\left(\frac{t}{x}\right)$$

for $v \in V$. Also, just as in [HL1], formula (3.20), for a generalized V-module (W, Y), write

$$Y^o(v, x) = Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})$$

for $v \in V$. (Here we are using the notation Y^o, as in [HLZ1] and [HLZ2], rather than the original notation Y^* of [HL1].) Also, let $\iota_+ : \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}] \to \mathbb{C}((t))$ (the space of formal Laurent series in t with only finitely many negative powers of t) be the natural map. As in formula (13.2) of [HL3], we define a linear action $\tau_{P(z)}$ of the space

$$V \otimes \iota_+\mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}] \tag{2.7}$$

on $(W_1 \otimes W_2)^*$ by

$$\begin{aligned}
\left(\tau_{P(z)}\left(x_0^{-1}\delta\left(\frac{x_1^{-1} - z}{x_0}\right) Y_t(v, x_1)\right) \lambda\right) (w_{(1)} \otimes w_{(2)}) = \\
= z^{-1}\delta\left(\frac{x_1^{-1} - x_0}{z}\right) \lambda(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
+ x_0^{-1}\delta\left(\frac{z - x_1^{-1}}{-x_0}\right) \lambda(w_{(1)} \otimes Y_2^o(v, x_1)w_{(2)})
\end{aligned} \tag{2.8}$$

for $v \in V$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. As in [HL3], this formula does indeed give a well-defined linear action (in generating-function form) of the space (2.7); see Section 3 of [HL1]. This action (2.8) restricts in particular to an action of $V \otimes \mathbb{C}[t, t^{-1}]$ on $(W_1 \otimes W_2)^*$, given in generating-function form by $\tau_{P(z)}(Y_t(v, x))$; one takes the residue with respect to x_0 of both sides of (2.8).

We will write $W_1 \boxed{Z}\, W_2$, which can be read "W_1 warning W_2", for the largest weak V-module inside $(W_1 \otimes W_2)^*$ with respect to the action $\tau_{P(z)}$ of $V \otimes \mathbb{C}[t, t^{-1}]$. (Here we omit the information about $P(z)$ from the notation.) It is clear that $W_1 \boxed{Z}\, W_2$ does exist and equals the sum (or union) of all weak V-modules inside $(W_1 \otimes W_2)^*$. Of course, all the elements of $W_1 \boxed{Z}\, W_2$ satisfy the lower truncation condition and the Jacobi identity with respect to the action $\tau_{P(z)}$. (Warning: This space $W_1 \boxed{Z}\, W_2$ can be strictly larger than the subspace $W_1 \boxbackslash_{P(z)} W_2$ of $(W_1 \otimes W_2)^*$ defined in formula (13.13) of [HL3], as we will show below.)

Denote by W' the contragredient module of a generalized V-module W, defined by exactly the same procedure as was carried out in [FHL] for ordinary (as opposed to generalized) modules. For a linear map F from $W_1 \otimes W_2$ to $\overline{W}_3$, define a linear map $F^\vee : W_3' \to (W_1 \otimes W_2)^*$ by

$$\langle F^\vee(\alpha), w_{(1)} \otimes w_{(2)} \rangle_{W_1 \otimes W_2} = \langle \alpha, F(w_{(1)} \otimes w_{(2)}) \rangle_{\overline{W}_3} \tag{2.9}$$

for $\alpha \in W_3'$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. (The subscripts of course designate the pairings; sometimes we will omit such subscripts.) For the case in which W_1, W_2 and W_3 are ordinary modules, it was observed in [HL3], Proposition 13.1 (see also [HL1], Proposition 5.3) that F is a $P(z)$-intertwining map of type $\binom{W_3}{W_1\, W_2}$ if and only if $F^\vee$ intertwines the two actions of the space (2.7) on W_3' (on which a monomial $v \otimes t^n$ acts as v_n and a general element acts according to the action of each of its monomials) and on $(W_1 \otimes W_2)^*$ (by formula (2.8)). The same observation still holds for the case of generalized modules (cf. [HLZ2]). For quasi-$P(z)$-intertwining maps, we shall prove:

Proposition 2.10 *The map F is a quasi-$P(z)$-intertwining map of type $\binom{W_3}{W_1\, W_2}$ if and only if $F^\vee$ intertwines the two actions of $V \otimes \mathbb{C}[t, t^{-1}]$ on W_3' and $(W_1 \otimes W_2)^*$.*

Remark 2.11 In the statement of Proposition 2.10 we have avoided saying that $F^\vee$ is a V-homomorphism since the target space, $(W_1 \otimes W_2)^*$, is rarely a (generalized) V-module.

Before giving the proof, we first write the action (2.8) in an alternative form, which will be more convenient in this paper, as follows:

$$\begin{aligned}\langle \tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) Y_t^o(v, x_1) \right) \lambda, w_{(1)} \otimes w_{(2)} \rangle = \\ = \langle \lambda, z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_1(v, x_0) w_{(1)} \otimes w_{(2)} \\ + x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) w_{(1)} \otimes Y_2(v, x_1) w_{(2)} \rangle\end{aligned} \tag{2.10}$$

for $v \in V$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, where $Y_t^o(v, x_1)$ is defined by

$$Y_t^o(v, x) = Y_t(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) = e^{xL(1)}(-x^{-2})^{L(0)}v \otimes x\delta\left(\frac{t}{x^{-1}}\right),$$

which in turn equals

$$(-1)^{\mathrm{wt}\, v} \sum_{m \in \mathbb{N}} \frac{1}{m!}(L(1)^m v) \otimes t^{-m-2+2\mathrm{wt}\, v} x^{-1}\delta\Big(\frac{t^{-1}}{x}\Big)$$

in case v is homogeneous, by formulas (3.25), (3.30), (3.32) and (3.38) of [HL1]. The equivalence of (2.8) and (2.10) can be seen by first replacing x_1 by x_1^{-1} and then replacing v by $e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v$ in either direction (recall Proposition 5.3.1 of [FHL]).

Proof of Proposition 2.10 For any linear map F from $W_1 \otimes W_2$ to $\overline{W}_3$, the condition that the map $F^\vee$ defined by (2.9) intertwines the two actions of $V \otimes \mathbb{C}[t, t^{-1}]$ on W_3' and $(W_1 \otimes W_2)^*$ means exactly that for any $\alpha \in W_3'$, $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$,

$$\langle F^\vee((Y_3')^o(v, x_1)\alpha), w_{(1)} \otimes w_{(2)}\rangle_{W_1 \otimes W_2} = \langle \tau_{P(z)}(Y_t^o(v, x_1))F^\vee(\alpha), w_{(1)} \otimes w_{(2)}\rangle_{W_1 \otimes W_2}. \quad (2.11)$$

The left-hand side of (2.11) is

$$\begin{aligned}\langle F^\vee((Y_3')^o(v, x_1)\alpha), w_{(1)} \otimes w_{(2)}\rangle_{W_1 \otimes W_2} &= \langle (Y_3')^o(v, x_1)\alpha, F(w_{(1)} \otimes w_{(2)})\rangle_{\overline{W}_3} \\ &= \langle \alpha, Y_3(v, x_1)F(w_{(1)} \otimes w_{(2)})\rangle_{\overline{W}_3},\end{aligned}$$

while by setting $\lambda = F^\vee(\alpha)$ in (2.10) and then taking Res_{x_0}, we see that the right-hand side of (2.11) is

$$\begin{aligned}&\langle \tau_{P(z)}(Y_t^o(v, x_1))F^\vee(\alpha), w_{(1)} \otimes w_{(2)}\rangle_{W_1 \otimes W_2} = \\ &\quad = \langle F^\vee(\alpha), \mathrm{Res}_{x_0} z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) Y_1(v, x_0)w_{(1)} \otimes w_{(2)} \\ &\qquad + w_{(1)} \otimes Y_2(v, x_1)w_{(2)}\rangle_{W_1 \otimes W_2} \\ &\quad = \langle \alpha, \mathrm{Res}_{x_0} z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) F(Y_1(v, x_0)w_{(1)} \otimes w_{(2)}) \\ &\qquad + F(w_{(1)} \otimes Y_2(v, x_1)w_{(2)})\rangle_{\overline{W}_3}.\end{aligned}$$

The proposition follows immediately. □

Remark 2.12 Note that for fixed λ, $\tau_{P(z)}(Y_t(v, x))\lambda$ is lower truncated (with respect to x) for any $v \in V$ if and only if $\tau_{P(z)}(Y_t^o(v, x))\lambda$ is upper truncated for any $v \in V$. Moreover, in this case, the Jacobi identity for $\tau_{P(z)}(Y_t(\cdot, x))$ holds on λ if and only if the opposite Jacobi identity for $\tau_{P(z)}(Y_t^o(\cdot, x))$ (see formula (3.23) in [HL1]) holds on λ. Indeed, first assume that the Jacobi identity for $\tau_{P(z)}(Y_t(\cdot, x))$ holds on λ. An examination of the proof of Theorem 5.2.1 in [FHL], which asserts that the contragredient of a module is indeed a module, in fact proves the desired opposite Jacobi identity. (A similar observation was made in reference to formula (3.23) in [HL1].) For the converse, one sees that the relevant steps in the proof of Theorem 5.2.1 in [FHL] are reversible.

Theorem 2.13 *Let W_1, W_2 be generalized V-modules and W_3 be an ordinary (respectively, generalized) V-module. Let F be a quasi-$P(z)$-intertwining map of type $\binom{W_3}{W_1\,W_2}$. Then for any $\alpha \in W_3'$, $F^\vee(\alpha) \in (W_1 \otimes W_2)^*$ satisfies the lower truncation condition and the Jacobi identity with respect to the action $\tau_{P(z)}$. In particular, $F^\vee(W_3') \subset (W_1 \otimes W_2)^*$ is an ordinary (respectively, generalized) V-module and $F^\vee : W_3' \to F^\vee(W_3')$ is a module map (respectively, a map of generalized modules). Conversely, every ordinary (respectively, generalized) V-module inside $(W_1 \otimes W_2)^*$ arises in this way.*

Proof Let F be as in the assumption. Then for any $\alpha \in W_3'$, by Proposition 2.10 we have

$$\tau_{P(z)}(Y_t^o(v, x_1))F^\vee(\alpha) = F^\vee((Y_3')^o(v, x_1)\alpha) \tag{2.12}$$

for any $v \in V$. Since the right-hand side of (2.12) is upper truncated in x_1, so is the left-hand side. Hence we have the lower truncation condition with respect to the action $\tau_{P(z)}$. The Jacobi identity on $F^\vee(\alpha)$ follows from (2.12) and the fact that α satisfies the Jacobi identity on W_3' (recall Remark 2.12). Also, $\tau_{P(z)}(Y_t(\mathbf{1}, x)) = 1$ from the definitions. Therefore, $F^\vee(W_3')$ is a weak V-module. But as an image of the ordinary (respectively, generalized) V-module W_3', it must be an ordinary (respectively, generalized) V-module itself.

Conversely, let M be a subspace of $(W_1 \otimes W_2)^*$ that becomes an ordinary (respectively, generalized) V-module under the action $\tau_{P(z)}$ of $V \otimes \mathbb{C}[t, t^{-1}]$. Take $W_3 = M'$, the contragredient module of M, and define $F : W_1 \otimes W_2 \to \overline{W}_3$ by

$$\langle \alpha, F(w_{(1)} \otimes w_{(2)})\rangle_{\overline{W}_3} = \langle \alpha, w_{(1)} \otimes w_{(2)}\rangle_{W_1 \otimes W_2} \tag{2.13}$$

for any $\alpha \in W_3' = M \subset (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. By using Res_{x_0} of (2.10) we have

$$\begin{aligned}
&\langle \alpha, Y_3(v, x_1)F(w_{(1)} \otimes w_{(2)})\rangle_{\overline{W}_3}\\
&\quad = \langle Y_3'(v, x_1)\alpha, F(w_{(1)} \otimes w_{(2)})\rangle_{\overline{W}_3}\\
&\quad = \langle \tau_{P(z)}(Y_t^o(v, x_1))\alpha, F(w_{(1)} \otimes w_{(2)})\rangle_{\overline{W}_3}\\
&\quad = \langle \tau_{P(z)}(Y_t^o(v, x_1))\alpha, w_{(1)} \otimes w_{(2)}\rangle_{W_1 \otimes W_2}\\
&\quad = \langle \alpha, \mathrm{Res}_{x_0} z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) Y_1(v, x_0)w_{(1)} \otimes w_{(2)} + w_{(1)} \otimes Y_2(v, x_1)w_{(2)}\rangle_{W_1 \otimes W_2}\\
&\quad = \langle \alpha, \mathrm{Res}_{x_0} z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) F(Y_1(v, x_0)w_{(1)} \otimes w_{(2)}) + F(w_{(1)} \otimes Y_2(v, x_1)w_{(2)})\rangle_{\overline{W}_3}
\end{aligned}$$

for any $\alpha \in M$, $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. This shows that F is a quasi-$P(z)$-intertwining map. In addition, by (2.13) we also have that

$$\langle F^\vee(\alpha), w_{(1)} \otimes w_{(2)}\rangle_{W_1 \otimes W_2} = \langle \alpha, F(w_{(1)} \otimes w_{(2)})\rangle_{\overline{W}_3} = \langle \alpha, w_{(1)} \otimes w_{(2)}\rangle_{W_1 \otimes W_2}$$

for any $\alpha \in W_3' = M$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, so that $F^\vee$ is the identity map on $W_3' = M$, and $M = F^\vee(W_3')$. $\square$

An immediate consequence is:

Corollary 2.14 *Suppose W_4 is also a generalized V-module (in addition to W_1, W_2 and W_3). Let F_1 and F_2 be quasi-$P(z)$-intertwining maps of types $\binom{W_3}{W_1\,W_2}$ and $\binom{W_4}{W_1\,W_2}$, respectively. Assume that both W_3 and W_4 are irreducible. Then the generalized V-modules $F_1^\vee(W_3')$ and $F_2^\vee(W_4')$ inside $(W_1 \otimes W_2)^*$ are irreducible and in particular either coincide with each other or have intersection* 0. $\square$

Remark 2.15 In the first half of Theorem 2.13, let $W \subset W_3$ be the space spanned by the homogeneous components of the elements $F(w_{(1)} \otimes w_{(2)}) \in \overline{W}_3$ for all $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. Then by formula (2.4) it is clear that W is closed under the action of each v_n for $v \in V$ and $n \in \mathbb{Z}$. Hence, in the case that W_3 is an ordinary (respectively, generalized) V-module, W is an ordinary (respectively, generalized) V-submodule of W_3. Furthermore, $F^\vee(W_3')$ is naturally isomorphic to W' as an ordinary (respectively, generalized) V-module; indeed, both V-homomorphisms $W_3' \to F^\vee(W_3')$ and $W_3' \to W'$ are surjective and have the common kernel

$$\{\alpha \in W_3' \,|\, \langle \alpha, W\rangle_{W_3} = 0\}$$

(recall (2.9)).

Remark 2.16 Consider the special case of Remark 2.15 in which W_1 and W_2 are ordinary modules, W_3 is $W_1 \boxtimes_{P(z)} W_2 = (W_1 \boxdot_{P(z)} W_2)'$ if it exists, and F is the canonical $P(z)$-intertwining map $W_1 \otimes W_2 \to \overline{W_1 \boxtimes_{P(z)} W_2}$ coming from the canonical injection $F^\vee$: $W_1 \boxdot_{P(z)} W_2 \to (W_1 \otimes W_2)^*$. Then the map $W_3' \to F^\vee(W_3') = W_3'$ is the identity and so the map $W_3' \to W'$ is an isomorphism of modules. In particular, $W = W_3$, and we have recovered Lemma 14.9 of [H2]: The homogeneous components of the tensor product elements $w_{(1)} \boxtimes_{P(z)} w_{(2)}$ $(= F(w_{(1)} \otimes w_{(2)}))$ span the tensor product module $W_1 \boxtimes_{P(z)} W_2$.

Recall from [HL3], formulas (13.12) and (13.16), that an element $\lambda \in (W_1 \otimes W_2)^*$ is said to satisfy the *$P(z)$-compatibility condition* if λ satisfies the lower truncation condition with respect to the action $\tau_{P(z)}$ and for any $v \in V$, the following formula holds:

$$\tau_{P(z)}\left(x_0^{-1}\delta\left(\frac{x_1^{-1}-z}{x_0}\right)Y_t(v, x_1)\right)\lambda = x_0^{-1}\delta\left(\frac{x_1^{-1}-z}{x_0}\right)\tau_{P(z)}(Y_t(v, x_1))\lambda; \tag{2.14}$$

that is, the action of the space (2.7) on λ given by (2.8) is compatible with the action of the space $V \otimes \mathbb{C}[t, t^{-1}]$ on λ given by restricting (2.8) to the elements $Y_t(v, x_1)$. Recall also that a subspace of $(W_1 \otimes W_2)^*$ (in particular, a generalized V-module inside $(W_1 \otimes W_2)^*$) is said to be *$P(z)$-compatible* if all of its elements satisfy the $P(z)$-compatibility condition. By Theorem 2.13 we have:

Corollary 2.17 *Let F be a quasi-$P(z)$-intertwining map of type $\binom{W_3}{W_1\,W_2}$. Assume that W_3 is irreducible. Then if $F^\vee(W_3')$ is not $P(z)$-compatible, none of its nonzero elements satisfies the $P(z)$-compatibility condition.*

Proof If $F^\vee(W_3') = 0$ the conclusion is trivial. Otherwise, by Theorem 2.13 and the irreducibility of W_3' we see that $F^\vee(W_3')$ is an irreducible generalized V-module. The statement now follows from the fact that the set of elements satisfying the $P(z)$-compatibility condition is stable under the action $\tau_{P(z)}$ (see Theorem 13.9 of [HL3], Proposition 6.2 of [HL1], and their generalization in [HLZ2] for the case of generalized modules). $\square$

We have:

Theorem 2.18 *Let F be a quasi-$P(z)$-intertwining map of type $\binom{W_3}{W_1\,W_2}$. Then the generalized module (ordinary if W_3 is ordinary) $F^\vee(W_3')$ is $P(z)$-compatible if and only if F is in fact a $P(z)$-intertwining map.*

Proof For convenience we use an equivalent form of (2.14) as follows:

$$\tau_{P(z)}\left(x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right)Y_t^o(v,x_1)\right)\lambda = x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right)\tau_{P(z)}(Y_t^o(v,x_1))\lambda \tag{2.15}$$

(recall the equivalence between (2.8) and (2.10)). Let F be a quasi-$P(z)$-intertwining map. In (2.15), setting $\lambda = F^\vee(\alpha)$ and applying to $w_{(1)}\otimes w_{(2)}$ for $\alpha\in W_3'$, $w_{(1)}\in W_1$ and $w_{(2)}\in W_2$, we see that the left-hand side becomes

$$\begin{aligned}
&\langle\tau_{P(z)}\left(x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right)Y_t^o(v,x_1)\right)F^\vee(\alpha), w_{(1)}\otimes w_{(2)}\rangle =\\
&\quad= \langle F^\vee(\alpha), z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)Y_1(v,x_0)w_{(1)}\otimes w_{(2)}\\
&\qquad\qquad +x_0^{-1}\delta\left(\frac{z-x_1}{-x_0}\right)w_{(1)}\otimes Y_2(v,x_1)w_{(2)}\rangle\\
&\quad= \langle\alpha, z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)F(Y_1(v,x_0)w_{(1)}\otimes w_{(2)})\\
&\qquad\qquad +x_0^{-1}\delta\left(\frac{z-x_1}{-x_0}\right)F(w_{(1)}\otimes Y_2(v,x_1)w_{(2)})\rangle,
\end{aligned}\tag{2.16}$$

while the right-hand side becomes

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right)\langle\tau_{P(z)}(Y_t^o(v,x_1))F^\vee(\alpha), w_{(1)}\otimes w_{(2)}\rangle =\\
&\quad= x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right)\langle F^\vee(\alpha), \mathrm{Res}_{x_0}z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)Y_1(v,x_0)w_{(1)}\otimes w_{(2)}\\
&\qquad\qquad +w_{(1)}\otimes Y_2(v,x_1)w_{(2)}\rangle\\
&\quad= x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right)\langle\alpha, \mathrm{Res}_{x_0}z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)F(Y_1(v,x_0)w_{(1)}\otimes w_{(2)})\\
&\qquad\qquad +F(w_{(1)}\otimes Y_2(v,x_1)w_{(2)})\rangle\\
&\quad= \langle\alpha, x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right)Y_3(v,x_1)F(w_{(1)}\otimes w_{(2)})\rangle,
\end{aligned}\tag{2.17}$$

where in the last step, we have used (2.4). Thus $F^\vee(W_3')$ is $P(z)$-compatible if and only if for any $\alpha\in W_3'$, the right-hand side of (2.16) is equal to the right-hand side of (2.17) for any $w_{(1)}\in W_1$ and $w_{(2)}\in W_2$, that is, if and only if (2.5) is true for any $w_{(1)}\in W_1$ and $w_{(2)}\in W_2$, which is equivalent to F being a $P(z)$-intertwining map. $\square$

Remark 2.19 Suppose that an element λ of $(W_1 \otimes W_2)^*$ satisfies the $P(z)$-compatibility condition and generates a generalized V-module W_λ under the action $\tau_{P(z)}$ (cf. the $P(z)$-local grading-restriction condition in [HL3]). Then W_λ is compatible, just as in the proof of Corollary 2.17. By Theorem 2.13, $W_\lambda = F^\vee(W')$ for some generalized V-module W and quasi-$P(z)$-intertwining map F of type $\binom{W}{W_1\, W_2}$. Then Theorem 2.18 ensures that F is in fact a $P(z)$-intertwining map. In particular, λ lies in the image of $F^\vee$ for the $P(z)$-intertwining map F.

Remark 2.20 Theorem 2.18 and Example 2.8 together provide examples of non-$P(z)$-compatible modules.[1] It is instructive to write down the details of these (counter)examples: Take V to be the vertex operator algebra constructed from a finite-dimensional commutative associative algebra with identity as in Example 2.8. For any V-modules W_1 and W_2 (that is, finite-dimensional modules W_1 and W_2 for V viewed as an associative algebra), the action $\tau_{P(z)}$ of $V \otimes \mathbb{C}[t, t^{-1}]$ on $(W_1 \otimes W_2)^*$ is given by:

$$(\tau_{P(z)}(Y_t(v, x))\lambda)(w_{(1)} \otimes w_{(2)}) = \lambda(w_{(1)} \otimes (v \cdot w_{(2)}))$$

for $v \in V$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. For every $\lambda \in (W_1 \otimes W_2)^*$, the lower truncation condition and the Jacobi identity clearly hold, and $\tau_{P(z)}(Y_t(\mathbf{1}, x))\lambda = \lambda$. Hence the whole (finite-dimensional) space $(W_1 \otimes W_2)^*$ is a V-module, and $W_1 \boxtimes W_2 = (W_1 \otimes W_2)^*$. This is in fact just the contragredient module of the V-module $W_1 \otimes W_2$ defined in Example 2.8. We know from Example 2.8 that the identity map on $(W_1 \otimes W_2)^*$ is a quasi-$P(z)$-intertwining map that is not in general a $P(z)$-intertwining map. In Theorem 2.18, take F to be this identity map, so that $F^\vee$ is the identity map on $(W_1 \otimes W_2)^*$. The proof of Theorem 2.18 immediately shows that $\lambda \in (W_1 \otimes W_2)^*$ satisfies the $P(z)$-compatibility condition if and only if

$$z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right)\lambda((v \cdot w_{(1)}) \otimes w_{(2)}) + x_0^{-1}\delta\left(\frac{z - x_1}{-x_0}\right)\lambda(w_{(1)} \otimes (v \cdot w_{(2)}))$$

[1]These examples show in particular that the construction of the tensor product functor in math.QA/0309350—the formula below formula (4.1)—appears to be incorrect: As a consequence of the definition of $P(z)$-tensor product (adopted from [HL1]), the contragredient of the $P(z)$-tensor product of modules W_1 and W_2 is the union (or sum) of all $P(z)$-*compatible* modules, rather than *all* modules, inside $(W_1 \otimes W_2)^*$. The examples we give here and below show that the space defined in math.QA/0309350 is sometimes strictly larger than the correct contragredient module of the tensor product module. In particular, the arguments in math.QA/0309350 purport to establish an assertion equivalent to associativity for *quasi-intertwining* operators, which is not true. The correct result, proved (in the logarithmic context) in [HLZ1], [HLZ2], is the associativity for *intertwining* operators; this work generalizes the arguments in [HL1]–[HL3], [H2] and of course is based on the compatibility condition. For the examples in the present remark, even when W_1 and W_2 are taken to be V itself, the construction in math.QA/0309350 results in a space strictly larger (in the sense of homogeneous subspaces) than the correct tensor product, V itself. All of this illustrates why the compatibility condition of [HL1]–[HL3] is crucial and cannot in general be omitted. As we have mentioned, the compatibility condition remains crucial in the construction of the natural associativity isomorphisms among triple tensor products in [H2], and the proofs of their fundamental properties. In the tensor product theory in [HL1]–[HL3] and [H2], the compatibility condition on elements of $(W_1 \otimes W_2)^*$ was not a restriction on the applicability of the theory; rather, it was a necessary condition for obtaining the (correct) theory, and the same is certainly true for the still more subtle logarithmic generalization of the theory in [HLZ1]–[HLZ2].

$$= \delta\left(\frac{x_1 - z}{x_0}\right) \lambda(w_{(1)} \otimes (v \cdot w_{(2)})),$$

or equivalently,

$$\lambda((v \cdot w_{(1)}) \otimes w_{(2)}) = \lambda(w_{(1)} \otimes (v \cdot w_{(2)}))$$

for any $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, which of course is not true in general (cf. Example 2.8). That is, $W_1 \boxtimes W_2$ typically has a lot of non-$P(z)$-compatible elements. In fact, the space of compatible elements in $(W_1 \otimes W_2)^*$ naturally identifies with $(W_1 \otimes_V W_2)^*$, the dual space of the tensor product, over the commutative associative algebra V, of the V-modules W_1 and W_2. This space of compatible elements is naturally a V-module (with V viewed either as a commutative associative algebra or as a vertex operator algebra)—the contragredient module (with V viewed either way) of $W_1 \otimes_V W_2$, which of course is naturally a quotient space of $W_1 \otimes W_2$. In particular, take W_1 and W_2 to be the commutative associative algebra V itself, viewed as a module. Then $V \boxtimes V = (V \otimes V)^*$, while the space of compatible elements λ is naturally identified with $(V \otimes_V V)^* = V^*$. The contragredient module of $(V \otimes V)^*$ cannot equal the correct tensor product module, namely, V, unless V is 1-dimensional.

3 Further examples and $g(V)_{\geq 0}$-homomorphisms

This section and the next are independent of Section 2, except for the definition of the notion of (logarithmic) quasi-intertwining operator.

In this section we will give further examples of quasi-intertwining operators that are not intertwining operators, by using a canonical Lie algebra associated with a vertex algebra and modules for this Lie algebra. Just as in Section 2, such examples give examples of non-$P(z)$-compatible modules. Motivated by these examples, we study some further properties of modules for this canonical Lie algebra, which will lead us to other results about quasi-intertwining operators.

Let $(V, Y, \mathbf{1})$ be a vertex algebra. Recall the canonical Lie algebra $g(V)$ associated with V (see [B], [FFR], [Li2], [MP]):

$$g(V) = (V \otimes \mathbb{C}[t, t^{-1}])/\mathrm{Im}\,(\mathcal{D} \otimes 1 + 1 \otimes d/dt),$$

where $\mathcal{D}$ is given by $\mathcal{D}u = u_{-2}\mathbf{1}$ for $u \in V$, with the bracket defined by means of representatives by:

$$[u \otimes t^m, v \otimes t^n] = \sum_{i \geq 0} \binom{m}{i} (u_i v \otimes t^{m+n-i}).$$

Denote by π the natural quotient map

$$\pi : V \otimes \mathbb{C}[t, t^{-1}] \to g(V).$$

For $v \in V,\ n \in \mathbb{Z}$, we set

$$v(n) = \pi(v \otimes t^n) \in g(V).$$

Note that $\mathbf{1}(-1)$ is a nonzero central element of $g(V)$. A $g(V)$-module on which $\mathbf{1}(-1)$ acts as a scalar ℓ is said to be of *level* ℓ. Recall that as in the case of affine Lie algebras, a $g(V)$-module W is said to be *restricted* if for any $w \in W$ and $v \in V$, we have $v(n)w = 0$ for n sufficiently large. A V-module is automatically a restricted $g(V)$-module of level 1 on which $v(n)$ acts as v_n. (See [B], [FFR], [Li2], [MP] for these and other standard notations and properties of $g(V)$ and its modules.)

We set

$$\begin{aligned} g(V)_{\geq 0} &= \pi(V \otimes \mathbb{C}[t]) \subset g(V), \\ g(V)_{<0} &= \pi(V \otimes t^{-1}\mathbb{C}[t^{-1}]) \subset g(V). \end{aligned}$$

Clearly, these are Lie subalgebras of $g(V)$ and

$$g(V) = g(V)_{<0} \oplus g(V)_{\geq 0}.$$

The following observation, due to [DLM2] and [P], will be used in the Appendix:

Proposition 3.1 *The linear map from V to $g(V)_{<0}$ sending v to $v(-1)$ $(= \pi(v \otimes t^{-1}))$ is a linear isomorphism.* $\square$

In addition to the examples in Example 2.8 (and Remark 2.9), we now give another way of constructing examples of quasi-intertwining operators that are not intertwining operators. We have:

Proposition 3.2 *Let V be a vertex operator algebra, let (W_1, Y_1) and (W_2, Y_2) be weak V-modules, and let θ be a linear map from W_1 to W_2. We define a linear map $\mathcal{Y}$ from W_1 to* $\mathrm{Hom}\,(V, W_2((x)))$ *by*

$$\mathcal{Y}(w, x)v = e^{xL(-1)} Y_2(v, -x)\theta(w) \quad \textit{for } w \in W_1,\ v \in V.$$

Then $\mathcal{Y}$ is a quasi-intertwining operator (of type $\binom{W_2}{W_1\, V}$) if and only if θ is a $g(V)_{\geq 0}$-homomorphism. Furthermore, $\mathcal{Y}$ is an intertwining operator if and only if θ is a V-homomorphism.

Proof Define

$$Y_2^t(w, x)v = e^{xL(-1)} Y_2(v, -x)w \quad \text{for } w \in W_2,\ v \in V.$$

Then we have

$$\mathcal{Y}(w, x)v = Y_2^t(\theta(w), x)v \quad \text{for } w \in W_1,\ v \in V.$$

From [FHL], Y_2^t is an intertwining operator of type $\binom{W_2}{W_2\, V}$. Assume that θ is a $g(V)_{\geq 0}$-homomorphism. For any V-module (W, Y_W) and any $u \in V$, we use $Y_W^-(u, x)$ to denote $\sum_{n \geq 0} u_n x^{-n-1}$, where $Y_W(u, x) = \sum_{n \in \mathbb{Z}} u_n x^{-n-1}$. For any $u, v \in V$, $w \in W_1$, using the fact that θ is a $g(V)_{\geq 0}$-homomorphism we have

$$\begin{aligned}
&Y_2(u,x_1)\mathcal{Y}(w,x_2)v-\mathcal{Y}(w,x_2)Y(u,x_1)v\\
&\quad=Y_2(u,x_1)Y_2^t(\theta(w),x_2)v-Y_2^t(\theta(w),x_2)Y(u,x_1)v\\
&\quad=\mathrm{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_2^t(Y_2(u,x_0)\theta(w),x_2)v\\
&\quad=\mathrm{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_2^t(Y_2^-(u,x_0)\theta(w),x_2)v\\
&\quad=\mathrm{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_2^t(\theta(Y_1^-(u,x_0)w),x_2)v\\
&\quad=\mathrm{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_2^t(\theta(Y_1(u,x_0)w),x_2)v\\
&\quad=\mathrm{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}(Y_1(u,x_0)w,x_2)v. \qquad (3.1)
\end{aligned}$$

For $w\in W_1$, noticing that $\theta(L(-1)w)=L(-1)\theta(w)$ we have

$$\begin{aligned}
&\mathcal{Y}(L(-1)w,x)=Y_2^t(\theta(L(-1)w),x)=Y_2^t(L(-1)\theta(w),x)=\frac{d}{dx}Y_2^t(\theta(w),x)\\
&=\frac{d}{dx}\mathcal{Y}(w,x).
\end{aligned}$$

Thus $\mathcal{Y}$ is a quasi-intertwining operator.

Conversely, assume that $\mathcal{Y}$ is a quasi-intertwining operator. Then the outside equality of (3.1) holds. Using the first three and the last two equalities in (3.1) we see that

$$\begin{aligned}
&\mathrm{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_2^t(Y_2^-(u,x_0)\theta(w),x_2)v\\
&\quad=\mathrm{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_2^t(\theta(Y_1^-(u,x_0)w),x_2)v.
\end{aligned}$$

For any $n\geq 0$, we have

$$\begin{aligned}
&\mathrm{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)x_0^nY_2^t(Y_2^-(u,x_0)\theta(w),x_2)v\\
&\quad=\mathrm{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)(x_1-x_2)^nY_2^t(Y_2^-(u,x_0)\theta(w),x_2)v\\
&\quad=\mathrm{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)(x_1-x_2)^nY_2^t(\theta(Y_1^-(u,x_0)w),x_2)v\\
&\quad=\mathrm{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)x_0^nY_2^t(\theta(Y_1^-(u,x_0)w),x_2)v.
\end{aligned}$$

Taking Res_{x_1} we get

$$Y_2^t(u_n\theta(w),x_2)v=Y_2^t(\theta(u_nw),x_2)v.$$

Setting $v = \mathbf{1}$ in this formula and using the definition of Y_2^t, we obtain $u_n\theta(w) = \theta(u_n w)$. This proves that θ is a $g(V)_{\geq 0}$-homomorphism.

Using the whole Jacobi identity instead of the commutator formula one shows analogously that $\mathcal{Y}$ is an intertwining operator if and only if θ is a $g(V)$-homomorphism. $\square$

Partly due to this proposition, we are interested in looking for $g(V)_{\geq 0}$-module maps that are not V-module maps. We will study this problem in the general context of vertex algebras. We now fix a vertex algebra $(V, Y, \mathbf{1})$.

Recall the following result ([DLM1], [LL], Proposition 4.5.7):

Proposition 3.3 *Let W be a V-module and let $u, v \in V$, $p, q \in \mathbb{Z}$ and $w \in W$. Let l be a nonnegative integer such that*

$$u_n w = 0 \quad \text{for } n \geq l$$

and let m be a nonnegative integer such that

$$v_n w = 0 \quad \text{for } n > m + q.$$

Then

$$u_p v_q w = \sum_{i=0}^{m} \sum_{j=0}^{l} \binom{p-l}{i} \binom{l}{j} (u_{p-l-i+j} v)_{q+l+i-j} w. \tag{3.2}$$

$\square$

The following result, due to [DM] and [Li2], is an immediate consequence of Proposition 3.3:

Proposition 3.4 *Let W be a V-module. Then for any $w \in W$, the linear span of the vectors $v_n w$ for $v \in V$, $n \in \mathbb{Z}$, namely, $g(V)w$, is a V-submodule of W. Furthermore, for any subspace or subset U of W, $g(V)U$ is the V-submodule of W generated by U.* $\square$

Examining Proposition 3.3 more closely we have:

Proposition 3.5 *Let W be a V-module. Then $g(V)_{\geq 0} w$ is a V-submodule of W for every $w \in W$, and so is $g(V)_{\geq 0} W$. Furthermore, $g(V)_{\geq 0} V$ is a (two-sided) ideal of V.*

Proof It follows directly from the formula (3.2) with $q \geq 0$ that for any $w \in W$, $g(V)_{\geq 0} w$ is a V-submodule of W. Thus $g(V)_{\geq 0} W$ is a V-submodule. Taking $W = V$, we see that $g(V)_{\geq 0} V$ is a left ideal of V. Since $[\mathcal{D}, v_n] = -n v_{n-1}$ for $v \in V$, $n \in \mathbb{Z}$, we have that $[\mathcal{D}, g(V)_{\geq 0}] \subset g(V)_{\geq 0}$, acting on V. Therefore $\mathcal{D} g(V)_{\geq 0} V \subset g(V)_{\geq 0} V$. From Remark 3.9.8 in [LL], $g(V)_{\geq 0} V$ is an ideal. $\square$

We have:

Proposition 3.6 *The following statements are equivalent:*

1. *There exists a V-module W such that $g(V)_{\geq 0}W \neq W$.*
2. *$g(V)_{\geq 0}V \neq V$, or equivalently $\mathbf{1} \notin g(V)_{\geq 0}V$.*

Proof Since $g(V)_{\geq 0}V$ is an ideal (Proposition 3.5), the two conditions in the second statement are equivalent. We need only prove that if there exists a V-module W such that $g(V)_{\geq 0}W \neq W$, then $g(V)_{\geq 0}V \neq V$. Let W be such a V-module. Then we have a nonzero V-module $\widetilde{W} = W/g(V)_{\geq 0}W$ (see Proposition 3.5) such that $g(V)_{\geq 0}\widetilde{W} = 0$. Since $Y(\mathbf{1}, x) = 1$ on $\widetilde{W}$, the annihilating ideal of V

$$\mathrm{Ann}_V(\widetilde{W}) = \{v \in V \mid Y(v, x)\widetilde{W} = 0\}$$

(see Proposition 4.5.11 of [LL]) is proper. For $u, v \in V,\ n \geq 0$, we have

$$Y(u_n v, x) = \mathrm{Res}_{x_1}(x_1 - x)^n [Y(u, x_1), Y(v, x)] = \sum_{i=0}^{n} \binom{n}{i} (-x)^i [u_{n-i}, Y(v, x)]$$

(recall (3.8.14) in [LL]). Since $g(V)_{\geq 0}\widetilde{W} = 0$, it follows that $u_n v \in \mathrm{Ann}_V(\widetilde{W})$ for $u, v \in V,\ n \geq 0$. This proves that $g(V)_{\geq 0}V \subset \mathrm{Ann}_V(\widetilde{W})$, a proper subspace of V. Consequently, $g(V)_{\geq 0}V \neq V$. $\square$

Furthermore, we have:

Proposition 3.7 *Suppose that $(V, Y, \mathbf{1}, \omega)$ is a vertex operator algebra of central charge $c \neq 0$. Then $g(V)_{\geq 0}W = W$ for any weak V-module W. In particular, $g(V)_{\geq 0}V = V$.*

Proof We have the following relation in V:

$$L(2)\omega = L(2)L(-2)\mathbf{1} = \frac{1}{2}c\mathbf{1}.$$

Since $c \neq 0$, we have $\mathbf{1} = (2/c)L(2)\omega \in g(V)_{\geq 0}V$. By Proposition 3.6, $g(V)_{\geq 0}W = W$ for any weak V-module W. $\square$

The following example shows that in Proposition 3.7, the condition $c \neq 0$ is necessary:

Example 3.8 Let V be the minimal vertex operator algebra $V_{Vir}(0, 0)$ associated with the Virasoro algebra $\mathcal{L}$ of central charge $c = 0$ (cf. Section 6.1 of [LL]). We are going to show that $V_{(0)} \not\subset g(V)_{\geq 0}V$. We know that $V_{(0)} = \mathbb{C}\mathbf{1}$, $V_{(1)} = 0$ and $V_{(2)} = \mathbb{C}\omega$, where ω is the conformal vector. Set $V_+ = \coprod_{n\geq 1} V_{(n)}$. We will show that V_+ is an ideal. If this is proved, we will have

$$g(V)_{\geq 0}V = g(V)_{\geq 0}\mathbf{1} + g(V)_{\geq 0}V_+ = g(V)_{\geq 0}V_+ \subset V_+,$$

which immediately implies that $\mathbf{1} \notin g(V)_{\geq 0}V$. Note that from Section 6.1 of [LL], $U(\mathcal{L})\omega$ is a left ideal of V (and hence, by Remark 3.9.8 of [LL], a (two-sided) ideal of V). It suffices to prove that $V_+ = U(\mathcal{L})\omega$. Since V is an $\mathcal{L}$-module with $\mathbf{1}$ as a generator, V_+ is spanned by the vectors

$$L(-m_1)\cdots L(-m_k)\mathbf{1}$$

for $k \geq 1,\ m_1 \geq \cdots \geq m_k \geq 2$. Using the formula $Y(\omega, x)\mathbf{1} = e^{xL(-1)}\omega$ we have

$$L(n)\mathbf{1} \in U(\mathcal{L})\omega \quad \text{for } n \in \mathbb{Z}.$$

It follows that $V_+ \subset U(\mathcal{L})\omega$. On the other hand, for $n \geq 1$ we have

$$L(n)\omega = L(n)L(-2)\mathbf{1} = (n+2)L(n-2)\mathbf{1} + \delta_{n,2}\frac{1}{2}c\mathbf{1} = 0,$$

since the central charge c is zero. That is, ω is a lowest weight vector for the Virasoro algebra $\mathcal{L}$. Thus

$$U(\mathcal{L})\omega \subset V_+.$$

Therefore we have $V_+ = U(\mathcal{L})\omega$, proving that V_+ is an ideal of V, and hence proving that $V_{(0)} \not\subset g(V)_{\geq 0}V$. Using an analogous argument we easily also see that for $V = V_{\hat{\mathfrak{g}}}(0,0)$, associated with an affine Kac-Moody Lie algebra $\hat{\mathfrak{g}}$ of level 0 (cf. Section 6.2 of [LL]), we have $\mathbf{1} \notin g(V)_{\geq 0}V$.

We have:

Proposition 3.9 *Let V be a vertex algebra and let W_1 and W_2 be V-modules. If $g(V)_{\geq 0}W_1 = W_1$, then any $g(V)_{\geq 0}$-homomorphism from W_1 to W_2 is a V-homomorphism.*

Proof Let f be a $g(V)_{\geq 0}$-homomorphism from W_1 to W_2. Let $u, v \in V,\ p, q \in \mathbb{Z},\ w \in W_1$. Let l, m be nonnegative integers such that

$$\begin{aligned} &u_n w = 0, \quad u_n f(w) = 0 \quad \text{for } n \geq l,\\ &v_n w = 0, \quad v_n f(w) = 0 \quad \text{for } n > m + q. \end{aligned}$$

By Proposition 3.3 we have

$$u_p v_q w = \sum_{i=0}^{m}\sum_{j=0}^{l}\binom{p-l}{i}\binom{l}{j}(u_{p-l-i+j}v)_{q+l+i-j}w,$$

$$u_p v_q f(w) = \sum_{i=0}^{m}\sum_{j=0}^{l}\binom{p-l}{i}\binom{l}{j}(u_{p-l-i+j}v)_{q+l+i-j}f(w).$$

Notice that if $q \geq 0$, we have $q + l + i - j \geq 0$ for $i \geq 0,\ 0 \leq j \leq l$. Then for $q \geq 0$, using (3.3) and (3.3) and the fact that f is a $g(V)_{\geq 0}$-homomorphism, we get

$$f(u_p v_q w) = \sum_{i=0}^{m}\sum_{j=0}^{l}\binom{p-l}{i}\binom{l}{j}f\left((u_{p-l-i+j}v)_{q+l+i-j}w\right)$$

$$
\begin{aligned}
&= \sum_{i=0}^{m}\sum_{j=0}^{l}\binom{p-l}{i}\binom{l}{j}(u_{p-l-i+j}v)_{q+l+i-j}f(w)\\
&= u_p v_q f(w)\\
&= u_p f(v_q w).
\end{aligned}
$$

This shows that

$$f(u_p w') = u_p f(w') \quad \text{for } u \in V,\ p \in \mathbb{Z},\ w' \in g(V)_{\geq 0}W_1.$$

Since we are assuming that $W_1 = g(V)_{\geq 0}W_1$, f is a V-homomorphism. $\square$

Example 3.10 Here we give concrete examples of $g(V)_{\geq 0}$-homomorphisms between V-modules that are not V-homomorphisms, for suitable vertex operator algebras V. Then by Proposition 3.2 we obtain quasi-intertwining operators that are not intertwining operators, and as in Section 2, this gives examples of non-$P(z)$-compatible modules. Let V be the vertex operator algebra $V_{Vir}(0,0)$ associated to the Virasoro algebra of central charge 0, or the vertex operator algebra $V_{\hat{\mathfrak{g}}}(0,0)$ associated with an affine Lie algebra $\hat{\mathfrak{g}}$ of level 0. Recall from Example 3.8 that $V = V_+ \oplus \mathbb{C}\mathbf{1}$, where $V_+ = \coprod_{n>0} V_{(n)}$ is an ideal of V. Consequently, $V/V_+ \simeq \mathbb{C}$ is a (one-dimensional) nontrivial V-module, on which $Y(v,x)$ acts as zero for $v \in V_+$ (and $Y(\mathbf{1},x)$ acts as the identity). Define a linear map

$$\theta : \mathbb{C} \to V;\ \ \alpha \mapsto \alpha\mathbf{1} \quad \text{for } \alpha \in \mathbb{C}.$$

Clearly, θ is a $g(V)_{\geq 0}$-homomorphism. But θ is not a V-homomorphism, since for any nonzero $v \in V_+$, v_{-1} acts on $\mathbb{C}$ as zero but $v_{-1}\mathbf{1} = v \neq 0$.

4 Necessary conditions for the existence of examples

In this section we let V be a vertex operator algebra. We will show that for weak V-modules W_1, W_2 and W_3, the condition $W_1 = g(V)_{\geq 0}W_1$ implies that any quasi-intertwining operator of type $\binom{W_3}{W_1\,W_2}$ is an intertwining operator; in other words, the condition $W_1 \neq g(V)_{\geq 0}W_1$ is necessary for the existence of a quasi-intertwining operator of type $\binom{W_3}{W_1\,W_2}$ that is not an intertwining operator. We conclude that if V has nonzero central charge, then any quasi-intertwining operator among any weak V-modules is an intertwining operator.

Recall the following definition from [Li3] and [Li4]:

Definition 4.1 Let $(W_2, Y_2), (W_3, Y_3)$ be weak V-modules. Denote by $\mathcal{H}(W_2, W_3)$ the vector subspace of $(\mathrm{Hom}\,(W_2, W_3))\{x\}$ consisting of the formal series $\phi(x)$ satisfying the following conditions: Writing

$$\phi(x)w_{(2)} = \sum_{n\in\mathbb{C}} w_{(3)}^{(n)} x^{-n-1} \quad \left(\text{where } w_{(3)}^{(n)} \in W_3\right)$$

for $w_{(2)} \in W_2$, we have

$$w_{(3)}^{(n)} = 0 \text{ for } n \text{ whose real part is sufficiently large;}$$

$$[L(-1), \phi(x)] = \frac{d}{dx}\phi(x);$$

and for any $v \in V$, there exists a nonnegative integer k such that

$$(x_1 - x_2)^k (Y_3(v, x_1)\phi(x_2) - \phi(x_2)Y_2(v, x_1)) = 0.$$

We also define a vertex operator map

$$Y_{\mathcal{H}}(\cdot, x_0) : V \to (\mathrm{End}\,\mathcal{H}(W_2, W_3))[[x_0, x_0^{-1}]]$$

by

$$\begin{aligned} & Y_{\mathcal{H}}(v, x_0)\phi(x) \\ & = \mathrm{Res}_{x_1} \left(x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right) Y_3(v, x_1)\phi(x) - x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right) \phi(x)Y_2(v, x_1) \right). \end{aligned} \tag{4.1}$$

The following was essentially proved in [Li2]:

Theorem 4.2 *Let W_2 and W_3 be weak V-modules. Then $(\mathcal{H}(W_2, W_3), Y_{\mathcal{H}})$ carries the structure of a weak V-module. Furthermore, for any weak V-module W_1, giving an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$ is equivalent to giving a V-homomorphism $\psi = \psi_x$ from W_1 to $\mathcal{H}(W_2, W_3)$, where the relation between $\mathcal{Y}$ and ψ is given by*

$$\psi_x(w_{(1)}) = \mathcal{Y}(w_{(1)}, x) \quad \textit{for } w_{(1)} \in W_1.$$

$\square$

Remark 4.3 For V-modules W_2, W_3, the space $\mathcal{H}(W_2, W_3)$ was defined in [Li3] and [Li4], where it was denoted by $G(W_2, W_3)$. Theorem 4.2 was proved in [Li3] and [Li4] with W_2, W_3 being V-modules, but the proof did not use the grading.

Let W_1, W_2 and W_3 be V-modules and $\mathcal{Y}$ a quasi-intertwining operator of type $\binom{W_3}{W_1 W_2}$. Since $L(-1) = \omega_0$, by using the commutator formula and the $L(-1)$-derivative property we get

$$[L(-1), \mathcal{Y}(w_{(1)}, x)] = \mathcal{Y}(L(-1)w_{(1)}, x) = \frac{d}{dx}\mathcal{Y}(w_{(1)}, x) \quad \text{for } w_{(1)} \in W_1.$$

For $v \in V,\ w_{(1)} \in W_1$, let k be a nonnegative integer such that $v_n w_{(1)} = 0$ for $n \geq k$. Using the commutator formula we get

$$(x_1 - x_2)^k [Y(v, x_1), \mathcal{Y}(w_{(1)}, x_2)] = 0.$$

Thus we have

$$\mathcal{Y}(w_{(1)}, x) \in \mathcal{H}(W_2, W_3) \quad \text{for } w_{(1)} \in W_1.$$

In view of this, we may and should consider $\mathcal{Y} = \mathcal{Y}(\cdot, x)$ as a linear map from W to $\mathcal{H}(W_2, W_3)$. Furthermore, for $v \in V,\ n \geq 0,\ w_{(1)} \in W_1$, using the definition of the action of v_n on $\mathcal{H}(W_2, W_3)$ and the commutator formula for $\mathcal{Y}$ we get

$$\begin{aligned}
v_n(\mathcal{Y}(w_{(1)}, x)) &= \mathrm{Res}_{x_1}(x_1 - x)^n [Y(v, x_1), \mathcal{Y}(w_{(1)}, x)] \\
&= \mathrm{Res}_{x_1}\mathrm{Res}_{x_0}(x_1 - x)^n x_1^{-1}\delta\left(\frac{x + x_0}{x_1}\right)\mathcal{Y}(Y(v, x_0)w_{(1)}, x) \\
&= \mathrm{Res}_{x_0} x_0^n \mathcal{Y}(Y(v, x_0)w_{(1)}, x) \\
&= \mathcal{Y}(v_n w_{(1)}, x).
\end{aligned}$$

Thus $\mathcal{Y}$ is a $g(V)_{\geq 0}$-homomorphism. Therefore we have proved:

Proposition 4.4 *Let W_1, W_2 and W_3 be weak V-modules and $\mathcal{Y}$ be a quasi-intertwining operator of type $\binom{W_3}{W_1 W_2}$. Then*

$$\mathcal{Y}(w_{(1)}, x) \in \mathcal{H}(W_2, W_3) \quad \textit{for } w_{(1)} \in W_1.$$

Furthermore, the linear map ψ_x from W_1 to $\mathcal{H}(W_2, W_3)$ defined by

$$\psi_x(w_{(1)}) = \mathcal{Y}(w_{(1)}, x)$$

is a $g(V)_{\geq 0}$-homomorphism, i.e.,

$$\psi_x(v_n w_{(1)}) = v_n \psi_x(w_{(1)})$$

for $v \in V,\ n \geq 0,\ w_{(1)} \in W_1$. $\square$

In view of Theorem 4.2 and Proposition 4.4, Proposition 3.9 gives:

Proposition 4.5 *Let W_1, W_2 and W_3 be weak V-modules. If $g(V)_{\geq 0} W_1 = W_1$, then any quasi-intertwining operator of type $\binom{W_3}{W_1 W_2}$ is an intertwining operator.*

Proof Let $\mathcal{Y}$ be a quasi-intertwining operator of type $\binom{W_3}{W_1 W_2}$. By Proposition 4.4, there exists a $g(V)_{\geq 0}$-homomorphism ψ_x from W_1 to the weak V-module $\mathcal{H}(W_2, W_3)$ such that

$$\mathcal{Y}(w, x) = \psi_x(w) \quad \text{for } w \in W_1.$$

From Theorem 4.2, $\mathcal{Y}$ is an intertwining operator if and only if ψ_x is a V-homomorphism, i.e., a $g(V)$-homomorphism. But by Proposition 3.9, ψ_x is indeed a V-homomorphism. Thus $\mathcal{Y}$ is an intertwining operator. $\square$

Combining Propositions 3.6, 3.9 and 4.5, we immediately have the first assertion of the following theorem:

Theorem 4.6 *Suppose that $g(V)_{\geq 0}V = V$. Then any $g(V)_{\geq 0}$-homomorphism between weak V-modules is a V-homomorphism and any quasi-intertwining operator among any weak V-modules is an intertwining operator. On the other hand, if $g(V)_{\geq 0}V \neq V$ and* $\dim V > 1$*, then there exists a $g(V)_{\geq 0}$-homomorphism between V-modules that is not a V-homomorphism and there exists a quasi-intertwining operator among V-modules that is not an intertwining operator.*

Proof Assume $g(V)_{\geq 0}V \neq V$ and $\dim V > 1$. We modify the construction given in Example 3.10 as follows. Set $W = V/g(V)_{\geq 0}V$. Since $g(V)_{\geq 0}V$ is an ideal of V (by Proposition 3.5), W is a nonzero module for the vertex operator algebra V, and we have $g(V)_{\geq 0}W = 0$. Let f be any nonzero linear functional on W. Define

$$\theta_f : W \to V, \;\; w \mapsto f(w)\mathbf{1}.$$

With $g(V)_{\geq 0}W = 0$ and $g(V)_{\geq 0}\mathbf{1} = 0$, it is clear that θ_f is a $g(V)_{\geq 0}$-homomorphism. Let $v \in V \backslash \mathbb{C}\mathbf{1}$ (as $\dim V > 1$) and let $w \in W$ be such that $f(w) = 1$. We have

$$\begin{aligned} v_{-1}\theta_f(w) &= v_{-1}f(w)\mathbf{1} = v_{-1}\mathbf{1} = v \notin \mathbb{C}\mathbf{1}, \\ \theta_f(v_{-1}w) &= f(v_{-1}w)\mathbf{1} \in \mathbb{C}\mathbf{1}, \end{aligned}$$

proving that θ_f is not a V-homomorphism. By Proposition 3.2 this yields to a quasi-intertwining operator (of type $\binom{V}{W\,V}$) that is not an intertwining operator. $\square$

Remark 4.7 If $\dim V = 1$, i.e., $V = \mathbb{C}\mathbf{1}$ with $\mathbf{1} \neq 0$, then $g(V)_{\geq 0}V = 0 \neq V$. In this case, a V-homomorphism is simply a linear map, so that any $g(V)_{\geq 0}$-homomorphism is a V-homomorphism.

Combining Proposition 3.7 with Theorem 4.6 we immediately have:

Corollary 4.8 *Suppose that the central charge of V is not 0. Then any $g(V)_{\geq 0}$-homomorphism between weak V-modules is a V-homomorphism and any quasi-intertwining operator among any weak V-modules is an intertwining operator.* $\square$

Remark 4.9 Let W_2 and W_3 be weak V-modules. By analogy with the space $\mathcal{H}(W_2, W_3)$, define $\mathcal{H}^{\log}(W_2, W_3)$ to be the vector subspace of $(\mathrm{Hom}\,(W_2, W_3))\{x\}[[\log x]]$ consisting of the formal series $\phi(x)$ satisfying the following conditions: Writing

$$\phi(x)w_{(2)} = \sum_{n\in\mathbb{C}}\sum_{l\in\mathbb{N}} w_{(3)}^{(n;\,l)}x^{-n-1}(\log x)^l \quad (\text{where } \; w_{(3)}^{(n;\,l)} \in W_3)$$

for $w_{(2)} \in W_2$, we have

$$w_{(3)}^{(n;\,l)} = 0$$

if either $l \in \mathbb{N}$ is sufficiently large or the real part of n is sufficiently large;

$$[L(-1), \phi(x)] = \frac{d}{dx}\phi(x);$$

and for any $v \in V$, there exists a nonnegative integer k such that

$$(x_1 - x_2)^k [Y(v, x_1), \phi(x_2)] = 0.$$

Define a vertex operator map $Y_{\mathcal{H}}$ from V to $(\mathrm{End}\, \mathcal{H}^{\log}(W_2, W_3))[[x, x^{-1}]]$ by (4.1). Note that the coefficients of the powers of $\log x$ in logarithmic intertwining operators (recall Definition 2.3) satisfy the Jacobi identity. Following the proof of Theorem 4.2 in [Li2], we have that the coefficients of the powers of $\log x$ in elements of $\mathcal{H}^{\log}(W_2, W_3)$ satisfy the Jacobi identity and thus $(\mathcal{H}^{\log}(W_2, W_3), Y_{\mathcal{H}})$ carries the structure of a weak V-module. Then the same proof as that for Proposition 4.4 shows that a logarithmic intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$ gives a natural V-homomorphism ψ from W_1 to $\mathcal{H}^{\log}(W_2, W_3)$ and that a logarithmic quasi-intertwining operator of type $\binom{W_3}{W_1 W_2}$ (recall Definition 2.5) gives a natural $g(V)_{\geq 0}$-homomorphism from W_1 to $\mathcal{H}^{\log}(W_2, W_3)$. Hence we see that the statements of Proposition 4.5, Theorem 4.6 and Corollary 4.8 also hold with "quasi-intertwining operator" replaced by "logarithmic quasi-intertwining operator," and "intertwining operator" replaced by "logarithmic intertwining operator."

Appendix: The Jacobi identity vs. the commutator formula for modules

In the definition of the notion of module for a vertex (operator) algebra, is the commutator formula enough? That is, does the commutator formula (see (A.1) below) imply the Jacobi identity? The answer is no, as one would expect. In fact the easiest counterexample is quite simple. The following is taken from Remark 4.4.6 of [LL]:

Example A.1 Let V be the 2-dimensional commutative associative algebra with a basis $\{1, a\}$ such that $a^2 = 1$. Then V has a vertex operator algebra structure with $Y(u, x)v = uv$ for $u, v \in V$ and with $\mathbf{1} = 1$ and $\omega = 0$. Equip the 1-dimensional space $W = \mathbb{C}w$ with a linear map $Y_W : V \to \mathrm{Hom}\,(W, W((x)))$ determined by $Y_W(1, x)w = w$, $Y_W(a, x)w = 0$. Then (W, Y_W) satisfies all the axioms for a V-module except the Jacobi identity, and the commutator formula certainly holds (trivially). In fact the Jacobi identity fails since $Y_W(Y(a, x_0)a, x_2)w = w \neq 0 = Y_W(a, x_0 + x_2)Y_W(a, x_2)w$.

We now give some less trivial counterexamples.

Let V be a vertex operator algebra. Let (W, Y_W) be a pair that satisfies all the axioms in the definition of the notion of module for V viewed as a vertex algebra (see Definition 4.1.1 in [LL]) except that the Jacobi identity is replaced by the commutator formula:

$$[Y_W(u, x_1), Y_W(v, x_2)] = \mathrm{Res}_{x_0} x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y_W(Y(u, x_0)v, x_2) \qquad (\text{A.1})$$

for $u, v \in V$, and in addition assume that the $L(-1)$-derivative property also holds on W. Then W is naturally a restricted $g(V)$-module of level 1. Conversely, let W be a restricted $g(V)$-module of level 1. Define a linear map $Y_W : V \to \mathrm{Hom}\,(W, W((x)))$ by

$$Y_W(v, x) = \sum_{n\in\mathbb{Z}} v(n)x^{-n-1}.$$

Then (W, Y_W) satisfies all the axioms in the definition of weak V-module except that the Jacobi identity is replaced by the commutator formula, and in addition, the $L(-1)$-derivative property holds. We have:

Proposition A.2 *Unless the vertex operator algebra V is one-dimensional, there exists a restricted $g(V)$-module of level 1 that is not a weak V-module. Furthermore, such an example can be chosen to satisfy the two grading restriction conditions in the definition of the notion of V-module if V has no elements of negative weight and $V_{(0)} = \mathbb{C}\mathbf{1}$. In particular, for any such vertex operator algebra V, the Jacobi identity cannot be replaced by the commutator formula in the definition of the notion of module.*

Proof In view of the creation property and vacuum property, $\mathbb{C}\mathbf{1}$ is a $(g(V)_{\geq 0} \oplus \mathbb{C}\mathbf{1}(-1))$-submodule of V, with $g(V)_{\geq 0}$ acting trivially and $\mathbf{1}(-1)$ acting as the identity. Form the induced $g(V)$-module

$$W = U(g(V)) \otimes_{U(g(V)_{\geq 0}\oplus\mathbb{C}\mathbf{1}(-1))} \mathbb{C}\mathbf{1},$$

which is of level 1. It follows by induction that W is a restricted $g(V)$-module (of level 1). By the Poincaré-Birkhoff-Witt theorem and Proposition 3.1 we have

$$W = U(g(V)_{<0}/\mathbb{C}\mathbf{1}(-1)) \simeq S(V/\mathbb{C}\mathbf{1}) \tag{A.2}$$

as a vector space. Notice that $\mathrm{wt}\, v_{-n} = \mathrm{wt}\, v + n - 1 > 0$ for homogeneous vector v of positive weight and for $n \geq 1$. If V has no elements of negative weight and $V_{(0)} = \mathbb{C}\mathbf{1}$, W satisfies the two grading restriction conditions in the definition of the notion of V-module.

Now, we claim that W is not a weak V-module if $\dim V > 1$. Otherwise, with $g(V)_{\geq 0}(1 \otimes \mathbf{1}) = 0$, the standard generator $1 \otimes \mathbf{1}$ of W is a vacuum-like vector and we have a V-homomorphism from V into W sending v to $v_{-1}(1\otimes\mathbf{1})$ $(= v(-1)\otimes\mathbf{1})$ for $v \in V$ (see [Li1]; cf. Section 4.7 of [LL]). The image of this map is $g(V)_{<0}(1\otimes\mathbf{1}) = g(V)(1\otimes\mathbf{1})$ by Proposition 3.1, and this space is a V-submodule of W by Proposition 3.4. Thus the map is surjective. That is,

$$W = \{v(-1)\otimes\mathbf{1} \mid v \in V\}.$$

But (A.2) implies that $\{v(-1)\otimes\mathbf{1} \mid v \in V\}$ is a proper subspace of W unless $V/\mathbb{C}\mathbf{1} = 0$. Thus W is not a weak V-module if $\dim V > 1$. $\square$

For a vertex operator algebra V, the Lie algebra $g(V)$ is naturally a $\mathbb{Z}$-graded Lie algebra $g(V) = \coprod_{n\in\mathbb{Z}} g(V)_{(n)}$, where the $\mathbb{Z}$-grading is given by $L(0)$-weights:

$$\mathrm{wt}\,(u \otimes t^m) = \mathrm{wt}\, u - m - 1$$

for homogeneous u and for $m \in \mathbb{Z}$. For any $n \in \mathbb{Z}$, we have

$$\begin{aligned} g(V)_{(n)} &= \sum_{m\in\mathbb{Z}} \pi(V_{(m)} \otimes t^{m-1-n}) \\ &= \operatorname{span}\{v(m-1-n) \mid v \in V_{(m)},\ m \in \mathbb{Z}\}. \end{aligned}$$

We set

$$g(V)_{(\pm)} = \coprod_{n>0} g(V)_{(\pm n)}.$$

(Note the distinction between $g(V)_{(-)}$ and $g(V)_{<0}$.) Clearly, the Lie subalgebras $g(V)_{\geq 0}$ and $g(V)_{<0}$ are also graded subalgebras of $g(V)$. Note that if V has no elements of negative weight, i.e., if $V = \coprod_{n\geq 0} V_{(n)}$, then $g(V)_{(-)}$ is a subalgebra of $g(V)_{\geq 0}$. If in addition $V_{(0)} = \mathbb{C}\mathbf{1}$, we also have $g(V)_{(0)} \subset g(V)_{\geq 0} \oplus \mathbb{C}\mathbf{1}(-1)$.

Remark A.3 Here we give another construction of counterexamples. Let V be any nonzero vertex operator algebra. If the conformal vector ω is zero, then $V = \operatorname{Ker} L(-1) = V_{(0)}$ is simply a finite-dimensional commutative associative algebra with identity. If $\dim V = 1$, a weak V-module simply amounts to a vector space. If $\dim V > 1$, we have already seen that a restricted $g(V)$-module of level 1 is not necessarily a V-module. Now assume that $\omega \neq 0$. Let $V = \coprod_{n\geq 0} V_{(r+n)}$ with $V_{(r)} \neq 0$. Then $V_{(r)}$ is naturally a module for the Lie subalgebra $g(V)_{(0)} \oplus g(V)_{(-)}$ of $g(V)$. Consider the induced $g(V)$-module

$$M = U(g(V)) \otimes_{U(g(V)_{(0)}\oplus g(V)_{(-)})} V_{(r)}.$$

Clearly, M is a $\mathbb{Z}$-graded $g(V)$-module of level 1 where the grading is given by the $L(0)$-eigenspaces. By the Poincaré-Birkhoff-Witt theorem,

$$M \simeq U(g(V)_{(+)}) \otimes V_{(r)}, \tag{A.3}$$

which implies that $M_{(n)} = 0$ for $n < r$ and $M_{(r)} = V_{(r)}$. Consequently, M is a restricted $g(V)$-module (of level 1) and hence M satisfies all the axioms in the definition of weak V-module except that the Jacobi identity is replaced by the commutator formula. We claim that M is not a weak V-module. Otherwise, by Proposition 3.4 we would have

$$M = g(V)V_{(r)} = g(V)_{(+)}V_{(r)} + V_{(r)}.$$

Combining this with (A.3) we must have $g(V)_{(+)} = 0$. But $0 \neq L(-2) \in g(V)_{(+)}$, since $L(-2)\mathbf{1} = \omega \neq 0$, a contradiction. Thus M is not a weak V-module.

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