# Full field algebras

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**Abstract**

We introduce a notion of full field algebra which is essentially an algebraic formulation of the notion of genus-zero full conformal field theory. For any vertex operator algebras $V^L$ and $V^R$, $V^L \otimes V^R$ is naturally a full field algebra and we introduce a notion of full field algebra over $V^L \otimes V^R$. We study the structure of full field algebras over $V^L \otimes V^R$ using modules and intertwining operators for $V^L$ and $V^R$. For a simple vertex operator algebra $V$ satisfying certain natural finiteness and reductivity conditions needed for the Verlinde conjecture to hold, we construct a bilinear form on the space of intertwining operators for $V$ and prove the nondegeneracy and other basic properties of this form. The proof of the nondegenracy of the bilinear form depends not only on the theory of intertwining operator algebras but also on the modular invariance for intertwining operator algebras through the use of the results obtained in the proof of the Verlinde conjecture by the first author. Using this nondegenerate bilinear form, we construct a full field algebra over $V \otimes V$ and an invariant bilinear form on this algebra.

## 0 Introduction

In the present paper, we solve the problem of constructing a genus-zero full conformal field theory (a conformal field theory on genus-zero Riemann surfaces containing both chiral and antichiral parts) from representations of a simple vertex operator algebra $V$ satisfying the following conditions: (i) $V_{(n)} = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$, and $W_{(0)} = 0$ for any irreducible $V$-module $W$ which is not equivalent to $V$. (ii) Every $\mathbb{N}$-gradable weak $V$-module is completely reducible. (iii) $V$ is $C_2$-cofinite. Note that the last two conditions are equivalent to a single condition that every weak $V$-module is completely reducible (see [L] and [ABD]).

Conformal field theories in its original form, as formulated by Belavin, Polyakov and Zamolodchikov [BPZ] and by Kontsevich and Segal [S1] [S2] [S3], have both chiral and antichiral parts. The fundamental work [MS1] [MS2] of Moore and Seiberg is also based on the existence of such full conformal field theories with both chiral and antichiral parts. In mathematics, however, it is mostly chiral conformal field theories that are constructed and studied. To use conformal field theory to solve mathematical problems and to understand mathematical results such as mirror symmetry, we need full conformal field theories, not just chiral or antichiral ones.

In the case of conformal field theories associated to tori, Tsukada first constructed and studied these theories mathematically in his Ph. D. thesis under the direction of I. Frenkel (see [Ts]). Assuming the existence of the structure of a modular tensor category on the category of modules for a vertex operator algebra, the existence of conformal blocks with monodromies compatible with the modular tensor category and all the necessary convergence properties, Felder, Fröhlich, Fuchs and Schweigert [FFFS] and Fuchs, Runkel, Schweigert and Fjelstad [FRS1] [FRS2] [FRS3] [FFRS] studied open-closed conformal field theories (in particular full (closed) conformal field theories) using the theory of tensor categories and three-dimensional topological field theories. They constructed correlation functions as states in some three-dimensional topological field theories and they showed the existence of consistent operator product expansion coefficients for bulk operators. However, since these works were based on the fundamental assumptions mentioned above, an explicit construction of the corresponding full conformal field theories, even in the genus-zero case, is still needed.

In [KO], Kapustin and Orlov studied full conformal field theories associated to tori. They introduced a notion of vertex algebra which is more general than the original notion of vertex algebra [Bo] or vertex operator algebra [FLM] by allowing both chiral and antichiral parts. In [R1] and [R2], Rosellen studied these algebras in details. However, the construction of the full conformal field theories associated to affine Lie algebras (the WZNZ models) and to the Virasoro algebra (the minimal models) was still an open problem. More generally, we would like to construct full conformal field theories from the representations of a vertex operator algebra satisfying reasonable conditions. Also, since the braid group representations obtained from the representations of these vertex operator algebras are not one dimensional in general, it seems that the corresponding full conformal field theories in general do not satisfy the axioms for the algebras introduced and studied in [KO], [R1] and [R2].

In [H7], [H8] and [H10], the first author constructed genus-zero chiral theories, genus-one chiral theories and modular tensor categories from the representations of simple vertex operator algebras satisfying the three conditions above. Since modular tensor categories give modular functors (see [Tu] and [BK]), these results also give modular functors. One of the remaining problems is to construct a full conformal field theory from a chiral theory and an antichiral theory obtained from the chiral theory. In the present paper, we solve this problem in the genus-zero case by constructing a full conformal field theory corresponding to what physicists call a diagonal theory (see, for example, [MS3]) from the representations of a simple vertex operator algebra satisfying the conditions above. The same full conformal field theory is also constructed by the second author using the theory of tensor categories in [K]. The genus-one case and the higher-genus case can be obtained from the construction of the genus-zero theories in this paper and the properties of genus-one and higher-genus chiral theories. These will be discussed in future publications.

Technically, we construct a genus-zero full conformal field theory as follows: We first introduce a notion of full field algebra and several variants, which are essentially algebraic formulations of the notion of genus-zero full conformal field theory (for a precise discussion of the equivalence of this notion of full field algebra and its variants with geometric formulations of genus-zero conformal field theories in terms of operads, see [K]). For a simple vertex operator algebra satisfying the three conditions above, by the results in [H7], we have an intertwining operator algebra, which is equivalent to a genus-zero chiral conformal field theory (see [H3] and [H4]). The genus-zero chiral conformal field theory also gives a genus-zero antichiral conformal field theory. We construct a nondegenerate bilinear form on the space of intertwining operators and use this bilinear form to put the genus-zero chiral and antichiral conformal field theories together. We show that the resulting mathematical object is a full field algebra satisfying additional properties and thus gives a genus-zero full conformal field theory. One interesting aspect of our construction is that our construction (actually the proof of the nondegeneracy of the bilinear form on the space of the intertwining operators) needs the theorem proved in [H9] (see also [H6]) stating that the Verlinde conjecture holds for such a vertex operator algebra. This theorem in [H9], and thus also our construction of genus-zero full conformal field theories, depend not only on genus-zero chiral theories constructed in [H7], but also on genus-one chiral theories constructed in [H8].

This paper is organized as follows: In Section 1, we introduce the notion of full field algebra and several variants and discuss their basic properties. In Section 2, we discuss basic relations between intertwining operator algebras and full field algebras. This is a section preparing for our construction in Section 3. Our construction of full field algebras is given in Section 3. We also construct invariant bilinear forms on these full field algebras in the same section.

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# 1 Definitions and basic properties

Let $\mathbb{F}_n(\mathbb{C}) = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ if } i \neq j\}$. For an $\mathbb{R}$-graded vector space $F = \coprod_{r\in\mathbb{R}} F_{(r)}$, we let $\overline{F} = \prod_{r\in\mathbb{R}} F_{(r)}$ be the algebraic completion of $F$. For $r \in \mathbb{R}$, let $P_r$ be the projection from $F$ or $\overline{F}$ to $F_{(r)}$. A series $\sum f_n$ in $\overline{F}$ is said to be *absolutely convergent* if for any $f' \in F'$, $\sum |\langle f', f_n\rangle|$ is convergent. The sums $\sum |\langle f', f_n\rangle|$ for $f' \in F'$ define a linear functional on $F'$. We call this linear functional the *sum* of the series and denote it by the same notation $\sum f_n$. If the homogeneous subspaces of $F$ are all finite-dimensional, then $\overline{F} = (F')^*$ and, in this case, the sum of an absolutely convergent series is always in $\overline{F}$. When the sum is in $\overline{F}$, we say that the series is *absolutely convergent in* $\overline{F}$.

**Definition 1.1** A *full field algebra* is an $\mathbb{R}$-graded vector space $F = \coprod_{r\in\mathbb{R}} F_{(r)}$ (graded by *total conformal weight* or simply *total weight*), equipped with *correlation function maps*

$$\begin{array}{rrcl} m_n : & F^{\otimes n} \times \mathbb{F}_n(\mathbb{C}) & \to & \overline{F} \\ & (u_1 \otimes \cdots \otimes u_n, (z_1, \dots, z_n)) & \mapsto & m_n(u_1, \dots, u_n; z_1, \bar{z}_1, \dots, z_n, \bar{z}_n), \end{array}$$

for $n \in \mathbb{Z}_+$ and a distinguished element $\mathbf{1}$ called *vacuum* satisfying the following axioms:

1. For $n \in \mathbb{Z}_+$, $m_n(u_1, \dots, u_n; z_1, \bar{z}_1, \dots, z_n, \bar{z}_n)$ is linear in $u_1, \dots, u_n$ and smooth in the real and imaginary parts of $z_1, \dots, z_n$.

2. For $u \in F$, $m_1(u; 0, 0) = u$.

3. For $n \in \mathbb{Z}_+$, $u_1, \dots, u_n \in F$,

$$\begin{aligned} & m_{n+1}(u_1, \dots, u_n, \mathbf{1}; z_1, \bar{z}_1, \dots, z_n, \bar{z}_n, z_{n+1}, \bar{z}_{n+1}) \\ & \quad = m_n(u_1, \dots, u_n; z_1, \bar{z}_1, \dots, z_n, \bar{z}_n). \end{aligned}$$

4. The *convergence property*: For $k, l_1, \dots, l_k \in \mathbb{Z}_+$ and $u_1^{(1)}, \dots, u_{l_1}^{(1)}, \dots, u_1^{(k)}, \dots, u_{l_k}^{(k)} \in F$, the series

$$\sum_{r_1,\dots,r_k} m_k(P_{r_1} m_{l_1}(u_1^{(1)}, \dots, u_{l_1}^{(1)}; z_1^{(1)}, \bar{z}_1^{(1)}, \dots, z_{l_1}^{(1)}, \bar{z}_{l_1}^{(1)}), \dots, \\ P_{r_k} m_{l_k}(u_1^{(k)}, \dots, u_{l_k}^{(k)}, z_1^{(k)}, \bar{z}_1^{(k)}, \dots, z_{l_k}^{(k)}, \bar{z}_{l_k}^{(k)}); z_1^{(0)}, \bar{z}_1^{(0)}, \dots, z_k^{(0)}, \bar{z}_k^{(0)}) \tag{1.1}$$

converges absolutely to

$$m_{l_1+\cdots+l_k}(u_1^{(1)}, \dots, u_{l_k}^{(k)}; z_1^{(1)} + z_1^{(0)}, \bar{z}_1^{(1)} + \bar{z}_1^{(0)}, \dots, z_{l_1}^{(1)} + z_1^{(0)}, \\ \bar{z}_{l_1}^{(1)} + \bar{z}_1^{(0)}, \dots, z_1^{(k)} + z_k^{(0)}, \bar{z}_1^{(k)} + \bar{z}_k^{(0)}, \dots, z_{l_k}^{k} + z_k^{(0)}, \bar{z}_{l_k}^{k} + \bar{z}_k^{(0)}). \tag{1.2}$$

when $|z_p^{(i)}| + |z_q^{(j)}| < |z_i^{(0)} - z_j^{(0)}|$ for $i, j = 1, \dots, k$, $i \neq j$ and for $p = 1, \dots, l_i$ and $q = 1, \dots, l_j$.

5. The *permutation property*: For any $n \in \mathbb{Z}_+$ and any $\sigma \in S_n$, we have

$$m_n(u_1, \dots, u_n; z_1, \bar{z}_1, \dots, z_n, \bar{z}_n) \\ = m_n(u_{\sigma(1)}, \dots, u_{\sigma(n)}; z_{\sigma(1)}, \bar{z}_{\sigma(1)}, \dots, z_{\sigma(n)}, \bar{z}_{\sigma(n)}) \tag{1.3}$$

for $u_1, \dots, u_n \in F$ and $(z_1, \dots, z_n) \in \mathbb{F}_n(\mathbb{C})$.

6. Let $\mathbf{d}$ be the grading operator, that is, the operator defined by $\mathbf{d}f = rf$ for $f \in F_{(r)}$. Then for $n \in \mathbb{Z}_+$, $a \in \mathbb{R}$, $u_1, \dots, u_n \in F$,

$$e^{a\mathbf{d}} m_n(u_1, \dots, u_n; z_1, \bar{z}_1, \dots, z_n, \bar{z}_n) \\ = m_n(e^{a\mathbf{d}} u_1, \dots, e^{a\mathbf{d}} u_n; e^a z_1, e^a \bar{z}_1, \dots, e^a z_n, e^a \bar{z}_n).$$

We denote the full field algebra defined above by $(F, m, \mathbf{1})$ or simply by $F$. In the definition above, we use the notations

$$m_n(u_1, \dots, u_n; z_1, \bar{z}_1 \dots, z_n, \bar{z}_n)$$

instead of

$$m_n(u_1, \dots, u_n; z_1, \dots, z_n)$$

to emphasis that these are in general not holomorphic in $z_1, \ldots, z_n$. For $u' \in F'$, $u_1, \ldots, u_n \in F$,

$$\langle u', m_n(u_1, \ldots, u_n; z_1, \bar{z}_1 \ldots, z_n, \bar{z}_n)\rangle$$

as a function of $z_1, \ldots, z_n$ is called a *correlation function*. *Homomorphisms* and *isomorphisms* for full field algebras are defined in the obvious way.

**Remark 1.2** Note that in the convergence property, we require that the multisum is absolutely convergent. This is stronger than the following convergence property: For $k, l \in \mathbb{Z}_+$ and $u_1, \ldots, u_{k-1}, v_1, \ldots, v_l \in F$, the series

$$\sum_r m_k(u_1, \ldots, u_{k-1}, P_r m_l(v_1, \ldots, v_l; z_1^{(k)}, \bar{z}_1^{(k)}, \ldots, z_l^{(k)}, \bar{z}_l^{(k)}); \qquad z_1^{(0)}, \bar{z}_1^{(0)}, \ldots, z_k^{(0)}, \bar{z}_k^{(0)})$$

converges absolutely to

$$m_{k+l}(u_1, \ldots, u_{k-1}, v_1, \ldots, v_l; z_1^{(0)}, \bar{z}_1^{(0)}, \ldots, z_{k-1}^{(0)}, \bar{z}_{k-1}^{(0)}, \ldots, z_1^{(k)} + z_k^{(0)}, \bar{z}_1^{(k)} + \bar{z}_k^{(0)}, \ldots, z_l^k + z_k^{(0)}, \bar{z}_l^k + \bar{z}_k^{(0)}).$$

when $z_i^{(0)} \neq z_j^{(0)}$ for $i, j = 1, \ldots, k$ and $|z_p^{(k)}| < |z_k^{(0)} - z_i^{(0)}|$ for $i = 1, \ldots, k-1$, and for $p = 1, \ldots, l$. However, for the purpose of constructing genus-zero conformal field theory satisfying geometric axioms, this version of the convergence property is actually enough.

Let $(F, m, \mathbf{1})$ be a full field algebra and let

$$\begin{array}{rrcl} \mathbb{Y}: & F^{\otimes 2} \times \mathbb{C}^\times & \to & \overline{F} \\ & (u \otimes v, z, \bar{z}) & \mapsto & \mathbb{Y}(u; z, \bar{z})v \end{array}$$

be given by

$$\mathbb{Y}(u; z, \bar{z})v = m_2(u \otimes v; z, \bar{z}, 0, 0)$$

for $u, v \in F$. The map $\mathbb{Y}$ is called the *full vertex operator map* and for $u \in F$, $\mathbb{Y}(u; z, \bar{z})$ is called the *full vertex operator* associated to $u$. We have the following immediate consequences of the definition:

**Proposition 1.3** *1. The identity property:* $\mathbb{Y}(\mathbf{1}; z, \bar{z}) = I_F$.

*2. The creation property:* $\lim_{z\to 0}\mathbb{Y}(u;z,\bar z)\mathbf{1}=u$.

*3. For* $f\in F$,

$$[\mathbf{d},\mathbb{Y}(u;z,\bar z)]=\left(z\frac{\partial}{\partial z}+\bar z\frac{\partial}{\partial \bar z}\right)\mathbb{Y}(u;z,\bar z)+\mathbb{Y}(\mathbf{d}u;z,\bar z) \tag{1.4}$$

*4. The total weight of the vacuum* $\mathbf{1}$ *is* 0*, that is,* $\mathbf{d1}=0$.

*Proof.* For $u\in F$,

$$\begin{aligned}\mathbb{Y}(\mathbf{1};z,\bar z)u &= m_2(\mathbf{1},u;z,\bar z,0,0)\\ &= m_2(u,\mathbf{1};0,0,z,\bar z)\\ &= m_1(u;0,0)\\ &= u.\end{aligned}$$

For $u\in F$,

$$\begin{aligned}\lim_{z\to 0}\mathbb{Y}(u;z,\bar z)\mathbf{1} &= \lim_{z\to 0} m_2(u,\mathbf{1};z,\bar z,0,0)\\ &= \lim_{z\to 0} m_1(u;z,\bar z)\\ &= m_1(u;0,0)\\ &= u.\end{aligned}$$

For $u,v\in F$ and $a\in\mathbb{R}$,

$$e^{a\mathbf{d}}\mathbb{Y}(u;z,\bar z)v = \mathbb{Y}(e^{a\mathbf{d}}u;e^a z,e^a\bar z)e^{a\mathbf{d}}v. \tag{1.5}$$

Taking derivatives of both sides of (1.5) with respect to $a$, letting $a=0$ and noticing that $v$ is arbitrary, we obtain (1.4).

From the identity property,

$$\left(z\frac{\partial}{\partial z}+\bar z\frac{\partial}{\partial \bar z}\right)\mathcal{Y}(\mathbf{1};z,\bar z)=0.$$

Then by (1.4), we obtain

$$[\mathbf{d},\mathbb{Y}(\mathbf{1};z,\bar z)]=\mathbb{Y}(\mathbf{d1};z,\bar z). \tag{1.6}$$

Since $\mathbb{Y}(\mathbf{1};z,\bar z)=I_F$, (1.6) gives

$$\mathbb{Y}(\mathbf{d1};z,\bar z)=0. \tag{1.7}$$

Applying (1.7) to $\mathbf{1}$, taking the limit $z \to 0$ on both sides of the resulting formula and using the creation property, we obtain $\mathbf{d}\mathbf{1} = 0$. So the total weight of $\mathbf{1}$ is 0. ∎

Now we discuss two important properties of full field algebras which follow also immediately from the definition.

**Proposition 1.4 (Associativity)** *For $u_1, u_2, u_3 \in F$,*

$$\mathbb{Y}(u_1; z_1, \bar{z}_1)\mathbb{Y}(u_2; z_2, \bar{z}_2)u_3 = \mathbb{Y}(\mathbb{Y}(u_1; z_1 - z_2, \bar{z}_1 - \bar{z}_2)u_2; z_2, \bar{z}_2)u_3 \qquad (1.8)$$

*when $|z_1| > |z_2| > |z_1 - z_2| > 0$.*

*Proof.* The convergence property says, in particular, that the series

$$\mathbb{Y}(u_1; z_1, \bar{z}_1)\mathbb{Y}(u_2; z_2, \bar{z}_2)u_3 = \sum_{n \in \mathbb{R}} \mathbb{Y}(u_1; z_1, \bar{z}_1)P_n\mathbb{Y}(u_2; z_2, \bar{z}_2)u_3, \qquad (1.9)$$

(a *product* of full vertex operators) converges absolutely in $\overline{F}$ for $u_1, u_2, u_3 \in F$ when $|z_1| > |z_2| > 0$. The convergence property also says, in particular, that the series

$$\begin{aligned}\mathbb{Y}(\mathbb{Y}(u_1; z_1 - z_2, \bar{z}_1 - \bar{z}_2)u_2, z_2, \bar{z}_2)u_3 \\ = \sum_{n \in \mathbb{R}} \mathbb{Y}(P_n\mathbb{Y}(u_1; z_1 - z_2, \bar{z}_1 - \bar{z}_2)u_2, z_2, \bar{z}_2)u_3\end{aligned} \qquad (1.10)$$

(an *iterate* of full vertex operators) converges absolutely for $u_1, u_2, u_3 \in F$ when $|z_2| > |z_1 - z_2| > 0$. Moreover, the convergence property also says that both (1.9) and (1.10) converge absolutely to

$$m_3(u_1, u_2, u_3; z_1, \bar{z}_1, z_2, \bar{z}_2, 0, 0).$$

This proves the associativity. ∎

**Proposition 1.5 (Commutativity)** *For $u_1, u_2, u_3 \in F$,*

$$\mathbb{Y}(u_1; z_1, \bar{z}_1)\mathbb{Y}(u_2; z_2, \bar{z}_2)u_3, \qquad (1.11)$$

$$\mathbb{Y}(u_2; z_2, \bar{z}_2)\mathbb{Y}(u_1; z_1, \bar{z}_1)u_3, \qquad (1.12)$$

*are the expansions of*

$$m_3(u_1, u_2, u_3; z_1, \bar{z}_1, z_2, \bar{z}_2, 0, 0).$$

*in the sets given by $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively.*

*Proof.* By the convergence property, we know that (1.11) and (1.12) converge absolutely to

$$m_3(u_1, u_2, u_3; z_1, \bar{z}_1, z_2, \bar{z}_2, 0, 0)$$

and

$$m_3(u_2, u_1, u_3; z_2, \bar{z}_2, z_1, \bar{z}_1, 0, 0),$$

respectively, when $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively. By the permutation property,

$$m_3(u_1, u_2, u_3; z_1, \bar{z}_1, z_2, \bar{z}_2, 0, 0) = m_3(u_2, u_1, u_3; z_2, \bar{z}_2, z_1, \bar{z}_1, 0, 0).$$

Thus (1.11) and (1.12) converge absolutely to

$$m_3(u_1, u_2, u_3; z_1, \bar{z}_1, z_2, \bar{z}_2, 0, 0)$$

when $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively. So they are the expansions of

$$m_3(u_1, u_2, u_3; z_1, \bar{z}_1, z_2, \bar{z}_2, 0, 0).$$

in the sets given by $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively. ∎

Before proving more properties, we would like to discuss first the problem of constructing full field algebras. It is clear that vertex operator algebras have structures of full field algebras. Let $(V^L, Y^L, \mathbf{1}^L, \omega^L)$ and $(V^R, Y^R, \mathbf{1}^R, \omega^R)$ be two vertex operator algebras. Consider the graded vector space $V^L \otimes V^R$ equipped with the correlation function maps, the vacuum and the operator $\mathbf{d}$ given as follows: For $n \in \mathbb{Z}_+$, $u_1^L, \dots, u_n^L \in V^L$ and $u_1^R, \dots, u_n^R \in V^R$, $m_n(u_1^L \otimes u_1^R, \dots, u_n^L \otimes u_n^R; z_1, \bar{z}_1, \dots, z_n, \bar{z}_n)$ are given by the analytic extensions of

$$(Y^L(u_1^L, z_1) \otimes Y^R(u_1^R, \bar{z}_1)) \cdots (Y^L(u_n^L, z_n) \otimes Y^R(u_n^R, \bar{z}_n))\mathbf{1}.$$

Then we take the vacuum $\mathbf{1} = \mathbf{1}^L \otimes \mathbf{1}^R$ and the operators $\mathbf{d} = L^L(0) \otimes I_{V^R} + I_{V^L} \otimes L^R(0)$. In particular, the full vertex operators are given by

$$\mathbb{Y}(u^L \otimes u^R, z)v^L \otimes v^R = Y^L(u^L, z)v^L \otimes Y^R(u^R, \bar{z})v^R.$$

for $u^L, v^L \in V^L, u^R, v^R \in V^R$ and $z \in \mathbb{C}^\times$.

We have:

**Proposition 1.6** *The vector space $V^L \otimes V^R$ equipped with the correlation function maps and the vacuum $\mathbf{1}$ given above is a full field algebra.*

*Proof.* The proof is a straightforward and easy verification. ∎

Note that there is also a vertex operator algebra structure on $V^L \otimes V^R$. For simplicity, we shall use the notation $V^L \otimes V^R$ to denote both the vertex operator algebra and the full field algebra structure. It should be easy to see which structure we will be using in the remaining part of this paper.

The full field algebra $V^L \otimes V^R$ in general does not give a genus-one theory, that is, suitable $q$-traces, even in the case that they are convergent, of the full vertex operators in general are not modular invariant. For chiral theories, we know from [H8] that if we consider the intertwining operator algebras constructed from irreducible modules for suitable vertex operator algebras, we do have modular invariance. So it is then natural to look for full field algebras from suitable extensions of $V^L \otimes V^R$ by $V^L \otimes V^R$-modules.

Note that $V^L \otimes V^R$ has an $\mathbb{Z} \times \mathbb{Z}$-grading with grading operators being $L^L(0) \otimes I_{V^R}$ and $I_{V^L} \otimes L^R(0)$. If a full field algebra is an extension of $V^L \otimes V^R$ by $V^L \otimes V^R$-modules, it has an $\mathbb{R} \times \mathbb{R}$-grading.

For any $\mathbb{R} \times \mathbb{R}$-graded vector space $F = \coprod_{(m,n)\in\mathbb{R}\times\mathbb{R}} F_{(m,n)}$, we have a *left grading operator* $\mathbf{d}^L$ and a *right grading operator* $\mathbf{d}^R$ defined by $\mathbf{d}^L u = mu, \mathbf{d}^R u = nu$ for $u \in E_{(m,n)}$, where $m$ $(n)$ is called the left (right) weight of $u$ and is denoted by $\mathrm{wt}^L u$ $(\mathrm{wt}^R u)$. For $m, n \in \mathbb{R}$, let $P_{m,n}$ be the projection from $F \to F_{(m,n)}$. We still use $F'$ and $\overline{F}$ to denote the graded dual and the algebraic completion of $F$, but note that they are with respect to the $\mathbb{R} \times \mathbb{R}$-grading, not any $\mathbb{R}$-grading induced from the $\mathbb{R} \times \mathbb{R}$-grading.

**Definition 1.7** An $\mathbb{R}\times\mathbb{R}$-graded full field algebra is a full field algebra $(F, m, \mathbf{1})$ equipped with an $\mathbb{R} \times \mathbb{R}$-grading on $F$ (graded by *left conformal weight* or *left weight* and *right conformal weight* or *right weight* and thus equipped with left and right grading operators $\mathbf{d}^L$ and $\mathbf{d}^R$) and operators $D^L$ and $D^R$ satisfying the following conditions:

1. The *grading compatibility*: $\mathbf{d} = \mathbf{d}^L + \mathbf{d}^R$.

2. The *single-valuedness property*: $e^{2\pi i(\mathbf{d}^L - \mathbf{d}^R)} = I_F$.

3. The *convergence property*: For $k, l_1, \ldots, l_k \in \mathbb{Z}_+$ and $u_1^{(1)}, \ldots, u_{l_1}^{(1)}, \ldots, u_1^{(k)}, \ldots, u_{l_k}^{(k)} \in F$, the series

$$\sum_{p_1,q_1,\ldots,p_k,q_k} m_k(P_{p_1,q_1} m_{l_1}(u_1^{(1)}, \ldots, u_{l_1}^{(1)}; z_1^{(1)}, \bar{z}_1^{(1)}, \ldots, z_{l_1}^{(1)}, \bar{z}_{l_1}^{(1)}), \ldots,$$

$$P_{p_k,q_k} m_{l_k}(u_1^{(k)},\dots,u_{l_k}^{(k)}; z_1^{(k)}, \bar{z}_1^{(k)},\dots, z_{l_k}^{(k)}, \bar{z}_{l_k}^{(k)}); z_1^{(0)}, \bar{z}_1^{(0)},\dots, z_k^{(0)}, \bar{z}_k^{(0)}) \tag{1.13}$$

converges absolutely to (1.2) when $|z_p^{(i)}| + |z_q^{(j)}| < |z_i^{(0)} - z_j^{(0)}|$ for $i, j = 1,\dots,k$, $i \neq j$ and for $p = 1,\dots,l_i$ and $q = 1,\dots,l_j$.

4. *The* $\mathbf{d}^L$*- and* $\mathbf{d}^R$*-bracket properties*: For $u \in F$,

$$\left[\mathbf{d}^L, \mathbb{Y}(u; z, \bar{z})\right] = z\frac{\partial}{\partial z}\mathbb{Y}(u; z, \bar{z}) + \mathbb{Y}(\mathbf{d}^L u; z, \bar{z}) \tag{1.14}$$

$$\left[\mathbf{d}^R, \mathbb{Y}(u; z, \bar{z})\right] = \bar{z}\frac{\partial}{\partial \bar{z}}\mathbb{Y}(u; z, \bar{z}) + \mathbb{Y}(\mathbf{d}^R u; z, \bar{z}). \tag{1.15}$$

5. The $D^L$*- and* $D^R$*-derivative property*: For $u \in F$,

$$\left[D^L, \mathbb{Y}(u; z, \bar{z})\right] = \mathbb{Y}(D^L u; z, \bar{z}) = \frac{\partial}{\partial z}\mathbb{Y}(u; z, \bar{z}), \tag{1.16}$$

$$\left[D^R, \mathbb{Y}(u; z, \bar{z})\right] = \mathbb{Y}(D^R u; z, \bar{z}) = \frac{\partial}{\partial \bar{z}}\mathbb{Y}(u; z, \bar{z}). \tag{1.17}$$

We denote the $\mathbb{R}\times\mathbb{R}$-graded full field algebra defined above by $(F, m, \mathbf{1}, D^L, D^R)$ or simply by $F$. But note that there is a refined grading on $F$ now.

**Remark 1.8** Note that for $\mathbb{R} \times \mathbb{R}$-graded full field algebra, there is also a weaker convergence property similar to the one in Remark 1.2.

**Remark 1.9** The single-valuedness property actually says that $F$ is graded by a subgroup $\{(m, n) \in \mathbb{R} \times \mathbb{R} \mid m - n \in \mathbb{Z}\}$ of $\mathbb{R} \times \mathbb{R}$. This single-valuedness indeed corresponds to a certain single-valuedness condition in the geometric axioms for full conformal field theories.

We have the following immediate consequences of the definition above.

**Proposition 1.10** *1. The pair (left weight, right weight) for* $\mathbf{1}$ *is* $(0, 0)$*, that is,* $\mathbf{d}^L\mathbf{1} = \mathbf{d}^R\mathbf{1} = 0$.

*2. The pairs (left weight, right weight) for* $D^L$ *and* $D^R$ *are* $(1, 0)$ *and* $(0, 1)$*, respectively, that is,*

$$\begin{aligned} [\mathbf{d}^L, D^L] &= D^L, \\ [\mathbf{d}^R, D^L] &= 0, \\ [\mathbf{d}^L, D^R] &= 0, \\ [\mathbf{d}^R, D^R] &= D^R. \end{aligned}$$

*3.* $D^L\mathbf{1} = D^R\mathbf{1} = 0$.

*Proof.* From the identity property,

$$z\frac{\partial}{\partial z}\mathbb{Y}(\mathbf{1}; z, \bar{z}) = 0.$$

Then by (1.14), we obtain

$$\left[\mathbf{d}^L, \mathbb{Y}(\mathbf{1}; z, \bar{z})\right] = \mathbb{Y}(\mathbf{d}^L\mathbf{1}; z, \bar{z}). \tag{1.18}$$

Since $\mathbb{Y}(\mathbf{1}; z, \bar{z}) = I_F$, we obtain

$$\mathbb{Y}(\mathbf{d}^L\mathbf{1}; z, \bar{z}) = 0 \tag{1.19}$$

from (1.18). Applying (1.19) to $\mathbf{1}$, taking the limit $z \to 0$ on both sides of (1.19) and using the creation property, we obtain $\mathbf{d}^L\mathbf{1} = 0$. So the left weight of $\mathbf{1}$ is 0. Similarly, we can prove that the right weight of $\mathbf{1}$ is 0.

Applying both sides of (1.14) to $\mathbf{1}$, taking the limit $z \to 0$ and then using the creation property and the fact $\mathbf{d}^L\mathbf{1} = 0$ we have just proved, we obtain

$$\lim_{z\to 0} z\frac{\partial}{\partial z}\mathbb{Y}(u; z, \bar{z})\mathbf{1} = 0 \tag{1.20}$$

for $u \in F$. Applying $\frac{\partial}{\partial z}$ to both sides of (1.14) and using the $D^L$-derivative property, we obtain

$$\begin{aligned}
&\left[\mathbf{d}^L, \mathbb{Y}(D^L u; z, \bar{z})\right] \\
&\quad = \frac{\partial}{\partial z} z\frac{\partial}{\partial z}\mathbb{Y}(u; z, \bar{z}) + \mathbb{Y}(D^L\mathbf{d}^L u; z, \bar{z}) \\
&\quad = z\frac{\partial}{\partial z}\frac{\partial}{\partial z}\mathbb{Y}(u; z, \bar{z}) + \frac{\partial}{\partial z}\mathbb{Y}(u; z, \bar{z}) + \mathbb{Y}(D^L\mathbf{d}^L u; z, \bar{z}) \\
&\quad = z\frac{\partial}{\partial z}\mathbb{Y}(D^L u; z, \bar{z}) + \mathbb{Y}(D^L u; z, \bar{z}) + \mathbb{Y}(D^L\mathbf{d}^L u; z, \bar{z}).
\end{aligned} \tag{1.21}$$

Applying (1.21) to $\mathbf{1}$, taking the limit $z \to 0$ on both sides of (1.21) and using the creation property, (1.20) and $\mathbf{d}^L\mathbf{1} = 0$, we obtain

$$\mathbf{d}^L D^L u = D^L u + D^L\mathbf{d}^L u,$$

proving that the left weight of $D^L$ is 1. Similarly, we can prove that the right weight of $D^L$ is 0, the left weight of $D^R$ is 0 and the right weight of $D^R$ is 1.

Using the $D^L$- and $D^R$-derivative properties and the creation property, we see immediately that $D^L\mathbf{1} = D^R\mathbf{1} = 0$. ∎

For an $\mathbb{R}\times\mathbb{R}$-graded full field algebra $(F, m, \mathbf{1}, D^L, D^R)$, we now introduce a formal vertex operator map. We shall use the convention that for any $z \in \mathbb{C}^\times$, $\log z = \log|z| + \sqrt{-1}\arg z$ where $0 \leq \arg z < 2\pi$. For $u \in F$, we use $\mathrm{wt}^L\, u$ and $\mathrm{wt}^R\, u$ to denote the left and right weights, respectively, of $u$. Let $u, v \in F$ and $w' \in F'$ be homogeneous elements. We have

$$\begin{aligned} \langle w', [\mathbf{d}^L, \mathbb{Y}(u; z, \bar{z})]v\rangle &= \langle (\mathbf{d}^L)'w', \mathbb{Y}(u; z, \bar{z})v\rangle - \langle w', \mathbb{Y}(u; z, \bar{z})\mathbf{d}^L v\rangle \\ &= (\mathrm{wt}^L\, w' - \mathrm{wt}^L\, v)\langle w', \mathbb{Y}(u; z, \bar{z})v\rangle \end{aligned} \tag{1.22}$$

where $(\mathbf{d}^L)'$ is the adjoint of $\mathbf{d}^L$. On the other hand,

$$\begin{aligned} &\langle w', \mathbb{Y}(\mathbf{d}^L u; z, \bar{z})v + z\frac{\partial}{\partial z}\mathbb{Y}(u; z, \bar{z})v\rangle \\ &\quad = \left(\mathrm{wt}^L\, u + z\frac{\partial}{\partial z}\right)\langle w', \mathbb{Y}(u; z, \bar{z})v\rangle. \end{aligned} \tag{1.23}$$

Let $f(z, \bar{z}) = \langle w', \mathbb{Y}(u; z, \bar{z})v\rangle$. Then by (1.14), (1.22) and (1.23), we have

$$z\frac{\partial}{\partial z}f(z, \bar{z}) = (\mathrm{wt}^L\, w' - \mathrm{wt}^L\, u - \mathrm{wt}^L\, v)f(z, \bar{z}). \tag{1.24}$$

Similarly, using (1.15), we have

$$\bar{z}\frac{\partial}{\partial \bar{z}}f(z, \bar{z}) = (\mathrm{wt}^R\, w' - \mathrm{wt}^R\, u - \mathrm{wt}^R\, v)f(z, \bar{z}). \tag{1.25}$$

The general solution of the system (1.24) and (1.25) is

$$Cz^{\mathrm{wt}^L w' - \mathrm{wt}^L u - \mathrm{wt}^L v}\bar{z}^{\mathrm{wt}^R w' - \mathrm{wt}^R u - \mathrm{wt}^R v} \tag{1.26}$$

where $C \in \mathbb{C}$. Note that $f(z, \bar{z})$ is a single-valued function and that by the single-valuedness of the full field algebra $F$,

$$(\mathrm{wt}^L\, w' - \mathrm{wt}^L\, u - \mathrm{wt}^L\, v) - (\mathrm{wt}^R\, w' - \mathrm{wt}^R\, u - \mathrm{wt}^R\, v) \in \mathbb{Z}.$$

This means that if we choose any branches of $z^{\mathrm{wt}^L w' - \mathrm{wt}^L u - \mathrm{wt}^L v}$ and $\bar{z}^{\mathrm{wt}^R w' - \mathrm{wt}^R u - \mathrm{wt}^R v}$, then there must be a unique constant $C$ such that $f(z, \bar{z})$ is equal to (1.26). We choose the branches of $z^{\mathrm{wt}^L w' - \mathrm{wt}^L u - \mathrm{wt}^L v}$ and $\bar{z}^{\mathrm{wt}^R w' - \mathrm{wt}^R u - \mathrm{wt}^R v}$ to be $e^{(\mathrm{wt}^L w' - \mathrm{wt}^L u - \mathrm{wt}^L v)\log z}$ and $e^{(\mathrm{wt}^R w' - \mathrm{wt}^R u - \mathrm{wt}^R v)\overline{\log z}}$, respectively. So there is a unique $C \in \mathbb{C}$ such that

$$f(z, \bar{z}) = Ce^{(\mathrm{wt}^L w' - \mathrm{wt}^L u - \mathrm{wt}^L v)\log z}e^{(\mathrm{wt}^R w' - \mathrm{wt}^R u - \mathrm{wt}^R v)\overline{\log z}}. \tag{1.27}$$

Hence $P_{p,q}\mathbb{Y}(u;z,\bar{z})P_{m,n}$, $m,n,p,q\in\mathbb{R}$ can be written as

$$u_{m,n}^{p,q}e^{(p-\mathrm{wt}^L\,u-m)\log z}e^{(q-\mathrm{wt}^R\,u-n)\,\overline{\log z}},$$

where $u_{m,n}^{p,q}$ are linear maps from $F_{(m,n)}\to F_{(p,q)}$ for $m,n,p,q\in\mathbb{R}$. Thus we have the following expansion:

$$\mathbb{Y}(u;z,\bar{z})=\sum_{r,s\in\mathbb{R}}\mathbb{Y}_{l,r}(u)e^{(-l-1)\log z}e^{(-r-1)\,\overline{\log z}} \tag{1.28}$$

where $\mathbb{Y}_{l,r}(u)\in\mathrm{End}\ F$ with $\mathrm{wt}^L\mathbb{Y}_{l,r}(u)=\mathrm{wt}^Lu-l-1$ and $\mathrm{wt}^R\mathbb{Y}_{l,r}(u)=\mathrm{wt}^Ru-r-1$. Moreover, the expansion above is unique. Let $x$ and $\bar{x}$ be independent and commuting formal variables. We define the *formal full vertex operator* $\mathbb{Y}_f$ associated to $u\in F$ by

$$\mathbb{Y}_f(u;x,\bar{x})=\sum_{l,r\in\mathbb{R}}\mathbb{Y}_{l,r}(u)x^{-l-1}\bar{x}^{-r-1}. \tag{1.29}$$

These formal full vertex operators give a *formal full vertex operator map*

$$\mathbb{Y}_f:F\otimes F\mapsto F\{x,\bar{x}\}.$$

For nonzero complex numbers $z$ and $\zeta$, we can substitute $e^{r\log z}$ and $e^{s\,\overline{\log\zeta}}$ for $x^r$ and $\bar{x}^s$, respectively, in $\mathbb{Y}_f(u;x,\bar{x})$ to obtain a map $\mathbb{Y}_{\mathrm{an}}(u;z,\zeta):F\to\overline{F}$ called the *analytic full vertex operator map.*

The following propositions are clear:

**Proposition 1.11** *For $u\in F$ and $z,\zeta\in\mathbb{C}^\times$, we have*

$$\mathbb{Y}_{\mathrm{an}}(u;z,\zeta)=z^{\mathbf{d}^L}\zeta^{\mathbf{d}^R}\mathbb{Y}(z^{-\mathbf{d}^L}\zeta^{-\mathbf{d}^R}u;1,1)z^{-\mathbf{d}^L}\zeta^{-\mathbf{d}^R}. \tag{1.30}$$

*For formal full vertex operators, we have*

$$\mathbb{Y}_f(u;x,\bar{x})=x^{\mathbf{d}^L}\bar{x}^{\mathbf{d}^R}\mathbb{Y}(x^{-\mathbf{d}^L}\bar{x}^{-\mathbf{d}^R}u;1,1)x^{-\mathbf{d}^L}\bar{x}^{-\mathbf{d}^R}. \tag{1.31}$$

**Proposition 1.12** *For $u\in F$,*

$$\mathbb{Y}_f(\mathbf{1};x,\bar{x})u = u, \tag{1.32}$$

$$\lim_{x\to0,\bar{x}\to0}\mathbb{Y}_f(u;x,\bar{x})\mathbf{1} = u, \tag{1.33}$$

*where $\lim_{x\to0,\bar{x}\to0}$ means taking the constant term of a power series in $x$ and $\bar{x}$. In particular, $\mathbb{Y}_{l,r}(u)\mathbf{1}=0$ for all $l,r\in\mathbb{R}$ and $\mathbb{Y}_{-1,-1}(u)\mathbf{1}=u$.*

**Proposition 1.13** *For $u \in F$, we have*

$$\begin{aligned} \big[D^L, \mathbb{Y}_f(w;x,\bar{x})\big] &= \mathbb{Y}_f(D^L w;x,\bar{x}) = \frac{\partial}{\partial x}\mathbb{Y}_f(w;x,\bar{x}), & (1.34)\\ \big[D^R, \mathbb{Y}_f(w;x,\bar{x})\big] &= \mathbb{Y}_f(D^R w;x,\bar{x}) = \frac{\partial}{\partial \bar{x}}\mathbb{Y}_f(w,x,\bar{x}). & (1.35)\end{aligned}$$

*In particular, we have $D^L\mathbf{1} = D^R\mathbf{1} = 0$ and for $l, r \in \mathbb{R}$,*

$$\begin{aligned} \big[D^L, \mathbb{Y}_{l,r}(u)\big] &= \mathbb{Y}_{l,r}(D^L u) = -l\mathbb{Y}_{l-1,r}(u), & (1.36)\\ \big[D^R, \mathbb{Y}_{l,r}(u)\big] &= \mathbb{Y}_{l,r}(D^R u) = -r\mathbb{Y}_{l,r-1}(u), & (1.37)\end{aligned}$$

We need the following strong version of the creation property:

**Lemma 1.14** *For $u \in F$,*

$$\mathbb{Y}_f(u;x,\bar{x})\mathbf{1} = e^{xD^L + \bar{x}D^R}u. \tag{1.38}$$

*Proof.* Using (1.34) and (1.35), we have

$$\mathbb{Y}_f(e^{x_0 D^L + \bar{x}_0 D^R}u;x,\bar{x}) = \mathbb{Y}_f(u;x+x_0,\bar{x}+\bar{x}_0). \tag{1.39}$$

Now let both sides of (1.39) act on the vacuum $\mathbf{1}$. Since $\mathbb{Y}_f(u;x+x_0,\bar{x}+\bar{x}_0)\mathbf{1}$ involves only nonnegative integer powers of $x+x_0$ and $\bar{x}+\bar{x}_0$, we can take the limit $x \to 0, \bar{x} \to 0$. Then replacing $x_0$ and $\bar{x}_0$ by $x$ and $\bar{x}$, we obtain (1.38). ∎

**Proposition 1.15 (Skew symmetry)** *For any $u, v \in F$ and $z \in \mathbb{C}^\times$, we have*

$$\mathbb{Y}(u;z,\bar{z})v = e^{zD^L + \bar{z}D^R}\mathbb{Y}(v;-z,\overline{-z})u \tag{1.40}$$

*and*

$$\mathbb{Y}_f(u;x,\bar{x})v = e^{xD^L + \bar{x}D^R}\mathbb{Y}_f(v;e^{\pi i}x, e^{-\pi i}\bar{x})u. \tag{1.41}$$

*Proof.* From the convergence property, it is clear that, for any $u, v \in F$,

$$\mathbb{Y}(\mathbb{Y}(u;z_1-z_2,\bar{z}_1-\bar{z}_2)v;z_2,\bar{z}_2)\mathbf{1} \tag{1.42}$$

converges absolutely to $m_3(u,v,\mathbf{1};z_1,\bar{z}_1,z_2,\bar{z}_2,0,0)$ when $|z_2| > |z_1 - z_2| > 0$, and

$$\mathbb{Y}(\mathbb{Y}(v;z_2-z_1,\bar{z}_2-\bar{z}_1)u;z_1,\bar{z}_1)\mathbf{1} \tag{1.43}$$

converges absolutely when $|z_1| > |z_1 - z_2| > 0$ to $m_3(v, u, \mathbf{1}; z_2, \bar{z}_2, z_1, \bar{z}_1, 0, 0)$ which is equal to $m_3(u, v, \mathbf{1}; z_1, \bar{z}_1, z_2, \bar{z}_2, 0, 0)$ by the permutation property. Hence, using (1.38), we obtain

$$e^{z_2 D^L + \bar{z}_2 D^R} \mathbb{Y}(u; z_1 - z_2, \bar{z}_1 - \bar{z}_2)v = e^{z_1 D^L + \bar{z}_1 D^R} \mathbb{Y}(v; z_2 - z_1, \bar{z}_2 - \bar{z}_1)u \quad (1.44)$$

when $|z_2| > |z_1 - z_2| > 0$ and $|z_1| > |z_1 - z_2| > 0$. We change the variables from $z_1, z_2$ to $z = z_1 - z_2$ and $z_2$. Then (1.44) gives

$$e^{z_2 D^L + \bar{z}_2 D^R} \mathbb{Y}(u; z, \bar{z})v = e^{(z_2+z) D^L + (\bar{z}_2 + \bar{z}) D^R} \mathbb{Y}(v; -z, \overline{-z})u \quad (1.45)$$

when $|z_2| > |z| > 0$ and $|z_2 + z| > |z| > 0$.

Notice that for fixed $z \neq 0$ and $w' \in F'$,

$$\langle w', e^{z_2 D^L + \bar{z}_2 D^R} \mathbb{Y}(u; z, \bar{z})v \rangle \quad (1.46)$$

involves only positive integral powers of $z_2, \bar{z}_2$ and thus is a power series in $z_2$ and $\bar{z}_2$ absolutely convergent when $|z_2| > |z| > 0$. From complex analysis, we know that a power series in two variables $z_2$ and $\zeta_2$ convergent at $z_2 = z_2^0$ and $\zeta_2 = \zeta_2^0$ must be convergent absolutely when $|z_2| < |z_2^0|$ and $|\zeta_2| < |\zeta_2^0|$. In particular, when $\zeta_2^0 = \bar{z}_2^0$, such a power series must be absolutely convergent when $|z_2| < |z_2^0|$ and $\zeta_2 = \bar{z}_2$. In our case, since for any fixed $z$, (1.46) is absolutely convergent when $|z_2| > |z| > 0$, we conclude that (1.46) converges absolutely for all $z_2$. Since $w'$ is arbitrary, we see that the left-hand side of (1.45) is absolutely convergent in $\overline{F}$ for all $z_2$. Since $z$ is also arbitrary, by the convergence property again, we see that the left-hand side of (1.45) is absolutely convergent in $\overline{F}$ for all $z$ and $z_2$ such that $z \neq 0$. Similarly, the right hand side of (1.45) also converges absolutely in $\overline{F}$ for all $z$ and $z_2$ such that $z \neq 0$.

If $e^{-z_2 D^L - \bar{z}_2 D^R}$ gives a linear operator on $\overline{F}$, then we can just multiply both sides of (1.45) by $e^{-z_2 D^L - \bar{z}_2 D^R}$ to obtain (1.40). In the case that the total weights of $F$ is lower-truncated, $e^{-z_2 D^L - \bar{z}_2 D^R}$ is indeed a linear operator on $\overline{F}$. In the most general case, this might not be true. But we can still obtain (1.40) as follows: Consider the formal series

$$\begin{aligned} &e^{x_1 D^L + \bar{x}_1 D^R} e^{x_2 D^L + \bar{x}_2 D^R} \mathbb{Y}(u; x, \bar{x})v \\ &\quad = e^{(x_1 + x_2) D^L + (\bar{x}_1 + \bar{x}_2) D^R} \mathbb{Y}(u; x, \bar{x})v \end{aligned} \quad (1.47)$$

where $x$, $\bar{x}$, $x_1$, $\bar{x}_1$, $x_2$ and $\bar{x}_2$ are commuting formal variables. Since $\mathbb{Y}(u; z, \bar{z})v$ is absolutely convergent in $\overline{F}$ when $z \neq 0$, we can substitute $z$, $\bar{z}$, $-z_2$, $-\bar{z}_2$, $z_2$

and $\bar{z}_2$ for $x$, $\bar{x}$, $x_1$, $\bar{x}_1$, $x_2$ and $\bar{x}_2$ on the right-hand side of (1.47), respectively, and the resulting series is absolutely convergent in $\overline{F}$. So we can do the same substitution on the left-hand side of (1.47) and the resulting series is absolutely convergent in $\overline{F}$. Similarly, consider the formal series

$$\begin{aligned} & e^{x_1 D^L + \bar{x}_1 D^R} e^{(x_2+x)D^L + (\bar{x}_2+\bar{x})D^R} \mathbb{Y}(v; e^{\pi i}x, e^{-\pi i}\bar{x})u \\ & \quad = e^{(x_1+x_2+x)D^L + (\bar{x}_1+\bar{x}_2+\bar{x})D^R} \mathbb{Y}(v; e^{\pi i}x, e^{-\pi i}\bar{x})u. \end{aligned} \tag{1.48}$$

Since $e^{zD^L+\bar{z}D^R}\mathbb{Y}(v; -z, \overline{-z})u$ is absolutely convergent in $\overline{F}$ when $z \neq 0$, we can substitute $z$, $\bar{z}$, $-z_2$, $-\bar{z}_2$, $z_2$ and $\bar{z}_2$ for $x$, $\bar{x}$, $x_1$, $\bar{x}_1$, $x_2$ and $\bar{x}_2$ on the right-hand side and thus also on the left-hand side of (1.48) and the resulting series is absolutely convergent in $\overline{F}$. The convergence of these series and (1.45) with suitably chosen $z_2$ gives (1.40)

Now (1.41) follows immediately: On the one hand, by (1.31), we have

$$x^{\mathbf{d}^L}\bar{x}^{\mathbf{d}^L}\mathbb{Y}(x^{-\mathbf{d}^L}\bar{x}^{-\mathbf{d}^L}u; 1, 1)x^{-\mathbf{d}^L}\bar{x}^{-\mathbf{d}^L}v = \mathbb{Y}_f(u; x, \bar{x})v. \tag{1.49}$$

On the other hand, we have

$$\begin{aligned} & x^{\mathbf{d}^L}\bar{x}^{\mathbf{d}^L}e^{D^L+D^R}\mathbb{Y}(x^{-\mathbf{d}^L}\bar{x}^{-\mathbf{d}^L}v; -1, -1)x^{-\mathbf{d}^L}\bar{x}^{-\mathbf{d}^L}u \\ & \quad = e^{xD^L+\bar{x}D^R}\mathbb{Y}_f(v; e^{\pi i}x, e^{-\pi i}\bar{x})u. \end{aligned} \tag{1.50}$$

Using skew symmetry (1.40), (1.49) and (1.50), we obtain (1.41). ∎

**Definition 1.16** An $\mathbb{R}\times\mathbb{R}$-graded full field algebra $(F, m, \mathbf{1}, D^L, D^R)$ is called *grading restricted* if it satisfies the following grading-restriction conditions:

1. There exists $M \in \mathbb{R}$ such that $F_{(m,n)} = 0$ if $n < M$ or $m < M$.
2. $\dim F_{(m,n)} < \infty$ for $m, n \in \mathbb{R}$.

We say that $F$ is *lower truncated* if $F$ satisfies the first grading restriction condition.

In this case, for $u \in F$ and $k \in \mathbb{R}$, we have $\sum_{l+r=k} \mathbb{Y}_{l,r}(u) \in \mathrm{End}\, F$ with total weight wt $u - k - 2$. We denote $\sum_{l+r=k} \mathbb{Y}_{l,r}(u)$ by $\mathbb{Y}_{k-1}(u)$. Then we have the expansion

$$\mathbb{Y}_f(u; x, x) = \sum_{k\in\mathbb{R}} \mathbb{Y}_k(u)x^{-k-1}, \tag{1.51}$$

where wt $\mathbb{Y}_k(u) =$ wt $u - k - 1$. For given $u, v \in F$, we have $\mathbb{Y}_k(u)w = 0$ for sufficiently large $k$.

Let $(V^L, Y^L, \mathbf{1}^L, \omega^L)$ and $(V^R, Y^R, \mathbf{1}^R, \omega^R)$ be vertex operator algebras. Let $\rho$ be an injective homomorphism from the full field algebra $V^L \otimes V^R$ to $F$. Then we have $\mathbf{1} = \rho(\mathbf{1}^L \otimes \mathbf{1}^R)$, $\mathbf{d}^L \circ \rho = \rho \circ (L^L(0) \otimes I_{V^R})$, $\mathbf{d}^R \circ \rho = \rho \circ (I_{V^L} \otimes L^R(0)))$, $D^L \circ \rho = \rho \circ (L^L(-1) \otimes I_{V^R})$ and $D^R \circ \rho = \rho \circ (I_{V^L} \otimes L^R(-1))$. Moreover, $F$ has a *left conformal element* $\rho(\omega^L \otimes \mathbf{1}^R)$ and an *right conformal element* $\rho(\mathbf{1}^L \otimes \omega^R)$. We have the following operators on $F$:

$$\begin{aligned} L^L(0) &= \mathrm{Res}_x \mathrm{Res}_{\bar{x}} \bar{x}^{-1} \mathbb{Y}_f(\rho(\omega^L \otimes \mathbf{1}^R); x, \bar{x}), \\ L^R(0) &= \mathrm{Res}_x \mathrm{Res}_{\bar{x}} x^{-1} \mathbb{Y}_f(\rho(\mathbf{1}^L \otimes \omega^L); x, \bar{x}), \\ L^L(-1) &= \mathrm{Res}_x \mathrm{Res}_{\bar{x}} x \bar{x}^{-1} \mathbb{Y}_f(\rho(\omega^L \otimes \mathbf{1}^R); x, \bar{x}), \\ L^R(-1) &= \mathrm{Res}_x \mathrm{Res}_{\bar{x}} x^{-1} \bar{x} \mathbb{Y}_f(\rho(\mathbf{1}^L \otimes \omega^L); x, \bar{x}). \end{aligned}$$

Since these operators are operators on $F$, it should be easy to distinguish them from those operators with the same noptation but acting on $V^L$ or $V^R$.

**Definition 1.17** Let $(V^L, Y^L, \mathbf{1}^L, \omega^L)$ and $(V^R, Y^R, \mathbf{1}^R, \omega^R)$ be vertex operator algebras. A *full field algebra over* $V^L \otimes V^R$ is a grading-restricted $\mathbb{R} \times \mathbb{R}$-graded full field algebra $(F, m, \mathbf{1}, D^L, D^R)$ equipped with an injective homomorphism $\rho$ from the full field algebra $V^L \otimes V^R$ to $F$ such that $\mathbf{d}^L = L^L(0)$, $\mathbf{d}^R = L^R(0)$, $D^L = L^L(-1)$ and $D^R = L^R(-1)$.

We shall denote the full field algebra over $V^L \otimes V^R$ defined above by $(F, m, \rho)$ or simply by $F$.

The following result allows us to construct full field algebras using the representation theory of vertex operator algebras:

**Theorem 1.18** *Let $(V^L, Y^L, \mathbf{1}^L, \omega^L)$ and $(V^R, Y^R, \mathbf{1}^R, \omega^R)$ be vertex operator algebras. Let $(F, m, \rho)$ be a full field algebra over $V^L \otimes V^R$. Then $F$ is a module for $V^L \otimes V^R$ viewed as a vertex operator algebra. Moreover, $\mathbb{Y}_f(\cdot, x, x)$, is an intertwining operator of type $\binom{F}{FF}$.*

*Proof.* Let $\mathbb{Y}^{L,R}$ be the vertex operator map for the full field algebra $\rho(V^L \otimes V^R)$. Then we have

$$\mathbb{Y}^{L,R}(\rho(u^L \otimes u^R); z, \bar{z}) \rho(v^L \otimes v^R) = \rho(Y^L(u^L, z) v^L \otimes Y^R(u^R, \bar{z}) v^R) \quad (1.52)$$

for $u^L, v^L \in V^L$, $u^R, v^R \in V^R$ and $z \in \mathbb{C}^\times$.

Now we show that a splitting formula similar to (1.52) holds for vertex operators of the form $\mathbb{Y}(\rho(u^L \otimes u^R); z, \bar{z}) : F \to \overline{F}$. By the associativity of $\mathbb{Y}$, we have

$$\begin{aligned}
&\langle w', \mathbb{Y}(\rho(u^L \otimes u^R); z_1, \bar{z}_1)\mathbb{Y}(\rho(v^L \otimes v^R); z_2, \bar{z}_2)w\rangle \\
&\quad = \langle w', \mathbb{Y}(\mathbb{Y}^{L,R}(\rho(u^L \otimes u^R); z_1 - z_2, \bar{z}_1 - \bar{z}_2)\rho(v^L \otimes v^R); z_2, \bar{z}_2)w\rangle
\end{aligned} \tag{1.53}$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$ for $u^L, v^L \in V^L$, $u^R, v^R \in V^R$, $w \in F$ and $w' \in F'$. Take $v^L = \mathbf{1}^L$ and $u^R, v^R = \mathbf{1}^R$. Then we have

$$\begin{aligned}
&\langle w', \mathbb{Y}(\rho(u^L \otimes \mathbf{1}^R); z_1, \bar{z}_1)w\rangle \\
&\quad = \langle w', \mathbb{Y}(\rho(u^L \otimes \mathbf{1}^R); z_1, \bar{z}_1)\mathbb{Y}(\mathbf{1}; z_2, \bar{z}_2)w\rangle \\
&\quad = \langle w', \mathbb{Y}(\mathbb{Y}^{L,R}(\rho(u^L \otimes \mathbf{1}^R); z_1 - z_2, \bar{z}_1 - \bar{z}_2)\rho(\mathbf{1}^L \otimes \mathbf{1}^R); z_2, \bar{z}_2)w\rangle \\
&\quad = \langle w', \mathbb{Y}(\rho((Y^L(u^L, z_1 - z_2)\mathbf{1}^L) \otimes \mathbf{1}^R); z_2, \bar{z}_2)w\rangle
\end{aligned} \tag{1.54}$$

Since the right-hand side of (1.54) is independent of $\bar{z}_1$, so is the left-hand side. Thus we see that $\mathbb{Y}(\rho(u^L \otimes \mathbf{1}^R), z, \bar{z})$ depends only on $z$ for all $u^L \in V^L$ and we shall also denote it by $Y^L(u^L, z)$. (Since it acts on $F$, there should be no confusion with the vertex operator $Y^L(u^L, z)$ acting on $V^L$.) So $Y^L(u^L, z)$ is a series in powers of $z$. But $Y^L(u^L, z)$ is also single valued. So by (1.28), there exists $u^L_n \in \text{End } F$ for $n \in \mathbb{Z}$ such that $\text{wt}^L\ u^L_n = \text{wt } u^L - n - 1$, $\text{wt}^R\ u^L_n = 0$ and

$$Y^L(u^L, z) = \sum_{n\in\mathbb{Z}} u^L_n z^{-n-1}.$$

Similarly, $\mathbb{Y}(\rho(\mathbf{1}^L \otimes u^R); z, \bar{z})$ depends only on $\bar{z}$ and will also be denoted by $Y^R(u^R, \bar{z})$. (There should also be no confusion with the vertex operator $Y^R(u^R, z)$ acting on $V^R$.) For $u^R \in V^R$, there exists $u^R_n \in \text{End } F$ for $n \in \mathbb{Z}$ such that $\text{wt}^R\ u^R_n = \text{wt } u^R - n - 1$, $\text{wt}^L\ u^R_n = 0$ and

$$Y^R(u^R, z) = \sum_{n\in\mathbb{Z}} u^R_n z^{-n-1}.$$

We also have the formal vertex operator maps, denoted using the same notations $Y^L$ and $Y^R$, associated to $Y^L$ and $Y^R$ given by

$$\begin{aligned}
Y^L(u^L, x) &= \sum_{n\in\mathbb{Z}} u^L_n x^{-n-1}, \\
Y^R(u^R, \bar{x}) &= \sum_{n\in\mathbb{Z}} u^R_n \bar{x}^{-n-1}
\end{aligned}$$

for $u^L \in V^L$ and $u^R \in V^R$.

For $u^L \in V^L, u^R \in V^R$ and $w \in F, w' \in F'$,

$$\langle w', Y^L(u^L, z_1) Y^R(u^R, \bar{z}_2) w\rangle = \langle w', \mathbb{Y}(\rho(u^L \otimes \mathbf{1}^R); z_1, \bar{z}_1) \mathbb{Y}(\rho(\mathbf{1}^L \otimes u^R); z_2, \bar{z}_2) w\rangle \tag{1.55}$$

is absolutely convergent when $|z_1| > |z_2| > 0$, and

$$\langle w', Y^R(u^R, \bar{z}_2) Y^L(u^L, z_1) w\rangle = \langle w', \mathbb{Y}(\rho(\mathbf{1}^L \otimes u^R); z_2, \bar{z}_2) \mathbb{Y}(\rho(u^L \otimes \mathbf{1}^R); z_1, \bar{z}_1) w\rangle \tag{1.56}$$

is absolutely convergent when $|z_2| > |z_1| > 0$. They are both analytic in $z_1$ and $\bar{z}_2$. By the convergence property for full field algebras, both side of (1.55) and (1.56) can be extended to a same smooth function on $\{(z_1, \bar{z}_2) \in (\mathbb{C}^\times)^2 | z_1 \neq z_2\}$. Since the complement of the union of the sets of convergence of (1.55) and (1.56) in $\{(z_1, \bar{z}_2) \in (\mathbb{C}^\times)^2 | z_1 \neq z_2\}$ is of lower dimension, by the properties of analytic functions, it is clear that the extended smooth function is actually analytic on $\{(z_1, \bar{z}_2) \in (\mathbb{C}^\times)^2 | z_1 \neq z_2\}$.

By associativity, we have

$$\langle w', Y^L(u^L, z_1) Y^R(u^R, \bar{z}_2) w\rangle = \langle w', \mathbb{Y}(\rho((Y^L(u^L, z_1 - z_2)\mathbf{1}^L) \otimes u^R); z_2, \bar{z}_2) w\rangle \tag{1.57}$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$. The right-hand side of (1.57) has a well-defined limit as $z_1$ goes to $z_2$. Therefore (1.55) and (1.56) can be further extended to a single analytic function on $\{(z_1, \bar{z}_2) \in (\mathbb{C}^\times)^2\}$. This absence of singularity further implies that the left-hand sides of (1.55) and (1.56) are absolutely convergent and are equal for all $z_1, \bar{z}_2 \in \mathbb{C}^\times$. Let $z_1 = z_2 = z$ in (1.55), (1.56) and (1.57). Use the discussion above and the creation property for the vertex operator map $Y^L$, we obtain

$$\mathbb{Y}(\rho(u^L \otimes u^R); z, \bar{z}) = Y^L(u^L, z) Y^R(u^R, \bar{z}) = Y^R(u^R, \bar{z}) Y^L(u^L, z), \tag{1.58}$$

or equivalently, in terms of formal vertex operator,

$$\mathbb{Y}_f(\rho(u^L \otimes u^R); x, \bar{x}) = Y^L(u^L, x) Y^R(u^R, \bar{x}) = Y^R(u^R, \bar{x}) Y^L(u^L, x) \tag{1.59}$$

for all $u^L \in V^L$ and $u^R \in V^R$. In particular, we have $[u^L_m, u^R_n] = 0$ for all $u^L \in V^L$ and $u^R \in V^R$.

Since $F$ is lower truncated, we have

$$\mathbb{Y}_f(\rho(u^L \otimes u^R); x, x) v \in (\text{End}\, F)((x)). \tag{1.60}$$

for $u^L \in V^L$, $u^R \in V^R$ and $v \in F$.

The associativity (1.53) together with (1.59) and (1.60) implies the associativity for the vertex operator map $\mathbb{Y}_f(\rho(\cdot); x, x)\cdot$. Together with the identity property this associativity implies that $F$ is a module for the vertex operator algebra $V^L \otimes V^R$.

Next we show that $\mathbb{Y}_f(\cdot; x, x)$ is an intertwining operator of type $\binom{F}{FF}$.

Since For given $u, v \in F$, we have $\mathbb{Y}_k(u)w = 0$ for sufficient large $k$, the lower-truncation property of $\mathbb{Y}_f(\cdot, x, x)$ holds. For $u \in F$, We also have

$$\begin{aligned}
\mathbb{Y}_f((D^L + D^R)u; x, x) &= \mathbb{Y}_f((D^L + D^R)u; x, \bar{x})|_{\bar{x}=x} \\
&= \left.\left(\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial \bar{x}}\right) \mathbb{Y}_f(u; x, \bar{x})\right)\right|_{\bar{x}=x} \\
&= \frac{d}{dx}\mathbb{Y}_f(u; x, x),
\end{aligned}$$

proving the $D$-derivative property of $\mathbb{Y}_f(\cdot; x, x)$.

Now, we prove the Jacobi identity for $\mathbb{Y}_f(\cdot; x, x)$. For any fixed $r \in \mathbb{R}$, using the associativity for the full vertex operator map $\mathbb{Y}$ twice, we obtain

$$\begin{aligned}
&\langle w', Y^L(u^L, z_1)Y^R(u^R, \bar{z}_2)\mathbb{Y}(u; r, r)w\rangle \\
&\quad = \langle w', \mathbb{Y}(\rho(u^L \otimes \mathbf{1}^L); z_1, \bar{z}_1)\mathbb{Y}(\rho(\mathbf{1}^L \otimes u^R); z_2, \bar{z}_2)\mathbb{Y}(u; r, r)w\rangle \\
&\quad = \langle w', \mathbb{Y}(\mathbb{Y}(\rho(u^L \otimes \mathbf{1}^L); z_1 - r, \bar{z}_1 - r)\mathbb{Y}(\rho(\mathbf{1}^L \otimes u^R); z_2 - r, \bar{z}_2 - r)u; r, r)w\rangle \\
&\quad = \langle w', \mathbb{Y}(Y^L(u^L, z_1 - r)Y^R(u^R, \bar{z}_2 - r)u; r, r)w\rangle
\end{aligned} \tag{1.61}$$

when $|z_1|, |z_2| > r > |z_1 - r|, |z_2 - r| > 0$ for all $u^L \in V^L$, $u^R \in V^R$, $u, w \in F$ and $w' \in F'$. By the commutativity for the full vertex operator map $\mathbb{Y}$,

$$\langle w', Y^L(u^L, z_1)Y^R(u^R, \bar{z}_2)\mathbb{Y}(u; r, r)w\rangle \tag{1.62}$$

and

$$\langle w', \mathbb{Y}(u; r, r)Y^L(u^L, z_1)Y^R(u^R, \bar{z}_2)w\rangle \tag{1.63}$$

are absolutely convergent in the regions $|z_1|, |z_2| > r > 0$ and $r > |z_1|, |z_2| > 0$, respectively, to the correlation function

$$\langle w', m_4(u^L \otimes \mathbf{1}^L, \mathbf{1}^R \otimes u^R, u, w; z_1, \bar{z}_1, z_2, \bar{z}_2, r, r, 0, 0). \tag{1.64}$$

By our discussion above, we know that the right-hand side of (1.61), (1.62) and (1.63) are all analytic in $z_1$ and $\bar{z}_2$ and that we can take $z_1 = \bar{z}_2$ in the right-hand side of (1.61), (1.62) and (1.63). Thus after taking $z_1 = \bar{z}_2$, the right-hand side of (1.61), (1.62) and (1.63) are analytic in $z = z_1 = \bar{z}_2$. Since the right-hand side of (1.61), (1.62) and (1.63) are the expansions of (1.64) in the regions $r > |z_1 - r|, |z_2 - r| > 0$, $|z_1|, |z_2| > r > 0$ and $r > |z_1|, |z_2| > 0$, respectively, we see that we can also let $z_1 = \bar{z}_2$ in (1.64) and the result is also analytic in $z = z_1 = \bar{z}_2$. Thus we have proved that

$$\langle w', \mathbb{Y}(\mathbb{Y}_f(\rho(u^L \otimes u^R); z - r, z - r)u; r, r)w\rangle,$$
$$\langle w', \mathbb{Y}_f(\rho(u^L \otimes u^R); z, z)\mathbb{Y}(u; r, r)w\rangle,$$
$$\langle w', \mathbb{Y}(u; r, r)\mathbb{Y}_f(\rho(u^L \otimes u^R); z, z)w\rangle$$

are absolutely convergent to

$$\langle w', m_4(\rho(u^L \otimes \mathbf{1}^L), \rho(\mathbf{1}^R \otimes u^R), u, w; z, \bar{z}, \bar{z}, z, r, r, 0, 0)$$

which is in fact analytic in $z$. Using the Cauchy formula for contour integrals, we obtain the Cauchy-Jacobi identity

$$\begin{aligned}
&\mathrm{Res}_{z=\infty} f(z)\langle w', \mathbb{Y}_f(\rho(u^L \otimes u^R); z, z)\mathbb{Y}(u; r, r)w\rangle \\
&\quad -\mathrm{Res}_{z=0} f(z)\langle w', \mathbb{Y}(u; r, r)\mathbb{Y}_f(\rho(u^L \otimes u^R); z, z)w\rangle \\
&\quad = \mathrm{Res}_{z=r} f(z)\langle w', \mathbb{Y}(\mathbb{Y}_f(\rho(u^L \otimes u^R); z - r, z - r)u; r, r)w\rangle, \quad (1.65)
\end{aligned}$$

where $f(z)$ is a rational function of $z$ with the only possible poles at $z = 0, r, \infty$. Since $w$ and $w'$ are arbitrary, this Cauchy-Jacobi identity gives us identities for the components of the vertex operator $\mathbb{Y}_f(u; x, x)$. These identities are the component form of the Jacobi identity for $\mathbb{Y}_f(u; x, x)$. ∎

**Definition 1.19** Let $c^L, c^R \in \mathbb{C}$. A *conformal full field algebra of central charges* $(c^L, c^R)$ is a grading-restricted $\mathbb{R}\times\mathbb{R}$-graded full field algebra $(F, m, \mathbf{1}, D^L, D^R)$ equipped with elements $\omega^L$ and $\omega^R$ called *left conformal element* and *right conformal element*, respectively, satisfying the following conditions:

1. The formal full vertex operators $\mathbb{Y}_f(\omega^L; x, \bar{x})$ and $\mathbb{Y}_f(\omega^R; x, \bar{x})$ are Laurent series in $x$ and $\bar{x}$, respectively, that is,

$$\begin{aligned}
\mathbb{Y}_f(\omega^L; x, \bar{x}) &= \sum_{n\in\mathbb{Z}} L^L(n)x^{-n-2}, \\
\mathbb{Y}_f(\omega^R; x, \bar{x}) &= \sum_{n\in\mathbb{Z}} L^R(n)\bar{x}^{-n-2}.
\end{aligned}$$

2. The *Virasoro relations*: For $m, n \in \mathbb{Z}$,

$$\begin{aligned}
[L^L(m), L^L(n)] &= (m-n)L^L(m+n) + \frac{c^L}{12}(m^3 - m)\delta_{m+n,0},\\
[L^R(m), L^R(n)] &= (m-n)L^R(m+n) + \frac{c^R}{12}(m^3 - m)\delta_{m+n,0},\\
[L^L(m), L^R(n)] &= 0.
\end{aligned}$$

3. $d^L = L^L(0)$, $d^R = L^R(0)$, $D^L = L^L(-1)$ and $D^R = L^R(-1)$.

We shall denote the conformal full field algebra by $(F, m, \mathbf{1}, \omega^L, \omega^R)$ or simply by $F$.

We have:

**Proposition 1.20** *Let $(F, m, \mathbf{1}, \omega^L, \omega^R)$ be a conformal full field algebra. Then the following commutator formula for Virasoro operators and formal full vertex operators hold: For $u \in F$,*

$$\begin{aligned}
&[\mathbb{Y}_f(\omega^L; x_1, \bar{x}_1), \mathbb{Y}_f(u; x_2, \bar{x}_2)]\\
&\quad = \mathrm{Res}_{x_0} x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\mathbb{Y}_f(\mathbb{Y}_f(\omega^L; x_0, \bar{x}_0)u; x_2, \bar{x}_2), \qquad (1.66)\\
&[\mathbb{Y}_f(\omega^R; x_1, \bar{x}_1), \mathbb{Y}_f(u; x_2, \bar{x}_2)]\\
&\quad = \mathrm{Res}_{\bar{x}_0} \bar{x}_2^{-1}\delta\left(\frac{\bar{x}_1 - \bar{x}_0}{\bar{x}_2}\right)\mathbb{Y}_f(\mathbb{Y}_f(\omega^R; x_0, \bar{x}_0)u; x_2, \bar{x}_2). \qquad (1.67)
\end{aligned}$$

*Proof.* For any $v' \in F'$, $u, v \in F$, we consider

$$\langle v', m_3(\omega^L, u, v; z_1, \bar{z}_1, z_2, \bar{z}_2, 0, 0)\rangle. \qquad (1.68)$$

Using the convergence property and the permutation property for conformal full field algebras, we know that it is equal to

$$\langle v', \mathbb{Y}(\omega^L; z_1, \bar{z}_1)\mathbb{Y}(u, z_2, \bar{z}_2)v\rangle, \qquad (1.69)$$
$$\langle v', \mathbb{Y}(u; z_2, \bar{z}_2)\mathbb{Y}(\omega^L, z_1, \bar{z}_1)v\rangle, \qquad (1.70)$$

in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, respectively. By the definition of conformal full field algebra, we know that for any fixed $z_2 \neq 0$, (1.69) and (1.70) are analytic as functions of $z_1$ in the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively. So (1.68) is analytic as a function of $z_1$ in the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$. But we know that (1.68) is smooth as a function of $z_1$ in $\mathbb{C} \setminus \{z_2, 0\}$. Thus (1.68) must be analytic in $\mathbb{C} \setminus \{z_2, 0\}$.

We know that (1.68) is equal to (1.69), (1.70) and

$$\langle v', \mathbb{Y}(\mathbb{Y}(\omega^L; z_1 - z_2, \bar{z}_1 - \bar{z}_2)u; z_2, \bar{z}_2)v\rangle$$

in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$ and $z_2| > |z_1 - z_2| > 0$, respectively. Since $F$ is lower truncated, using the Virasoro relation, we see that $\mathbb{Y}_f(\omega^L; x, \bar{x})u$ and $\mathbb{Y}_f(\omega^L; x, \bar{x})v$ have only finitely many terms in negative powers of $x$. Also using the lower-truncation property of $F$ and the Virasoro relation, we see that for any $w \in F$, $\langle v', \mathbb{Y}_f(\omega^L; x, \bar{x})w\rangle$ has only finitely many terms in positive powers of $x$. Using these facts, we see that the singularities $z_1 = z_2, 0, \infty$ of (1.68) are all poles. Using the Cauchy formula, we obtain the component form (1.66).

Similarly, we can prove (1.67). ∎

The following is clear from the definition and Theorem 1.18:

**Proposition 1.21** *Let $(V^L, Y^L, \mathbf{1}^L, \omega^L)$ and $(V^L, Y^L, \mathbf{1}^L, \omega^L)$ be vertex operator algebras of central charges $c^L$ and $c^R$, respectively. A full field algebra $(F, m, \rho)$ over $V^L \otimes V^R$ equipped with the left and right conformal elements $\rho(\omega^L \otimes \mathbf{1}^R)$ and $\rho(\mathbf{1}^L \otimes \omega^R)$ is a conformal full field algebra.*

In view of this proposition, we shall call the conformal full field algebra in the proposition above, that is, a full field algebra $(F, m, \rho)$ over $V^L \otimes V^R$ equipped with the left and right conformal elements $\rho(\omega^L \otimes \mathbf{1}^R)$ and $\rho(\mathbf{1}^L \otimes \omega^R)$, a *conformal full field algebra over $V^L \otimes V^R$* and denote it by $(F, m, \rho)$ or simply by $F$.

# 2 Intertwining operator algebras and full field algebras

Let $V^L$ and $V^R$ be vertex operator algebras. In the preceding section, we have shown that a conformal full field algebra $(F, m, \rho)$ over $V^L \otimes V^R$ is a module for the vertex operator algebra $V^L \otimes V^R$ and the $\mathbb{Y}_f(\cdot; x, x)$ is an intertwining operator of type $\binom{F}{FF}$. This result suggests a method to construct conformal full field algebras from intertwining operator algebras, which are algebras of intertwining operators for vertex operator algebras and were introduced and studied in [H1], [H2], [H3], [H4], [H5] and [H7] by the first author.

Let $V$ be a vertex operator algebra and for a $V$-module $W$, let $C_1(W)$ be the subspace of $V$ spanned by $u_{-1}w$ for $u \in V_+ = \coprod_{n \in \mathbb{Z}_+} V_{(n)}$ and $w \in W$.

We consider the following conditions for a vertex operator algebra $V$:

1. Every $\mathbb{C}$-graded generalized $V$-module is a direct sum of $\mathbb{C}$-graded irreducible $V$-modules.

2. There are only finitely many inequivalent $\mathbb{C}$-graded irreducible $V$-modules and they are all $\mathbb{R}$-graded.

3. Every $\mathbb{R}$-graded irreducible $V$-module $W$ satisfies the $C_1$-cofiniteness condition, that is, $\dim W/C_1(W) < \infty$.

In this section, we fix vertex operator algebras $(V^L, Y^L, \mathbf{1}^L, \omega^L)$ and $(V^R, Y^R, \mathbf{1}^R, \omega^R)$ satisfying these conditions. Let $\mathcal{A}^L$ and $\mathcal{A}^R$ be the sets of equivalent classes of irreducible modules for $V^L$ and for $V^R$, respectively. Let $\{W^{L;a} \mid a \in \mathcal{A}^L\}$ be a complete set of representatives of the equivalence classes in $\mathcal{A}^L$ and $\{W^{R;b} \mid b \in \mathcal{A}^R\}$ a complete set of representatives of the equivalence classes in $\mathcal{A}^R$.

**Proposition 2.1** *The vertex operator algebra $V^L \otimes V^R$ also satisfy the conditions above.*

*Proof.* Let $W$ be a generalized $V^L \otimes V^R$-module. Then $W$ is a generalized $V^L$-module. So there exist vector spaces $M^a$ for $a \in \mathcal{A}^L$ such that $W$ is equivalent to the generalized $V^L$-module $\coprod_{a \in \mathcal{A}^L} (W^{L;a} \otimes M^a)$. Since $W$ is also a generalized $V^R$-module, $M^a$ must be $V^R$-modules. So they can be written as direct sums of irreducible $V^R$-modules $W^{R;b}$, $b \in \mathcal{A}^R$. So $W$ is equivalent to $\coprod_{a \in \mathcal{A}^L, b \in \mathcal{A}^R} N_{ab}(W^{L;b} \otimes W^{R;a})$ where $N_{ab} \in \mathbb{N}$ for $a \in \mathcal{A}^L$, $b \in \mathcal{A}^R$. By Proposition 4.7.2 of [FHL], $W^{L;a} \otimes W^{R;b}$ are irreducible $V^L \otimes V^R$-modules. So $V^L \otimes V^R$ satisfies Condition 1. The second condition follows from Theorem 4.7.4 of [FHL]. The $C_1$-cofiniteness follows immediately from the fact that

$$C_1(W^{L;a}) \otimes W^{R;b} \oplus W^{L;a} \otimes C_1(W^{R;b}) \subset C_1(W^{L;a} \otimes W^{R;b}).$$

∎

This result immediately gives:

**Corollary 2.2** *Let $F$ be a module for the vertex operator algebra $V^L \otimes V^R$. Then as a module for the vertex operator algebra $V^L \otimes V^R$, $F$ is isomorphic to*

$$\coprod_{a\in\mathcal{A}^L} \coprod_{b\in\mathcal{A}^R} \coprod_{m_{ab}=1}^{h_{ab}} (W^{L;a})^{(m_{ab})} \otimes (W^{R;b})^{(m_{ab})} \tag{2.1}$$

Let $F$ be a module for the vertex operator algebra $V^L \otimes V^R$ and let $\gamma$ be an isomorphism from (2.1) to $F$. Then there exist operators $L^L(0)$ and $L^R(0)$ on $F$ given by

$$\begin{aligned} L^L(0)\rho(w^L \otimes w^R) &= \rho((L^L(0)w^L) \otimes w^R), \\ L^R(0)\rho(w^L \otimes w^R) &= \rho(w^L \otimes (L^R(0)w^R)) \end{aligned}$$

for $w^L \in (W^{L;a})^{(m_{ab})}$ and $w^R \in (W^{R;b})^{(m_{ab})}$. Clearly $L^L(0)$ and $L^R(0)$ commute with each other.

Let $\mathcal{Y}$ be an intertwining operator of type $\binom{F}{FF}$ and $\gamma$ an isomorphism from (2.1) to $F$. Let

$$\begin{aligned} \mathbb{Y}^{\mathcal{Y}} : (F \otimes F) \times \mathbb{C}^\times &\to \overline{F} \\ (u \otimes v, z) &\mapsto \mathbb{Y}^{\mathcal{Y}}(u; z, \bar{z})v \end{aligned}$$

and

$$\begin{aligned} \mathbb{Y}^{\mathcal{Y}}_f : F \otimes F &\to F\{x, \bar{x}\} \\ u \otimes v &\mapsto \mathbb{Y}^{\mathcal{Y}}_f(u; x, \bar{x})v \end{aligned}$$

be linear maps given by

$$\mathbb{Y}^{\mathcal{Y}}(u; z, \bar{z})v = z^{L^L(0)}\bar{z}^{L^R(0))}\mathcal{Y}(u, 1)z^{-L^L(0)}\bar{z}^{-L^R(0)}$$

and

$$\mathbb{Y}^{\mathcal{Y}}_f(u; x, \bar{x})v = x^{L^L(0)}\bar{x}^{L^R(0)}\mathcal{Y}(u, 1)x^{-L^L(0)}\bar{x}^{-L^R(0)},$$

respectively, for $u \in F$. We call $\mathbb{Y}^{\mathcal{Y}}$ and $\mathbb{Y}^{\mathcal{Y}}_f$ the *splitting* and *formal splitting* of $\mathcal{Y}$, respectively.

**Proposition 2.3** *Let $\mathcal{Y}$ be an intertwining operator of type $\binom{F}{FF}$, $\mathbb{Y}^{\mathcal{Y}}$ and $\mathbb{Y}^{\mathcal{Y}}_f$, the splitting and formal splitting of $\mathcal{Y}$, respectively, and $\gamma$ an isomorphism from (2.1) to $F$. Then for any $a_1, a_2 \in \mathcal{A}^L$, $b_1, b_2 \in \mathcal{A}^R$, $1 \le m_{a_1b_1} \le h_{a_1b_1}$ and $1 \le m_{a_2b_2} \le h_{a_2b_2}$, there exist intertwining operators $\mathcal{Y}_{a_1a_2}^{L;m_{a_3b_3};a_3}$*

*and* $\mathcal{Y}_{b_1b_2}^{R;m_{a_3b_3};b_3}$ *for* $a_3 \in \mathcal{A}^L$, $b_3 \in \mathcal{A}^R$ *and* $m_{a_3b_3} = 1, \dots, h_{a_3b_3}$ *of types* $\binom{(W^{L;a_3})^{(m_{a_3b_3})}}{(W^{L;a_1})^{(m_{a_1b_1})}\ (W^{L;a_2})^{(m_{a_2b_2})}}$ *and* $\binom{(W^{R;b_3})^{(m_{a_3b_3})}}{(W^{R;b_1})^{(m_{a_1b_1})}\ (W^{R;b_2})^{(m_{a_2b_2})}}$, *respectively, such that for* $u^L \otimes u^R \in (W^{L;a_1})^{(m_{a_1b_1})} \otimes (W^{R;b_1})^{(m_{a_1b_1})}$ *and* $v^L \otimes v^R \in (W^{L;a_2})^{(m_{a_2b_2})} \otimes (W^{R;b_2})^{(m_{a_2b_2})}$, *we have*

$$\begin{aligned}
&\mathbb{Y}^{\mathcal{Y}}(\gamma(u^L \otimes u^R); z, \bar{z})\gamma(v^L \otimes v^R)\\
&\quad = \sum_{a_3 \in \mathcal{A}^L} \sum_{b_3 \in \mathcal{A}^R} \sum_{m_{a_3b_3}=1}^{h_{a_3b_3}} \gamma(\mathcal{Y}_{a_1a_2}^{L;m_{a_3b_3};a_3}(u^L, z)v^L \otimes \mathcal{Y}_{b_1b_2}^{R;m_{a_3b_3};b_3}(u^R, \bar{z})v^R).
\end{aligned} \tag{2.2}$$

*Similarly, for the formal full vertex operator, we have*

$$\begin{aligned}
&\mathbb{Y}_f^{\mathcal{Y}}(\gamma(u^L \otimes u^R); x, \bar{x})\gamma(v^L \otimes v^R)\\
&\quad = \sum_{a_3 \in \mathcal{A}^L} \sum_{b_3 \in \mathcal{A}^R} \sum_{m_{a_3b_3}=1}^{h_{a_3b_3}} \gamma(\mathcal{Y}_{a_1a_2}^{L;m_{a_3b_3};a_3}(u^L, x)v^L \otimes \mathcal{Y}_{b_1b_2}^{R;m_{a_3b_3};b_3}(u^R, \bar{x})v^R).
\end{aligned} \tag{2.3}$$

*Proof.* Since $\mathbb{Y}^{\mathcal{Y}}$ restricted to

$$\gamma((W^{L;a_1})^{(m_{a_1b_1})} \otimes (W^{R;b_1})^{(m_{a_1b_1})}) \otimes \gamma((W^{L;a_2})^{(m_{a_2b_2})} \otimes (W^{R;b_2})^{(m_{a_2b_2})})$$

is an intertwining operator of type

$$\binom{F}{\gamma((W^{L;a_1})^{(m_{a_1b_1})} \otimes (W^{R;b_1})^{(m_{a_1b_1})})\ \ \gamma((W^{L;a_2})^{(m_{a_2b_2})} \otimes (W^{R;b_2})^{(m_{a_2b_2})})},$$

it was proved in [DMZ] that (2.2) is true when $z = \bar{z} = r > 0$. Then we have

$$\begin{aligned}
&\mathbb{Y}^{\mathcal{Y}}(\gamma(u^L \otimes u^R); z, \bar{z})\gamma(v^L \otimes v^R)\\
&\quad = z^{L^L(0)}\bar{z}^{L^R(0)}\mathcal{Y}(\gamma(u^L \otimes u^R), 1)z^{-L^L(0)}\bar{z}^{-L^R(0)}\gamma(v^L \otimes v^R)\\
&\quad = \sum_{a_3 \in \mathcal{A}^L} \sum_{b_3 \in \mathcal{A}^R} \sum_{m_{a_3b_3}=1}^{h_{a_3b_3}} \gamma((z^{L^L(0)}\mathcal{Y}_{a_1a_2}^{L;m_{a_3b_3};a_3}(u^L, 1)\bar{z}^{-L^L(0)}v^L)\\
&\qquad\qquad \otimes (z^{L^R(0)}\mathcal{Y}_{b_1b_2}^{R;m_{a_3b_3};b_3}(u^R, 1)\bar{z}^{-L^R(0)}v^R))\\
&\quad = \sum_{a_3 \in \mathcal{A}^L} \sum_{b_3 \in \mathcal{A}^R} \sum_{m_{a_3b_3}=1}^{h_{a_3b_3}} \gamma(\mathcal{Y}_{a_1a_2}^{L;m_{a_3b_3};a_3}(u^L, z)v^L \otimes \mathcal{Y}_{b_1b_2}^{R;m_{a_3b_3};b_3}(u^R, \bar{z})v^R).
\end{aligned}$$

The proof of (2.3) is completely the same. ∎

**Corollary 2.4** *Let $(F, m, \rho)$ be a conformal full field algebra over $V^L \otimes V^R$. Then as a module for the vertex operator algebra $V^L \otimes V^R$, $F$ is isomorphic to (2.1). Moreover, if $\gamma$ is an isomorphism from (2.1) to $F$, then for any $a_1, a_2 \in \mathcal{A}^L$, $b_1, b_2 \in \mathcal{A}^R$, $1 \le m_{a_1b_1} \le h_{a_1b_1}$ and $1 \le m_{a_2b_2} \le h_{a_2b_2}$, there exist intertwining operators $\mathcal{Y}_{a_1a_2}^{L;m_{a_3b_3};a_3}$ and $\mathcal{Y}_{b_1b_2}^{R;m_{a_3b_3};b_3}$ for $a_3 \in \mathcal{A}^L$, $b_3 \in \mathcal{A}^R$ and $m_{a_3b_3} = 1, \dots, h_{a_3b_3}$ of types $\binom{(W^{L;a_3})^{(m_{a_3b_3})}}{(W^{L;a_1})^{(m_{a_1b_1})}\,(W^{L;a_2})^{(m_{a_2b_2})}}$ and $\binom{(W^{R;b_3})^{(m_{a_3b_3})}}{(W^{R;b_1})^{(m_{a_1b_1})}\,(W^{R;b_2})^{(m_{a_2b_2})}}$, respectively, such that for $u^L \otimes u^R \in (W^{L;a_1})^{(m_{a_1b_1})} \otimes (W^{R;b_1})^{(m_{a_1b_1})}$ and $v^L \otimes v^R \in (W^{L;a_2})^{(m_{a_2b_2})} \otimes (W^{R;b_2})^{(m_{a_2b_2})}$, the formulas (2.2) and (2.3) hold when $\mathbb{Y}^{\mathcal{Y}}$ and $\mathbb{Y}_f^{\mathcal{Y}}$ are replaced by $\mathbb{Y}$ and $\mathbb{Y}_f$, respectively.*

*Proof.* The first conclusion follows immediately from Corollary 2.2. Now if we consider the intertwining operator $\mathbb{Y}_f(\cdot; x, x)$, then the second conclusion follows immediately from Proposition 2.3. ∎

For either the map $\mathbb{Y}_f^{\mathcal{Y}}$ in Proposition 2.3 or the formal full vertex operator map $\mathbb{Y}_f$ for a conformal full field algebra over $V^L \otimes V^R$, we can substitute $z$ and $\zeta$ for the formal variables $x$ and $\bar{x}$ in $\mathbb{Y}_f^{\mathcal{Y}}(\cdot; x, \bar{x})$ or $\mathbb{Y}_f(\cdot; x, \bar{x})$ (that is, substitute $e^{r \log z}$ and $e^{s\,\overline{\log \zeta}}$ for $x^r$ and $\bar{x}^s$, respectively, for $r, s \in \mathbb{R}$) to obtain $\mathbb{Y}_{\rm an}^{\mathcal{Y}}(\cdot; z, \zeta)$ (called *analytic splitting of $\mathcal{Y}$*) or $\mathbb{Y}_{\rm an}(\cdot; z, \zeta)$. Then by (2.3), we have:

**Corollary 2.5** *For the analytic splitting $\mathbb{Y}_{\rm an}^{\mathcal{Y}}$ of $\mathcal{Y}$ in Proposition 2.3, we have*

$$\begin{aligned}
&\mathbb{Y}_{\rm an}^{\mathcal{Y}}(\gamma(u^L \otimes u^R); z, \zeta)\gamma(v^L \otimes v^R) \\
&\quad = \sum_{a_3 \in \mathcal{A}^L} \sum_{b_3 \in \mathcal{A}^R} \sum_{m_{a_3b_3}=1}^{h_{a_3b_3}} \gamma(\mathcal{Y}_{a_1a_2}^{L;m_{a_3b_3};a_3}(u^L, z)v^L \otimes \mathcal{Y}_{b_1b_2}^{R;m_{a_3b_3};b_3}(u^R, \zeta)v^R)
\end{aligned} \tag{2.4}$$

*for $u^L \otimes u^R \in (W^{L;a_1})^{(m_{a_1b_1})} \otimes (W^{R;b_1})^{(m_{a_1b_1})}$ and $v^L \otimes v^R \in (W^{L;a_2})^{(m_{a_2b_2})} \otimes (W^{R;b_2})^{(m_{a_2b_2})}$. The same is also true for the analytic full vertex operator map $\mathbb{Y}_{\rm an}$ for a conformal full field algebra over $V^L \otimes V^R$.*

This corollary allows us to treat the left and right variables $z$ and $\bar{z}$ in $\mathbb{Y}^{\mathcal{Y}}(\cdot; z, \bar{z})$ or $\mathbb{Y}(\cdot; z, \bar{z})$ independently. In particular, we have the following strong versions of associativity and commutativity for conformal full field algebra over $V^L \otimes V^R$:

**Proposition 2.6 (Associativity)** *Let $(F, m, \rho)$ be a conformal full field algebra over $V^L \otimes V^R$. Then for $u, v, w \in F$ and $w' \in F'$,*

$$\begin{aligned} &\langle w', \mathbb{Y}_{\mathrm{an}}(u; z_1, \zeta_1)\mathbb{Y}_{\mathrm{an}}(v; z_2, \zeta_2)w\rangle \\ &\qquad = \langle w', \mathbb{Y}_{\mathrm{an}}(\mathbb{Y}_{\mathrm{an}}(u; z_1 - z_2, \zeta_1 - \zeta_2)v; z_2, \zeta_2)w\rangle \end{aligned} \tag{2.5}$$

*when $|z_1| > |z_2| > |z_1 - z_2| > 0$ and $|\zeta_1| > |\zeta_2| > |\zeta_1 - \zeta_2| > 0$.*

*Proof.* Using (2.4) and the convergence result proved by the first author in [H7] for vertex operator algebras satisfying the conditions assumed for $V^L$ and $V^R$ in the beginning of this section, the left-hand side of (2.5) converges absolutely when $|z_1| > |z_2| > 0$ and $|\zeta_1| > |\zeta_1| > 0$, and the right-hand side of (2.5) converges absolutely when $|z_2| > |z_1 - z_2| > 0$ and $|\zeta_2| > |\zeta_1 - \zeta_2| > 0$. By the associativity (1.8), (2.5) is true when $\zeta_1 = \bar{z}_1$ and $\zeta_2 = \bar{z}_2$ for all $u, v, w \in F$ and $w' \in F'$. In particular, replacing $u$ by $(L^L(-1))^k(L^R(-1))^l u$, $v$ by $(L^L(-1))^m L^R(-1))^n v$, for $k, l, m, n \in \mathbb{N}$ and using the $L^L(-1)$- and $L^R(-1)$-derivative properties, we obtain

$$\begin{aligned} &\frac{\partial^k}{\partial z_1^k}\frac{\partial^l}{\partial \zeta_1^l}\frac{\partial^m}{\partial z_2^m}\frac{\partial^n}{\partial \zeta_2^n}\langle w', \mathbb{Y}_{\mathrm{an}}(u; z_1, \zeta_1)\mathbb{Y}_{\mathrm{an}}(v; z_2, \zeta_2)w\rangle\Big|_{\zeta_1=\bar{z}_1, \zeta_2=\bar{z}_2} \\ &\quad = \frac{\partial^k}{\partial z_1^k}\frac{\partial^l}{\partial \zeta_1^l}\frac{\partial^m}{\partial z_2^m}\frac{\partial^n}{\partial \zeta_2^n}\langle w', \mathbb{Y}_{\mathrm{an}}(\mathbb{Y}_{\mathrm{an}}(u; z_1 - z_2, \zeta_1 - \zeta_2)v; z_2, \zeta_2)w\rangle\Big|_{\zeta_1=\bar{z}_1, \zeta_2=\bar{z}_2} \end{aligned} \tag{2.6}$$

for all $k, l, m, n \in \mathbb{N}$, when $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$. We know that both sides of (2.5) give branches of some multivalued analytic functions in the region given by $|z_1| > |z_2| > 0$, $|\zeta_1| > |\zeta_1| > 0$, $|z_2| > |z_1 - z_2| > 0$ and $|\zeta_2| > |\zeta_1 - \zeta_2| > 0$. From (2.6), we know that the power series expansions of these branches are equal in the neighborhood of those points satisfying $\zeta_1 = \bar{z}_1$, $\zeta_2 = \bar{z}_2$. Thus (2.6) holds in the region $|z_1| > |z_2| > 0$, $|\zeta_1| > |\zeta_1| > 0$, $|z_2| > |z_1 - z_2| > 0$ and $|\zeta_2| > |\zeta_1 - \zeta_2| > 0$. ∎

**Proposition 2.7 (Commutativity)** *Let $(F, m, \rho)$ be a conformal full field algebra over $V^L \otimes V^R$. Then for $u, v, w \in F$ and $w' \in F'$,*

$$\langle w', \mathbb{Y}_{\mathrm{an}}(u; z_1, \zeta_1)\mathbb{Y}_{\mathrm{an}}(v; z_2, \zeta_2)w\rangle \tag{2.7}$$

*and*

$$\langle w', \mathbb{Y}_{\mathrm{an}}(v; z_2, \zeta_2)\mathbb{Y}_{\mathrm{an}}(u; z_1, \zeta_1)w\rangle \tag{2.8}$$

*are absolutely convergent when* $|z_1| > |z_2| > 0$, $|\zeta_1| > |\zeta_2| > 0$ *and when* $|z_2| > |z_1| > 0$, $|\zeta_2| > |\zeta_1| > 0$, *respectively, and can both be analytically extended to a same multivalued analytic function of* $(z_1, z_2; \zeta_1, \zeta_2)$ *for* $(z_1, z_2; \zeta_1, \zeta_2) \in M^2 \times M^2$, *where* $M^2 = \{(z_1, z_2) \in (\mathbb{C}^\times)^2 \mid z_1 \neq z_2\}$.

*Proof.* The convergence and the existence of analytic extensions follow immediately from Corollary 2.5 and the convergence and the existence of analytic extensions of products of intertwining operators for the vertex operator algebras $V^L$ and $V^R$.

By Proposition 1.5, we know that these two multivalued analytic functions obtained by analytically extending (2.7) and (2.8) have equal values at points of the form $(z_1, \bar{z}_1, \dots, z_n, \bar{z}_n)$ for $(z_1, \dots, z_n) \in \mathbb{F}_n(\mathbb{C})$. Using the $L^L(-1)$- and $L^R(-1)$-conjugation properties for full vertex operators, we see that these two analytic functions are actually the same, that is, they are analytic extensions of each other. ∎

For $(z_1, \dots, z_n), (\zeta_1, \dots, \zeta_n) \in \mathbb{F}_n(\mathbb{C})$, we denote the corresponding elements of $\mathbb{F}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C})$ by $(z_1, \zeta_1, \dots, z_n, \zeta_n)$ instead of $(z_1, \dots, z_n, \zeta_1, \dots, \zeta_n)$. We have the following analyticity of the correlation functions:

**Proposition 2.8** *Let* $(F, m, \rho)$ *be a conformal full field algebra over* $V^L \otimes V^R$. *For any* $n \in \mathbb{Z}_+$ *and* $u_1, \dots, u_n$, *there exists a multivalued analytic function of* $(z_1, \zeta_1, \dots, z_n, \zeta_n) \in \mathbb{F}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C})$ *such that for* $(z_1, \dots, z_n) \in \mathbb{F}_n(\mathbb{C})$, *the values*

$$m_n(u_1, \dots, u_n; z_1, \bar{z}_1, \dots, z_n, \bar{z}_n)$$

*of the correlation function is a value of this multivalued analytic function above at the point* $(z_1, \bar{z}_1, \dots, z_n, \bar{z}_n)$. *Moreover, these multivalued analytic functions are determined uniquely by the products of analytic full vertex operators in their regions of convergence.*

*Proof.* The proof of this result is basically the same as the proof of the generalized rationality for intertwining operator algebras in [H5]. We have proved the above strong versions of associativity and commutativity for analytic full vertex operators. Using these strong versions of associativity and commutativity, we see that the multivalued analytic functions in various regions obtained from all kinds of products and iterates of analytic full vertex operators are analytic extensions of each other. Thus we have such a global multivalued analytic function. Clearly these multivalued analytic functions are determined uniquely by the products of analytic full vertex operators in their regions of convergence. ∎

By the results above, we see that for a conformal full field algebra over $V^L \otimes V^R$, the correlation function maps are determined uniquely by the products of analytic full vertex operators in their regions of convergence, and thus are determined uniquely by the full vertex operator map. In view of this fact, we shall use also $(F, \mathbb{Y}, \rho)$ to denote a conformal full field algebra over $V^L \otimes V^R$.

We shall use

$$E(m)_n(u_1, \dots, u_n; z_1, \zeta_1, \dots, z_n, \zeta_n) \tag{2.9}$$

to denote the analytic extension obtained in the proposition above together with the prefered values

$$m_n(u_1, \dots, u_n; z_1, \bar{z}_1, \dots, z_n, \bar{z}_n)$$

at the special points of the form $(z_1, \bar{z}_1, \dots, z_n, \bar{z}_n)$. For $u_1, \dots, u_n \in F$ and a path

$$\begin{aligned} \gamma : [0, 1] &\to \mathbb{F}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C}) \\ t &\mapsto (z_1(t), \zeta_1(t), \dots, z_n(t), \zeta_n(t)) \end{aligned}$$

starting from a point of the form $(z_1, \bar{z}_1, \dots, z_n, \bar{z}_n)$, we shall use

$$E(m)_n(u_1, \dots, u_n; z_1(t), \zeta_1(t), \dots, z_n(t), \zeta_n(t))$$

to denote the value of (2.9) at the point $\gamma(t)$ obtained by analytically extend the preferred value of (2.9) at the starting point $\gamma(0)$ of $\gamma$ along the path $\gamma$ to the point $\gamma(t)$.

**Corollary 2.9** *Let $(F, \mathbb{Y}, \rho)$ be a conformal full field algebra over $V^L \otimes V^R$. Let*

$$\begin{aligned} \gamma : [0, 1] &\to \mathbb{F}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C}) \\ t &\mapsto (z_1(t), \zeta_1(t), \dots, z_n(t), \zeta_n(t)) \end{aligned}$$

*be a path starting from a point of the form $(z_1, \bar{z}_1, \dots, z_n, \bar{z}_n)$. Then we have the following permutation property: For $u_1, \dots, u_n \in F$ and $\sigma \in S_n$,*

$$\begin{aligned} &E(m)_n(u_1, \dots, u_n; z_1(t), \zeta_1(t), \dots, z_n(t), \zeta_n(t)) \\ &\quad = E(m)_n(u_{\sigma(1)}, \dots, u_{\sigma(n)}; z_{\sigma(1)}(t), \zeta_{\sigma(1)}(t), \dots, z_{\sigma(n)}(t), \zeta_{\sigma(n)}(t)). \end{aligned} \tag{2.10}$$

*Proof.* This follows immediately from the permutation property for full field algebras and the uniqueness of analytic extensions. ∎

**Corollary 2.10** *Let $(F, \mathbb{Y}, \rho)$ be a conformal full field algebra over $V^L \otimes V^R$. Let $r_1, r_2 \in \mathbb{R}$ satisfying $r_2 > r_1 > 0$. Then for $u, v, w \in F$ and $w' \in F'$,*

$$\langle w', \mathbb{Y}_{\rm an}(v; r_2, r_2)\mathbb{Y}_{\rm an}(u; r_1, r_1)w\rangle, \tag{2.11}$$

*can be obtained by analytically extending the analytic function (which is a branch of a multivalued function)*

$$\langle w', \mathbb{Y}_{\rm an}(u; z_1, \zeta_1)\mathbb{Y}_{\rm an}(v; z_2, \zeta_2)w\rangle, \tag{2.12}$$

*defined near the point $z_1 = \zeta_1 = r_2$, $z_2 = \zeta_2 = r_1$, in the region $|z_1| > |z_2| > 0$ and $|\zeta_1| > |\zeta_2| > 0$, along the path given by*

$$\begin{aligned} [0,1] &\to M^2 \times M^2 \\ t &\mapsto ((z_1(t), z_2(t)), (\zeta_1(t), \zeta_2(t))), \end{aligned}$$

*where*

$$\begin{aligned} z_1(t) &= \frac{r_1 + r_2}{2} + e^{i\pi t}\frac{r_2 - r_1}{2}, \\ z_2(t) &= \frac{r_1 + r_2}{2} - e^{i\pi t}\frac{r_2 - r_1}{2}, \\ \zeta_1(t) &= \frac{r_1 + r_2}{2} + e^{-i\pi t}\frac{r_2 - r_1}{2}, \\ \zeta_2(t) &= \frac{r_1 + r_2}{2} - e^{-i\pi t}\frac{r_2 - r_1}{2}, \end{aligned}$$

*to the region $|z_2| > |z_1| > 0$ and $|\zeta_2| > |\zeta_1| > 0$ and then evaluated at $z_1 = \zeta_1 = r_1$ and $z_2 = \zeta_2 = r_2$.*

*Proof.* By Proposition 2.7, we know that (2.11) can indeed be obtained from (2.12) by analytic extension. What we need to show now is that the analytic extension along the path given above gives precisely (2.11).

Since $\mathbb{Y}(\cdot; z, \bar{z}) = \mathbb{Y}_{\rm an}(\cdot; z, \zeta)|_{\zeta=\bar{z}}$ and $\zeta_1(t) = \overline{z_1(t)}$ and $\zeta_2(t) = \overline{z_2(t)}$, we see that (2.11) is equal to $\langle w', m_3(v, u, w; r_2, r_2, r_1, r_1, 0, 0)\rangle$ and that

$$\langle w', \mathbb{Y}_{\rm an}(u; z_1(t), \zeta_1(t))\mathbb{Y}_{\rm an}(v; z_2(t), \zeta_2(t))w\rangle \tag{2.13}$$

when $|z_2(t)| > |z_1(t)| > 0$ and

$$\langle w', \mathbb{Y}_{\rm an}(v; z_2(t), \zeta_2(t))\mathbb{Y}_{\rm an}(u; z_1(t), \zeta_1(t))w\rangle \tag{2.14}$$

when $|z_1(t)| > |z_2(t)| > 0$ are equal to

$$\langle w', E(m)_3(u, v, w; z_1(t), \zeta_1(t), z_2(t), \zeta_2(t), 0, 0)\rangle$$

and

$$\langle w', E(m)_3(v, u, w; z_2(t), \zeta_2(t), z_1(t), \zeta_1(t), 0, 0)\rangle,$$

respectively. By the permutation property of full field algebras and Corollary 2.9), we see that (2.11), (2.13) and (2.14) are equal to

$$\langle w', m_3(u, v, w; r_1, r_1, r_2, r_2, 0, 0)\rangle,$$

$$\langle w', E(m)_3(u, v, w; z_1(t), \zeta_1(t), z_2(t), \zeta_2(t), 0, 0)\rangle$$

and

$$\langle w', E(m)_3(u, v, w; z_1(t), \zeta_1(t), z_2(t), \zeta_2(t), 0, 0)\rangle,$$

respectively. From this fact, we see that indeed the analytic extension of (2.12) near the point $z_1 = \zeta_1 = r_2$, $z_2 = \zeta_2 = r_1$, along the path given above gives (2.11).

This result can also be proved directly using the associativity (Proposition 2.6) and the skew-symmetry (1.41) (see [K] for details). ∎

**Theorem 2.11** *A conformal full field algebra over $V^L \otimes V^R$ is equivalent to a module $F$ for the vertex operator algebra $V^L \otimes V^R$ equipped with an intertwining operator $\mathcal{Y}$ of type $\binom{F}{FF}$ and an injective linear map $\rho : V^L \otimes V^R \to F$, satisfying the following conditions:*

1. *The identity property: $\mathcal{Y}(\rho(\mathbf{1}^L \otimes \mathbf{1}^R), x) = I_F$.*
2. *The creation property: For $u \in F$, $\lim_{x\to 0} \mathcal{Y}(u, x)\rho(\mathbf{1}^L \otimes \mathbf{1}^R) = u$.*
3. *The associativity: The equality (2.5) holds when $|z_1| > |z_2| > 0$ and $|\zeta_1| > |\zeta_2| > 0$.*
4. *The single-valuedness property:*

$$e^{2\pi i(L^L(0) - L^R(0))} = I_F. \tag{2.15}$$

5. *The skew symmetry:*

$$\mathbb{Y}^{\mathcal{Y}}(u;1,1)v = e^{L^L(-1)+L^R(-1)}\mathbb{Y}^{\mathcal{Y}}(v;e^{\pi i},e^{-\pi i})u. \tag{2.16}$$

*Proof.* If $(F,\mathbb{Y},\rho)$ is a conformal full field algebra over $V^L\otimes V^R$, then the results in Section 1 shows that $F$ is a $V^L\otimes V^R$-module, $\mathbb{Y}_f(\cdot;x,x)$ is an intertwining operator of type $\binom{F}{FF}$ and the five conditions are all satisfied. We now prove the converse.

Let $F$ be a module for $V^L\otimes V^R$, $\mathcal{Y}$ an intertwining operator of type $\binom{F}{FF}$ and $\rho: V^L\otimes V^R\to F$ an injective linear map, satisfying the five conditions above. We take the splitting $\mathbb{Y}^{\mathcal{Y}}$ of $\mathcal{Y}$ to be the full vertex operator map. For simplicity, we shall denote $\mathbb{Y}^{\mathcal{Y}}$ simply by $\mathbb{Y}$. We now want to construct the maps $m_n$ for $n\in\mathbb{N}$ and to verify the convergence property.

Using (2.4) and the convergence property of the intertwining operators for the vertex operator algebras $V^L$ and $V^R$, we know that for $u_1,\dots,u_n\in F$ and $w'\in F'$,

$$\langle w', \mathbb{Y}(u_1;z_1,\zeta_1)\cdots\mathbb{Y}(u_n;z_n,\zeta_n)\mathbf{1}\rangle \tag{2.17}$$

is absolutely convergent when $|z_1|>\cdots>|z_n|>0$, $|\zeta_1|>\cdots>|\zeta_n|>0$, and can be analytically extended to a (possibly multivalued) analytic function of $z_1,\dots,z_n,\zeta_1,\dots,\zeta_n$ in the region given by $z_i\neq z_j$, $z_i\neq 0$, $\zeta_i\neq\zeta_j$, $\zeta_i\neq 0$. We use

$$E(m)_n(w',u_1,\dots,u_n;z_1,\zeta_1,\dots,z_n,\zeta_n) \tag{2.18}$$

to denote this function. This is a function of $z_1,\zeta_1,\dots,z_n,\zeta_n$ where $(z_1,\dots,z_n)$, $(\zeta_1,\dots,\zeta_n)\in\mathbb{F}_n(\mathbb{C})$. So we can view this function as a function on $\mathbb{F}_n(\mathbb{C})\times\mathbb{F}_n(\mathbb{C})$. In general, this function is multivalued. Using analytic extension, a value of this function at a point $\mathbb{P}_1\in\mathbb{F}_n(\mathbb{C})\times\mathbb{F}_n(\mathbb{C})$ and a path $\gamma$ in $\mathbb{F}_n(\mathbb{C})\times\mathbb{F}_n(\mathbb{C})$ from $\mathbb{P}_1$ to $\mathbb{P}_2\in\mathbb{F}_n(\mathbb{C})\times\mathbb{F}_n(\mathbb{C})$, determines uniquely a value of the function at the point $\mathbb{P}_2$. Moreover, this value depends only on the homotopy class of the path $\gamma$. We shall call the value of the function (2.18) at $\mathbb{P}_2$ obtained this way the *value of (2.18) at* $\mathbb{P}_2$ *obtained by analytically extending the value of (2.18) at* $\mathbb{P}_1$ *along* $\gamma$.

We choose the correlation function

$$\langle w', m_n(u_1,\dots,u_n;z_1,\bar{z}_1,\dots,z_n,\bar{z}_n)\rangle \tag{2.19}$$

as follows: For $z_1=n,\dots,z_n=1$, we define (2.19) to be (2.17) with $z_1=\zeta_1=n,\dots,z_n=\zeta_n=1$. For general $(z_1,\dots,z_n)\in\mathbb{F}_n(\mathbb{C})$, we choose a path $\gamma$ from $(n,\dots,1)$ to $(z_1,\dots,z_n)$. Then we have a path $\gamma\times\overline{\gamma}$ from

$$((n,\dots,1),(n,\dots,1))\in\mathbb{F}_n(\mathbb{C})\times\mathbb{F}_n(\mathbb{C})$$

to

$$((z_1, \dots, z_n), (\bar{z}_1, \dots, \bar{z}_n)) \in \mathbb{F}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C}).$$

We define (2.19) at to be the value of (2.18) obtained by analytically extending the value of (2.18) at $((n, \dots, 1), (n, \dots, 1))$ along $\gamma \times \overline{\gamma}$.

The first thing we have to prove is that the correlation function we just defined is indeed independent of the path $\gamma$. To prove this fact, we need only prove that if $\gamma$ is a loop in $\mathbb{F}_n(\mathbb{C})$ based at $(n, \dots, 1)$ , then the value of (2.18) at $((n, \dots, 1), (n, \dots, 1))$ obtained by analytically extending the value (2.19) of (2.18) at $((n, \dots, 1), (n, \dots, 1))$ along the loop $\gamma \times \overline{\gamma}$ is equal to the original value (2.19) of (2.18) at $((n, \dots, 1), (n, \dots, 1))$. In other words, we need only prove that the *monodromy along the path* $\gamma \times \overline{\gamma}$ *is trivial*. Note that the group of the homotopy classes of based loops in $\mathbb{F}_n(\mathbb{C})$ , that is, the fundamental group of $\mathbb{F}_n(\mathbb{C})$, is the pure braid group of $n$ strands (see [Bi]). This group is generated by the homotopy classes of the loops given by fixing $z_1, \dots, z_{j-1}$, $z_{j+1}, \dots, z_n$ to be $n, \dots, n-(j-2)$, $n-j, \dots, 1$, respectively, and moving $z_j$ starting from $z_j = n-(j-1)$ around $z_i = n-(i-1)$ once (but not around other points above) in the counter clockwise direction, for $i \neq j$, $i, j = 1, \dots, n$. Hence we need only prove that the monodromy along the path $\gamma \times \overline{\gamma}$ is trivial for (the homotopy class of) such a loop $\gamma$.

We now prove that the monodromy along the path $\gamma \times \bar{\gamma}$ is trivial for (the homotopy classes of) such a loop $\gamma$. Let $r$ be a positive real number satisfying $n-(i-1) > r > n-i$. Note that $r$ satisfies $r > n-(i-1)-r > 0$. We know that

$$\begin{aligned} \langle w', \mathbb{Y}(u_1; n, n) \cdots \mathbb{Y}(u_i; n-(i-1), n-(i-1)) \mathbb{Y}(u_j; r, r) \cdot \\ \cdot \mathbb{Y}(u_{i+1}; n-i, n-i) \cdots \mathbb{Y}(u_i; n-(j-2), n-(j-2)) \cdot \\ \cdot \mathbb{Y}(u_{i+1}; n-j, n-j)) \cdots \mathbb{Y}(u_n; 1, 1)\mathbf{1} \rangle \end{aligned} \tag{2.20}$$

can be obtained by analytically extending the value

$$\langle w', \mathbb{Y}(u_1; n, n) \cdots \mathbb{Y}(u_n; 1, 1)\mathbf{1} \rangle.$$

along a path from $((n, \dots, 1), (n, \dots, 1))$ to

$$((n, \dots, n-(j-2), r, n-j, \dots, 1), (n, \dots, n-(j-2), r, n-j, \dots, 1)). \tag{2.21}$$

Such a path can always be taken to be of the form $\gamma_0 \times \bar{\gamma}_0$ where $\gamma_0$ is a path in $\mathbb{F}_n(\mathbb{C})$ from $(n, \dots, 1)$ to $(n, \dots, n-(j-2), r, n-j, \dots, 1)$. This path $\gamma_0$ induces an isomorphism from the fundamental group of $\mathbb{F}_n(\mathbb{C})$ based at $(n, \dots, 1)$ to that based at $(n,\dots,n-(j-2),r,n-j,\dots,1)$. It is clear that the monodromy along a loop based at $(n,\dots,1)$ is trivial if and only if the monodromy along the corresponding loop based at $(n,\dots,n-(j-2),r,n-j,\dots,1)$ is trivial. So we need only prove that the monodromy along a loop of the form $\gamma\times\bar{\gamma}$ is trivial where $\gamma$ is a loop based at $(n,\dots,n-(j-2),r,n-j,\dots,1)$ given by fixing $z_1,\dots,z_{j-1}$, $z_{j+1},\dots,z_n$ to be $n,\dots,n-(j-2)$, $n-j,\dots,1$, respectively, and moving $z_j$ starting from $z_j=r$ around $z_i=n-(i-1)$ once (but not around other points above) in the counter clockwise direction. By the definition of $\gamma_0$, the value of (2.18) at the point (2.21) obtained by analytically extending the value (2.19) of (2.18) at the point $((n,\dots,1),(n,\dots,1))$ along $\gamma_0$ is (2.20). Since we also have $r>n-(i-1)-r>0$, by associativity, (2.20) is equal to

$$\begin{aligned}&\langle w',\mathbb{Y}(u_1;n,n)\cdots\mathbb{Y}(u_{i-1};n-(i-2),n-(i-2))\cdot\\&\quad\cdot\mathbb{Y}(\mathbb{Y}(u_i;n-(i-1)-r,n-(i-1)-r)u_j;r,r)\cdot\\&\qquad\cdot\mathbb{Y}(u_{i+1};n-i,n-i))\cdots\mathbb{Y}(u_i;n-(j-2),n-(j-2))\cdot\\&\qquad\quad\cdot\mathbb{Y}(u_{i+1};n-j,n-j))\cdots\mathbb{Y}(u_n;r_n,r_n)\mathbf{1}\rangle.\end{aligned}$$

Now let $\gamma:[0,1]\to\mathbb{F}_n(\mathbb{C})$ be the loop given by

$$t\mapsto(n,\dots,n-(i-2),r+e^{2\pi it}(n-(i-1)-r),n-i,\dots,n-(j-2),r,n-j,\dots,1).$$

Then the value of (2.18) at (2.21) obtained by analytically extending the original value (2.20) of (2.18) at the point (2.21) along $\gamma$ is

$$\begin{aligned}&\langle w',\mathbb{Y}(u_1;n,n)\cdots\mathbb{Y}(u_{i-1};n-(i-2),n-(i-2))\cdot\\&\quad\cdot\mathbb{Y}(\mathbb{Y}(u_i;e^{2\pi i}(n-(i-1)-r),e^{-2\pi i}(n-(i-1)-r))u_j;r,r)\cdot\\&\qquad\cdot\mathbb{Y}(u_{i+1};n-i,n-i)\cdots\mathbb{Y}(u_i;n-(j-2),n-(j-2))\cdot\\&\qquad\quad\cdot\mathbb{Y}(u_{i+1};n-j,n-j))\cdots\mathbb{Y}(u_n;r_n,r_n)\mathbf{1}\rangle.\end{aligned}\tag{2.22}$$

But by the $L^L(0)$- and $L^R(0)$-conjugation properties and the siungle-valuedness property, we have

$$\begin{aligned}&\mathbb{Y}(u_i;e^{2\pi i}(n-(i-1)-r),e^{-2\pi i}(n-(i-1)-r))\\&\quad=e^{2\pi i(L^L(0)-L^R(0))}\mathbb{Y}(e^{-2\pi i(L^L(0)-L^R(0))}u_i;n-(i-1)-r,n-(i-1)-r)\cdot\\&\qquad\cdot e^{-2\pi i(L^L(0)-L^R(0))}\\&\quad=\mathbb{Y}(u_i;n-(i-1)-r,n-(i-1)-r).\end{aligned}\tag{2.23}$$

Using (2.23) and the associativity again, we see that (2.22) is equal to (2.20). Thus the analytic extension along this loop indeed gives trivial monodromy.

Now the correlation functions and thus the maps $m_n$ for $n \in \mathbb{N}$ are defined. The only remaining thing to be shown is the convergence property. We need to show that for any $k \in \mathbb{Z}_+$, $l_1, \dots, l_k \in \mathbb{Z}_+$, $(z_1, \dots, z_k) \in \mathbb{F}_n(\mathbb{C})$, $(z_1^{(i)}, \dots, z_{l_i}^{(i)}) \in \mathbb{F}_{l_i}(\mathbb{C})$, $i = 1, \dots, k$, the series (1.13) converges absolutely to (1.2) when $|z_p^{(i)}| + |z_q^{(j)}| < |z_i^{(0)} - z_j^{(0)}|$ for $i \neq j$, $i, j = 1, \dots, k$, $p = 1, \dots, l_i$ and $q = 1, \dots, l_j$.

We use induction on $k$. We first prove the special case in which $k = 2$ and $z_2^{(0)} = \bar{z}_2^{(0)} = 0$. The case $k = 1$ is in fact a special case. By the definition of $\mathbb{Y}$, (1.13) becomes

$$\sum_{p_1, q_1, p_2, q_2} \mathbb{Y}(P_{p_1, q_1} m_{l_1}(u_1^{(1)}, \dots, u_{l_1}^{(1)}; z_1^{(1)}, \bar{z}_1^{(1)}, \dots, z_{l_1}^{(1)}, \bar{z}_{l_1}^{(1)}); z_1^{(0)}, \bar{z}_1^{(0)}) \cdot \\ \cdot P_{p_2, q_2} m_{l_2}(u_1^{(2)}, \dots, u_{l_2}^{(2)}; z_1^{(2)}, \bar{z}_1^{(2)}, \dots, z_{l_2}^{(2)}, \bar{z}_{l_2}^{(2)})). \quad (2.24)$$

We use induction on $l_1$. When $l_1 = 1$, (2.24) becomes

$$\sum_{p_1, q_1, p_2, q_2} \mathbb{Y}(P_{p_1, q_1} \mathbb{Y}(u_1^{(1)}; z_1^{(1)}, \bar{z}_1^{(1)}) \mathbf{1}; z_1^{(0)}, \bar{z}_1^{(0)}) \cdot \\ \cdot P_{p_2, q_2} m_{l_2}(u_1^{(2)}, \dots, u_{l_2}^{(2)}; z_1^{(2)}, \bar{z}_1^{(2)}, \dots, z_{l_2}^{(2)}, \bar{z}_{l_2}^{(2)})). \quad (2.25)$$

Using the construction of $\mathbb{Y}$ in terms of intertwining operators, the properties of intertwining operators and noticing that our condition $|z_1^{(1)}| + |z_j^{(2)}| < |z_1^{(0)}|$ implies $|z_1^{(1)}| < |z_1^{(0)}|$ and $|z_1^{(1)} + z_1^{(0)}| > |z_j^{(2)}|$, we know that

$$\sum_{p_1, q_1} \mathbb{Y}(P_{p_1, q_1} \mathbb{Y}(u_1^{(1)}; z_1^{(1)}, \bar{z}_1^{(1)}) \mathbf{1}; z_1^{(0)}, \bar{z}_1^{(0)}) \cdot \\ \cdot P_{p_2, q_2} m_{l_2}(u_1^{(2)}, \dots, u_{l_2}^{(2)}; z_1^{(2)}, \bar{z}_1^{(2)}, \dots, z_{l_2}^{(2)}, \bar{z}_{l_2}^{(2)}))$$

is absolutely convergent to

$$\begin{aligned} &\mathbb{Y}(u_1^{(1)}; z_1^{(1)} + z_1^{(0)}, \bar{z}_1^{(1)} + \bar{z}_1^{(0)}) \mathbb{Y}(\mathbf{1}; z_1^{(0)}, \bar{z}_1^{(0)}) \cdot \\ &\qquad \cdot P_{p_2, q_2} m_{l_2}(u_1^{(2)}, \dots, u_{l_2}^{(2)}; z_1^{(2)}, \bar{z}_1^{(2)}, \dots, z_{l_2}^{(2)}, \bar{z}_{l_2}^{(2)})) \\ &= \mathbb{Y}(u_1^{(1)}; z_1^{(1)} + z_1^{(0)}, \bar{z}_1^{(1)} + \bar{z}_1^{(0)}) \cdot \\ &\qquad \cdot P_{p_2, q_2} m_{l_2}(u_1^{(2)}, \dots, u_{l_2}^{(2)}; z_1^{(2)}, \bar{z}_1^{(2)}, \dots, z_{l_2}^{(2)}, \bar{z}_{l_2}^{(2)})). \end{aligned}$$

Then by the construction of the correlation function maps and, in particular, by the fact that the correlation functions are values of multivalued analytic functions at certain particular points, we know that the right-hand side of (2.25) is absolutely convergent to

$$m_{1+l_2}(u_1^{(1)}, u_1^{(2)}, \ldots, u_{l_2}^{(2)}; z_1^{(1)} + z_1^{(0)}, \bar{z}_1^{(1)} + \bar{z}_1^{(0)}, z_1^{(2)}, \bar{z}_1^{(2)}, \ldots, z_{l_2}^{(2)}, \bar{z}_{l_2}^{(2)}).$$

Here we have used the fact that if a certain iterated sum of a series in powers of these complex variables is convergent to an analytic functions in the region above, then the multisum must also be absolutely convergent.

Now we assume that for $l_1 < l$, the conclusion holds. We want to prove the conclusion for the case $l_1 = l$. We first assume that $|z_1^{(1)}|, \ldots, |z_l^{(1)}|$ are all different from each other. Then in particular there exists $t$ such that $|z_t^{(1)}| > |z_1^{(1)}|, \ldots, \widehat{|z_t^{(1)}|}, \ldots, |z_l^{(1)}|$, where and also below we use $\hat{}$ to denote that the item under $\hat{}$ is missing. Then (2.24) in this case is equal to

$$\begin{aligned}\sum_{p_1,q_1,p_2,q_2} \sum_{r,s} \mathbb{Y}(P_{p_1,q_1}\mathbb{Y}(u_t^{(1)}; z_t^{(1)}, \bar{z}_t^{(1)})\cdot &\\ \cdot P_{r,s} m_{l-1}(u_1^{(1)}, \ldots, \widehat{u_t^{(1)}}, \ldots, u_l^{(1)}; z_1^{(1)}, \bar{z}_1^{(1)}, \ldots, \widehat{z_t^{(1)}}, \widehat{\bar{z}_t^{(1)}}, \ldots, z_l^{(1)}, \bar{z}_l^{(1)});&\\ z_1^{(0)}, \bar{z}_1^{(0)})\cdot&\\ \cdot P_{p_2,q_2} m_{l_2}(u_1^{(2)}, \ldots, u_{l_2}^{(2)}; z_1^{(2)}, \bar{z}_1^{(2)}, \ldots, z_{l_2}^{(2)}, \bar{z}_{l_2}^{(2)})).&\end{aligned} \tag{2.26}$$

Using the construction of $\mathbb{Y}$ in terms of intertwining operators and the properties of intertwining operators, and noticing that our condition $|z_t^{(1)}| + |z_j^{(2)}| < |z_1^{(0)}|$ implies $|z_t^{(1)}| < |z_1^{(0)}|$ and $|z_t^{(1)} + z_1^{(0)}| > |z_j^{(2)}|$, we know that

$$\begin{aligned}\sum_{p_1,q_1} \mathbb{Y}(P_{p_1,q_1}\mathbb{Y}(u_t^{(1)}; z_t^{(1)}, \bar{z}_t^{(1)})\cdot &\\ \cdot P_{r,s} m_{l-1}(u_1^{(1)}, \ldots, \widehat{u_t^{(1)}}, \ldots, u_l^{(1)}; z_1^{(1)}, \bar{z}_1^{(1)}, \ldots, \widehat{z_t^{(1)}}, \widehat{\bar{z}_t^{(1)}}, \ldots, z_l^{(1)}, \bar{z}_l^{(1)});&\\ z_1^{(0)}, \bar{z}_1^{(0)})\cdot&\\ \cdot P_{p_2,q_2} m_{l_2}(u_1^{(2)}, \ldots, u_{l_2}^{(2)}; z_1^{(2)}, \bar{z}_1^{(2)}, \ldots, z_{l_2}^{(2)}, \bar{z}_{l_2}^{(2)}))&\end{aligned} \tag{2.27}$$

is convergent absolutely and, when $|z_t^{(1)} + z_1^{(0)}| > |z_1^{(0)}|$, it is absolutely convergent to

$$\begin{aligned}\mathbb{Y}(u_t^{(1)}; z_t^{(1)} + z_1^{(0)}, \bar{z}_t^{(1)} + \bar{z}_1^{(0)})\cdot &\\ \cdot \mathbb{Y}(P_{r,s} m_{l-1}(u_1^{(1)}, \ldots, \widehat{u_t^{(1)}}, \ldots, u_l^{(1)}; z_1^{(1)}, \bar{z}_1^{(1)}, \ldots, \widehat{z_t^{(1)}}, \widehat{\bar{z}_t^{(1)}}, \ldots, z_l^{(1)}, \bar{z}_l^{(1)});&\\ z_1^{(0)}, \bar{z}_1^{(0)})\cdot&\\ \cdot P_{p_2,q_2} m_{l_2}(u_1^{(2)}, \ldots, u_{l_2}^{(2)}; z_1^{(2)}, \bar{z}_1^{(2)}, \ldots, z_{l_2}^{(2)}, \bar{z}_{l_2}^{(2)})).&\end{aligned} \tag{2.28}$$

By the induction assumption,

$$\sum_{r,s,p_2,q_2} \mathbb{Y}(u_t^{(1)}; z_t^{(1)} + z_1^{(0)}, \bar{z}_t^{(1)} + \bar{z}_1^{(0)})\cdot$$
$$\cdot\mathbb{Y}(P_{r,s}m_{l-1}(u_1^{(1)}, \dots, \widehat{u_t^{(1)}}, \dots, u_l^{(1)}; z_1^{(1)}, \bar{z}_1^{(1)}, \dots, \widehat{z_t^{(1)}}, \widehat{\bar{z}_t^{(1)}}, \dots, z_l^{(1)}, \bar{z}_l^{(1)});$$
$$z_1^{(0)}, \bar{z}_1^{(0)})\cdot$$
$$\cdot P_{p_2,q_2}m_{l_2}(u_1^{(2)}, \dots, u_{l_2}^{(2)}; z_1^{(2)}, \bar{z}_1^{(2)}, \dots, z_{l_2}^{(2)}, \bar{z}_{l_2}^{(2)})) \qquad (2.29)$$

is absolutely convergent to

$$\mathbb{Y}(u_t^{(1)}; z_t^{(1)} + z_1^{(0)}, \bar{z}_t^{(1)} + \bar{z}_1^{(0)})\cdot$$
$$\cdot m_{l+l_2-1}(u_1^{(1)}, \dots, \widehat{u_t^{(1)}}, \dots, u_l^{(1)}, u_1^{(2)}, \dots, u_{l_2}^{(2)}; z_1^{(1)} + z_1^{(0)}, \bar{z}_1^{(1)} + \bar{z}_1^{(0)}, \dots,$$
$$\widehat{z_t^{(1)} + z_1^{(0)}}, \widehat{\bar{z}_t^{(1)} + \bar{z}_1^{(0)}}, \dots, z_l^{(1)} + z_1^{(0)}, \bar{z}_l^{(1)} + \bar{z}_1^{(0)} z_1^{(2)}, \bar{z}_1^{(2)}, \dots, z_{l_2}^{(2)}, \bar{z}_{l_2}^{(2)})$$
$$= m_{l+l_2}(u_1^{(1)}, \dots, u_l^{(1)}, u_1^{(2)}, \dots, u_{l_2}^{(2)}; z_1^{(1)} + z_1^{(0)}, \bar{z}_1^{(1)} + \bar{z}_1^{(0)}, \dots,$$
$$z_l^{(1)} + z_1^{(0)}, \bar{z}_l^{(1)} + \bar{z}_1^{(0)} z_1^{(2)}, \bar{z}_1^{(2)}, \dots, z_{l_2}^{(2)}, \bar{z}_{l_2}^{(2)}). \qquad (2.30)$$

We know that the right-hand side of (2.29) is a value of the multivalued analytic function

$$E(m)_{l+l_2}(u_1^{(1)}, \dots, u_l^{(1)}, u_1^{(2)}, \dots, u_{l_2}^{(2)}; z_1^{(1)} + z_1^{(0)}, \zeta_1^{(1)} + \zeta_1^{(0)}, \dots,$$
$$z_l^{(1)} + z_1^{(0)}, \zeta_l^{(1)} + \zeta_1^{(0)} z_1^{(2)}, \zeta_1^{(2)}, \dots, z_{l_2}^{(2)}, \zeta_{l_2}^{(2)})$$

at the points satisfying $\zeta_1^{(0)} = \bar{z}_1^{(0)}$, $\zeta_p^{(i)} = \bar{z}_p^{(i)}$ for $p = 1, \dots, l_i$, $i = 1, 2$. Since both the sum of (2.29) and the right-hand side of (2.30) are values of multivalued analytic functions in the same region and we have proved that their values are equal when $|z_t^{(1)} + z_1^{(0)}| > |z_1^{(0)}|$, (2.29) must be convergent absolutely to the right-hand side of (2.30) even when $|z_t^{(1)} + z_1^{(0)}| > |z_1^{(0)}|$ is not satisfied. By the properties of analytic functions, we know that (2.26) as a sum in a different order is also convergent absolutely to the right-hand side of (2.30).

Now we discuss the case that some of $|z_1^{(1)}|, \dots, |z_l^{(1)}|$ are equal. Let $N(z_1^{(1)}, \dots, z_l^{(1)})$ be the subset of $\{z_1^{(1)}, \dots, z_l^{(1)}\}$ consisting of those elements whose absolute values are equal to the absolute values of some other elements of $\{z_1^{(1)}, \dots, z_l^{(1)}\}$. We use induction on the number of elements of $N(z_1^{(1)}, \dots, z_l^{(1)})$. When the number is 0, this is the case discussed above. Now assume that when the number is equal to $n$, the conclusion holds. When this number is equal to $n+1$, let $\epsilon$ be a complex number such that the number of elements of $N(z_1^{(1)}+\epsilon, \dots, z_l^{(1)}+\epsilon)$ is $n$ and $|z_p^{(1)}+\epsilon|+|z_q^{(2)}+\epsilon| < |z_1^{(0)}|$ for $p=1,\dots,l$ and $q=1,\dots,l_2$. Note that we can always find such an $\epsilon$ and we can take such an $\epsilon$ with $|\epsilon|$ to be arbitrarily small.

By induction assumption,

$$\sum_{p_1,q_1,p_2,q_2} \mathbb{Y}(P_{p_1,q_1} m_l(u_1^{(1)}, \dots, u_l^{(1)}; z_1^{(1)}+\epsilon, \bar{z}_1^{(1)}+\bar{\epsilon}, \dots, z_l^{(1)}+\epsilon, \bar{z}_l^{(1)}+\bar{\epsilon}); z_1^{(0)}, \bar{z}_1^{(0)}) \cdot \\ \cdot P_{p_2,q_2} m_{l_2}(u_1^{(2)}, \dots, u_{l_2}^{(2)}; z_1^{(2)}+\epsilon, \bar{z}_1^{(2)}+\bar{\epsilon}, \dots, z_{l_2}^{(2)}+\epsilon, \bar{z}_{l_2}^{(2)}+\bar{\epsilon}) \tag{2.31}$$

is absolutely convergent to

$$m_{l+l_2}(u_1^{(1)}, \dots, u_l^{(1)}, u_1^{(2)}, \dots, u_{l_2}^{(2)}; z_1^{(1)}+z_1^{(0)}+\epsilon, \bar{z}_1^{(1)}+\bar{z}_1^{(0)}+\bar{\epsilon}, \dots, \\ z_l^{(1)}+z_1^{(0)}+\epsilon, \bar{z}_l^{(1)}+\bar{z}_1^{(0)}+\bar{\epsilon} z_1^{(2)}+\epsilon, \bar{z}_1^{(2)}+\bar{\epsilon}, \dots, z_{l_2}^{(2)}+\epsilon, \bar{z}_{l_2}^{(2)}+\bar{\epsilon}). \tag{2.32}$$

We have

$$\sum_{r_1,s_1,r_2,s_2} \sum_{p_1,q_1,p_2,q_2} \langle e^{-\epsilon L^L(1)-\bar{\epsilon} L^R(1)} u', \mathbb{Y}(P_{r_1,s_1} e^{\epsilon L^L(-1)+\bar{\epsilon} L^R(-1)} \cdot \\ \cdot P_{p_1,q_1} m_l(u_1^{(1)}, \dots, u_l^{(1)}; z_1^{(1)}, \bar{z}_1^{(1)}, \dots, z_l^{(1)}, \bar{z}_l^{(1)}); z_1^{(0)}, \bar{z}_1^{(0)}) \cdot \\ \cdot P_{r_1,s_1} e^{\epsilon L^L(-1)+\bar{\epsilon} L^R(-1)} P_{p_2,q_2} m_{l_2}(u_1^{(2)}, \dots, u_{l_2}^{(2)}; z_1^{(2)}, \bar{z}_1^{(2)}, \dots, z_{l_2}^{(2)}, \bar{z}_{l_2}^{(2)})\rangle \\ = \sum_{r_1,s_1,r_2,s_2} \langle e^{-\epsilon L^L(1)-\bar{\epsilon} L^R(1)} u', \\ \mathbb{Y}(P_{r_1,s_1} m_l(u_1^{(1)}, \dots, u_l^{(1)}; z_1^{(1)}+\epsilon, \bar{z}_1^{(1)}+\bar{\epsilon}, \dots, z_l^{(1)}+\epsilon, \bar{z}_l^{(1)}+\bar{\epsilon}); \\ z_1^{(0)}, \bar{z}_1^{(0)}) \cdot \\ \cdot P_{r_1,s_1} m_{l_2}(u_1^{(2)}, \dots, u_{l_2}^{(2)}; z_1^{(2)}+\epsilon, \bar{z}_1^{(2)}+\bar{\epsilon}, \dots, z_{l_2}^{(2)}+\epsilon, \bar{z}_{l_2}^{(2)}+\bar{\epsilon})\rangle. \tag{2.33}$$

So the right-hand side and thus also the left-hand side of (2.33) is absolutely convergent to

$$\langle e^{-\epsilon L^L(1)-\bar{\epsilon} L^R(1)} u', m_{l+l_2}(u_1^{(1)}, \dots, u_l^{(1)}, u_1^{(2)}, \dots, u_{l_2}^{(2)}; z_1^{(1)}+z_1^{(0)}+\epsilon,$$

$$\bar{z}_1^{(1)} + \bar{z}_1^{(0)} + \bar{\epsilon}, \dots, z_l^{(1)} + z_1^{(0)} + \epsilon, \bar{z}_l^{(1)} + \bar{z}_1^{(0)} + \bar{\epsilon},$$
$$z_1^{(2)} + \epsilon, \bar{z}_1^{(2)} + \bar{\epsilon}, \dots, z_{l_2}^{(2)} + \epsilon, \bar{z}_{l_2}^{(2)} + \bar{\epsilon})\rangle$$
$$= \langle u', e^{-\epsilon L^L(-1)-\bar{\epsilon}L^R(-1)} m_{l+l_2}(u_1^{(1)}, \dots, u_l^{(1)}, u_1^{(2)}, \dots, u_{l_2}^{(2)}; z_1^{(1)} + z_1^{(0)} + \epsilon,$$
$$\bar{z}_1^{(1)} + \bar{z}_1^{(0)} + \bar{\epsilon}, \dots, z_l^{(1)} + z_1^{(0)} + \epsilon, \bar{z}_l^{(1)} + \bar{z}_1^{(0)} + \bar{\epsilon},$$
$$z_1^{(2)} + \epsilon, \bar{z}_1^{(2)} + \bar{\epsilon}, \dots, z_{l_2}^{(2)} + \epsilon, \bar{z}_{l_2}^{(2)} + \bar{\epsilon})\rangle$$
$$= \langle u', m_{l+l_2}(u_1^{(1)}, \dots, u_l^{(1)}, u_1^{(2)}, \dots, u_{l_2}^{(2)}; z_1^{(1)} + z_1^{(0)}, \bar{z}_1^{(1)} + \bar{z}_1^{(0)}, \dots,$$
$$z_l^{(1)} + z_1^{(0)}, \bar{z}_l^{(1)} + \bar{z}_1^{(0)} z_1^{(2)}, \bar{z}_1^{(2)}, \dots, z_{l_2}^{(2)}, \bar{z}_{l_2}^{(2)})\rangle. \tag{2.34}$$

Since the left-hand side of (2.34) is a value of a multivalued analytic function, any of its expansion must be absolutely convergent. In particular, the left-hand side of (2.33) as an expansion of the left-hand side of (2.34) is absolutely convergent. Thus we can exchange the order of the two summation signs such that the resulting series is still absolutely convergent to the left-hand side of (2.34) and thus to the right-hand side of (2.34).

But for $u' \in F'$,

$$\sum_{p_1,q_1,p_2,q_2} \langle u', \mathbb{Y}(P_{p_1,q_1} m_l(u_1^{(1)}, \dots, u_l^{(1)}; z_1^{(1)}, \bar{z}_1^{(1)}, \dots, z_l^{(1)}, \bar{z}_l^{(1)});$$
$$z_1^{(0)}, \bar{z}_1^{(0)}) \cdot$$
$$\cdot P_{p_2,q_2} m_{l_2}(u_1^{(2)}, \dots, u_{l_2}^{(2)}; z_1^{(2)}, \bar{z}_1^{(2)}, \dots, z_{l_2}^{(2)}, \bar{z}_{l_2}^{(2)})\rangle$$
$$= \sum_{p_1,q_1,p_2,q_2} \langle e^{-\epsilon L^L(1)-\bar{\epsilon}L^R(1)} u', e^{\epsilon L^L(-1)+\bar{\epsilon}L^R(-1)} \cdot$$
$$\cdot \mathbb{Y}(P_{p_1,q_1} m_l(u_1^{(1)}, \dots, u_l^{(1)}; z_1^{(1)}, \bar{z}_1^{(1)}, \dots, z_l^{(1)}, \bar{z}_l^{(1)}); z_1^{(0)}, \bar{z}_1^{(0)}) \cdot$$
$$\cdot P_{p_2,q_2} m_{l_2}(u_1^{(2)}, \dots, u_{l_2}^{(2)}; z_1^{(2)}, \bar{z}_1^{(2)}, \dots, z_{l_2}^{(2)}, \bar{z}_{l_2}^{(2)})\rangle$$
$$= \sum_{p_1,q_1,p_2,q_2} \langle e^{-\epsilon L^L(1)-\bar{\epsilon}L^R(1)} u', \mathbb{Y}(e^{\epsilon L^L(-1)+\bar{\epsilon}L^R(-1)} \cdot$$
$$\cdot P_{p_1,q_1} m_l(u_1^{(1)}, \dots, u_l^{(1)}; z_1^{(1)}, \bar{z}_1^{(1)}, \dots, z_l^{(1)}, \bar{z}_l^{(1)}); z_1^{(0)}, \bar{z}_1^{(0)}) \cdot$$
$$\cdot e^{\epsilon L^L(-1)+\bar{\epsilon}L^R(-1)} P_{p_2,q_2} m_{l_2}(u_1^{(2)}, \dots, u_{l_2}^{(2)}; z_1^{(2)}, \bar{z}_1^{(2)}, \dots, z_{l_2}^{(2)}, \bar{z}_{l_2}^{(2)})\rangle$$
$$= \sum_{p_1,q_1,p_2,q_2} \sum_{r_1,s_1,r_2,s_2} \langle e^{-\epsilon L^L(1)-\bar{\epsilon}L^R(1)} u', \mathbb{Y}(P_{r_1,s_1} e^{\epsilon L^L(-1)+\bar{\epsilon}L^R(-1)} \cdot$$
$$\cdot P_{p_1,q_1} m_l(u_1^{(1)}, \dots, u_l^{(1)}; z_1^{(1)}, \bar{z}_1^{(1)}, \dots, z_l^{(1)}, \bar{z}_l^{(1)}); z_1^{(0)}, \bar{z}_1^{(0)}) \cdot$$
$$\cdot P_{r_1,s_1} e^{\epsilon L^L(-1)+\bar{\epsilon}L^R(-1)} P_{p_2,q_2} m_{l_2}(u_1^{(2)}, \dots, u_{l_2}^{(2)}; z_1^{(2)}, \bar{z}_1^{(2)}, \dots, z_{l_2}^{(2)}, \bar{z}_{l_2}^{(2)})\rangle.$$

$$(2.35)$$

We have shown that the right-hand side of (2.35) is absolute convergent to the right-hand side of (2.34). Thus the left-hand side of (2.35) is also absolute convergent to the right-hand side of (2.34). So we have proved the convergence property when the number of elements of $N(z_1^{(1)}, \ldots, z_l^{(1)})$ is $n+1$. Thus the convergence property is proved when some of $|z_1^{(1)}|, \ldots, |z_l^{(1)}|$ are equal.

By induction principle, we have proved the convergence property in this special case.

We now assume that when $k < K$, (1.13) converges absolutely to (1.2) when $z_p^{(0)} \neq z_q^{(0)}$ for $p, q = 1, \ldots, K$, $z_p^{(i)} \neq z_q^{(i)}$ for $p, q = 1, \ldots, l_i$ and $i = 1, \ldots, K$ $1 \leq p, q \leq l_i$, and $|z_p^{(i)}| + |z_q^{(j)}| < |z_i^{(0)} - z_j^{(0)}|$ for $p = 1, \ldots, l_i$, $q = 1, \ldots, l_j$, $i, j = 1, \ldots, K$, $i \neq j$. Now we consider the case $k = K$. We first consider the case that $z_p^{(i)} \in \mathbb{R}_+ \cup \{0\}$ for $p = 1, \ldots, l_i$ and $i = 0, \ldots, K$ and $z_1^{(0)} > \cdots > z_K^{(0)}$. By the definition of the correlation function maps, we know that (1.13) in this case is equal to

$$\begin{aligned}
\sum_{p_1, q_1, \ldots, p_K, q_K} \sum_{r,s} \mathbb{Y}(P_{p_1,q_1} m_{l_1}(u_1^{(1)}, \ldots, u_{l_1}^{(1)}; z_1^{(1)}, \bar{z}_1^{(1)}, \ldots, z_{l_1}^{(1)}, \bar{z}_{l_1}^{(1)}); z_1^{(0)}, \bar{z}_1^{(0)}) \cdot \\
\cdot P_{r,s} m_{K-1}(P_{p_2,q_2} m_{l_2}(u_1^{(2)}, \ldots, u_{l_2}^{(2)}; z_1^{(2)}, \bar{z}_1^{(2)}, \ldots, z_{l_2}^{(2)}, \bar{z}_{l_2}^{(2)}), \ldots, \\
P_{p_K,q_K} m_{l_K}(u_1^{(K)}, \ldots, u_{l_K}^{(K)}; z_1^{(K)}, \bar{z}_1^{(K)}, \ldots, z_{l_K}^{(K)}, \bar{z}_{l_K}^{(K)}); \\
z_2^{(0)}, \bar{z}_2^{(0)}, \ldots, z_K^{(0)}, \bar{z}_K^{(0)}).
\end{aligned} \tag{2.36}$$

Using the induction assumption, we have

$$\begin{aligned}
\sum_{r,s,p_1,q_1} \sum_{p_2,q_2,\ldots,p_K,q_K} \mathbb{Y}(P_{p_1,q_1} m_{l_1}(u_1^{(1)}, \ldots, u_{l_1}^{(1)}; z_1^{(1)}, \bar{z}_1^{(1)}, \ldots, z_{l_1}^{(1)}, \bar{z}_{l_1}^{(1)}); z_1^{(0)}, \bar{z}_1^{(0)}) \cdot \\
\cdot P_{r,s} m_{K-1}(P_{p_2,q_2} m_{l_2}(u_1^{(2)}, \ldots, u_{l_2}^{(2)}; z_1^{(2)}, \bar{z}_1^{(2)}, \ldots, z_{l_2}^{(2)}, \bar{z}_{l_2}^{(2)}), \ldots, \\
P_{p_K,q_K} m_{l_K}(u_1^{(K)}, \ldots, u_{l_K}^{(K)}; z_1^{(K)}, \bar{z}_1^{(K)}, \ldots, z_{l_K}^{(K)}, \bar{z}_{l_K}^{(K)}); \\
z_2^{(0)}, \bar{z}_2^{(0)}, \ldots, z_K^{(0)}, \bar{z}_K^{(0)}) \\
= \sum_{r,s,p_1,q_1} \mathbb{Y}(P_{p_1,q_1} m_{l_1}(u_1^{(1)}, \ldots, u_{l_1}^{(1)}; z_1^{(1)}, \bar{z}_1^{(1)}, \ldots, z_{l_1}^{(1)}, \bar{z}_{l_1}^{(1)}); z_1^{(0)}, \bar{z}_1^{(0)}) \cdot \\
\cdot P_{r,s} m_{l_2+\cdots+l_K}(u_1^{(2)}, \ldots, u_{l_2}^{(2)}, \ldots, u_1^{(K)}, \ldots, u_{l_K}^{(K)}; z_1^{(2)} + z_2^{(0)}, \\
\bar{z}_1^{(2)} + \bar{z}_2^{(0)}, \ldots, z_{l_2}^{(2)} + z_2^{(0)}, \bar{z}_{l_2}^{(2)} + \bar{z}_2^{(0)}, \ldots,
\end{aligned}$$

$$z_1^{(K)} + z_K^{(0)}, \bar{z}_1^{(K)} + \bar{z}_K^{(0)}, \ldots, z_{l_K}^{(K)} + z_K^{(0)}, \bar{z}_{l_K}^{(K)} + \bar{z}_K^{(0)}). \tag{2.37}$$

Since $z_p^{(1)} + z_q^{(i)} < z_1^{(0)} - z_i^{(0)}$ for $p = 1, \ldots, l_1$, $q = 1, \ldots, l_i$ and $i = 2, \ldots, K$, we have $z_p^{(1)} + (z_q^{(i)} + z_i^{(0)}) < z_1^{(0)} - 0$. Thus by the special case we proved above, the right-hand side of (2.37) is absolutely convergent to

$$\begin{aligned} m_{l_1+\cdots+l_K}(u_1^{(1)}, \ldots, u_{l_1}^{(1)}, \ldots, u_1^{(K)}, \ldots, u_{l_K}^{(K)}; z_1^{(1)} + z_1^{(0)},& \\ \bar{z}_1^{(1)} + \bar{z}_1^{(0)}, \ldots, z_{l_1}^{(1)} + z_1^{(0)}, \bar{z}_{l_1}^{(1)} + \bar{z}_1^{(0)}, \ldots, z_1^{(K)} + z_K^{(0)},& \\ \bar{z}_1^{(K)} + \bar{z}_K^{(0)}, \ldots, z_{l_K}^{(K)} + z_K^{(0)}, \bar{z}_{l_K}^{(K)} + \bar{z}_K^{(0)}).& \end{aligned} \tag{2.38}$$

Note that (2.38) is a value of the $\overline{F}$-valued multivalued analytic function

$$\begin{aligned} E(m)_{l_1+\cdots+l_K}(u_1^{(1)}, \ldots, u_{l_1}^{(1)}, \ldots, u_1^{(K)}, \ldots, u_{l_K}^{(K)}; z_1^{(1)} + z_1^{(0)},& \\ \zeta_1^{(1)} + \zeta_1^{(0)}, \ldots, z_{l_1}^{(1)} + z_1^{(0)}, \zeta_{l_1}^{(1)} + \zeta_1^{(0)}, \ldots, z_1^{(K)} + z_K^{(0)},& \\ \zeta_1^{(K)} + \zeta_K^{(0)}, \ldots, z_{l_K}^{(K)} + z_K^{(0)}, \zeta_{l_K}^{(K)} + \zeta_K^{(0)}).& \end{aligned} \tag{2.39}$$

at the point $z_i^{(j)} = z_i^{(j)}$, $\zeta_i^{(j)} = \bar{z}_i^{(j)}$. Thus its expansions, no matter in which ways, must be convergent absolutely. In particular, (2.36) as one expansion of (2.38) must be convergent absolutely to (2.38), proving the convergence in this special case of the case $k = K$.

We know that for a series in powers of several variables, if it is absolutely convergent when these variables are equal to some real numbers, then it is also convergent when the variables are equal to complex numbers whose absolute values are equal to these real numbers. Using this property, we see that

$$\begin{aligned} \sum_{p_1,q_1,\ldots,p_K,q_K} \sum_{r,s} \mathbb{Y}(P_{p_1,q_1} E(m)_{l_1}(u_1^{(1)}, \ldots, u_{l_1}^{(1)}; z_1^{(1)}, \zeta_1^{(1)}, \ldots, z_{l_1}^{(1)}, \zeta_{l_1}^{(1)}); z_1^{(0)}, \zeta_1^{(0)})\cdot& \\ \cdot P_{r,s} E(m)_{K-1}(P_{p_2,q_2} E(m)_{l_2}(u_1^{(2)}, \ldots, u_{l_2}^{(2)}; z_1^{(2)}, \zeta_1^{(2)}, \ldots, z_{l_2}^{(2)}, \zeta_{l_2}^{(2)}), \ldots,& \\ P_{p_K,q_K} E(m)_{l_K}(u_1^{(K)}, \ldots, u_{l_K}^{(K)}; z_1^{(K)}, \zeta_1^{(K)}, \ldots, z_{l_K}^{(K)}, \zeta_{l_K}^{(K)});& \\ z_2^{(0)}, \zeta_2^{(0)}, \ldots, z_K^{(0)}, \zeta_K^{(0)}).& \end{aligned} \tag{2.40}$$

is convergent absolutely to a branch of (2.39) when $z_p^{(0)} \neq z_q^{(0)}$, $\zeta_p^{(0)} \neq \zeta_q^{(0)}$ for $p, q = 1, \ldots, K$, $z_p^{(i)} \neq z_q^{(i)}$, $\zeta_p^{(i)} \neq \zeta_q^{(i)}$ for $p, q = 1, \ldots, l_i$, $i = 1, \ldots, K$ $|z_p^{(i)}| + |z_q^{(j)}| < ||z_i^{(0)}| - |z_j^{(0)}||$, $|\zeta_p^{(i)}| + |\zeta_q^{(j)}| < ||\zeta_i^{(0)}| - |\zeta_j^{(0)}||$, for $p = 1, \ldots, l_i$,

$q = 1, \dots, l_j$, $i, j = 1, \dots, K$, $i \neq j$, and $|z_1^{(0)}| > \cdots > |z_K^{(0)}|$, $|\zeta_1^{(0)}| > \cdots > |\zeta_K^{(0)}|$. Using the permutation property for the correlation functions, we obtain that (2.40) is convergent absolutely to a branch of (2.39) when $|z_{\sigma(1)}^{(0)}| > \cdots > |z_{\sigma(K)}^{(0)}|$ and $|\zeta_{\sigma(1)}^{(0)}| > \cdots > |\zeta_{\sigma(K)}^{(0)}|$ for some $\sigma \in S_K$.

Now for fixed $z_1^{(0)}, \dots, z_K^{(0)}$, $\zeta_1^{(0)}, \dots, \zeta_K^{(0)}$ satisfying $|z_{\sigma(1)}^{(0)}| > \cdots > |z_{\sigma(K)}^{(0)}|$ and $|\zeta_{\sigma(1)}^{(0)}| > \cdots > |\zeta_{\sigma(K)}^{(0)}|$ for some $\sigma \in S_K$, any branch of (2.39) can be expanded as a series in powers of $z_p^{(i)}$ and $\zeta_p^{(i)}$, $p = 1, \dots, l_i$, $i = 1, \dots, K$ in the region

$$
\begin{aligned}
\{(z_1^{(1)}, \dots, z_{l_K}^{(K)}, \zeta_1^{(1)}, \dots, \zeta_{l_K}^{(K)}) \mid z_p^{(i)} \neq z_q^{(i)}, \zeta_p^{(i)} \neq \zeta_q^{(i)} \\
\text{for } p, q = 1, \dots, l_i, i = 1, \dots, K, \\
|z_p^{(i)}| + |z_q^{(j)}| < |z_i^{(0)} - z_j^{(0)}|, |\zeta_p^{(i)}| + |\zeta_q^{(j)}| < |\zeta_i^{(0)} - \zeta_j^{(0)}|, \\
\text{for } p = 1, \dots, l_i, q = 1, \dots, l_j, i, j = 1, \dots, K, i \neq j\}.
\end{aligned}
\tag{2.41}
$$

But in the region

$$
\begin{aligned}
\{(z_1^{(1)}, \dots, z_{l_K}^{(K)}, \zeta_1^{(1)}, \dots, \zeta_{l_K}^{(K)}) \mid z_p^{(i)} \neq z_q^{(i)}, \zeta_p^{(i)} \neq \zeta_q^{(i)} \\
\text{for } p, q = 1, \dots, l_i, i = 1, \dots, K, \\
|z_p^{(i)}| + |z_q^{(j)}| < ||z_i^{(0)}| - |z_j^{(0)}||, |\zeta_p^{(i)}| + |\zeta_q^{(j)}| < ||\zeta_i^{(0)}| - |\zeta_j^{(0)}||, \\
\text{for } p = 1, \dots, l_i, q = 1, \dots, l_j, i, j = 1, \dots, K, i \neq j\},
\end{aligned}
\tag{2.42}
$$

we have proved that one branch of (2.39) can be expanded as the series (2.40), which can be further expanded as a series in powers of $z_p^{(i)}$ and $\zeta_p^{(i)}$, $p = 1, \dots, l_i$, $i = 1, \dots, K$ in this region. Since the region (2.42) is contained in the region (2.41) and the coefficients of the expansion can be determined completely using the values of the branch in the region (2.42), we see that the restriction to the region (2.42) of the expansion in the region (2.41) is the same as the expansion in the region (2.41). Thus, the series (2.40) is convergent absolutely to a branch of (2.39) in the region (2.41).

In the region (2.41), when $\zeta_p^{(0)} = z_p^{(0)} \in \mathbb{R}$ for $p = 1, \dots, K$, $\zeta_p^{(i)} = z_p^{(i)} \in \mathbb{R}$ for $p = 1, \dots, l_i$, $i = 1, \dots, K$, we have proved that (1.13) in this case is convergent absolutely to the right-hand side of (2.37). Thus in the region (2.41), (1.13) with $k = K$ is convergent absolutely to the right-hand side of (2.37), the value of a branch of (2.39).

Finally we consider the case that some of $|z_1^{(0)}|, \dots, |z_K^{(0)}|$ are equal. Recall the subset $N(z_1^{(0)}, \dots, z_K^{(0)})$ of $\{z_1^{(0)}, \dots, z_K^{(0)}\}$ consisting of those elements whose absolute values are equal to the absolute values of some other elements of $\{z_1^{(0)}, \dots, z_K^{(0)}\}$. We use induction on the number of elements of $N(z_1^{(0)}, \dots, z_K^{(0)})$. When the number is 0, this is the case discussed above. Now assume that when the number is equal to $n$, the conclusion holds. When this number is equal to $n+1$, let $\epsilon$ be a complex number such that the number of elements of $N(z_1^{(0)}+\epsilon, \dots, z_K^{(0)}+\epsilon)$ is $n$ and that the other conditions are still satisfied. Note that we can always find such an $\epsilon$ and we can take such an $\epsilon$ with $|\epsilon|$ to be arbitrary small. By induction assumption,

$$\sum_{p_1,q_1,\dots,p_K,q_K} m_K(P_{p_1,q_1} m_{l_1}(u_1^{(1)}, \dots, u_{l_1}^{(1)}; z_1^{(1)}, \bar{z}_1^{(1)}, \dots, z_{l_1}^{(1)}, \bar{z}_{l_1}^{(1)}), \dots,$$
$$P_{p_K,q_K} m_{l_K}(u_1^{(K)}, \dots, u_{l_K}^{(K)}; z_1^{(K)}, \bar{z}_1^{(K)}, \dots, z_{l_K}^{(K)}, \bar{z}_{l_K}^{(K)});$$
$$z_1^{(0)}+\epsilon, \bar{z}_1^{(0)}+\bar{\epsilon}, \dots, z_K^{(0)}+\epsilon, \bar{z}_K^{(0)}+\bar{\epsilon})$$

is absolutely convergent to

$$m_{l_1+\cdots+l_K}(u_1^{(1)}, \dots, u_{l_1}^{(1)}, \dots, u_1^{(K)}, \dots, u_{l_K}^{(K)}; z_1^{(1)}+z_1^{(0)}+\epsilon,$$
$$\bar{z}_1^{(1)}+\bar{z}_1^{(0)}+\bar{\epsilon}, \dots, z_{l_1}^{(1)}+z_1^{(0)}+\epsilon, \bar{z}_{l_1}^{(1)}+\bar{z}_1^{(0)}+\bar{\epsilon}, \dots, z_1^{(K)}+z_K^{(0)}+\epsilon,$$
$$\bar{z}_1^{(K)}+\bar{z}_K^{(0)}+\bar{\epsilon}, \dots, z_{l_K}^{(K)}+z_K^{(0)}+\epsilon, \bar{z}_{l_K}^{(K)}+\bar{z}_K^{(0)}+\bar{\epsilon}).$$

Thus for $u' \in F$,

$$\sum_{p_1,q_1,\dots,p_K,q_K} \langle u', m_K(P_{p_1,q_1} m_{l_1}(u_1^{(1)}, \dots, u_{l_1}^{(1)}; z_1^{(1)}, \bar{z}_1^{(1)}, \dots, z_{l_1}^{(1)}, \bar{z}_{l_1}^{(1)}), \dots,$$
$$P_{p_K,q_K} m_{l_K}(u_1^{(K)}, \dots, u_{l_K}^{(K)}; z_1^{(K)}, \bar{z}_1^{(K)}, \dots, z_{l_K}^{(K)}, \bar{z}_{l_K}^{(K)});$$
$$z_1^{(0)}, \bar{z}_1^{(0)}, \dots, z_K^{(0)}, \bar{z}_K^{(0)})\rangle$$
$$= \sum_{p_1,q_1,\dots,p_K,q_K} \langle e^{\epsilon L^L(1)+\bar{\epsilon} L^R(1)} u', e^{-\epsilon L^L(-1)-\bar{\epsilon} L^R(-1)} \cdot$$
$$\cdot m_K(P_{p_1,q_1} m_{l_1}(u_1^{(1)}, \dots, u_{l_1}^{(1)}; z_1^{(1)}, \bar{z}_1^{(1)}, \dots, z_{l_1}^{(1)}, \bar{z}_{l_1}^{(1)}), \dots,$$
$$P_{p_K,q_K} m_{l_K}(u_1^{(K)}, \dots, u_{l_K}^{(K)}; z_1^{(K)}, \bar{z}_1^{(K)}, \dots, z_{l_K}^{(K)}, \bar{z}_{l_K}^{(K)});$$
$$z_1^{(0)}, \bar{z}_1^{(0)}, \dots, z_K^{(0)}, \bar{z}_K^{(0)})\rangle$$
$$= \sum_{p_1,q_1,\dots,p_K,q_K} \langle e^{\epsilon L^L(1)+\bar{\epsilon} L^R(1)} u',$$

$$m_K(P_{p_1,q_1}m_{l_1}(u_1^{(1)},\dots,u_{l_1}^{(1)};z_1^{(1)},\bar{z}_1^{(1)},\dots,z_{l_1}^{(1)},\bar{z}_{l_1}^{(1)}),\dots,$$
$$P_{p_K,q_K}m_{l_K}(u_1^{(K)},\dots,u_{l_K}^{(K)};z_1^{(K)},\bar{z}_1^{(K)},\dots,z_{l_K}^{(K)},\bar{z}_{l_K}^{(K)});$$
$$z_1^{(0)}+\epsilon,\bar{z}_1^{(0)}+\bar{\epsilon},,\dots,z_K^{(0)}+\epsilon,,\bar{z}_K^{(0)}+\bar{\epsilon})\rangle.$$

is absolutely convergent to

$$\langle e^{\epsilon L^L(1)+\bar{\epsilon}L^R(1)}u', m_{l_1+\cdots+l_K}(u_1^{(1)},\dots,u_{l_1}^{(1)},\dots,u_1^{(K)},\dots,u_{l_K}^{(K)};z_1^{(1)}+z_1^{(0)}+\epsilon,$$
$$\bar{z}_1^{(1)}+\bar{z}_1^{(0)}+\bar{\epsilon},\dots,z_{l_1}^{(1)}+z_1^{(0)}+\epsilon,\bar{z}_{l_1}^{(1)}+\bar{z}_1^{(0)}+\bar{\epsilon},\dots,z_1^{(K)}+z_K^{(0)}+\epsilon,$$
$$\bar{z}_1^{(K)}+\bar{z}_K^{(0)}+\bar{\epsilon},\dots,z_{l_K}^{(K)}+z_K^{(0)}+\epsilon,\bar{z}_{l_K}^{(K)}+\bar{z}_K^{(0)}+\bar{\epsilon})\rangle$$
$$=\langle u', m_{l_1+\cdots+l_K}(u_1^{(1)},\dots,u_{l_1}^{(1)},\dots,u_1^{(K)},\dots,u_{l_K}^{(K)};z_1^{(1)}+z_1^{(0)},$$
$$\bar{z}_1^{(1)}+\bar{z}_1^{(0)},\dots,z_{l_1}^{(1)}+z_1^{(0)},\bar{z}_{l_1}^{(1)}+\bar{z}_1^{(0)},\dots,z_1^{(K)}+z_K^{(0)},$$
$$\bar{z}_1^{(K)}+\bar{z}_K^{(0)},\dots,z_{l_K}^{(K)}+z_K^{(0)},\bar{z}_{l_K}^{(K)}+\bar{z}_K^{(0)})\rangle.$$

Since $u'$ is arbitrary, we have proved that (1.13) with $k=K$ is convergent absolutely to the right-hand side of (2.37) in the case that the number of elements of $N(z_1^{(0)},\dots,z_K^{(0)})$ is $n+1$. Thus we have proved this conclusion in the case that some of $|z_1^{(0)}|,\dots,|z_K^{(0)}|$ are equal.

By the principle of induction, the convergence property is proved. ∎

**Remark 2.12** Although the definition of full field algebra in Definition 1.1 is very general, it is not easy to verify all the axioms directly. Theorem 2.11 gives an equivalent definition of conformal full field algebra over $V^L\otimes V^R$ and the axioms in this definition are much easier to verify than those in Definitions 1.1, 1.7 and 1.16. In our construction of full field algebras in the next section, we shall use this definition to verify the structure we construct is indeed a full field algebra.

# 3 A construction of full field algebras with nondegenerate invariant bilinear forms

Let $V$ be a simple vertex operator algebra and $C_2(V)$ the subspace of $V$ spanned by $u_{-2}v$ for $u,v\in V$. In this section, we assume that $V$ satisfies the following conditions:

1. $V_{(n)} = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ and $W_{(0)} = 0$ for any irreducible $V$-module $W$ which is not equivalent to $V$.

2. Every $\mathbb{N}$-gradable weak $V$-module is completely reducible.

3. $V$ is $C_2$-cofinite, that is, $\dim V/C_2(V) < \infty$.

(Note that by results of Li [L] and Abe, Buhl and Dong [ABD], Conditions 2 and 3 can be replaced by a single condition that every weak $V$-module is completely reducible.)

Since $V$ satisfies the conditions above, all the results in [H9] can be used. We shall use all the notations, conventions and choices used in this paper. In particular, we use the following notations and choices: $\mathcal{A}$ is the (finite) set of equivalence classes of irreducible $V$-modules; $e$ is the equivalence class containing $V$; $' : \mathcal{A} \to \mathcal{A}$ is the map induced from the functor given by taking contragredient modules; for $a \in \mathcal{A}$, $W^a$ is a representative of $a$; $(\cdot, \cdot)$ is the nondegenerate bilinear form on $V$ normalized by $(\mathbf{1}, \mathbf{1}) = 1$; for $a_1, a_2, a_3 \in \mathcal{A}$, $\mathcal{V}_{a_1 a_2}^{a_3}$ are the spaces of intertwining operators of type $\binom{W^{a_3}}{W^{a_1} W^{a_2}}$; $\sigma_{12}$ and $\sigma_{23}$ are actions of (12) and (23) on $\mathcal{V}$ and they generate an action of $S_3$ on

$$\mathcal{V} = \coprod_{a_1, a_2, a_3 \in \mathcal{A}} \mathcal{V}_{a_1 a_2}^{a_3};$$

for any bases $\mathcal{Y}_{a_1 a_2; i}^{a_3; (p)}$, $i = 1, \dots, N_{a_1 a_2}^{a_3}$, $p = 1, 2, 3, 4, 5, 6, \dots$ and $a_1, a_2, a_3 \in \mathcal{A}$, of $\mathcal{V}_{a_1 a_2}^{a_3}$,

$$F(\mathcal{Y}_{a_1 a_5; i}^{a_4; (1)} \otimes \mathcal{Y}_{a_2 a_3; j}^{a_5; (2)}; \mathcal{Y}_{a_6 a_3; l}^{a_4; (3)} \otimes \mathcal{Y}_{a_1 a_2; k}^{a_6; (4)}) \in \mathbb{C}$$

are matrix elements of the fusing isomorphism; for $a \in \mathcal{A}$, $\mathcal{Y}_{ea;1}^{a}$, $\mathcal{Y}_{ae;1}^{a}$ and $\mathcal{Y}_{aa';1}^{e}$ are bases of $\mathcal{V}_{ea}^{a}$, $\mathcal{V}_{ae}^{a}$ and $\mathcal{V}_{aa'}^{e}$ chosen in [H9]; for $a \in \mathcal{A}$, there exists $h_a \in \mathbb{Q}$ such that $W^a = \coprod_{n \in h_a + \mathbb{N}} W_{(n)}$.

For $a_1, a_2, a_3 \in \mathcal{A}$, we now want to introduce a pairing between $\mathcal{V}_{a_1 a_2}^{a_3}$ and $\mathcal{V}_{a_1' a_2'}^{a_3'}$.

For $a \in \mathcal{A}$, $w_a \in W^a$ and $w_a' \in (W^a)'$, we shall use $\tilde{w}_a$ and $\tilde{w}_a'$ to denote $e^{-L(1)} w_a$ and $e^{-L(1)} w_a'$, respectively. Then we have

$$\langle e^{L(1)} \tilde{w}_a', e^{L(1)} \tilde{w}_a \rangle = \langle w_a', w_a \rangle.$$

We have:

**Lemma 3.1** *For $a \in \mathcal{A}$, $w_a \in W^a$ and $w_a' \in (W^a)'$,*

$$\mathrm{Res}_{z=0} z^{-1} \mathcal{Y}_{a'a;1}^{e}(z^{L(0)} e^{\pi i (L(0) - h_a)} \tilde{w}_a', z) z^{L(0)} \tilde{w}_a = \langle w_a', w_a \rangle \mathbf{1}.$$

*Proof.* Since $V_{(0)} = \mathbb{C}\mathbf{1}$ and $\mathcal{Y}^{e}_{a'a;1} = \sigma_{23}(\mathcal{Y}^{a'}_{a'e;1})$,

$$\begin{aligned}
&\mathrm{Res}_{z=0} z^{-1}\mathcal{Y}^{e}_{a'a;1}(z^{L(0)}e^{\pi i(L(0)-h_a)}\tilde{w}'_a, z)z^{L(0)}\tilde{w}_a \\
&\quad = (\mathbf{1}, \mathcal{Y}^{e}_{a'a;1}(1^{L(0)}e^{\pi i(L(0)-h_a)}\tilde{w}'_a, 1)\tilde{w}_a)\mathbf{1} \\
&\quad = (\mathbf{1}, \sigma_{23}(\mathcal{Y}^{a'}_{a'e;1})(e^{\pi i(L(0)-h_a)}\tilde{w}'_{a_i}, 1)\tilde{w}_a)\mathbf{1} \\
&\quad = \langle e^{\pi i h_a}\mathcal{Y}^{a'}_{a'e;1}(e^{L(1)}e^{-\pi i L(0)}e^{\pi i(L(0)-h_a)}\tilde{w}'_a, 1)\mathbf{1}, \tilde{w}_a\rangle\mathbf{1} \\
&\quad = \langle \mathcal{Y}^{a'}_{a'e;1}(e^{L(1)}\tilde{w}'_a, 1)\mathbf{1}, \tilde{w}_a\rangle\mathbf{1} \\
&\quad = \langle e^{L(-1)}e^{L(1)}\tilde{w}'_a, \tilde{w}_a\rangle\mathbf{1} \\
&\quad = \langle e^{L(1)}\tilde{w}'_a, e^{L(1)}\tilde{w}_a\rangle\mathbf{1} \\
&\quad = \langle w'_a, w_a\rangle\mathbf{1}.
\end{aligned}$$

■

For a single-valued branch $f_1(z_1, z_2)$ of a multivalued analytic function in a region $A$, we use $E(f_1(z_1, z_2))$ to denote the multivalued analytic extension together with the preferred branch $f_1(z_1, z_2)$. Let $w_1 = w_1(z_1, z_2)$ and $w_2 = w_2(z_1, z_2)$ be a change of variables and $f_2(z_1, z_2)$ a branch of $E(f_1(z_1, z_2))$ in a region $B$ containing $w_1(z_1, z_2) = 0$ and $w_2(z_1, z_2) = 0$ such that $A \cap B \neq \emptyset$ and $f_1(z_1, z_2) = f(z_1, z_2)$ for $(z_1, z_2) \in A \cap B$. Then we use

$$\mathrm{Res}_{w_1=0 \mid w_2} E(f_1(z_1, z_2))$$

to denote the coefficient of $w_1^{-1}$ in the expansion of $f_2(z_1, z_2)$ as a series in powers of $w_1$ whose coefficients are analytic functions of $w_2$. By definition, we have

$$\mathrm{Res}_{w_1=0 \mid C_1 w_2 + C_2} E(f_1(z_1, z_2)) = \mathrm{Res}_{w_1=0 \mid w_2} E(f_1(z_1, z_2)) \tag{3.1}$$

for any $C_1 \in \mathbb{C}^\times, C_2 \in \mathbb{C}$ independent of $z_1$ and $z_2$. We have:

**Proposition 3.2** *For $a_1, a_2, a_3 \in \mathcal{A}$, $w_{a_1} \in W^{a_1}$, $w_{a_2} \in W^{a_2}$, $w_{a'_1} \in (W^{a_1})'$, $w'_{a_2} \in (W^{a_1})'$, $\mathcal{Y}_1 \in \mathcal{V}^{a_3}_{a_1 a_2}$ and $\mathcal{Y}_2 \in \mathcal{V}^{a'_3}_{a'_1 a'_2}$, there exists a constant $\langle \mathcal{Y}_1, \mathcal{Y}_2\rangle_{\mathcal{V}^{a_3}_{a_1 a_2}} \in \mathbb{C}$ such that*

$$\begin{aligned}
&\mathrm{Res}_{1-z_1-z_2=0 \mid z_2}(1-z_1-z_2)^{-1}E(\langle e^{L(1)}\mathcal{Y}_2((1-z_1-z_2)^{L(0)}\tilde{w}'_{a_1}, z_1)\tilde{w}'_{a_2}, \\
&\qquad\qquad e^{L(1)}\mathcal{Y}_1((1-z_1-z_2)^{L(0)}\tilde{w}_{a_1}, z_2)\tilde{w}_{a_2}\rangle) \\
&\quad = \langle w'_{a_1}, w_{a_1}\rangle\langle w'_{a_2}, w_{a_2}\rangle\langle \mathcal{Y}_1, \mathcal{Y}_2\rangle_{\mathcal{V}^{a_3}_{a_1 a_2}}.
\end{aligned} \tag{3.2}$$

*Explicitly, for any bases* $\{\mathcal{Y}^{a_3;(1)}_{a_1a_2;i} \mid i=1,\dots,N^{a_3}_{a_1a_2}\}$ *and* $\{\mathcal{Y}^{a_3';(2)}_{a_1'a_2';i} \mid i=1,\dots,N^{a_3'}_{a_1'a_2'}\}$ *of* $\mathcal{V}^{a_3}_{a_1a_2}$ *and* $\mathcal{V}^{a_3'}_{a_1'a_2'}$*, respectively, and for* $m,n,k,l\in\mathbb{Z}_+$*,* $i=1,\dots,N^{a_3}_{a_1a_2}$ *and* $j=1,\dots,N^{a_3'}_{a_1'a_2'}$*, we have*

$$\begin{aligned}\langle \mathcal{Y}^{a_3;(1)}_{a_1a_2;j}, \mathcal{Y}^{a_3';(2)}_{a_1'a_2';i}\rangle_{\mathcal{V}^{a_3}_{a_1a_2}} &= F(\sigma_{23}(\mathcal{Y}^{a_3';(2)}_{a_1'a_2';i})\otimes \mathcal{Y}^{a_3;(1)}_{a_1a_2;j}; \mathcal{Y}^{a_2}_{ea_2;1}\otimes \mathcal{Y}^{e}_{a_1'a_1;1})\\ &= F(\sigma_{23}(\mathcal{Y}^{a_3;(1)}_{a_1a_2;j})\otimes \mathcal{Y}^{a_3';(2)}_{a_1'a_2';i}; \mathcal{Y}^{a_2'}_{ea_2';1}\otimes \mathcal{Y}^{e}_{a_1a_1';1}). \quad (3.3)\end{aligned}$$

*Proof.* We prove (3.2) in the case $\mathcal{Y}_1=\mathcal{Y}^{a_3;(1)}_{a_1a_2;j}$ and $\mathcal{Y}_2=\mathcal{Y}^{a_3';(2)}_{a_1'a_2';i}$ for $i=1,\dots,N^{a_3'}_{a_1'a_2'}$ and $j=1,\dots,N^{a_3}_{a_1a_2}$, respectively, or equivalently, we prove (3.3). The general case follows immediately from the bilinearity in $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of the right-hand side of (3.2).

For $a_1,a_2\in\mathcal{A}$, $a_1,a_2\neq e$, let $\{\mathcal{Y}^{a_2}_{a_1a_2;i} \mid i=1,\dots,N^{a_2}_{a_1a_2}\}$ and $\{\mathcal{Y}^{a_2}_{a_1'a_1;i} \mid i=1,\dots,N^{a_2}_{a_1'a_1}\}$ be an arbitrary basis of $\mathcal{V}^{a_2}_{a_1a_2}$ and $\mathcal{V}^{a_2}_{a_1'a_1}$, respectively.

For $w_{a_1}\in W^{a_1}$, $w_{a_2}\in W^{a_2}$, $w_{a_1'}\in (W^{a_1})'$, $w'_{a_2}\in (W^{a_1})'$, we have

$$\begin{aligned}&\mathrm{Res}_{1-z_1-z_2=0\mid z_2}(1-z_1-z_2)^{-1}E(\langle e^{L(1)}\mathcal{Y}^{a_3';(2)}_{a_1'a_2';i}((1-z_1-z_2)^{L(0)}\tilde{w}'_{a_1},z_1)\tilde{w}'_{a_2},\\ &\qquad\qquad e^{L(1)}\mathcal{Y}^{a_3;(1)}_{a_1a_2;j}((1-z_1-z_2)^{L(0)}\tilde{w}_{a_1},z_2)\tilde{w}_{a_2}\rangle)\\ &= \mathrm{Res}_{1-z_1-z_2=0\mid z_2}(1-z_1-z_2)^{-1}\cdot\\ &\quad\cdot E(\langle \mathcal{Y}^{a_3';(2)}_{a_1'a_2';i}(e^{(1-z_1)L(1)}(1-z_1)^{-2L(0)}\cdot\\ &\qquad\cdot(1-z_1-z_2)^{L(0)}\tilde{w}'_{a_1},(1-z_1)^{-1})e^{L(-1)}e^{L(1)}\tilde{w}'_{a_2},\\ &\qquad\qquad \mathcal{Y}^{a_3;(1)}_{a_1a_2;j}((1-z_1-z_2)^{L(0)}\tilde{w}_{a_1},z_2)\tilde{w}_{a_2}\rangle)\\ &= \mathrm{Res}_{1-z_1-z_2=0\mid z_2}(1-z_1-z_2)^{-1}\\ &\quad E(\langle e^{-L(1)}e^{L(1)}\tilde{w}'_{a_2},\sigma_{23}(\mathcal{Y}^{a_3';(2)}_{a_1'a_2';i})((1-z_1-z_2)^{L(0)}e^{\pi i(L(0)-h_{a_1'})}\tilde{w}'_{a_1},1-z_1)\cdot\\ &\qquad\qquad\cdot\mathcal{Y}^{a_3;(1)}_{a_1a_2;j}((1-z_1-z_2)^{L(0)}\tilde{w}_{a_1},z_2)\tilde{w}_{a_2}\rangle)\\ &= \sum_{a_4\in\mathcal{A}}\sum_{p=1}^{N^{a_2}_{a_4a_2}}\sum_{q=1}^{N^{a_4}_{a_1'a_1}} F(\sigma_{23}(\mathcal{Y}^{a_3';(2)}_{a_1'a_2';i})\otimes\mathcal{Y}^{a_3;(1)}_{a_1a_2;j};\mathcal{Y}^{a_2}_{a_4a_2;p}\otimes\mathcal{Y}^{a_4}_{a_1'a_1;q})\cdot\\ &\quad\cdot\mathrm{Res}_{1-z_1-z_2=0\mid z_2}(1-z_1-z_2)^{-1}\cdot\\ &\qquad\cdot E(\langle e^{L(-1)}e^{L(1)}\tilde{w}'_{a_2},\mathcal{Y}^{a_2}_{a_4a_2;p}(\mathcal{Y}^{a_4}_{a_1'a_1;q}((1-z_1-z_2)^{L(0)}\cdot\\ &\qquad\qquad\cdot e^{\pi i(L(0)-h_{a_1'})}\tilde{w}'_{a_1},1-z_1-z_2)\cdot\\ &\qquad\qquad\qquad\cdot(1-z_1-z_2)^{L(0)}\tilde{w}_{a_1},z_2)\tilde{w}_{a_2}\rangle)\end{aligned}$$

$$
\begin{aligned}
&= F(\sigma_{23}(\mathcal{Y}^{a_3';(2)}_{a_1'a_2';i}) \otimes \mathcal{Y}^{a_3;(1)}_{a_1a_2;j}; \mathcal{Y}^{a_2}_{ea_2;p} \otimes \mathcal{Y}^{e}_{a_1'a_1;1}) \cdot \\
&\quad \cdot \mathrm{Res}_{1-z_1-z_2=0\,|\,z_2}(1-z_1-z_2)^{-1} \cdot \\
&\qquad \cdot E(\langle e^{L(-1)}e^{L(1)}\tilde{w}'_{a_2}, \mathcal{Y}^{a_2}_{ea_2;1}(\mathcal{Y}^{e}_{a_1'a_1;1}((1-z_1-z_2)^{L(0)} \cdot \\
&\qquad\qquad \cdot e^{\pi i(L(0)-h_{a_1'})}\tilde{w}'_{a_1}, 1-z_1-z_2) \cdot \\
&\qquad\qquad\qquad \cdot (1-z_1-z_2)^{L(0)}\tilde{w}_{a_1}, z_2)\tilde{w}_{a_2}\rangle) \\
&= F(\sigma_{23}(\mathcal{Y}^{a_3';(2)}_{a_1'a_2';i}) \otimes \mathcal{Y}^{a_3;(1)}_{a_1a_2;j}; \mathcal{Y}^{a_2}_{ea_2;p} \otimes \mathcal{Y}^{e}_{a_1'a_1;1}) \cdot \\
&\qquad\qquad \cdot \langle e^{L(-1)}e^{L(1)}\tilde{w}'_{a_2}, \mathcal{Y}^{a_2}_{ea_2;1}(\langle w'_{a_1}, w_{a_1}\rangle \mathbf{1}, z_2)\tilde{w}_{a_2}\rangle \\
&= \langle w'_{a_1}, w_{a_1}\rangle F(\sigma_{23}(\mathcal{Y}^{a_3';(2)}_{a_1'a_2';i}) \otimes \mathcal{Y}^{a_3;(1)}_{a_1a_2;j}; \mathcal{Y}^{a_2}_{ea_2;p} \otimes \mathcal{Y}^{e}_{a_1'a_1;1})\langle e^{L(1)}\tilde{w}'_{a_2}, e^{L(1)}\tilde{w}_{a_2}\rangle \\
&= \langle w'_{a_1}, w_{a_1}\rangle \langle w'_{a_2}, w_{a_2}\rangle F(\sigma_{23}(\mathcal{Y}^{a_3';(2)}_{a_1'a_2';i}) \otimes \mathcal{Y}^{a_3;(1)}_{a_1a_2;j}; \mathcal{Y}^{a_2}_{ea_2;p} \otimes \mathcal{Y}^{e}_{a_1'a_1;1}),
\end{aligned}
$$

where we have used the fact that $W^{a_4}_{(0)} = 0$ for $a_4 \neq e$. This proves (3.2) and also the first equality in (3.3). The second equality in (3.3) can be proved similarly or can be simply obtained using the first equality in (3.3) and symmetry. ■

Clearly, $\langle \mathcal{Y}_1, \mathcal{Y}_2\rangle_{\mathcal{V}^{a_3}_{a_1a_2}}$ is bilinear in $\mathcal{Y}_1$ and $\mathcal{Y}_2$. Thus we have a pairing $\langle \cdot, \cdot \rangle_{\mathcal{V}^{a_3}_{a_1a_2}} : \mathcal{V}^{a_3}_{a_1a_2} \otimes \mathcal{V}^{a_3'}_{a_1'a_2'} \to \mathbb{C}$.

We need the following lemma:

**Lemma 3.3** *For $a_1, a_2, a_3, a_4, a_5 \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}^{a_3'}_{a_1'a_2'}$, $\mathcal{Y}_2 \in \mathcal{V}^{a_3}_{a_4a_5}$, $w'_{a_1} \in (W^{a_1})'$, $w'_{a_2} \in (W^{a_2})'$, $w_{a_4} \in W^{a_1}$, $w_{a_4} \in W^{a_5}$, if $a_1 \neq a_4$ or $a_2 \neq a_5$, then*

$$
\begin{aligned}
&\mathrm{Res}_{1-z_1-z_2=0\,|\,z_2}(1-z_1-z_2)^{-1}E(\langle e^{L(1)}\mathcal{Y}_1((1-z_1-z_2)^{L(0)}\tilde{w}'_{a_1}, z_1)\tilde{w}'_{a_2}, \\
&\qquad\qquad e^{L(1)}\mathcal{Y}_2((1-z_1-z_2)^{L(0)}\tilde{w}_{a_4}, z_2)\tilde{w}_{a_5}\rangle) = 0.
\end{aligned}
$$

*Proof.* Using the $L(1)$- and $L(-1)$-conjugation formulas for intertwining operators, the definition of $\sigma_{23}$ and the associativity of intertwining operators, we know that there exist a $V$-module $W$ and intertwining operators $\mathcal{Y}_3$ and $\mathcal{Y}_4$ of types $\binom{W^{a_2}}{W W^{a_5}}$ and $\binom{W}{W^{a_1'} W^{a_4}}$, respectively, such that

$$
\begin{aligned}
&\mathrm{Res}_{1-z_1-z_2=0\,|\,z_2}(1-z_1-z_2)^{-1}E(\langle e^{L(1)}\mathcal{Y}_1((1-z_1-z_2)^{L(0)}\tilde{w}'_{a_1}, z_1)\tilde{w}'_{a_2}, \\
&\qquad\qquad e^{L(1)}\mathcal{Y}_2((1-z_1-z_2)^{L(0)}\tilde{w}_{a_4}, z_2)\tilde{w}_{a_5}\rangle) \\
&\quad = \mathrm{Res}_{1-z_1-z_2=0\,|\,z_2}(1-z_1-z_2)^{-1} \cdot \\
&\qquad \cdot E(\langle e^{L(-1)}e^{L(1)}\tilde{w}'_{a_2}, \sigma_{23}(\mathcal{Y}_1)((1-z_1-z_2)^{L(0)}\tilde{w}'_{a_1}, 1-z_1) \cdot \\
&\qquad\qquad \cdot \mathcal{Y}_2((1-z_1-z_2)^{L(0)}\tilde{w}_{a_4}, z_2)\tilde{w}_{a_5}\rangle)
\end{aligned}
$$

$$= \mathrm{Res}_{1-z_1-z_2=0\;|\;z_2}(1-z_1-z_2)^{-1}\cdot$$
$$\cdot E(\langle e^{L(-1)}e^{L(1)}\tilde{w}'_{a_2}, \mathcal{Y}_3(\mathcal{Y}_4((1-z_1-z_2)^{L(0)}\tilde{w}'_{a_1}, 1-z_1-z_2)\cdot$$
$$\cdot(1-z_1-z_2)^{L(0)}\tilde{w}_{a_4}, z_2)\tilde{w}_{a_5}\rangle). \qquad (3.4)$$

If $a_1 \neq a_4$, $W^{a_4}$ is not equivalent to $W^{a_1}$. Thus $\mathcal{V}^{e}_{a'_1 a_4} = 0$. So it is possible to find such a $V$-module $W$ which does not contain a summand equivalent to $V$. By the assumption on $V$, we have $W_{(0)} = 0$. So the right-hand side of (3.4) is 0, proving the lemma in this case. If $a_1 = a_4$, $\mathcal{V}^{e}_{a'_1 a_1}$ is one-dimensional. We can choose $W$ to contain one and only one copy of $V$. If $a_2 \neq a_5$, any intertwining operator of type $\binom{a_2}{e a_5}$ (that is, type $\binom{W^{a_2}}{V W^{a_5}}$) must be 0. So $\mathcal{Y}_3(\mathbf{1}, z_2) = 0$. Since $W_{(0)} = \mathbb{C}\mathbf{1}$, there exists $\lambda \in \mathbb{C}$ such that the right-hand side of (3.4) is equal to

$$\lambda\langle e^{L(-1)}e^{L(1)}\tilde{w}'_{a_2}, \mathcal{Y}_3(\mathbf{1}, z_2)\tilde{w}_{a_2}\rangle = 0,$$

proving the lemma in the case $a_2 \neq a_5$. ∎

As in [H9], we now choose a canonical basis of $\mathcal{V}^{a_3}_{a_1 a_2}$ for $a_1, a_2, a_3 \in \mathcal{A}$ when one of $a_1, a_2, a_3$ is $e$: For $a \in \mathcal{A}$, we choose $\mathcal{Y}^{a}_{ea;1}$ to be the vertex operator $Y_{W^a}$ defining the module structure on $W^a$ and we choose $\mathcal{Y}^{a}_{ae;1}$ to be the intertwining operator defined using the action of $\sigma_{12}$, or equivalently the skew-symmetry in this case,

$$\begin{aligned}\mathcal{Y}^{a}_{ae;1}(w_a, x)u &= \sigma_{12}(\mathcal{Y}^{a}_{ea;1})(w_a, x)u\\ &= e^{xL(-1)}\mathcal{Y}^{a}_{ea;1}(u, -x)w_a\\ &= e^{xL(-1)}Y_{W^a}(u, -x)w_a\end{aligned}$$

for $u \in V$ and $w_a \in W^a$. Since $V'$ as a $V$-module is isomorphic to $V$, we have $e' = e$. From [FHL], we know that there is a nondegenerate invariant bilinear form $(\cdot,\cdot)$ on $V$ such that $(\mathbf{1}, \mathbf{1}) = 1$. We choose $\mathcal{Y}^{e}_{aa';1} = \mathcal{Y}^{e'}_{aa';1}$ to be the intertwining operator defined using the action of $\sigma_{23}$ by

$$\mathcal{Y}^{e'}_{aa';1} = \sigma_{23}(\mathcal{Y}^{a}_{ae;1}),$$

that is,

$$(u, \mathcal{Y}^{e}_{aa';1}(w_a, x)w_{a'}) = e^{\pi i h_a}\langle \mathcal{Y}^{a}_{ae;1}(e^{xL(1)}(e^{-\pi i}x^{-2})^{L(0)}w_a, x^{-1})u, w_{a'}\rangle$$

for $u \in V$, $w_a \in W^a$ and $w_{a'} \in W^{a'}$. Since the actions of $\sigma_{12}$ and $\sigma_{23}$ generate the action of $S_3$ on $\mathcal{V}$, we have

$$\mathcal{Y}^{e}_{a'a;1} = \sigma_{12}(\mathcal{Y}^{e}_{aa';1})$$

for any $a \in \mathcal{A}$.

**Theorem 3.4** *The pairing* $\langle\cdot,\cdot\rangle_{\mathcal{V}_{a_1a_2}^{a_3}} : \mathcal{V}_{a_1a_2}^{a_3} \otimes \mathcal{V}_{a_1'a_2'}^{a_3'} \to \mathbb{C}$ *is nondegenerate. In particular,* $N_{a_1'a_2'}^{a_3'} = N_{a_1a_2}^{a_3}$.

*Proof.* For $a_1, a_2, a_3 \in \mathcal{A}$ such that one of $a_1, a_2, a_3$ is $e$, we have a canonical basis $\mathcal{Y}_{a_1a_2;1}^{a_3}$ given above. For $a_1, a_2, a_3 \neq e$, let $\mathcal{Y}_{a_1a_2;i}^{a_3}$, $i = 1, \dots, N_{a_1a_2}^{a_3}$, be an arbitrary basis of $\mathcal{V}_{a_1a_2}^{a_3}$.

For $a_1, a_2, a_3 \in \mathcal{A}$, let

$$\begin{aligned}\mathcal{Y}_{a_1a_2;j}^{a_3;(1)} &= \sigma_{123}(\mathcal{Y}_{a_2a_3';j}^{a_1'}),\\ \mathcal{Y}_{a_1'a_2';i}^{a_3';(2)} &= \sigma_{23}(\mathcal{Y}_{a_1'a_3;i}^{a_2}).\end{aligned}$$

Then the first equality of (3.3) gives

$$\begin{aligned}\langle\sigma_{23}(\mathcal{Y}_{a_1'a_3;i}^{a_2}), \sigma_{123}(\mathcal{Y}_{a_2a_3';j}^{a_1'})\rangle_{\mathcal{V}_{a_1a_2}^{a_3}} &= \langle\mathcal{Y}_{a_1'a_2';i}^{a_3';(2)}, \mathcal{Y}_{a_1a_2;j}^{a_3;(1)}\rangle_{\mathcal{V}_{a_1a_2}^{a_3}}\\ &= F(\mathcal{Y}_{a_1'a_3;i}^{a_2} \otimes \sigma_{123}(\mathcal{Y}_{a_2a_3';j}^{a_1'}); \mathcal{Y}_{ea_2;1}^{a_2} \otimes \mathcal{Y}_{a_1'a_1;1}^{e}).\end{aligned} \tag{3.5}$$

In [H9], the first author proved the following formula ((4.9) in [H9]):

$$\begin{aligned}&\sum_{k=1}^{N_{a_1'a_3}^{a_2}} F(\mathcal{Y}_{a_2e;1}^{a_2} \otimes \mathcal{Y}_{a_3'a_3;1}^{e}; \mathcal{Y}_{a_1'a_3;k}^{a_2} \otimes \mathcal{Y}_{a_2a_3';j}^{a_1'})\cdot\\ &\qquad\cdot F(\mathcal{Y}_{a_1'a_3;k}^{a_2} \otimes \sigma_{123}(\mathcal{Y}_{a_2a_3';i}^{a_1'}); \mathcal{Y}_{ea_2;1}^{a_2} \otimes \mathcal{Y}_{a_1'a_1;1}^{e})\\ &\quad= \delta_{ij} F(\mathcal{Y}_{a_2e;1}^{a_2} \otimes \mathcal{Y}_{a_2'a_2;1}^{e}; \mathcal{Y}_{ea_2;1}^{a_2} \otimes \mathcal{Y}_{a_2a_2';1}^{e}).\end{aligned} \tag{3.6}$$

In the same paper [H9], the first author also proved that

$$F(\mathcal{Y}_{a_2e;1}^{a_2} \otimes \mathcal{Y}_{a_2'a_2;1}^{e}; \mathcal{Y}_{ea_2;1}^{a_2} \otimes \mathcal{Y}_{a_2a_2';1}^{e}) \neq 0.$$

Thus from (3.5) and (3.6), we see that the matrix

$$(\alpha_{ij}) = (\langle\sigma_{23}(\mathcal{Y}_{a_1'a_3;i}^{a_2}), \sigma_{123}(\mathcal{Y}_{a_2a_3';j}^{a_1'})\rangle_{\mathcal{V}_{a_1a_2}^{a_3}}) \tag{3.7}$$

is left invertible. Note that when $a_1, a_2, a_3 \neq e$, $\mathcal{Y}_{a_1'a_3;i}^{a_2}$ and $\mathcal{Y}_{a_2a_3';j}^{a_1'}$ in (3.7) are arbitrary bases of $\mathcal{V}_{a_1'a_3}^{a_2}$ and $\mathcal{V}_{a_2a_3'}^{a_1'}$, respectively.

We now show that (3.7) is also right invertible. By definition, the bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{V}_{a_1 a_2}^{a_3}}$ is symmetric in the sense that

$$\langle \mathcal{Y}_1, \mathcal{Y}_2 \rangle_{\mathcal{V}_{a_1 a_2}^{a_3}} = \langle \mathcal{Y}_2, \mathcal{Y}_1 \rangle_{\mathcal{V}_{a_1' a_2'}^{a_3'}}$$

for $a_1, a_2, a_3 \in \mathcal{A}$. So

$$\begin{aligned}
&\langle \sigma_{23}(\mathcal{Y}_{a_1' a_3; i}^{a_2}), \sigma_{123}(\mathcal{Y}_{a_2 a_3'; j}^{a_1'}) \rangle_{\mathcal{V}_{a_1 a_2}^{a_3}} \\
&\quad = \langle \sigma_{123}(\mathcal{Y}_{a_2 a_3'; j}^{a_1'}), \sigma_{23}(\mathcal{Y}_{a_1' a_3; i}^{a_2}) \rangle_{\mathcal{V}_{a_1' a_2'}^{a_3'}} \\
&\quad = \langle \sigma_{23}(\sigma_{13}(\mathcal{Y}_{a_2 a_3'; j}^{a_1'})), \sigma_{123}(\sigma_{13}(\mathcal{Y}_{a_1' a_3; i}^{a_2})) \rangle_{\mathcal{V}_{a_1' a_2'}^{a_3'}}.
\end{aligned} \tag{3.8}$$

Note that for $a_1, a_2, a_3 \in \mathcal{A}$, $\sigma_{13}(\mathcal{Y}_{a_3' a_2; i}^{a_1'})$ is a basis of $\mathcal{V}_{a_1 a_2}^{a_3}$ such that when one of the elements $a_1, a_2, a_3 \in \mathcal{A}$ is $e$, these basis elements are equal to the special ones we chosen above. Thus by the result we obtained above, the matrix

$$(\beta_{ij}) = (\langle \sigma_{23}(\sigma_{13}(\mathcal{Y}_{a_2 a_3'; i}^{a_1'})), \sigma_{123}(\sigma_{13}(\mathcal{Y}_{a_1' a_3; j}^{a_2})) \rangle_{\mathcal{V}_{a_1' a_2'}^{a_3'}})$$

must be left invertible. So the transpose of $(\beta_{ij})$, that is, the matrix

$$(\gamma_{kl}) = (\langle \sigma_{23}(\sigma_{13}(\mathcal{Y}_{a_2 a_3'; l}^{a_1'})), \sigma_{123}(\sigma_{13}(\mathcal{Y}_{a_1' a_3; k}^{a_2})) \rangle_{\mathcal{V}_{a_1' a_2'}^{a_3'}}),$$

is right invertible. By (3.8), we see that (3.7) is also right invertible.

Now we have shown that the matrix (3.7) is in fact invertible. This is equivalent to the nondegeneracy of the bilinear form. It also implies $N_{a_1' a_2'}^{a_3'} = N_{a_1 a_2}^{a_3}$. ∎

For $a \in \mathcal{A}$, let

$$F_a = F(\mathcal{Y}_{ae;1}^{a} \otimes \mathcal{Y}_{a'a;1}^{e}; \mathcal{Y}_{ea;1}^{a} \otimes \mathcal{Y}_{aa';1}^{e}) \neq 0.$$

Then by (3.12) in [H9], $F_{a'} = F_a$ for $a \in \mathcal{A}$.

**Lemma 3.5** *If for $a \in \mathcal{A}$, $\mathcal{Y}_{ea;1}^{a}$, $\mathcal{Y}_{ae;1}^{a}$, $\mathcal{Y}_{aa';1}^{e}$ are the canonical bases of $\mathcal{V}_{ea}^{a}$, $\mathcal{V}_{ae}^{a}$, $\mathcal{V}_{aa'}^{e}$, respectively, chosen in [H9] and above, then their dual bases in $\mathcal{V}_{ea'}^{a'}$, $\mathcal{V}_{a'e}^{a'}$, $\mathcal{V}_{a'a}^{e}$ with respect to the pairing $\langle \cdot, \cdot \rangle_{\mathcal{V}_{ea}^{a}}$, $\langle \cdot, \cdot \rangle_{\mathcal{V}_{ae}^{a}}$, $\langle \cdot, \cdot \rangle_{\mathcal{V}_{aa'}^{e}}$, respectively, are equal to $\mathcal{Y}_{ea';1}^{a'}$, $\mathcal{Y}_{a'e;1}^{a}$, $\frac{\mathcal{Y}_{a'a;1}^{e}}{F_a}$.*

*Proof.* This result follows immediately from the definition of the canonical bases in [H9]. ∎

We have:

**Proposition 3.6** *For $a_1, a_2, a_3 \in \mathcal{A}$, let $\{\mathcal{Y}^{a_3}_{a_1a_2;i} \mid i = 1, \dots, N^{a_3}_{a_1a_2}\}$ be bases of $\mathcal{V}^{a_3}_{a_1a_2}$ and let $\{\mathcal{Y}'^{;a'_3}_{a'_1a'_2;i} \mid i = 1, \dots, N^{a'_3}_{a'_1a'_2}\}$ be the dual bases of $\{\mathcal{Y}^{a_3}_{a_1a_2;i} \mid i = 1, \dots, N^{a_3}_{a_1a_2}\}$ with respect to the pairing $\langle \cdot, \cdot \rangle_{\mathcal{V}^{a_3}_{a_1a_2}}$. Assume that for $a \in \mathcal{A}$, $\mathcal{Y}^{a}_{ea;1}$, $\mathcal{Y}^{a}_{ae;1}$, $\mathcal{Y}^{e}_{aa';1}$ are the canonical bases of $\mathcal{V}^{a}_{ea}$, $\mathcal{V}^{a}_{ae}$, $\mathcal{V}^{e}_{aa'}$, respectively, we have chosen. Then for $a_1, a_2, a_3, a_4 \in \mathcal{A}$,*

$$\sum_{a_5 \in \mathcal{A}} \sum_{p=1}^{N^{a_4}_{a_1a_5}} \sum_{q=1}^{N^{a_5}_{a_2a_3}} F(\mathcal{Y}^{a_4}_{a_1a_5;p} \otimes \mathcal{Y}^{a_5}_{a_2a_3;q}; \mathcal{Y}^{a_4}_{a_6a_3;m} \otimes \mathcal{Y}^{a_6}_{a_1a_2;k}) \cdot$$

$$\cdot F(\mathcal{Y}'^{;a'_4}_{a'_1a'_5;p} \otimes \mathcal{Y}'^{;a_5}_{a'_2a'_3;q}; \mathcal{Y}'^{;a'_4}_{a'_7a'_3;n} \otimes \mathcal{Y}'^{;a'_7}_{a'_1a'_2;l})$$

$$= \delta_{a_6a_7}\delta_{mn}\delta_{kl}.$$

*Proof.* For $a_1, a_2, a_3, a_4 \in \mathcal{A}$, $w_{a_i} \in W^{a_i}$ and $w'_{a_i} \in (W^{a_i})'$ satisfying $\langle w'_{a_i}, w_{a_i} \rangle = 1$ for $i = 1, 2, 3$, using (3.2) and Lemma 3.3, we have

$$\begin{aligned}
&\mathrm{Res}_{1-z_1-z_3=0 \mid z_3} \mathrm{Res}_{1-z_2-z_4=0 \mid z_4} (1 - z_1 - z_3)^{-1}(1 - z_2 - z_4)^{-1} \\
&\quad E(\langle e^{L(1)} \mathcal{Y}'^{;a'_4}_{a'_1a'_5;k}((1 - z_1 - z_3)^{L(0)} \tilde{w}'_{a_1}, z_1) \cdot \\
&\qquad\qquad \cdot \mathcal{Y}'^{;a'_5}_{a'_2a'_3;l}((1 - z_2 - z_4)^{L(0)} \tilde{w}'_{a_2;q}, z_2) \tilde{w}'_{a_3}, \\
&\quad e^{L(1)} \mathcal{Y}^{a'_4}_{a_1a_6;m}((1 - z_1 - z_3)^{L(0)} \tilde{w}_{a_1}, z_3) \cdot \\
&\qquad\qquad \cdot \mathcal{Y}^{a_6}_{a_2a_3;n}((1 - z_2 - z_4)^{L(0)} \tilde{w}_{a_2;q}, z_4) \tilde{w}_{a_3} \rangle) \\
&= \delta_{a_5a_6}\delta_{km} \mathrm{Res}_{1-z_2-z_4=0 \mid z_4} (1 - z_2 - z_4)^{-1} \\
&\quad E(\langle \mathcal{Y}'^{;a'_5}_{a'_2a'_3;l}((1 - z_2 - z_4)^{L(0)} \tilde{w}'_{a_2}, z_2) \tilde{w}'_{a_3}, \\
&\qquad\qquad \mathcal{Y}^{a_5}_{a_2a_3;n}((1 - z_2 - z_4)^{L(0)} \tilde{w}_{a_2}, z_4) \tilde{w}_{a_3} \rangle) \\
&= \delta_{a_5a_6}\delta_{km}\delta_{ln}.
\end{aligned} \tag{3.9}$$

On the other hand, by the associativity of intertwining operators and Lemma 3.3, we have

$$\begin{aligned}
&\mathrm{Res}_{1-z_1-z_3=0 \mid z_3} \mathrm{Res}_{1-z_2-z_4=0 \mid z_4} (1 - z_1 - z_3)^{-1}(1 - z_2 - z_4)^{-1} \\
&\quad E(\langle e^{L(1)} \mathcal{Y}'^{;a'_4}_{a'_1a'_5;k}((1 - z_1 - z_3)^{L(0)} \tilde{w}'_{a_1}, z_1) \cdot
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\cdot\mathcal{Y}'^{;a_5'}_{a_2'a_3';l}((1-z_2-z_4)^{L(0)}\tilde{w}'_{a_2}, z_2)\tilde{w}'_{a_3},\\
&e^{L(1)}\mathcal{Y}^{a_4'}_{a_1a_6;m}((1-z_1-z_3)^{L(0)}\tilde{w}_{a_1}, z_3)\cdot\\
&\qquad\qquad\cdot\mathcal{Y}^{a_6}_{a_2a_3;n}((1-z_2-z_4)^{L(0)}\tilde{w}_{a_2}, z_4)\tilde{w}_{a_3}\rangle)\\
&=\sum_{a_7,a_8\in\mathcal{A}}\sum_{i,j,s,t}F(\mathcal{Y}'^{;a_4'}_{a_1'a_5';k}\otimes\mathcal{Y}'^{;a_5'}_{a_2'a_3';l};\mathcal{Y}'^{;a_4'}_{a_7'a_3';i}\otimes\mathcal{Y}'^{;a_7'}_{a_1'a_2';j})\cdot\\
&\qquad\cdot F(\mathcal{Y}^{a_4}_{a_1a_6;m}\otimes\mathcal{Y}^{a_6}_{a_2a_3;n};\mathcal{Y}^{a_4}_{a_8a_3;s}\otimes\mathcal{Y}^{a_8}_{a_1a_2;t})\cdot\\
&\quad\cdot\mathrm{Res}_{1-z_1-z_3=0\mid z_3}\mathrm{Res}_{1-z_2-z_4=0\mid z_4}(1-z_1-z_3)^{-1}(1-z_2-z_4)^{-1}\\
&\quad E(\langle e^{L(1)}\mathcal{Y}'^{;a_4'}_{a_7'a_3';i}(\mathcal{Y}'^{;a_7'}_{a_1'a_2';j}((1-z_1-z_3)^{L(0)}\tilde{w}'_{a_1}, z_1-z_2)\cdot\\
&\qquad\qquad\cdot(1-z_2-z_4)^{L(0)}\tilde{w}'_{a_2}, z_2)\tilde{w}'_{a_3},\\
&\quad e^{L(1)}\mathcal{Y}^{a_4}_{a_8a_3;s}(\mathcal{Y}^{a_8}_{a_1a_2;t}((1-z_1-z_3)^{L(0)}\tilde{w}_{a_1}, z_3-z_4)\cdot\\
&\qquad\qquad\cdot(1-z_2-z_4)^{L(0)}\tilde{w}_{a_2}, z_4)\tilde{w}_{a_3}\rangle)\\
&=\sum_{a_7,a_8\in\mathcal{A}}\sum_{i,j,s,t}F(\mathcal{Y}'^{;a_4'}_{a_1'a_5';k}\otimes\mathcal{Y}'^{;a_5'}_{a_2'a_3';l};\mathcal{Y}'^{;a_4'}_{a_7'a_3';i}\otimes\mathcal{Y}'^{;a_7'}_{a_1'a_2';j})\cdot\\
&\qquad\cdot F(\mathcal{Y}^{a_4}_{a_1a_6;m}\otimes\mathcal{Y}^{a_6}_{a_2a_3;n};\mathcal{Y}^{a_4}_{a_8a_3;s}\otimes\mathcal{Y}^{a_8}_{a_1a_2;t})\cdot\\
&\quad\cdot\mathrm{Res}_{1-z_2-z_4=0\mid z_4}\mathrm{Res}_{1-z_1-z_3=0\mid z_3}(1-z_1-z_3)^{-1}(1-z_2-z_4)^{-1}\\
&\quad E\left(\left\langle e^{L(1)}\mathcal{Y}'^{;a_4'}_{a_7'a_3';i}\left((1-z_2-z_4)^{L(0)}\mathcal{Y}'^{;a_7'}_{a_1'a_2';j}\left(\left(\frac{1-z_1-z_3}{1-z_2-z_4}\right)^{L(0)}\cdot\right.\right.\right.\right.\\
&\qquad\qquad\left.\left.\cdot\tilde{w}'_{a_1}, \frac{z_1-z_2}{1-z_2-z_4}\right)\tilde{w}'_{a_2}, z_2\right)\tilde{w}'_{a_3},\\
&\quad e^{L(1)}\mathcal{Y}^{a_4}_{a_8a_3;s}\left((1-z_2-z_4)^{L(0)}\mathcal{Y}^{a_8}_{a_1a_2;t}\left(\left(\frac{1-z_1-z_3}{1-z_2-z_4}\right)^{L(0)}\cdot\right.\right.\\
&\qquad\qquad\left.\left.\left.\left.\cdot\tilde{w}_{a_1}, \frac{z_3-z_4}{1-z_2-z_4}\right)\tilde{w}_{a_2}, z_4\right)\tilde{w}_{a_3}\right\rangle\right). \quad (3.10)
\end{aligned}
$$

We now change the variables $z_1$ and $z_3$ to

$$z_5=\frac{z_1-z_2}{1-z_2-z_4}$$

and

$$z_6=\frac{z_3-z_4}{1-z_2-z_4}.$$

Then

$$1-z_5-z_6 = \frac{1-z_1-z_3}{1-z_2-z_4},$$

$$z_3 \;=\; (1-z_2-z_4)z_6+z_4.$$

For any branch $f(z_1, z_2, z_3, z_4)$ of a multivalued analytic function of $z_1$, $z_2$, $z_3$ and $z_4$ on a suitable region $A$ such that it is equal to the restriction to $A\cap B$ of a branch of the same analytic function on a region $B$ containing the point $1-z_1-z_3=0$, by definition, we have

$$\begin{aligned}&\mathrm{Res}_{1-z_1-z_3=0\;|\;z_3}E(f(z_1,z_2,z_3,z_4))\\&\quad=\mathrm{Res}_{1-z_5-z_6=0\;|\;(1-z_2-z_4)z_6+z_4}E(f(z_1,z_2,z_3,z_4))\frac{1-z_1-z_3}{1-z_5-z_6}.\end{aligned}\tag{3.11}$$

By (3.1), we have

$$\begin{aligned}&\mathrm{Res}_{1-z_5-z_6=0\;|\;(1-z_2-z_4)z_6+z_4}E(f(z_1,z_2,z_3,z_4))\frac{1-z_1-z_3}{1-z_5-z_6}\\&\quad=\mathrm{Res}_{1-z_5-z_6=0\;|\;z_6}E(f(z_1,z_2,z_3,z_4))\frac{1-z_1-z_3}{1-z_5-z_6}.\end{aligned}\tag{3.12}$$

From (3.11) and (3.12), we obtain

$$\begin{aligned}&\mathrm{Res}_{1-z_1-z_3=0\;|\;z_3}E(f(z_1,z_2,z_3,z_4))\\&\quad=\mathrm{Res}_{1-z_5-z_6=0\;|\;z_6}E(f(z_1,z_2,z_3,z_4))\frac{1-z_1-z_3}{1-z_5-z_6}.\end{aligned}\tag{3.13}$$

Using (3.13), the definition of the pairings $\langle\cdot,\cdot\rangle_{\mathcal{V}_{a_1a_2}^{a_3}}$, Lemma 3.3, and the fact that for $a_1, a_2, a_3\in\mathcal{A}$, $\{\mathcal{Y}'^{;a_3'}_{a_1'a_2';i}\mid i=1,\dots,N^{a_3'}_{a_1'a_2'}\}$ are the dual bases of $\{\mathcal{Y}^{a_3}_{a_1a_2;i}\mid i=1,\dots,N^{a_3}_{a_1a_2}\}$ with respect to the pairing $\langle\cdot,\cdot\rangle_{\mathcal{V}_{a_1a_2}^{a_3}}$, we see that the right-hand side of (3.10) is equal to

$$\begin{aligned}\sum_{a_7,a_8\in\mathcal{A}}\sum_{i,j,s,t}&F(\mathcal{Y}'^{;a_4'}_{a_1'a_5';k}\otimes\mathcal{Y}'^{;a_5'}_{a_2'a_3';l};\mathcal{Y}'^{;a_4'}_{a_7'a_3';i}\otimes\mathcal{Y}'^{;a_7'}_{a_1'a_2';j})\cdot\\&\qquad\cdot F(\mathcal{Y}^{a_4}_{a_1a_6;m}\otimes\mathcal{Y}^{a_6}_{a_2a_3;n};\mathcal{Y}^{a_4}_{a_8a_3;s}\otimes\mathcal{Y}^{a_8}_{a_1a_2;t})\cdot\\&\cdot\mathrm{Res}_{1-z_2-z_4=0\;|\;z_4}\mathrm{Res}_{1-z_5-z_6=0\;|\;z_6}(1-z_5-z_6)^{-1}(1-z_2-z_4)^{-1}\\&\quad E(\langle e^{L(1)}\mathcal{Y}'^{;a_4'}_{a_7'a_3';i}((1-z_2-z_4)^{L(0)}\cdot\\&\qquad\qquad\cdot\mathcal{Y}'^{;a_7'}_{a_1'a_2';j}((1-z_5-z_6)^{L(0)}\tilde{w}'_{a_1},z_5)\tilde{w}'_{a_2},z_2)\tilde{w}'_{a_3},\\&\qquad e^{L(1)}\mathcal{Y}^{a_4}_{a_8a_3;s}((1-z_2-z_4)^{L(0)}\cdot\\&\qquad\qquad\cdot\mathcal{Y}^{a_8}_{a_1a_2;t}((1-z_5-z_6)^{L(0)}\tilde{w}_{a_1},z_6)\tilde{w}_{a_2},z_4)\tilde{w}_{a_3}\rangle)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{a_7,a_8\in\mathcal{A}}\sum_{i,j,s,t} F(\mathcal{Y}'^{;a'_4}_{a'_1a'_5;k}\otimes\mathcal{Y}'^{;a'_5}_{a'_2a'_3;l};\mathcal{Y}'^{;a'_4}_{a'_7a'_3;i}\otimes\mathcal{Y}'^{;a'_7}_{a'_1a'_2;j})\cdot\\
&\qquad\qquad\cdot F(\mathcal{Y}^{a_4}_{a_1a_6;m}\otimes\mathcal{Y}^{a_6}_{a_2a_3;n};\mathcal{Y}^{a_4}_{a_7a_3;s}\otimes\mathcal{Y}^{a_7}_{a_1a_2;t})\cdot\\
&\quad\cdot\mathrm{Res}_{1-z_5-z_6=0\mid z_6}\mathrm{Res}_{1-z_2-z_4=0\mid z_4}(1-z_5-z_6)^{-1}(1-z_2-z_4)^{-1}\\
&\qquad E(\langle e^{L(1)}\mathcal{Y}'^{;a'_4}_{a'_7a'_3;i}((1-z_2-z_4)^{L(0)}\cdot\\
&\qquad\qquad\cdot\mathcal{Y}'^{;a'_7}_{a'_1a'_2;j}((1-z_5-z_6)^{L(0)}\tilde{w}'_{a_1},z_5)\tilde{w}'_{a_2},z_2)\tilde{w}'_{a_3},\\
&\qquad e^{L(1)}\mathcal{Y}^{a_4}_{a_8a_3;s}((1-z_2-z_4)^{L(0)}\cdot\\
&\qquad\qquad\cdot\mathcal{Y}^{a_7}_{a_1a_2;t}((1-z_5-z_6)^{L(0)}\tilde{w}_{a_1},z_6)\tilde{w}_{a_2},z_4)\tilde{w}_{a_3}\rangle)\\
&= \sum_{a_7\in\mathcal{A}}\sum_{i,j,s,t} F(\mathcal{Y}'^{;a'_4}_{a'_1a'_5;k}\otimes\mathcal{Y}'^{;a'_5}_{a'_2a'_3;l};\mathcal{Y}'^{;a'_4}_{a'_7a'_3;i}\otimes\mathcal{Y}'^{;a'_7}_{a'_1a'_2;j})\cdot\\
&\qquad\qquad\cdot F(\mathcal{Y}^{a_4}_{a_1a_6;m}\otimes\mathcal{Y}^{a_6}_{a_2a_3;n};\mathcal{Y}^{a_4}_{a_7a_3;s}\otimes\mathcal{Y}^{a_7}_{a_1a_2;t})\cdot\\
&\quad\cdot\mathrm{Res}_{1-z_5-z_6=0\mid z_6}(1-z_5-z_6)^{-1}\delta_{is}\\
&\qquad E(\langle e^{L(1)}\mathcal{Y}'^{;a'_7}_{a'_1a'_2;j}((1-z_5-z_6)^{L(0)}\tilde{w}'_{a_1},z_5)\tilde{w}'_{a_2},\\
&\qquad\qquad e^{L(1)}\mathcal{Y}^{a_8}_{a_1a_2;t}((1-z_5-z_6)^{L(0)}\tilde{w}_{a_1},z_6)\tilde{w}_{a_2}\rangle)\\
&= \sum_{a_7\in\mathcal{A}}\sum_{i,j,s,t} F(\mathcal{Y}'^{;a'_4}_{a'_1a'_5;k}\otimes\mathcal{Y}'^{;a'_5}_{a'_2a'_3;l};\mathcal{Y}'^{;a'_4}_{a'_7a'_3;i}\otimes\mathcal{Y}'^{;a'_7}_{a'_1a'_2;j})\cdot\\
&\qquad\qquad\cdot F(\mathcal{Y}^{a_4}_{a_1a_6;m}\otimes\mathcal{Y}^{a_6}_{a_2a_3;n};\mathcal{Y}^{a_4}_{a_8a_3;s}\otimes\mathcal{Y}^{a_8}_{a_1a_2;t})\delta_{is}\delta_{jt}\\
&= \sum_{a_7\in\mathcal{A}}\sum_{i,j} F(\mathcal{Y}'^{;a'_4}_{a'_1a'_5;k}\otimes\mathcal{Y}'^{;a'_5}_{a'_2a'_3;l};\mathcal{Y}'^{;a'_4}_{a'_7a'_3;i}\otimes\mathcal{Y}'^{;a'_7}_{a'_1a'_2;j})\cdot\\
&\qquad\qquad\cdot F(\mathcal{Y}^{a_4}_{a_1a_6;m}\otimes\mathcal{Y}^{a_6}_{a_2a_3;n};\mathcal{Y}^{a_4}_{a_7a_3;i}\otimes\mathcal{Y}^{a_7}_{a_1a_2;j}).
\end{aligned}
\tag{3.14}
$$

From (3.9)–(3.14), we see that the right inverse of the matrix with entries

$$F(\mathcal{Y}'^{;a'_4}_{a'_1a'_5;k}\otimes\mathcal{Y}'^{;a'_5}_{a'_2a'_3;l};\mathcal{Y}'^{;a'_4}_{a'_7a'_3;i}\otimes\mathcal{Y}'^{;a'_7}_{a'_1a'_2;j})$$

is the transpose of the matrix with entries

$$F(\mathcal{Y}^{a_4}_{a_1a_6;m}\otimes\mathcal{Y}^{a_6}_{a_2a_3;n};\mathcal{Y}^{a_4}_{a_7a_3;i}\otimes\mathcal{Y}^{a_7}_{a_1a_2;j}).$$

Since for square matrices, right inverses are also left inverses, the proposition is proved. ∎

For $a\in\mathcal{A}$, we use $\sqrt{F_a}$ to denote the square root $\sqrt{|F_a|}e^{\frac{i\arg F_a}{2}}$ of $F_a$. For $a_1,a_2,a_3\in\mathcal{A}$, consider the modified pairings

$$\frac{\sqrt{F_{a_3}}}{\sqrt{F_{a_1}}\sqrt{F_{a_2}}}\langle\cdot,\cdot\rangle_{\mathcal{V}^{a_3}_{a_1a_2}}.$$

These pairings give a nondegenerate bilinear form $(\cdot,\cdot)_{\mathcal{V}}$ on $\mathcal{V}$. For any $\sigma\in S_3$, $\{\sigma(\mathcal{Y}^{a_3}_{a_1a_2;i})\mid i=\dots,N^{a_3}_{a_1a_2}\}$ is a basis of $\sigma(\mathcal{V}^{a_3}_{a_1a_2})$.

We have:

**Proposition 3.7** *The nondegenerate bilinear form $(\cdot,\cdot)_{\mathcal{V}}$ is invariant with respect to the action of $S_3$ on $\mathcal{V}$, that is, for $a_1,a_2,a_3\in\mathcal{A}$, $\sigma\in S_3$, $\mathcal{Y}_1\in\mathcal{V}^{a_3}_{a_1a_2}$ and $\mathcal{Y}_2\in\mathcal{V}^{a_3'}_{a_1'a_2'}$,*

$$(\sigma(\mathcal{Y}_1),\sigma(\mathcal{Y}_2))_{\mathcal{V}}=(\mathcal{Y}_1,\mathcal{Y}_2)_{\mathcal{V}}.$$

*Equivalently, for $a_1,a_2,a_3\in\mathcal{A}$,*

$$\left\{\frac{\sqrt{F_{a_1}}\sqrt{F_{a_2}}\sqrt{F_{a_{\sigma^{-1}(3)}}}}{\sqrt{F_{a_{\sigma^{-1}(1)}}}\sqrt{F_{a_{\sigma^{-1}(2)}}}\sqrt{F_{a_3}}}\sigma(\mathcal{Y}'^{;a_3'}_{a_1'a_2';i})\;\middle|\; i=\dots,N^{a_3}_{a_1a_2}\right\}$$

*is the dual basis of $\{\sigma(\mathcal{Y}^{a_3}_{a_1a_2;i})\mid i=\dots,N^{a_3}_{a_1a_2}\}$.*

*Proof.* The equivalence of the first conclusion and the second conclusion is clear.

We first prove the result for $\sigma=\sigma_{12}$. In this case, we need to show that $\{\sigma_{12}(\mathcal{Y}^{a_3'}_{a_1'a_2';j})\mid i=\dots,N^{a_3}_{a_1a_2}\}$ is the dual basis of $\{\sigma_{12}(\mathcal{Y}^{a_3}_{a_1a_2;i})\mid i=\dots,N^{a_3}_{a_1a_2}\}$. For $i,j=1,\dots,N^{a_3}_{a_1a_2}$, by (3.3), we have

$$\langle\sigma_{12}(\mathcal{Y}'^{;a_3'}_{a_1'a_2';i}),\sigma_{12}(\mathcal{Y}^{a_3}_{a_1a_2;j})\rangle=F(\sigma_{23}(\sigma_{12}(\mathcal{Y}'^{;a_3'}_{a_1'a_2';i}))\otimes\sigma_{12}(\mathcal{Y}^{a_3}_{a_1a_2;j});\mathcal{Y}^{a_1}_{ea_1;1}\otimes\mathcal{Y}^{e}_{a_2'a_2;1}). \tag{3.15}$$

By Proposition 3.4 in [H9], the right-hand side of (3.7) is equal to

$$\begin{aligned}F(\sigma_{132}&(\sigma_{12}(\mathcal{Y}^{a_3}_{a_1a_2;j}))\otimes\sigma_{123}(\sigma_{23}(\sigma_{12}(\mathcal{Y}'^{;a_3'}_{a_1'a_2';i})));\sigma_{123}(\mathcal{Y}^{e}_{a_2'a_2;1})\otimes\sigma_{132}(\mathcal{Y}^{a_1}_{ea_1;1}))\\&=F(\sigma_{23}(\mathcal{Y}^{a_3}_{a_1a_2;j})\otimes\mathcal{Y}'^{;a_3'}_{a_1'a_2';i};\mathcal{Y}^{a_2'}_{ea_2';1}\otimes\mathcal{Y}^{e}_{a_1a_1';1})\\&=F(\sigma_{23}(\mathcal{Y}^{a_3''}_{a_1''a_2'';j})\otimes\mathcal{Y}'^{;a_3'}_{a_1'a_2';i};\mathcal{Y}^{a_2'}_{ea_2';1}\otimes\mathcal{Y}^{e}_{a_1''a_1';1}).\end{aligned} \tag{3.16}$$

By (3.3) again, the right-hand side of (3.16) is equal to

$$\langle\mathcal{Y}^{a_3''}_{a_1''a_2'';j},\mathcal{Y}'^{;a_3'}_{a_1'a_2';i}\rangle=\delta_{ij},$$

proving the case of $\sigma=\sigma_{12}$.

Next we prove the result for $\sigma=\sigma_{23}$. We need to find the relation between the matrices

$$\langle\sigma_{23}(\mathcal{Y}^{a_3'}_{a_1'a_2';i}),\sigma_{23}(\mathcal{Y}^{a_3}_{a_1a_2';j})\rangle_{\mathcal{V}^{a_2'}_{a_1a_3'}}$$

and

$$\langle \mathcal{Y}^{a_3'}_{a_1'a_2';i}, \mathcal{Y}^{a_3}_{a_1a_2';j}\rangle_{\mathcal{V}^{a_3}_{a_1a_2}}.$$

By definition, we need to find the relation between the matrices

$$F(\mathcal{Y}^{a_3'}_{a_1'a_2';i}\otimes \sigma_{23}(\mathcal{Y}^{a_3}_{a_1a_2;j}); \mathcal{Y}^{a_3'}_{ea_3';1}\otimes \mathcal{Y}^{e}_{a_1'a_1;1}) \tag{3.17}$$

and

$$F(\sigma_{23}(\mathcal{Y}^{a_3';(2)}_{a_1'a_2';i})\otimes \mathcal{Y}^{a_3;(1)}_{a_1a_2;j}; \mathcal{Y}^{a_2}_{ea_2;1}\otimes \mathcal{Y}^{e}_{a_1'a_1;1}). \tag{3.18}$$

From (4.9) in [H9] (or (3.6)), we see that the inverse of the matrix (3.17) is

$$\frac{F(\mathcal{Y}^{a_3'}_{a_3'e;1}\otimes \mathcal{Y}^{e}_{a_2a_2';1}; \mathcal{Y}^{a_3'}_{a_1'a_2';k}\otimes \sigma_{132}(\sigma_{23}(\mathcal{Y}^{a_3}_{a_1a_2;j}))))}{F_{a_3'}}. \tag{3.19}$$

By Proposition 3.4 in [H9] and the fact $F_{a'}=F_a$ for $a\in\mathcal{A}$, (3.19) is equal to

$$\begin{aligned}&\frac{F(\mathcal{Y}^{a_2'}_{a_2'e;1}\otimes \mathcal{Y}^{e}_{a_3a_3';1}; \sigma_{23}(\mathcal{Y}^{a_3}_{a_1a_2;j})\otimes \sigma_{132}(\mathcal{Y}^{a_3'}_{a_1'a_2';k}))}{F_{a_3}}\\ &\quad=\frac{F_{a_2}}{F_{a_3}}\frac{F(\mathcal{Y}^{a_2'}_{a_2'e;1}\otimes \mathcal{Y}^{e}_{a_3a_3';1}; \sigma_{23}(\mathcal{Y}^{a_3}_{a_1a_2;j})\otimes \sigma_{132}(\mathcal{Y}^{a_3'}_{a_1'a_2';k}))}{F_{a_2'}},\end{aligned}$$

which by (4.9) in [H9] (or (3.6)) again is equal to $\frac{F_{a_2}}{F_{a_3}}$ times the inverse of (3.18). So the inverse of the matrix (3.17) is equal to $\frac{F_{a_2}}{F_{a_3}}$ times the inverse of (3.18). Thus the matrix (3.17) is equal to $\frac{F_{a_3}}{F_{a_2}}$ times the matrix (3.18), or equivalently,

$$(\sigma_{23}(\mathcal{Y}^{a_3}_{a_1a_2;j}), \sigma_{23}(\mathcal{Y}^{a_3'}_{a_1'a_2';k}))_{\mathcal{V}} = (\mathcal{Y}^{a_3}_{a_1a_2;j}, \mathcal{Y}^{a_3'}_{a_1'a_2';k})_{\mathcal{V}}.$$

Since $S_3$ is generated by $\sigma_{12}$ and $\sigma_{23}$, the conclusion of the proposition follows. ∎

We are ready to construct a full field algebra using the bases of intertwining operators we have chosen. Let

$$F=\oplus_{a\in\mathcal{A}}W^a\otimes W^{a'}.$$

For $w_{a_1}\in W^{a_1}$, $w_{a_2}\in W^{a_2}$, $w_{a_1'}\in W^{a_1'}$ and $w_{a_2'}\in W^{a_2'}$, we define

$$\begin{aligned}&\mathbb{Y}((w_{a_1}\otimes w_{a_1'}), z,\zeta)(w_{a_2}\otimes w_{a_2'})\\ &\quad=\sum_{a_3\in\mathcal{A}}\sum_{p=1}^{N^{a_3}_{a_1a_2}}\mathcal{Y}^{a_3}_{a_1a_2;p}(w_{a_1},z)w_{a_2}\otimes \mathcal{Y}'^{;a_3'}_{a_1'a_2';p}(w_{a_1'},\zeta)w_{a_2'}.\end{aligned}$$

**Theorem 3.8** *The quadruple* $(F, \mathbb{Y}, \mathbf{1}\otimes\mathbf{1}, \omega\otimes\mathbf{1}, \mathbf{1}\otimes\omega)$ *is a conformal full field algebra over* $V\otimes V$.

*Proof.* The identity property, the creation property and the single-valuedness property are clear. We prove the associativity and the skew-symmetry here.

We prove associativity first. For $a_1, a_2 \in \mathcal{A}$, $w_{a_1} \in W^{a_1}$, $w_{a_2} \in W^{a_2}$, $w'_{a_1} \in (W^{a_1})'$, $w'_{a_2} \in (W^{a_2})'$, using the associativity of intertwining operators and Proposition 3.6, we have

$$
\begin{aligned}
&\mathbb{Y}((w_{a_1}\otimes w'_{a_1}), z_1, \zeta_1)\mathbb{Y}((w_{a_2}\otimes w'_{a_2}), z_2, \zeta_2)\\
&\quad= \sum_{a_3,a_4,a_5\in\mathcal{A}} \sum_{p=1}^{N_{a_1a_5}^{a_4}} \sum_{q=1}^{N_{a_2a_3}^{a_5}} (\mathcal{Y}_{a_1a_5;p}^{a_4}(w_{a_1}, z_1)\mathcal{Y}_{a_2a_3;q}^{a_5}(w_{a_2}, z_2))\\
&\qquad\qquad\otimes(\mathcal{Y}_{a'_1a'_5;p}^{\prime;a'_4}(w'_{a_1}, \zeta_1)\mathcal{Y}_{a'_2a'_3;q}^{\prime;a'_5}(w'_{a_2}, \zeta_2))\\
&\quad= \sum_{a_3,a_4,a_5,a_6,a_7\in\mathcal{A}} \sum_{p=1}^{N_{a_1a_5}^{a_4}} \sum_{q=1}^{N_{a_2a_3}^{a_5}} \sum_{m=1}^{N_{a_6a_3}^{a_5}} \sum_{n=1}^{N_{a_1a_2}^{a_6}} \sum_{k=1}^{N_{a'_7a'_3}^{a'_5}} \sum_{l=1}^{N_{a'_1a'_2}^{a'_7}}\\
&\qquad\cdot F(\mathcal{Y}_{a_1a_5;p}^{a_4}\otimes\mathcal{Y}_{a_2a_3;q}^{a_5}; \mathcal{Y}_{a_6a_3;m}^{a_4}\otimes\mathcal{Y}_{a_1a_2;n}^{a_6})\cdot\\
&\qquad\cdot F(\mathcal{Y}_{a'_1a'_5;p}^{\prime;a'_4}\otimes\mathcal{Y}_{a'_2a'_3;q}^{\prime;a'_5}; \mathcal{Y}_{a'_7a'_3;k}^{a'_4}\otimes\mathcal{Y}_{a'_1a'_2;l}^{a'_7})\cdot\\
&\qquad\cdot((\mathcal{Y}_{a_6a_3;m}^{a_4}(\mathcal{Y}_{a_1a_2;n}^{a_6}(w_{a_1}, z_1-z_2)w_{a_2}, z_2))\\
&\qquad\qquad\otimes(\mathcal{Y}_{a'_7a'_3;k}^{\prime;a'_4}(\mathcal{Y}_{a'_1a'_2;l}^{\prime;a'_7}(w'_{a_1}, \zeta_1-\zeta_2)w'_{a_2}, \zeta_2)))\\
&\quad= \sum_{a_3,a_4,a_6,a_7\in\mathcal{A}} \sum_{m=1}^{N_{a_6a_3}^{a_5}} \sum_{n=1}^{N_{a_1a_2}^{a_6}} \sum_{k=1}^{N_{a'_7a'_3}^{a'_5}} \sum_{l=1}^{N_{a'_1a'_2}^{a'_7}} \delta_{a_6a_7}\delta_{mk}\delta_{nl}\cdot\\
&\qquad\cdot(\mathcal{Y}_{a_6a_3;m}^{a_4}(\mathcal{Y}_{a_1a_2;n}^{a_6}(w_{a_1}, z_1-z_2)w_{a_2}, z_2))\\
&\qquad\qquad\otimes(\mathcal{Y}_{a'_7a'_3;k}^{\prime;a'_4}(\mathcal{Y}_{a'_1a'_2;l}^{\prime;a'_7}(w'_{a_1}, \zeta_1-\zeta_2)w'_{a_2}, \zeta_2))\\
&\quad= \sum_{a_3,a_4,a_6\in\mathcal{A}} \sum_{m=1}^{N_{a_6a_3}^{a_5}} \sum_{n=1}^{N_{a_1a_2}^{a_6}} (\mathcal{Y}_{a_6a_3;m}^{a_4}(\mathcal{Y}_{a_1a_2;n}^{a_6}(w_{a_1}, z_1-z_2)w_{a_2}, z_2))\\
&\qquad\qquad\otimes(\mathcal{Y}_{a'_6a'_3;m}^{\prime;a'_4}(\mathcal{Y}_{a'_1a'_2;n}^{\prime;a'_6}(w'_{a_1}, \zeta_1-\zeta_2)w'_{a_2}, \zeta_2))\\
&\quad= \mathbb{Y}(\mathbb{Y}((w_{a_1}\otimes w'_{a_1}), z_1-z_2, \zeta_1-\zeta_2)(w_{a_2}\otimes w'_{a_2}), z_2, \zeta_2).
\end{aligned}
$$

We now prove the skew-symmetry. By Proposition 3.7, $\{\sigma_{12}(\mathcal{Y}_{a'_1a'_2;i}^{\prime;a'_3}) \mid i = 1, \dots, N_{a_1a_2}^{a_3}\}$ is the dual basis of $\{\sigma_{12}(\mathcal{Y}_{a_1a_2;i}^{a_3}) \mid i = 1, \dots, N_{a_1a_2}^{a_3}\}$. Thus for

$a_1, a_2 \in \mathcal{A}$, $w_{a_1} \in W^{a_1}$, $w_{a_2} \in W^{a_2}$, $w'_{a_1} \in (W^{a_1})'$, $w'_{a_2} \in (W^{a_2})'$, we have

$$\begin{aligned}
&\mathbb{Y}((w_{a_2} \otimes w'_{a_2}), z, \zeta)(w_{a_1} \otimes w'_{a_1}) \\
&\quad = \sum_{a_3 \in \mathcal{A}} \sum_{p=1}^{N_{a_2 a_1}^{a_3}} \sigma_{12}(\mathcal{Y}_{a_1 a_2; p}^{a_3})(w_{a_2}, z) w_{a_1} \otimes \sigma_{12}(\mathcal{Y}_{a'_1 a'_2; p}^{\prime; a'_3})(w_{a'_2}, \zeta) w_{a'_1} \\
&\quad = \sum_{a_3 \in \mathcal{A}} \sum_{p=1}^{N_{a_1 a_2}^{a_3}} e^{-\pi i \Delta(\mathcal{Y}_{a_1 a_2; p}^{a_3})} e^{zL(-1)} \mathcal{Y}_{a_1 a_2; p}^{a_3}(w_{a_1}, e^{\pi i} z) w_{a_2} \\
&\qquad\qquad\qquad \otimes e^{\pi i \Delta(\mathcal{Y}_{a'_1 a'_2; p}^{a'_3})} e^{\zeta L(-1)} \mathcal{Y}_{a'_1 a'_2; p}^{\prime; a'_3})(w_{a'_1}, e^{-\pi i} \zeta) w_{a'_2} \\
&\quad = (e^{zL(-1)} \otimes e^{\zeta L(-1)}) \cdot \\
&\qquad\qquad \sum_{a_3 \in \mathcal{A}} \sum_{p=1}^{N_{a_1 a_2}^{a_3}} \mathcal{Y}_{a_1 a_2; p}^{a_3}(w_{a_1}, e^{\pi i} z) w_{a_2} \otimes \mathcal{Y}_{a'_1 a'_2; p}^{\prime; a'_3}(w_{a'_1}, e^{-\pi i} \zeta) w_{a'_2} \\
&\quad = (e^{zL(-1)} \otimes e^{\zeta L(-1)}) \mathbb{Y}((w_{a_1} \otimes w'_{a_1}), e^{\pi i} z, e^{-\pi i} \zeta)(w_{a_2} \otimes w'_{a_2}).
\end{aligned}$$

∎

**Definition 3.9** A nondegenerate bilinear form $(\cdot, \cdot)$ on a conformal full field algebra

$$(F, m, \mathbf{1}, \omega^L, \omega^R)$$

is said to be *invariant* if for $u, v, w \in F$,

$$\begin{aligned}
&(\mathbb{Y}(u; z, \bar{z})v, w) \\
&\quad = (v, \mathbb{Y}(e^{zL^L(1) + \bar{z}L^R(1)} e^{\pi i L^L(0) - \pi i L^R(0)} z^{-2L^L(0)} \bar{z}^{-2L^R(0)} u; z^{-1}, \bar{z}^{-1}) w).
\end{aligned}$$

The conformal full field algebra $F$ we constructed above has a natural nondegenerate bilinear form $(\cdot, \cdot)_F : F \otimes F \to \mathbb{C}$ given by

$$((w_{a_1} \otimes w'_{a_1}), (w_{a_2} \otimes w'_{a_2}))_F = \begin{cases} 0 & a_1 \neq a'_2 \\ F_{a_1} \langle w_{a_1}, w_{a_2} \rangle \langle w'_{a_1}, w'_{a_2} \rangle & a_1 = a'_2 \end{cases}$$

for $a_1, a_2 \in \mathcal{A}$, $w_{a_1} \in W^{a_1}$, $w_{a_2} \in W^{a_2}$, $w'_{a_1} \in (W^{a_1})'$, $w'_{a_2} \in (W^{a_2})'$. We have:

**Theorem 3.10** *The nondegenerate bilinear form $(\cdot, \cdot)_F$ is invariant.*

*Proof.* For $a_1, a_2, a_3 \in \mathcal{A}$, $w_{a_1} \in W^{a_1}$, $w_{a_2} \in W^{a_2}$, $w_{a_3} \in W^{a_3}$, $w'_{a_1} \in (W^{a_1})'$, $w'_{a_2} \in (W^{a_2})'$, $w'_{a_3} \in (W^{a_3})'$, using Proposition 3.7 for the case $\sigma = \sigma_{23}$, we have

$$
\begin{aligned}
&(\mathcal{Y}((w_{a_1} \otimes w'_{a_1}), z, \zeta)(w_{a_2} \otimes w'_{a_2}), (w_{a_3} \otimes w'_{a_3}))_F \\
&\quad = \sum_{a_4 \in \mathcal{A}} \sum_{p=1}^{N_{a_1 a_2}^{a_4}} ((\mathcal{Y}_{a_1 a_2; p}^{a_4}(w_{a_1}, z) w_{a_2} \otimes \mathcal{Y}_{a'_1 a'_2; p}^{\prime; a'_4}(w'_{a_1}, \zeta) w'_{a_2}), (w_{a_3} \otimes w'_{a_3}))_F \\
&\quad = \sum_{p=1}^{N_{a_1 a_2}^{a_3}} F_{a_3} \langle \mathcal{Y}_{a_1 a_2; p}^{a_3}(w_{a_1}, z) w_{a_2}, w_{a_3} \rangle \langle \mathcal{Y}_{a'_1 a'_2; p}^{\prime; a'_3}(w'_{a_1}, \zeta) w'_{a_2}, w'_{a_3} \rangle \\
&\quad = \sum_{p=1}^{N_{a_1 a_2}^{a_3}} F_{a_3} \langle \sigma_{23}(\sigma_{23}(\mathcal{Y}_{a_1 a_2; p}^{a_3})(w_{a_1}, z) w_{a_2}, w_{a_3} \rangle \langle \sigma_{23}(\sigma_{23}(\mathcal{Y}_{a'_1 a'_2; p}^{\prime; a'_3})(w'_{a_1}, \zeta) w'_{a_2}, w'_{a_3} \rangle \\
&\quad = \sum_{p=1}^{N_{a_1 a_2}^{a_3}} F_{a_3} \langle w_{a_2}, e^{\pi i h_{a_1}} \sigma_{23}(\mathcal{Y}_{a_1 a_2; p}^{a_3})(e^{zL(1)} e^{-\pi i L(0)} z^{-2L(0)} w_{a_1}, z^{-1}) w_{a_3} \rangle \cdot \\
&\qquad\qquad \cdot \langle w'_{a_2}, e^{-\pi i h_{a_1}} \sigma_{23}(\mathcal{Y}_{a'_1 a'_2; p}^{\prime; a'_3})(e^{\zeta L(1)} e^{\pi i L(0)} \zeta^{-2L(0)} w'_{a_1}, \zeta^{-1}) w'_{a_3} \rangle \\
&\quad = \sum_{p=1}^{N_{a_1 a_2}^{a_3}} F_{a_2} \langle w_{a_2}, \sigma_{23}(\mathcal{Y}_{a_1 a_2; p}^{a_3})(e^{zL(1)} e^{-\pi i L(0)} z^{-2L(0)} w_{a_1}, z^{-1}) w_{a_3} \rangle \cdot \\
&\qquad\qquad \cdot \left\langle w'_{a_2}, \left( \frac{F_{a_3}}{F_{a_2}} \sigma_{23}(\mathcal{Y}_{a'_1 a'_2; p}^{\prime; a'_3}) \right) (e^{\zeta L(1)} e^{\pi i L(0)} \zeta^{-2L(0)} w'_{a_1}, \zeta^{-1}) w'_{a_3} \right\rangle \\
&\quad = \sum_{p=1}^{N_{a_1 a_2}^{a_3}} ((w_{a_2} \otimes w'_{a_2}), (\sigma_{23}(\mathcal{Y}_{a_1 a_2; p}^{a_3})(e^{zL(1)} e^{-\pi i L(0)} z^{-2L(0)} w_{a_1}, z^{-1}) w_{a_3} \\
&\qquad\qquad \otimes \left( \frac{F_{a_3}}{F_{a_2}} \sigma_{23}(\mathcal{Y}_{a'_1 a'_2; p}^{\prime; a'_3}) \right) (e^{\zeta L(1)} e^{\pi i L(0)} \zeta^{-2L(0)} w'_{a_1}, \zeta^{-1}) w'_{a_3}))_F \\
&\quad = \sum_{a_4 \in \mathcal{A}} \sum_{p=1}^{N_{a_1 a_4}^{a_3}} ((w_{a_2} \otimes w'_{a_2}), (\sigma_{23}(\mathcal{Y}_{a_1 a_4; p}^{a_3})(e^{zL(1)} e^{-\pi i L(0)} z^{-2L(0)} w_{a_1}, z^{-1}) w_{a_3} \\
&\qquad\qquad \otimes \left( \frac{F_{a_3}}{F_{a_2}} \sigma_{23}(\mathcal{Y}_{a'_1 a'_4; p}^{\prime; a'_3}) \right) (e^{\zeta L(1)} e^{\pi i L(0)} \zeta^{-2L(0)} w'_{a_1}, \zeta^{-1}) w'_{a_3}))_F \\
&\quad = ((w_{a_2} \otimes w'_{a_2}), \mathcal{Y}((e^{zL(1)} e^{-\pi i L(0)} z^{-2L(0)} w_{a_1} \\
&\qquad\qquad \otimes (e^{\zeta L(1)} e^{\pi i L(0)} \zeta^{-2L(0)} w'_{a_1}), z^{-1}, \zeta^{-1})(w_{a_3} \otimes w'_{a_3}))_F,
\end{aligned}
$$

proving the invariance. ∎

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