Representations of vertex operator algebras and braided finite tensor categories

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Abstract

We discuss what has been achieved in the past twenty years on the construction and study of a braided finite tensor category structure on a suitable module category for a suitable vertex operator algebra. We identify the main difficult parts in the construction, discuss the methods developed to overcome these difficulties and present some further problems that still need to be solved. We also choose to discuss three among the numerous applications of the construction.

1 Introduction

Finite tensor categories, roughly speaking, are rigid tensor categories satisfying all reasonable finiteness conditions. The category of finite-dimensional representations in positive characteristic of a finite group is an example of a finite tensor category. Such a finite tensor category is symmetric. Another class of examples of finite tensor categories is constructed from representations of quantum groups at roots of unity. In this case, the finite tensor categories are not symmetric but are instead braided. Moreover, some of them are modular tensor categories [T1] [T2] which satisfy additional conditions, including, in particular, semisimplicity (of modules) and a nondegeneracy property. Finite tensor categories have been studied systematically by Etingof and Ostrik [EO]. In general, finite tensor categories are not necessarily semisimple.

In the semisimple case, modular tensor categories play an important role in the study of quantum groups, knot and three-manifold invariants, three-dimensional topological quantum field theories and rational conformal field theories. They arose first in the study of rational conformal field theories. In [MS1] [MS2], Moore and Seiberg derived a set of polynomial equations from an axiom system for a rational conformal field theory. Moreover, after Witten commented that one of their equations was analogous to Mac Lane's coherence property, they demonstrated in these papers a convincing analogy between the theory of such polynomial equations and the theory of tensor categories. Later the mathematical notion of modular tensor category based on the theory of tensor categories was formulated precisely by Turaev in [T1] and [T2]. Examples of modular tensor categories were constructed from representations of quantum groups but the problem of constructing modular tensor categories from candidates for conformal field theories, especially the proofs of the rigidity and the nondegeneracy property, was open for a long time. It was solved by the author in 2005 in [H12] (see also the announcement [H9] and the expositions [H10] and [Le]) using the representation theory of vertex operator algebras, including, in particular, the semisimple tensor product theory developed by Lepowsky and the author [HL1]–[HL5] [HL7] [H1] [H7] and the proof of the Verlinde conjecture by the author [H11], which in turn was based on the constructions and studies of genus-zero and genus-one correlation functions in [H7] and [H8].

The semisimplicity of the module categories in the work above simplifies many constructions and proofs (although, as the required material listed above indicates, the semisimple theory is already substantial and highly nontrivial). But the theory of semisimple finite tensor categories is far from the whole story. The study of finite tensor categories in [EO] is in fact motivated by generalizations of the results in the semisimple case to the general case. In the case of conformal field theories, nonsemisimple generalizations of rational conformal field theories are called "logarithmic conformal field theories." Logarithmic operator product expansions were first studied by Gurarie [G] and logarithmic conformal field theory has been developed rapidly in recent years by both physicists and mathematicians. It has been conjectured that certain candidates for logarithmic conformal field theories should give finite tensor categories. But not even a precise formulation of a general conjecture has been previously given in the literature.

In the present paper, we discuss what has been done and what still needs to be done for the problem of constructing braided finite tensor category structure on a suitable module category for a vertex operator algebra. We shall also identify clearly the main difficulties that we have encountered in establishing these results and the methods that we have developed to overcome them. We will also present problems that still need to be solved and discuss three applications. In particular, we give a general conjecture on the class of vertex operator algebras for which the categories of grading-restricted generalized modules have natural structures of finite tensor categories.

The present paper is organized as follows: We recall briefly some basic notions in the theory of tensor categories in Section 2. In Section 3, we discuss results on the construction of a modular tensor category from modules for a vertex operator algebra satisfying certain natural positive energy, finiteness and reductivity conditions. In Section 4, we discuss results on the construction of a braided tensor category from grading-restricted generalized modules for a vertex operator algebra satisfying certain positive energy and finiteness conditions (but not necessarily the reductivity condition). Conjectures and problems are also discussed in this section. Applications are discussed in Section 5.

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2 Finite tensor categories

We first recall the basic notions in the theory of tensor categories. The purpose is mainly to clarify the terminology because different terminologies exist in the literature. We shall be sketchy in describing these notions. See [T2], [BK] and [EO] for more details.

A *tensor category* is an abelian category with a monoidal category structure. A *braided tensor category* is a tensor category with a natural braiding isomorphism from the tensor product bifunctor to the composition of the tensor product bifunctor and the permutation functor on the direct product of two copies of the category, such that the two standard hexagon diagrams are commutative (see [T2] and [BK]). A tensor category with tensor product bifunctor $\boxtimes$ and unit object V is *rigid* if for every object W in the category, there are right and left dual objects W^* and ${}^*W$ together with morphisms $e_W : W^* \boxtimes W \to V$, $i_W : V \to W \boxtimes W^*$, $e'_W : W \boxtimes {}^*W \to V$ and $i'_W : V \to {}^*W \boxtimes W$ such that the compositions of the morphisms in the sequence

$$\begin{aligned} W \longrightarrow \quad V \boxtimes W \quad &\xrightarrow{i_W \boxtimes 1_W} (W \boxtimes W^*) \boxtimes W \longrightarrow \\ \longrightarrow W \boxtimes (W^* \boxtimes W) &\xrightarrow{1_W \boxtimes e_W} \quad W \boxtimes V \quad \longrightarrow W \end{aligned}$$

and three similar sequences are equal to the identity 1_W on W (see [T2] and [BK]). A rigid braided tensor category together with a twist (a natural isomorphism from the category to itself) satisfying natural conditions (see [T2] and [BK] for the precise conditions) is called a *ribbon category*.

An object W in an abelian category is *simple* if any monomorphism to W is either 0 or an isomorphism. An abelian category is said to be *semisimple* if every object is isomorphic to a direct sum of simple objects. An object W is of *finite length* if there exists a finite sequence of monomorphisms $0 \to W_n \to \cdots \to W_0 = W$ such that the cokernels of these monomorphisms are simple objects.

An object W in an abelian category is *projective* if for any objects W_1 and W_2, any morphism $p : W \to W_2$ and any epimorphism $q : W_1 \to W_2$, there exists a morphism $\tilde{p} : W \to W_1$ such that $q \circ \tilde{p} = p$. Let W be an object of the abelian category. A *projective cover of W in the category* is a projective object U and an epimorphism $p : U \to W$ such that for any projective object W_1 and any epimorphism $q : W_1 \to W$, there exists an epimorphism $\tilde{q} : W_1 \to U$ such that $p \circ \tilde{q} = q$.

A *finite tensor category* is a rigid tensor category such that every object is of finite length, every space of morphisms is finite-dimensional, there are only finitely many inequivalent simple objects, and every simple object has a projective cover. A *braided finite tensor category* is a finite tensor category which is also a braided tensor category.

Example 2.1. The category of finite-dimensional modules for a finite group is a finite tensor category. But the category of finite-dimensional modules for a simple finite-dimensional Lie algebra is not a finite tensor category.

3 The semisimple case

A semisimple ribbon category with finitely many inequivalent simple objects $W_1, \ldots, W_m$ and braiding isomorphism c is a *modular tensor category* if it has the following nondegeneracy property: The $m \times m$ matrix formed by the traces of the morphisms $c_{W_iW_j} \circ c_{W_jW_i}$ in the ribbon category for $i, j = 1, \ldots, m$ is invertible. See [T2] and [BK] for studies of modular tensor categories. In this semisimple case, simple objects are projective covers of themselves.

In 1988, as we mentioned in the introduction, Moore and Seiberg [MS1] [MS2] derived a set of polynomial equations from an axiom system for a rational conformal field theory. Inspired by a comment of Witten, they observed an analogy between the theory of these polynomial equations and the theory of tensor categories. The structures given by these Moore-Seiberg equations were called modular tensor categories by I. Frenkel. However, in the work of Moore and Seiberg, as they commented, tensor product and other structures were neither formulated nor constructed mathematically. Later, Turaev formulated a notion of modular tensor category in [T1] and [T2] and gave examples of such tensor categories from representations of quantum groups at roots of unity based on results obtained by many people on quantum groups and their representations, especially those in the pioneering work [RT1] and [RT2] by Reshetikhin and Turaev on the construction of knot and 3-manifold invariants from representations of quantum groups. The original structures given by the Moore-Seiberg equations can be obtained easily from modular tensor categories in this sense and are analogous to 6-j symbols in the representation theory of Lie algebras. This new and conceptual formulation of the notion of modular tensor category by Turaev led to the conjecture that a rational conformal field theory (if such a structure actually exists) gives naturally a modular tensor category in this sense of Turaev. Moreover, since the construction of rational conformal field theories is harder than the construction of modular tensor categories, a more appropriate problem is to construct directly modular tensor categories in the sense of Turaev from representations of vertex operator algebras, which are substructures of candidates for rational conformal field theories. In fact, it turns out that a series of results obtained by the author and collaborators in the construction of modular tensor categories from representations of vertex operator algebras are necessary steps (but already sufficient for many applications so far) in the author's program of constructing rational conformal field theories from representations of vertex operator algebras.

In physics, there have been known candidates for rational conformal field theories, for example, the Wess-Zumino-Novikov-Witten (WZNW) models and the minimal models. Until 2005, it was a well-known conjecture that the categories of suitable modules for affine Lie algebras of positive integral levels and for the Virasoro algebra of certain central charges and some other categories studied by physicists are indeed modular tensor categories.

The first mathematical construction of a rigid braided tensor category structure from representations of affine Lie algebras was given by Kazhdan and Lusztig [KaL1]–[KaL5]. But these tensor categories are not finite and do not correspond to rational conformal field theories. Under the assumption that the conjectured braided tensor category structure on the category of integrable highest weight modules of a positive integral level for an affine Lie algebra is rigid, Finkelberg [Fi1] [Fi2] showed that this conjectured braided tensor category structure can actually be obtained by transporting the corresponding braided tensor category structure constructed by Kazhdan and Lusztig to this category. There were also the works of Tsuchiya-Ueno-Yamada [TUY] and Beilinson-Feigin-Mazur [BFM], in which the WZNW models and minimal models were studied using algebro-geometric methods. The author was told by experts that the results obtained in these works can be used to construct braided tensor category structures on the corresponding module categories for WZNW models and minimal models. However, the rigidity and the nondegeneracy property of these braided tensor categories cannot be proved using the results and methods in [TUY] and [BFM]. The book [BK] gave a construction of the braided tensor categories for WZNW models but did not give a proof of the rigidity. So even in the case of WZNW and minimal models, the construction of the corresponding modular tensor category structures was still an unsolved open problem before 2005.

On the other hand, starting from 1991, Lepowsky and the author in [HL1]–[HL5], [HL7] and [H1] developed a tensor product theory for modules for a vertex operator algebra satisfying suitable finiteness and reductivity conditions. In particular, the braided tensor category structures for the WZNW and minimal models were constructed in [HL6] and [H3], respectively, using this general theory. In this tensor product theory, the hard part is the construction of the associativity isomorphism and the proof of the commutativity of the pentagon and hexagon diagrams (the main coherence properties). The method used required both algebra (especially the method of formal variables) and complex analysis. For example, one of the main steps in the construction of the associativity isomorphism and the proof of the coherence properties is the proof of the convergence of products of intertwining operators. The formal variable method is necessary because to prove the convergence, we have to prove that the formal series of products of intertwining operators satisfy differential equations with formal series as coefficients. On the other hand, we cannot construct the associativity isomorphism and prove the coherence properties without using some delicate complex analysis. The complex variable method is necessary and no algebraic method can be used to replace it. We not only have to prove the convergence, but also have to deal with very subtle issues for the convergent series. For example, if $\sum_{n\in D} a_n z^n = 0$, is it true that $a_n = 0$ for $n \in D$? If $D = \mathbb{C}$, then the answer is no. If D is a discrete subset of $\mathbb{R}$, then the answer is yes. In the original construction of the associativity isomorphism in [H1], D is assumed to be a strictly increasing sequence in $\mathbb{R}$. In particular, D in this case is discrete.

As in the special cases of WZNW and minimal models, the rigidity of these braided tensor categories was still a conjecture before 2005. We now know that the reason why the rigidity was so hard is that one needs the Verlinde conjecture to prove the rigidity. The proof of the Verlinde conjecture by the author in [H11] requires not only the genus-zero theory (the theory of intertwining operators) but also the genus-one theory (the theory of q-traces of intertwining operators and their modular invariance). Note that the statement of rigidity actually involves only the genus-zero theory but its proof in [H12] needs the genus-one theory. There must be something deep going on here.

The modular invariance result needed in the proofs of the Verlinde conjecture, the rigidity and the nondegeneracy property is the (strong) result for intertwining operators established in [H8]. The modular invariance proved by Zhu [Zhu1] [Zhu2] is only a very special case of this stronger result needed, and is far from enough for these purposes. The paper [H8] not only established the most general modular invariance result in the semisimple case, but also constructed all genus-one correlation functions of the corresponding chiral rational conformal field theories. After Zhu's modular invariance was proved in 1990, the modular invariance for products or iterates of more than one intertwining operator had been an open problem for a long time. In the case of products or iterates of at most one intertwining operator and any number of vertex operators for modules, a straightforward generalization of Zhu's result using the same method gives the modular invariance (see [M]). But for products or iterates of more than one intertwining operator, Zhu's method simply does not work. In fact, in this general case, even the theory of intertwining operators had not been fully developed before 2003. This is one of the main reasons that for about 15 years after 1990, there had been not much progress towards the proof of the rigidity and the nondegeneracy property.

This situation changed in 2003 when the author constructed chiral genus-zero correlation functions using intertwining operators [H7] and proved the modular invariance of the space of q-traces of products and iterates of intertwining operators [H8]. These results are for a simple vertex operator algebra V satisfying the following conditions (for basic definitions and terminology in the theory of vertex operator algebras, see [FLM], [FHL], [LL], [HLZ2] and [H13]):

I V is of positive energy ($V_{(0)} = \mathbb{C}\mathbf{1}$ and $V_{(n)} = 0$ for $n < 0$) and the contragredient V', as a V-module, is equivalent to V.

II Every $\mathbb{N}$-gradable weak V-module is a direct sum of irreducible V-modules. (In fact, the results proved in [H13] imply that this condition can be weakened to the condition that every grading-restricted generalized V-module is a direct sum of irreducible V-modules.)

III V is C_2-cofinite.

Using these results the author proved the Verlinde conjecture in [H11] in 2004 and the rigidity and nondegeneracy property for the braided tensor category of modules for a vertex operator algebra satisfying Conditions I–III in [H12] in 2005[1]. In particular, we have:

Theorem 3.1 ([H12]). *Let V be a simple vertex operator algebra satisfying Conditions I–III above. Then the category of V-modules has a natural structure of modular tensor category.*

Remark 3.2. From this result and [T2], we obtain a modular functor (including all genus) which in particular gives a representation of the mapping class group of a Riemann surface of any genus. On the other hand, for the vertex operator algebra V, chiral correlation functions (or conformal blocks) on Riemann surfaces of any genus can also be defined directly. These chiral correlation functions or conformal blocks also give a representation of the mapping class group of a Riemann surface of any genus. These two representations of the mapping class group of the same Riemann surface are certainly expected to be the same. But the proof of this fact needs a construction of

[1] In fact, the author proved the rigidity and nondegeneracy property before the summer of 2004 and discussed the proof in talks in two conferences in 2004. But the paper [H12] was posted to the arXiv in 2005.

the chiral correlation functions on higher-genus Riemann surfaces from intertwining operators. This construction is still an unsolved problem. In fact, the only unsolved part in this construction is a suitable convergence problem, which is now also the main unsolved problem in the author's program of constructing higher-genus chiral rational conformal field theories from representations of vertex operator algebras satisfying Conditions I–III.

4 The general (not necessarily semisimple) case

The semisimplicity of the categories under consideration, as we have discussed in the preceding section, simplifies many things. But it is not natural to study only the semisimple case. The semisimple case is also not general enough for further developments and applications. A satisfying theory should be a theory in the general case (not necessarily semisimple) and the theory in the semisimple case would become a special case of the general theory.

To achieve this, we need to remove Condition II discussed in the preceding section. In this general case, our strategy is still the same as in the semisimple case. The first step is to construct braided tensor category structures. The second step is to prove the rigidity. The last step is to formulate and prove a generalization of the nondegeneracy property in this general case. The first step has been carried out recently by the author in [H13] using the general logarithmic tensor product theory (in which the semisimple theory in [HL1]–[HL5] [HL7], [H1] and [H7] is indeed a special case) developed by Lepowsky, Zhang and the author [HLZ1] [HLZ2] and a number of results (see [H13] for details) in the representation theory of vertex operator algebras obtained by many people in the past twenty years. The second and third steps will need modular invariance in the general (not necessarily semisimple) case and are one of the research projects that the author is finishing.

In the semisimple case, we know from [H7] and [H12] that the C_1-cofiniteness conditions together with some other minor conditions are enough for the construction of braided tensor category structures while the C_2-cofiniteness conditions are needed only in the proof of the rigidity and the nondegeneracy property. In the general case, for braided tensor category structures, we also do not need the stronger C_2-cofiniteness condition; C_1-cofiniteness conditions together with some other conditions are enough. More precisely, we consider the following conditions for a vertex operator algebra V:

1 There exists a positive integer N such that the difference between the lowest weights of any two irreducible V-modules is less than N and the associative algebra $A_N(V)$ (see [DLM] and the explanations below) is finite dimensional.

2 V is C_1-cofinite in the sense of Li [Li].

3 Irreducible V-modules are $\mathbb{R}$-graded and are C_1-cofinite in the sense of [H7] (or quasi-rational in the sense of Nahm [N]).

Here are some explanations of the conditions above:

1 $A_N(V)$ is the natural generalization of Zhu's algebra (see [Zhu1] and [Zhu2]) by Dong-Li-Mason [DLM].

2 Li's C_1-cofiniteness condition in [Li] can also be defined for modules, but it is mainly useful for the algebra V. We believe that this should be the correct cofiniteness condition on V needed for genus-zero theories.

3 The C_1-cofiniteness condition for modules used in [H7] always holds for V. It was introduced first by Nahm in [N], where such modules are called quasi-rational. Clearly it cannot be used as a condition for the algebra. But this C_1-cofiniteness condition for modules is important for getting differential equations satisfied by intertwining operators. We believe that this is the correct cofiniteness condition for modules needed for genus-zero theories.

4 If V satisfies Conditions I and III in the preceding section, it satisfies Conditions 1–3 above.

The logarithmic tensor product theory for suitable categories of generalized modules for vertex operator algebras developed by Lepowsky, Zhang and the author [HLZ1] [HLZ2] says that if the vertex operator algebra V, generalized V-modules in a suitable category $\mathcal{C}$ and logarithmic intertwining operators satisfy certain conditions, including in particular the assumption that $\mathcal{C}$ is closed under a candidate for the tensor product bifunctor, then the category $\mathcal{C}$ has a natural structure of a braided tensor category. Using this general theory, the following result is obtained by the author:

Theorem 4.1 ([H13]). *Assume that V satisfies Conditions 1–3 above. Then every irreducible V-module has a projective cover and the category of grading-restricted generalized V-modules has a natural structure of a braided tensor category.*

The logarithmic tensor product theory in [HLZ1] and [HLZ2] reduces the proof of this theorem to the proof that the conditions needed to use the logarithmic tensor product theory are all satisfied. In this case, the proof of the convergence and extension properties are similar to the semisimple case. The hard part is the proof that the tensor product of two grading-restricted generalized V-modules is still a grading-restricted generalized V-module, or equivalently, the existence of the tensor product bifunctor. The crucial steps in this proof are the proof of the existence of projective covers using the theory of $A_N(V)$-algebras and the proof of the existence of the tensor product bifunctor using projective covers.

The theorem above does not say anything about the rigidity of the braided tensor category. The author believes that in general the rigidity is not true for these braided tensor categories. But we have the following:

Conjecture 4.2. *Assume that V is a simple vertex operator algebra satisfying Condition I and III in the preceding section; in particular, Conditions 1–3 above hold. Then the braided tensor category given in Theorem 4.1 is rigid.*

This conjecture immediately implies the following:

Corollary 4.3. *Assume that V is a simple vertex operator algebra satisfying Condition I and III in the preceding section. Then the category of grading-restricted generalized V-modules has a natural structure of braided finite tensor category. Moreover, equipped with a natural twisting, it is a ribbon category.*

The proof of Conjecture 4.2 is expected to be similar to Theorem 3.1 in the semisimple case. In that case, the rigidity was proved using the Verlinde conjecture which in turn was proved using the modular invariance and the genus-one associativity established in [H8]. So to prove Conjecture 4.2, we need first to establish a generalization of the modular invariance and the genus-one associativity and then to formulate and prove a generalization of the Verlinde conjecture relating fusion rules and modular transformations, both in this general case. The author has generalized his proofs of the modular invariance and genus-one associativity in the semisimple case to this general case and a generalization of the Verlinde conjecture is expected to be a consequence of these results. As in the semisimple case, the rigidity will be a consequence of these results. We know that in the semisimple case the nondegeneracy property also follows easily from the Verlinde conjecture. Therefore we expect that a proof of a suitable nondegeneracy property in this case can be obtained based on these results.

5 Applications

The tensor category structures discussed in the preceding two sections have many applications. In fact, these tensor category structures had been conjectured to exist for many years. Many results were obtained by physicists and mathematicians based on this postulated existence. These results are of their own importance and interest in different areas of mathematics and physics. Here we choose to discuss three of these applications in this section.

5.1 Open-closed conformal field theories

The first application is on algebras in modular tensor categories and open-closed conformal field theories. Various notions of algebra, including associative algebra, commutative associative algebra, Frobenius algebra and so on, can be defined in a braided tensor category. In the case that the modular tensor category is constructed from the category of modules for a vertex operator algebra in Theorem 3.1, these algebras are equivalent to substructures of open-closed conformal field theories. Therefore we can apply our theory of these modular tensor categories to the study of open-closed conformal field theories.

Comparing tensor categories constructed from representations of vertex operator algebras with tensor categories of vector spaces, we can see that vertex operator algebras in Theorems 3.1 and 4.1 in fact play a role analogous to the coefficient fields of vector spaces. Given a field, we have a symmetric tensor category of vector spaces over this field. The theory of all types of algebras is based on this symmetric tensor category structure. In our case, given a vertex operator algebra satisfying the conditions in Theorems 3.1 or 4.1, we have a braided tensor category. The general theory of algebras in braided tensor categories can now be applied to study algebras in this particular braided tensor category.

Under suitable assumptions, including, in particular, the existence of a modular tensor category structure on the category of modules for a vertex operator algebra, in a series of papers, Felder, Fröhlich, Fuchs, Schweigert, Fjelstad and Runkel [FFFS] [FFRS1] [FFRS2] [FRS1] [FRS2] developed an approach to open-closed conformal field theories using algebras in modular tensor categories and three-dimensional topological field theories constructed from such categories. Because of Theorem 3.1, the results on modular tensor categories and on algebras in these tensor categories in these papers are indeed equivalent to results in open-closed conformal field theory. In this approach, one starts with a modular tensor category and a symmetric Frobenius algebra in this category and constructs correlators for the corresponding open-closed conformal field theory from the category and the algebra.

There is another approach to open-closed conformal field theories developed by the author [H3]–[H6], by Kong and the author [HK1]–[HK3] and by Kong [Ko1]–[Ko3] using directly the representation theory of vertex operator algebras. In this approach, one starts from a vertex operator algebra satisfying suitable conditions and constructs correlation functions of the corresponding open-closed conformal field theory from representations of the vertex operator algebra. Using the tensor product theory developed by Lepowsky and the author [HL1]–[HL5] [HL7] [H1] [H7] and the author's construction of the modular tensor category structures in [H12], Kirillov, Lepowsky and the author [HKL], Kong and the author [HK1] and Kong [Ko1] established the equivalence of suitable algebras in suitable modular tensor categories with suitable vertex operator algebras, open-string vertex operator algebras or full field algebras, respectively. Using all these results, Kong [Ko3] introduced what he called Cardy algebras which are conjectured to be equivalent to open-closed conformal field theories.

In a recent paper [KoR], Kong and Runkel studied the relations between these two approaches discussed above and unified them in a single framework for open-closed conformal field theories.

The results discussed above on algebras in braided tensor categories are all given in the semisimple case, since the corresponding open-closed conformal field theories are rational. But many results can be easily generalized to the general case. It will be interesting to see how much of the theory in the semisimple case can be generalized to the not-necessarily-semisimple case.

5.2 Triplet W-algebras

The second application is on a conjectured equivalence between the braided finite tensor category of grading-restricted generalized modules for a triplet W-algebra and the braided finite tensor category of suitable modules for a restricted quantum group.

The triplet $\mathcal{W}$-algebras of central charge $1-6\frac{(p-1)^2}{p}$ were introduced first by Kausch [Ka1] and have been studied extensively by Flohr [Fl1] [Fl2], Gaberdiel-Kausch [GK1] [GK2], Kausch [Ka2], Fuchs-Hwang-Semikhatov-Tipunin [FHST], Abe [A], Feigin-Gaĭnutdinov-Semikhatov-Tipunin [FGST1] [FGST2] [FGST3], Carqueville-Flohr [CF], Flohr-Gaberdiel [FG], Fuchs [Fu], Adamović-Milas [AM1] [AM2], Flohr-Grabow-Koehn [FGK], Flohr-Knuth [FK] and Gaberdiel-Runkel [GR1] [GR1]. Based on the results of Feigin-Gaĭnutdinov-Semikhatov-Tipunin obtained [FGST1] and of Fuchs-Hwang-Semikhatov-Tipunin [FHST], Feigin, Gaĭnutdinov, Semikhatov and Tipunin conjectured [FGST2] the equivalence mentioned above and proved the conjecture in the simplest $p=2$ case. But their formulation of the conjecture also includes the statement that the categories of modules for the triplet $\mathcal{W}$-algebras considered in their paper are indeed braided tensor categories.

The triplet $\mathcal{W}$-algebras are vertex operator algebras satisfying the positive energy condition (Conditions I) and the C_2-cofiniteness condition (Condition III) but not Condition II. The C_2-cofiniteness condition was proved by Abe [A] in the simplest $p=2$ case and by Carqueville-Flohr [CF] and Adamović-Milas [AM2] in the general case. Condition II was proved to be not satisfied by these vertex operator algebras by Abe [A] in the simplest $p=2$ case and by Fuchs-Hwang-Semikhatov-Tipunin [FHST] and Adamović-Milas [AM2] in the general case. By Theorem 4.1, the category of grading-restricted generalized modules for a triplet W-algebra has a natural braided tensor category structure, not just a quasi-tensor category structure. Assuming that Conjecture 4.2 is true for a triplet W-algebra, we see that by Theorem 4.1 and Corollary 4.3, the category of grading-restricted generalized modules for a triplet W-algebra is a braided finite tensor category. Then the conjecture of Feigin, Gainutdinov, Semikhatov and Tipunin is now purely about the equivalence between the braided finite tensor category of grading-restricted generalized modules for a triplet W-algebra and the braided finite tensor category of suitable modules for the corresponding restricted quantum group. We expect that the logarithmic tensor product theory developed in [HLZ1], [HLZ2] and [H13] will be useful in proving and understanding this conjecture.

5.3 Knot and 3-manifold invariants

Finally we discuss the application to the construction and study of 3-dimensional topological field theories and knot and 3-manifold invariants. In [T2], Turaev constructed 3-dimensional topological field theories and knot and 3-manifold invariants from modular tensor categories. Combining this result of Turaev with Theorem 3.1, we immediately obtain a 3-dimensional topological field theory, and a knot and 3-manifold invariant from a simple vertex operator algebra satisfying the condition I–III needed in Theorem 3.1.

In [He], Hennings constructed a topological invariant of 3-manifolds from quantum groups in a manner similar to the Witten-Reshetikhin-Turaev invariant [W] [RT2]. But in this construction, instead of working with a semisimple part of the category of the representations of a quantum group, Hennings worked directly with the nonsemisimple theory. See also the refinement by Kauffman and Radford in [KaR]. In [KeL], Kerler and Lyubashenko constructed a 3-dimensional extended topological field theory from a modular bounded abelian ribbon category, which is a nonsemisimple generalization of a modular tensor category. In [Ke2], it was shown that underlying the Hennings invariant is exactly the nonsemisimple 3-dimensional extended topological field theories constructed in [KeL] (see also [Ke1]).

If Conjecture 4.2 and thus Corollary 4.3 is proved and the nondegeneracy property is formulated and proved, we might obtain a modular bounded abelian ribbon category in the sense of Kerler and Lyubashenko or a similar structure. It is reasonable to conjecture that we should be able to obtain a 3-dimensional extended topological field theory or some other natural generalization of a semisimple 3-dimensional topological field theory. It will be interesting to see whether we will be able to obtain new knot and 3-manifold invariants in this way.

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