Vertex operator algebras as a new type of symmetry

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1. What are symmetries?

Symmetries are properties that are invariant under the action of groups. So symmetries are described by groups.

In quantum theory, states are given by vectors of Hilbert spaces. If there is a symmetry of the theory, the state spaces must be representations of the corresponding group.

All such representaions of the group form what mathematicians call symmetric tensor categories.

Tannaka-Krein reconstruction theorem: Any symmetric tensor category satisfying some additional natural properties is the tensor category of representations of a certain group.

Therefore, such tensor categories can be used to replace groups and symmetries can be described by such symmetric tensor categories.

New symmetries: New types of tensor categories.

Examples: Braided tensor catogories, modular tensor categories and so on.

Instead of groups, different types of algebras give these new types of categories, that is, the categories of representations of these algebras are such categories.

Examples: Quantum groups (certain algebras, not groups), conformal nets (certain operator algebras in the traditional sense) and vertex operator algebras (not operator algebras in the traditional sense).

These algebras should be viewed as new types of symmetries.

More structures for vertex operator algebras: The categories of representations of suitable vertex operator algebras are what we call vertex tensor categories.

Vertex tensor categories give braided tensor categories. Semisimple, rigid and nondegenerate vertex tensor categories give modular tensor categories.

Vertex operator algebras give a new type of symmetries that have rich structures. They incorporate the moduli space of Riemann surfaces naturally into their structures.

Question: Can the topological order introduced by Xiao-Gang Wen be described by the breaking of such new type of symmetries in a certain sense?

2. What is a vertex operator algebra?

Vertex operator algebras are analogues of both commutative associative algebras and Lie algeberas

Commutative associative algebras:

A vector space: C
Multiplication: ab for $a, b \in C$
Identity: $1 \in C$
Associativity: $(ab)c = a(bc)$ for $a, b, c \in C$
Commutativity: $ab = ba$ for $a, b \in C$
Identity property: $1a = a1 = a$ for $a \in C$
Distribution property: $a(b+c) = ab + ac$ for $a, b, c \in C$

Examples:

(i) The space $\mathbb{C}[x_1, \cdots, x_n]$ of polynomials in $x_1, \cdots, x_n$.

(ii) The space of functions on a manifold, e.g., space-time.

Lie algebras:

A vector space: $\mathfrak{g}$
A Lie bracket: $[a, b]$ for $a, b \in \mathfrak{g}$,
Skew-symmetry: $[a, b] = -[b, a]$ for $a, b \in \mathfrak{g}$
Jacobi identity: $[a, [b, c]] - [b, [a, c]] = [[a, b], c]$ for $a, b, c \in \mathfrak{g}$

Distribution property: $[a, b+c] = [a, b] + [a, c]$.

Vertex operator algebras:

Analogues of both commutative associative algebras and Lie algebras.

A vector space: $V = \bigoplus_{n\in\mathbb{Z}} V_{(n)}$, $\dim V_{(n)} = 0$, $V_{(n)} = 0$ for n sufficiently negative.

Vertex operators: For $v \in V$, there is a vertex operator $Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}$, where v_n for $n \in \mathbb{Z}$ are operators on V

A vacuum: $\mathbb{1}$ or $|0\rangle \in V_{(0)}$

A conformal element: $\omega \in V_{(2)}$ corresponding to the stress-energy tensor, that is, $Y(\omega, z)$ is the stress-energy tensor $T(z)$

satisfying the following:

Properties for the vacuum: $Y(|0\rangle, z) = 1_V$

$\lim Y(v,z)|0\rangle = v$ for $v \in V$

Associativity (meromorphic operator product expansion):

For $u_1, u_2 \in V$, $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$.

$Y(u_1, z_1)\, Y(u_2, z_2) = Y(Y(u_1, z_1 - z_2)u_2, z_2)$.

More precisely, if we let $V' = \bigoplus_{n\in\mathbb{Z}} V_{(n)}^* \subset V^*$ and let $\langle\, , \rangle$ be the pairing between V' and V ($\langle\,|\,\rangle$ in physics), then

$$\langle v', Y(u_1, z_1) Y(u_2, z_2) v\rangle$$

and

$$\langle v', Y(Y(u_1, z_1 - z_2) u_2, z_2) v\rangle$$

are absolutely convergent when $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, and their sums are equal when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

(Operator product expansion: Since

$$Y(u_1, z_1 - z_2) = \sum_{n\in\mathbb{Z}} (u_1)_n (z_1 - z_2)^{-n-1},$$

we have

$$Y(u_1, z_1) Y(u_2, z_2) = Y(Y(u_1, z_1 - z_2) u_2, z_2)$$
$$= \sum_{n\in\mathbb{Z}} \frac{Y((u_1)_n u_2, z_2)}{(z_1 - z_2)^{n+1}},$$

that is, the product of two vertex operators can be expanded as a Laurent series in $z_1 - z_2$ with finitely many singular terms and with vertex operators as coefficients.

Commutativity: For $u_1, u_2 \in V$,

$$Y(u_1, z_1) Y(u_2, z_2) \sim Y(u_2, z_2) Y(u_1, z_1).$$

More precisely, for $v' \in V'$, $u_1, u_2, v \in V$,

$$\langle v', Y(u_1, z_1) Y(u_2, z_2) v \rangle$$

and

$$\langle v', Y(u_2, z_2) Y(u_1, z_1) v \rangle,$$

absolutely convergent when $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively, are analytic extensions of each other.

Properties for the conformal element or the stress-energy tensor:

Let $L(n)$ for $n \in \mathbb{Z}$ be given by $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$.

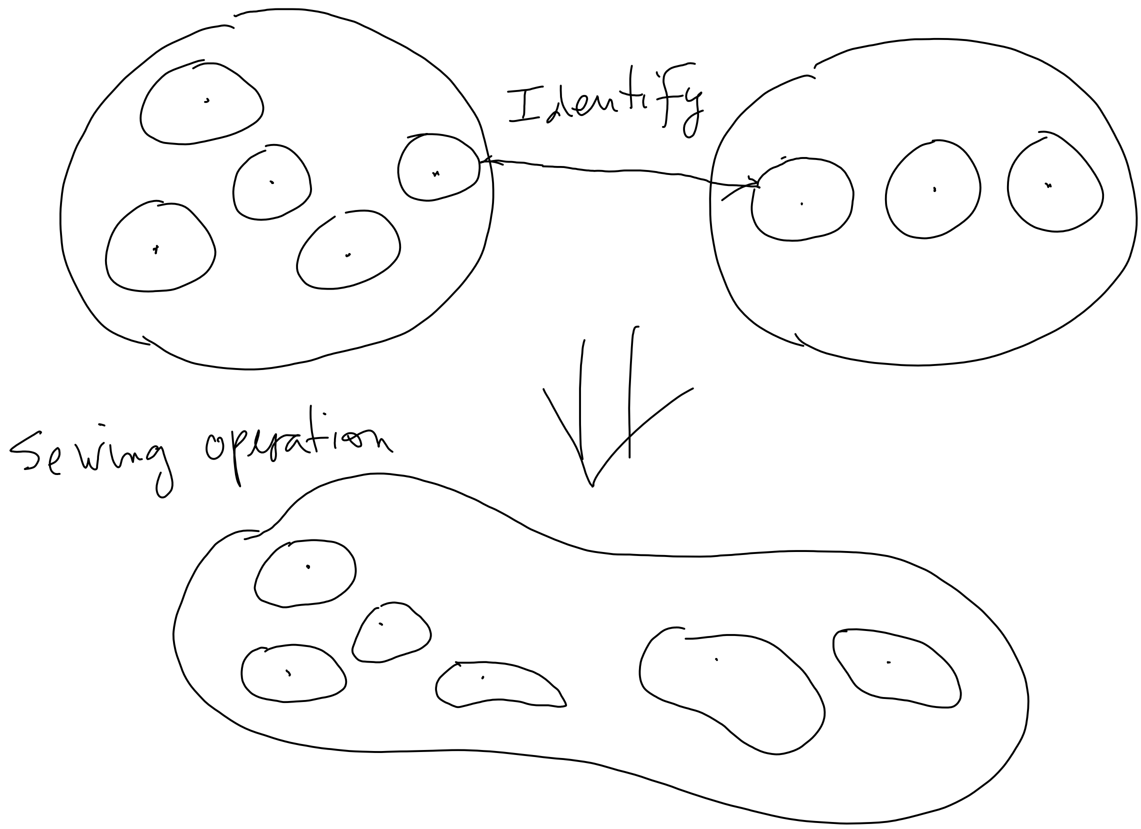
$$[L(m), L(n)] = (m-n) L(m+n) + \frac{c}{12}(m^3 - m)\delta_{m+n,0},$$

$$L(0) v = n v \quad \text{for } v \in V_{(n)},$$

$$\frac{d}{dz} Y(v, z) = Y(L(-1)v, z) \quad \text{for } v \in V.$$

Geometric meaning of vertex operator algebras:

Vertex operator algebras are meromorphic representations of the algebraic structure given by the moduli space of Riemann sphere with punctures and local coordnates.



The set of conformal equivalence classes of spheres with punctures and local coordinates with the sewing operation is an example of a type of algebraic structures called operads.

Vertex operator algebras are linear representations of this algebraic structure satisfying some meromorphicity conditions. In particular, the geometry of Riemann sphere is intrinsically incorporated into the structure of any vertex operator algebra.

Examples:

(i) Trivial examples: Commutative associative algebras

C: a commutative associative algebra. Let $V_{(0)} = C$, $V_{(n)} = 0$ for $n \neq 0$. $Y(a, z)b = ab$, $|0\rangle = 1$, $\omega = 0$.

(ii) Free boson:

Heisenberg algebra: $[a_m, a_n] = m\,\delta_{m+n,0}$

V: Fock space, the space spanned by vectors of the form $a_{-n_1} \cdots a_{-n_\ell}|0\rangle$, $n_1 \geq \cdots \geq n_\ell > 0$

V is a module or representation of the Heisenberg algebra.

$V_{(n)}$ subspace spanned by elements of the form above such that $n_1 + \cdots + n_\ell = n$.

Then $V = \bigoplus_{n \in \mathbb{Z}} V_{(n)}$.

$|0\rangle \in V_{(0)}$

$\omega = \frac{1}{2} a_{-1} a_{-1} |0\rangle \in V_{(2)}$.

Vertex operators : $Y(|0\rangle, z) = 1_V$

$$Y(a_{-1}|0\rangle, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} = a(z)$$

$$Y(a_{-n}|0\rangle, z) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} a(z)$$

$$Y(a_{-n_1} \cdots a_{-n_\ell}|0\rangle, z)$$
$$= {:}\frac{1}{(n_1-1)!} \frac{d^{n_1-1}}{dz^{n_1-1}} a(z) \cdots \frac{1}{(n_\ell-1)!} \frac{d^{n_\ell-1}}{dz^{n_\ell-1}} a(z){:}.$$

(ii) Wess-Zumino(-Novikov)-Witten models (affine Lie algebras):

$\mathfrak{g}$ finite-dimensional semisimple Lie algebra

Affine Lie algebra

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$$

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m(a, b)\delta_{m+n,0} K$$

$$[K, a \otimes t^m] = 0$$

Denote $a \otimes t^m = a(m)$

Let a_i for $i = 1, \cdots, \dim \mathfrak{g}$ be an orthonormal basis of $\mathfrak{g}$. Let V be the space spanned by elements of the form

$$a_{i_1}(-n_1) \cdots a_{i_\ell}(-n_\ell)|0\rangle, \quad \ell = 0, 1, 2, \cdots,$$
$$i_1 \geq \cdots \geq i_\ell$$
$$n_1 \geq \cdots \geq n_\ell > 0$$

Then V is a vertex operator algebra with K acting as k. V is also a module for $\hat{\mathfrak{g}}$. Let J be the maximal proper submodule for $\hat{\mathfrak{g}}$ of V. Then V/J is also a vertex operator algebra. When k is a positive integer, V/J is the vertex operator algebra for the Wess-Zumino-Witten model.

3. Representation theory:

Modules for a vertex operator algebra V:

$$W = \bigoplus_{n \in \mathbb{C}} W_{(n)}$$

with vertex operators $Y_W(v, z)$ for $v \in V$ satisfying all the properties for vertex operator algebras which still make sense.

Examples:

(i) Free boson:

Let $|k\rangle$ be an eigenstate of a_0 with eigenvalue k. Let W be the space spanned by elements of the form

$$a_{-n_1} \cdots a_{-n_\ell} |k\rangle \qquad n_1 \geq \cdots \geq n_\ell > 0.$$

Vertex operators: $Y_W(|0\rangle, z) = 1_W$

$$Y_W(a_{-1}|0\rangle, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} = a_W(z)$$

$$Y_W(a_{-n_1} \cdots a_{-n_\ell}|0\rangle, z)$$

$$= {}^\circ_\circ \frac{1}{(n_1-1)!} \frac{d^{n_1-1}}{dz^{n_1-1}} a_W(z) \cdots \frac{1}{(n_\ell-1)!} \frac{d^{n_\ell-1}}{dz^{n_\ell-1}} a_W(z) {}^\circ_\circ$$

(ii) Affine Lie algebras:

Let M be finite-dimensional module for $\mathfrak{g}$.

Let W be the space spanned by elements of the form

$$a_{i_1}(-n_1)\cdots a_{i_l}(-n_l)w, \quad w\in M, \quad l=0,1,2,\cdots$$
$$i_1\geq\cdots\geq i_l, \quad n_1\geq\cdots\geq n_l>0.$$

Then W has a structure of module for the vertex operator algebra V constructed from the affine Lie algebra.

Intertwining operator algebra: The algebra of the direct sum of all irreducible modules for a vertex operator algebra.

Let $W^1, \ldots, W^m$ be all irreducible modules for V.

Take $W=\bigoplus_{a=1}^{m} W^m$. Together with the most general type of vertex operators called intertwining operators, W has an algebraic structure called intertwining operator algebras.

Intertwining operator algebras

Intertwining operator algebras give field-theoretic description of nonabelian anyons:

The irreducible modules $W^1, \cdots, W^m$ are the state spaces of quasi-particles. For each element $w^a \in W^a$, there is a set of intertwining operators

$$\mathcal{Y}^{c}_{ab;i}(w^a, z)$$

which are series in powers of z with linear maps from W^b to W^c as coefficients. These operators satisfy

$$\mathcal{Y}^{c}_{ab;i}(w^a, z_1)\,\mathcal{Y}^{b}_{de;j}(w^d, z_2)$$

$$\sim \sum_{f=1}^{m} \sum_{k} \sum_{l} B^{c;f;kl}_{ade;b;i;j}\, \mathcal{Y}^{c}_{df;k}(w^d, z^2)\,\mathcal{Y}^{f}_{ae}(w^a, z_1)$$

where $(B^{c;f;kl}_{ade;b;i;j})$ give representations of the braid groups.

Intertwining operator algebras can also be used to give the fractional quantum Hall wave functions.

Theorem [H]: For a vertex operator algebra V satisfying certain semisimple and finiteness conditions, the category of modules for V is a modular vertex tensor category. In particular, it is a modular tensor category.

Nonsemisimple representation theory and logarithmic conformal field theory:

Logarithmic conformal field theories arose in the study of disorder systems and in the study of dimer model, sandpile and so on.

Theorem [H]: For a vertex operator algebra V satisfying the same finiteness conditions but not the semisimple condition, the category of modules for V is a vertex tensor category. In particular, it is a braided tensor category.