Cofiniteness conditions, projective covers and the logarithmic tensor product theory

Yi-Zhi Huang

Abstract

We construct projective covers of irreducible V-modules in the category of grading-restricted generalized V-modules when V is a vertex operator algebra satisfying the following conditions: 1. V is C_1-cofinite in the sense of Li. 2. There exists a positive integer N such that the differences between the real parts of the lowest conformal weights of irreducible V-modules are bounded by N and such that the associative algebra $A_N(V)$ is finite dimensional. This result shows that the category of grading-restricted generalized V-modules is a finite abelian category over $\mathbb{C}$. Using the existence of projective covers, we prove that if such a vertex operator algebra V satisfies in addition Condition 3, that irreducible V-modules are $\mathbb{R}$-graded and C_1-cofinite in the sense of the author, then the category of grading-restricted generalized V-modules is closed under operations $\boxtimes_{P(z)}$ for $z \in \mathbb{C}^{\times}$. We also prove that other conditions for applying the logarithmic tensor product theory developed by Lepowsky, Zhang and the author hold. Consequently, for such V, this category has a natural structure of braided tensor category. In particular, when V is of positive energy and C_2-cofinite, Conditions 1–3 are satisfied and thus all the conclusions hold.

0 Introduction

In the present paper, we construct projective covers of irreducible V-modules in the category of grading-restricted generalized V-modules and prove that the logarithmic tensor product theory developed in [HLZ1] and [HLZ2] can be applied to this category when V satisfies suitable natural cofiniteness and other conditions (see below). Consequently, for such V, this category is a finite abelian category over $\mathbb{C}$ and has a natural structure of braided tensor category. We refer the reader to [EO], [Fu] and [HLZ2] for detailed discussions on the importance and applications of projective covers and logarithmic tensor products.

For a vertex operator algebra V satisfying certain finite reductivity conditions, Lepowsky and the author have developed a tensor product theory for V-modules in [HL1]–[HL7], [H1] and [H6]. Consider a simple vertex operator algebra V satisfies the following slightly stronger conditions: (i) V is of positive energy (that is, $V_{(n)} = 0$ for $n < 0$ and $V_{(0)} = \mathbb{C}\mathbf{1}$) and V' is equivalent to V as V-modules. (ii) Every $\mathbb{N}$-gradable weak V-module is completely reducible. (iii) V is C_2-cofinite. Then the author further proved in [H11] (see also [H8], [H9]; cf. [Le]) that the category of V-modules for such V has a natural structure of modular tensor category. This result reduces a large part of the representation theory of such a vertex operator algebra to the study of the corresponding modular tensor category and allows us to employ the powerful homological-algebraic methods.

The representation theory of a vertex operator algebra satisfying the three conditions above corresponds to a chiral conformal field theory[1]. Such a chiral conformal field theory has all the properties of the chiral part of a rational conformal field theory. In fact, in view of the results in [H10] and [H11] (see also [H8], [H9]; cf. [Le]), one might even want to define a rational conformal field theory to be a conformal field theory whose chiral algebra is a vertex operator algebra satisfying the three conditions above (see, for example, [Fu]).

In the study of many problems in mathematics and physics, for example, problems in the studies of mirror symmetry, string theory, disorder systems and percolation, it is necessary to study irrational conformal field theories.

[1]In this introduction, though we often mention conformal field theories, we shall not discuss the precise mathematical formulation of conformal field theory and the problem of mathematically constructing conformal field theories. Instead, we use the term conformal field theories to mean certain conformal-field-theoretic structures and results, such as operator product expansions, modular invariance, fusion rules, Verlinde formula and so on. See [H2]–[H10] and [HK1]–[HK2] for the relationship between the representation theory of vertex operator algebras and conformal field theories and the results on the mathematical formulation and construction of conformal field theories in terms of representations of vertex operator algebras.

If we use the definition above as the definition of rational conformal field theory, then to study irrational conformal field theories means that we have to study the representation theory of vertex operator algebras for which at least one of the three conditions is not satisfied.

In the present paper, we study the representation theory of vertex operator algebras satisfying certain conditions weaker than Conditions (i)–(iii) above. In particular, we shall not assume the complete reducibility of $\mathbb{N}$-gradable V-modules or even V-modules. Since the complete reducibility is not assumed, one will not be able to generalize the author's proof in [H6] to show that in this case the analytic extensions of products of intertwining operators still do not have logarithmic terms. Thus in this case, the corresponding conformal field theories in physics must be logarithmic ones studied first by Gurarie in [G]. The triplet $\mathcal{W}$-algebras of central charge $1-6\frac{(p-1)^2}{p}$, introduced first by Kausch [K1] and studied extensively by Flohr [Fl1] [Fl2], Gaberdiel-Kausch [GK1] [GK2], Kausch [K2], Fuchs-Hwang-Semikhatov-Tipunin [FHST], Abe [A], Feigin-Gaĭnutdinov-Semikhatov-Tipunin [FGST1] [FGST2] [FGST3], Carqueville-Flohr [CF], Flohr-Gaberdiel [FG], Fuchs [Fu], Adamović-Milas [AM1] [AM2] Flohr-Grabow-Koehn [FGK] and Flohr-Knuth [FK], are examples of vertex operator algebras satisfying the positive energy condition and the C_2-cofiniteness condition, but not Condition (ii) above. For the proof of the C_2-cofiniteness condition, see [A] for the simplest $p=2$ case and [CF] and [AM2] for the general case. For the proof that Condition (ii) is not satisfied by these vertex operator algebras, see [A] for the simplest $p=2$ case and [FHST] and [AM2] for the general case. A family of $N=1$ triplet vertex operator superalgebras has been constructed and studied recently by Adamović and Milas in [AM3]. Among many results obtained in [AM3] are the C_2-cofiniteness of these vertex operator superalgebras and a proof that Condition (ii) is not satisfied by them.

In [HLZ1] and [HLZ2], Lepowsky, Zhang and the author generalized the tensor product theory of Lepowsky and the author [HL1]–[HL7] [H1] [H6] to a logarithmic tensor product theory for suitable categories of generalized modules for Möbius or conformal vertex algebras satisfying suitable conditions. In this theory, generalized modules in these categories are not required to be completely reducible, not even required to be completely reducible for the operator $L(0)$ (see also [Mil]). The general theory in [HLZ1] and [HLZ2] is quite flexible since it can be applied to any category of generalized modules such that the assumptions in [HLZ1] and [HLZ2] hold. One assumption is that the category should be closed under the $P(z)$-tensor product $\boxtimes_{P(z)}$ for some $z \in \mathbb{C}^\times$. Since the category is also assumed to be closed under the operation of taking contragredient, this assumption is equivalent to the assumption that the category is closed under an operation $\boxslash_{P(z)}$ (see [HLZ1] and [HLZ2]). There are also some other assumptions for the category to be a braided tensor category.

This logarithmic tensor product theory can be applied to a range of different examples. The original tensor product theory developed in [HL1]–[HL7], [H1] and [H2] becomes a special case. We also expect that this logarithmic tensor product theory will play an important role in the study of unitary conformal field theories which do not have logarithmic fields but are not necessarily rational. For a vertex operator algebra associated to modules for an affine Lie algebra of a non-positive integral level, Zhang [Zha1] [Zha2] proved that the category $\mathcal{O}_\kappa$ is closed under the operation $\boxslash_{P(z)}$ by reinterpreting, in the framework of [HLZ1] and [HLZ2], the result proved by Kazhdan and Lusztig [KazL1]–[KazL5] that the category $\mathcal{O}_\kappa$ is closed under their tensor product bifunctor. It is also easy to see that objects in the category $\mathcal{O}_\kappa$ satisfy the C_1-cofiniteness condition in the sense of [H6] (see [Zha2]) and the category $\mathcal{O}_\kappa$ satisfies the other conditions for applying the logarithmic tensor product theory in [HLZ1] and [HLZ2]. As a consequence, we obtain another construction of the Kazhdan-Lusztig braided tensor category structures.

In the present paper, we consider the following conditions for a vertex operator algebra V: 1. V is C_1-cofinite in the sense of Li [Li]. 2. There exists a positive integer N such that the differences of the real parts of the lowest conformal weights of irreducible V-modules are less than or equal to N and such that the associative algebra $A_N(V)$ introduced by Dong-Li-Mason [DLM1] (a generalization of the associative algebra $A_0(V)$ introduced by Zhu [Zhu2]) is finite dimensional. 3. Irreducible V-modules are $\mathbb{R}$-graded and are C_1-cofinite in the sense of [H6]. Note that the first part of Condition 2 is always satisfied when V has only finitely many irreducible V-modules and also note that if V is of positive energy and C_2-cofinite, V satisfies all the three conditions 1–3 (see Proposition 4.1).

When V satisfies Conditions 1 and 2, we prove that a generalized V-module is of finite-length if and only if it is grading restricted and if and only if it is quasi-finite dimensional. Grading-restricted generalized V-modules were first studied by Milas in [Mil] and were called logarithmic V-modules. When V satisfies these two conditions, we prove, among many basic properties of generalized V-modules, that any irreducible V-module W has a projective cover in the category of grading-restricted generalized V-modules. The existence of projective covers of irreducible V-modules is a basic assumption in [Fu] and, since Condition 2 implies that there are only finitely many irreducible V-modules, this existence says that the category of grading-restricted generalized V-modules is a finite abelian category over $\mathbb{C}$.

Using this existence of projective covers, we prove that if V satisfies Conditions 1–3, the category of grading-restricted generalized V-modules is closed under the operation $\boxslash_{P(z)}$ for $z \in \mathbb{C}^{\times}$ and thus is closed under the operation $\boxtimes_{P(z)}$. We also prove that other conditions for applying the logarithmic tensor product theory in [HLZ1] and [HLZ2] developed by Lepowsky, Zhang and the author hold. Consequently, this category has a natural structure of a braided tensor category.

Note that if, in addition, V is simple and the braided tensor category of grading-restricted generalized V-modules is rigid, then the results of the present paper show that this category is a "finite tensor category" in the sense of Etingof and Ostrik [EO] (cf. [Fu]). We shall discuss the rigidity in a future paper since it needs the generalization of the author's result [H7] on the modular invariance of the q-traces of intertwining operators in the finitely reductive case to a result on the modular invariance of the q-(pseudo-)traces of logarithmic intertwining operators in the nonreductive case.

Since a positive energy C_2-cofinite vertex operator algebra satisfies Conditions 1–3, these conclusions hold for such a vertex operator algebra[2].

For many results in this paper, we prove them under weaker assumptions so that these results might also be useful for other purposes. Some of the proofs can be simplified greatly if V satisfies stronger conditions such as the C_2-cofiniteness condition but we shall not discuss these simplifications. Almost all the results in the present paper hold also when the vertex operator algebra is a grading-restricted Möbius vertex algebra.

The present paper is organized as follows: In Section 1, we given basic definitions and properties for generalized modules. We study cofiniteness conditions for vertex operator algebras and their modules in Section 2. We

[2] In a preprint arXiv:math/0309350, incorrect "tensor product" operations in the category of finite length generalized modules for a vertex operator algebra satisfying the C_2-cofiniteness condition were introduced. The counterexamples given in the paper [HLLZ] show that the "tensor product" operations in that preprint do not have the basic properties a tensor product operation must have and are different from the tensor product operations defined in [HLZ1] and [HLZ2] and proved to exist in the present paper (see the paper [HLLZ] for details).

also discuss associative algebras introduced by Zhu and Dong-Li-Mason in this section. In Section 3, we prove that if V satisfies Conditions 1 and 2 above, then any irreducible V-module W has a projective cover in the category of grading-restricted generalized modules. In particular, we see that this category is a finite abelian category over $\mathbb{C}$. In Section 4, we prove that if V satisfies Conditions 1–3, the category of grading-restricted generalized V-modules is closed under the operation $\boxbslash_{P(z)}$ for any $z \in \mathbb{Z}^{\times}$ and thus is closed under the $P(z)$-tensor product $\boxtimes_{P(z)}$. The other assumptions needed in the logarithmic tensor product theory are also shown to hold in this section. Combining with the results of [HLZ1] and [HLZ2], we obtain the conclusion that this category is a braided tensor category.

Acknowledgment The author gratefully acknowledges the partial support from NSF grant DMS-0401302. The author is grateful to Antun Milas and Jim Lepowsky for comments.

1 Definitions and basic properties

In this paper, we shall assume that the reader is familiar with the basic notions and results in the theory of vertex operator algebras. In particular, we assume that the reader is familiar with weak modules, $\mathbb{N}$-gradable weak modules, contragredient modules and related results. Our terminology and conventions follow those in [FLM], [FHL] and [LL]. We shall use $\mathbb{Z}$, $\mathbb{Z}_{+}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{C}^{\times}$ to denote the (sets of) integers, positive integers, nonnegative integers, rational numbers, real numbers, complex numbers and nonzero complex numbers, respectively. For $n \in \mathbb{C}$, we use $\Re(n)$ and $\Im(n)$ to denote the real and imaginary parts of n.

We fix a vertex operator algebra $(V, Y, \mathbf{1}, \omega)$ in this paper. We first recall the definitions of generalized V-module and related notions in [HLZ1] and [HLZ2] (see also [Mil]):

Definition 1.1 A *generalized V-module* is a $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ equipped with a linear map

$$\begin{aligned} Y_W : V \otimes W &\rightarrow W((x)) \\ v &\mapsto Y_W(v, x) \end{aligned}$$

satisfying all the axioms for V-modules except that we do not require W satisfying the two grading-restriction conditions and that the $L(0)$-grading property is replaced by the following weaker version, still called the $L(0)$-*grading property*: For $n \in \mathbb{C}$, the homogeneous subspaces $W_{[n]}$ are the generalized eigenspaces of $L(0)$ with eigenvalues n, that is, for $n \in \mathbb{C}$, $w \in W_{[n]}$, there exists $K \in \mathbb{Z}_+$, depending on w, such that $(L(0) - n)^K w = 0$. *Homomorphisms* (or *module maps*) and *isomorphisms* (or *equivalence*) between generalized V-modules, *generalized V-submodules* and *quotient generalized V-modules* are defined in the obvious way.

The generalized modules we are mostly interested in the present paper are given in the following definition:

Definition 1.2 A generalized V-module W is *irreducible* if there is no generalized V-submodule of W which is neither 0 nor W itself. A generalized V-module is *lower truncated* if $W_{[n]} = 0$ when $\Re(n)$ is sufficiently negative. For a lower-truncated generalized V-module W, if there exists $n_0 \in \mathbb{C}$ such that $W_{[n_0]} \neq 0$ but $W_{[n]} = 0$ when $\Re(n) < \Re(n_0)$ or $\Re(n) = \Re(n_0)$ but $\Im(n) \neq \Im(n_0)$, then we say that W *has a lowest conformal weight*, or for simplicity, W *has a lowest weight*. In this case, n_0 is called the *lowest conformal weight* or *lowest weight* of W, the homogeneous subspace $W_{[n_0]}$ of W is called the *lowest weight space* or *lowest weight space* of W and elements of $W_{[n_0]}$ are called *lowest conformal weight vectors* or *lowest weight vectors* of W. A generalized V-module is *grading restricted* if W is lower truncated and $\dim W_{[n]} < \infty$ for $n \in \mathbb{C}$. A *quasi-finite-dimensional generalized V-module* is a generalized V-module such that for any real number R, $\dim \coprod_{\Re(n) \leq R} W_{[n]} < \infty$. A generalized V-module W is an *(ordinary) V-module* if W is grading restricted and $W_{[n]} = W_{(n)}$ for $n \in \mathbb{C}$, where for $n \in \mathbb{C}$, $W_{(n)}$ are the eigenspaces of $L(0)$ with eigenvalues n. A generalized V-module W is of *length* l if there exist generalized V-submodules $W = W_1 \supset \cdots \supset W_{l+1} = 0$ such that W_i/W_{i+1} for $i = 1, \ldots, l$ are irreducible V-modules. A *finite length generalized V-module* is a generalized V-module of length l for some $l \in \mathbb{Z}_+$. *Homomorphisms* and *isomorphisms* between grading-restricted or finite length generalized V-modules are homomorphisms and isomorphisms between the underlying generalized V-modules.

Remark 1.3 If W is an $\mathbb{R}$-graded lower-truncated generalized V-module or if W is lower-truncated and generated by one homogeneous element, then

W has a lowest weight. In particular, V or any irreducible lower-truncated generalized V-module has a lowest weights.

Remark 1.4 *The category of finite length generalized V-modules is clearly closed under the operation of direct sum, taking generalized V-submodules and quotient generalized V-submodules.*

Proposition 1.5 *The contragredient of a generalized V-module of length l is also of length l.*

Proof. Let $W = W_1 \supset \cdots \supset W_{l+1} = 0$ be a finite composition series of W. Then $(W/W_i)'$ can be naturally embedded into $(W/W_{i+1})'$. We view $(W/W_i)'$ as a generalized V-submodule of $(W/W_{i+1})'$. Then $(W/W_{l+1})' \supset (W/W_l)' \supset \cdots \supset (W/W_1)' = 0$. Moreover $(W/W_{i+1})'/(W/W_i)'$ is equivalent to $(W_i/W_{i+1})'$. Since W_i/W_{i+1} is irreducible, $(W_i/W_{i+1})'$ is irreducible (see [FHL]) and then $(W/W_{i+1})'/(W/W_i)'$ is irreducible. Thus $(W/W_{l+1})' \supset (W/W_l)' \supset \cdots \supset (W/W_1)' = 0$ is a composition series of length l. ∎

Proposition 1.6 *An irreducible grading-restricted generalized V-module is a V-module.*

Proof. Let W be an irreducible generalized V-module. Let $W_{(n)}$ be the subspace of $W_{[n]}$ containing all eigenvectors of $L(0)$ with eigenvalue n. Then $\coprod_{n\in\mathbb{C}} W_{(n)}$ is not 0 and is clearly a V-submodule of W. Since W is irreducible, $W = \coprod_{n\in\mathbb{C}} W_{(n)}$, that is, W is graded by eigenvalues of $L(0)$. ∎

Proposition 1.7 *A generalized V-module of length l is generated by l homogeneous elements whose weights are the lowest weights of irreducible V-modules.*

We omit the proof of Proposition 1.7 since it is standard.

Proposition 1.8 *A quasi-finite-dimensional generalized V-module is grading restricted. An irreducible V-module is quasi-finite dimensional.*

Proof. If a generalized V-module W is quasi-finite dimensional, then for any $n \in \mathbb{C}$,

$$\dim W_{[n]} \leq \dim \coprod_{\Re(m) \leq \Re(n)} W_{[m]} < \infty.$$

If for any $R \in \mathbb{R}$, there exists $n \in \mathbb{C}$ such that $\Re(n) \leq R$ and $W_{[n]} \neq 0$, then clearly $\dim \coprod_{\Re(m) \leq R} W_{[m]} = \infty$ for any $R \in \mathbb{R}$. Contradiction. So W must also be lower truncated.

If W is an irreducible V-module, then there exists $h \in \mathbb{C}$ such that $W = \coprod_{n \in h+\mathbb{N}} W_{(n)}$. Clearly W is quasi-finite dimensional. ∎

Proposition 1.9 *Every finite-length generalized V-module is quasi-finite dimensional.*

Proof. Let $W = W_1 \supset \cdots \supset W_n \supset W_{n+1} = 0$ be a finite composition series of W. Then for $R \in \mathbb{R}$, $\coprod_{\Re(m) \leq R} W_{[m]}$ is linearly isomorphic to

$$\coprod_{i=0}^{n} \coprod_{\Re(m) \leq R} (W_i/W_{i+1})_{[m]}.$$

Since W_i/W_{i+1} for $i = 0 \ldots, n+1$ are V-modules, that is, they are $L(0)$-semisimple and grading restricted, by Proposition 1.8, they are all quasi-finite dimensional. So

$$\dim \coprod_{\Re(m) \leq R} (W_i/W_{i+1})_{[m]} < \infty.$$

Thus $\dim \coprod_{\Re(m) \leq R} W_{[m]} < \infty$. ∎

2 Cofiniteness conditions and associative algebras

Definition 2.1 For a positive integer $n \geq 1$ and a weak V-module W, let $C_n(W)$ be the subspace of W spanned by elements of the form $u_n w$ where $u \in V_+ = \coprod_{n \in \mathbb{Z}_+} V_{(n)}$ and $w \in W$. We say that W is C_n-*cofinite* or satisfies the C_n-*cofiniteness condition* if $W/C_n(W)$ is finite dimensional. In particular, when $n \geq 2$ and $W = V$, we say that the vertex operator algebra V is C_n-cofinite.

Remark 2.2 The C_2-cofiniteness condition for the vertex operator algebra was first introduced (called "Condition C") by Zhu in [Zhu1] and [Zhu2] and was used by him to establish the modular invariance of the space of characters for the vertex operator algebra. The C_1-cofiniteness condition was first introduced by Nahm in [N] and was called quasi-rationality there. In [Li], generalizing Zhu's C_2-cofiniteness condition, Li introduced and studied the C_n-cofiniteness conditions. Note that in the definition above, when $n = 1$ and $W = V$, the cofiniteness condition is always satisfied.

In the case of $n = 1$ and $W = V$, there is in fact another version of cofiniteness condition introduced by Li in [Li]:

Definition 2.3 Let $C_1^a(V)$ be the subspace of V spanned by elements of the form $u_n v$ for $u, v \in V_+ = \coprod_{n\in\mathbb{Z}_+} V_{(n)}$ and $L(-1)v$ for $v \in V$. The vertex operator algebra V is said to be C_1^a*-cofinite* or satisfies the C_1^a*-cofiniteness condition* if $V/C_1^a(V)$ is finite dimensional. For the reason we explain in the remark below, we shall omit the superscript a in the notation, that is, we shall say V is C_1*-cofinite* or satisfies the C_1*-cofiniteness condition* instead of V is C_1^a*-cofinite* or satisfies the C_1^a*-cofiniteness condition*.

Remark 2.4 The C_1-cofiniteness condition in Definition 2.3 can also be defined for lower-truncated generalized V-modules (see [Li]). But it is now clear that this cofiniteness condition is mainly interesting for vertex operator algebras, not for modules, while the C_1-cofiniteness condition in Definition 2.1 is only interesting for weak modules, not for vertex operator algebras. This is the reason why in the rest of the present paper, we shall omit the superscript a (meaning *algebra*) in the term "C_1^a-cofinite" for V, that is, when we say that a vertex operator algebra is C_1-cofinite, we always mean that it is C_1^a-cofinite.

Definition 2.5 A vertex operator algebra V is said to be of *positive energy* if $V_{(n)} = 0$ when $n < 0$ and $V_{(0)} = \mathbb{C}\mathbf{1}$.

Remark 2.6 Positive energy vertex operator algebras are called vertex operator algebras of CFT type in some papers, for example, in [GN], [B] and [ABD]. We use the term "positive energy" in this paper because there are many other conformal-field-theoretic properties of vertex operator algebras and, more importantly, because the term "positive energy" gives precisely what this condition means: If the vertex operator algebra V is the operator product algebra of the meromorphic fields of a conformal field theory, then as an operator acting on this algebra, $\overline{L}(0) = 0$ or, equivalently, $L(0) = L(0) + \overline{L}(0)$. But we know that $L(0) + \overline{L}(0)$ is the energy operator of the conformal field theory. So the condition says that the energy of any non-vacuum state in the vertex operator algebra is positive and the energy of the vacuum is of course 0.

Remark 2.7 Using the $L(-1)$-derivative property, it is easy to see that the C_n-cofiniteness of a weak V-module W implies the C_m-cofiniteness of W for $1 \leq m \leq n$ and, when V is of positive energy, the C_2-cofiniteness of V implies the C_1-cofiniteness of V.

The following result is due to Gaberdiel and Neitzke [GN]:

Proposition 2.8 ([GN]) *Let V be of positive energy and C_2-cofinite. Then V is C_n-cofinite for $n \geq 2$.* ∎

Proposition 2.9 *Assume that all irreducible V-modules satisfy the C_n-cofiniteness condition. Then any finite length generalized module is C_n-cofinite.*

To prove this result, we need:

Lemma 2.10 *For a generalized V-module W_2 and a generalized V-submodule W_1 of W_2, $C_n(W_2/W_1) = (C_n(W_2) + W_1)/W_1$ as subspaces of W_2/W_1.*

Proof. Note that both $C_n(W_2/W_1)$ and $(C_n(W_2) + W_1)/W_1$ consists of elements of the form $\sum_{i=1}^{k} v^{(i)}_{-n} w^{(i)} + W_1$ for $v^{(i)} \in V_+$ and $w^{(i)} \in W_2$. So they are the same. ∎

Lemma 2.11 *If W_1 is a C_n-cofinite generalized V-submodule of a generalized V-module W_2 of finite length such that W_2/W_1 is C_n-cofinite, then W_2 is also C_n-cofinite.*

Proof. Let X_1 be a subspace of W_1 such that the restrictions to X_1 of the projection from W_1 to $W_1/(C_n(W_2) \cap W_1)$ is a linear isomorphism. Let X_2 be a subspace of W_2 such that the restriction to X_2 of the projection from W_2 to $W_2/(C_n(W_2) + W_1)$ is a linear isomorphism. Then we have $W_2 = C_n(W_2) + X_1 + X_2$ and $W_2/C_n(W_2)$ is isomorphic to $X_1 + X_2$.

Since $C_n(W_1) \subset (C_n(W_2) \cap W_1)$, $\dim W_1/(C_n(W_2) \cap W_1) \leq \dim W_1/C_1(W_1)$. By assumption, $\dim W_1/C_n(W_1) < \infty$ and thus $\dim W_1/(C_n(W_2) \cap W_1) < \infty$. By definition X_1 is linearly isomorphic to $W_1/(C_n(W_2) \cap W_1)$. So X_1 is also finite-dimensional.

By definition X_2 is linearly isomorphic to $W_2/(C_n(W_2)+W_1)$ and $W_2/(C_n(W_2)+W_1)$ is linearly isomorphic $(W_2/W_1)/((C_n(W_2)+W_1)/W_1)$. By Lemma 2.10, $(W_2/W_1)/((C_n(W_2)+W_1)/W_1) = (W_2/W_1)/C_n(W_2/W_1)$ which by assumption is finite dimensional. Thus X_2 is also finite-dimensional.

Since $W_2 = C_n(W_2)+X_1+X_2$ and both X_1 and X_2 are finite dimensional, $W_2/C_n(W_2)$ is finite dimensional. ∎

Proof of Proposition 2.9 Since W is of finite length, there exist generalized V-submodules $W = W_1 \supset \cdots \supset W_{n+1} = 0$ such that W_i/W_{i+1} for $i = 1, \dots, n$ are irreducible. By assumption, W_i/W_{i+1} for $i = 1, \dots, n$ and are C_n-cofinite. Using Lemma 2.11 repeatedly, we obtain that W is C_n-cofinite (and in fact W_i for $i = 0, \dots, n$ are also C_n-cofinite). ∎

In the next section, we shall need Zhu's algebra [Zhu1] [Zhu2] and its generalizations by Dong, Li and Mason [DLM1] associated to a vertex operator algebra. Here we study the relation between the cofiniteness conditions and these associative algebras. We first recall those definitions, constructions and results we need from [DLM1].

For $n \in \mathbb{N}$, define a product $*_n$ on V by

$$u *_n v = \sum_{m=0}^{n} (-1)^m \binom{m+n}{n} \mathrm{Res}_x x^{-n-m-1} Y((1+x)^{L(0)+n}u, x)v$$

for $u, v \in V$. let $O_n(V)$ be the subspace of V spanned by elements of the form $\mathrm{Res}_x x^{-2n-2} Y((1+x)^{L(0)+n}u, x)v$ for $u, v \in V$ and of the form $(L(-1)+L(0))u$ for $u \in V$.

Theorem 2.12 ([DLM1]) *The subspace O_n is a two-sided ideal of V under the product $*_n$ and the product $*_n$ induces a structure of associative algebra on the quotient $A_n(V) = V/O_n(V)$ with the identity $\mathbf{1} + O_n(V)$ and with $\omega + O_n(V)$ in the center of $A_n(V)$.* ∎

Remark 2.13 When $n = 0$, $A_0(V)$ is the associative algebra first introduced and studied by Zhu in [Zhu1] and [Zhu2].

We shall need the following result due to Dong-Li-Mason [DLM2] and Miyamoto [Miy]:

Proposition 2.14 ([DLM2], [Miy]) *For $n \in \mathbb{N}$, if V is C_{2n+2}-cofinite, then $A_n(V)$ is finite dimensional. Moreover,* $\dim A_n(V) \leq \dim V/C_{2n+2}(V)$.

Proof. For the first statement, the case $n = 0$ is exactly Proposition 3.6 in [DLM2]. Here we give a straightforward generalization of the proof of Proposition 3.6 in [DLM2]. It is slightly different from the proof of the general case in the proof of Theorem 2.5 in [Miy]. Our proof proves the stronger second statement.

By definition, $C_{2n+2}(V)$ are spanned by elements of the form $u_{-2n-2}v$ for $u, v \in V_+$. Since V is C_{2n+2}-cofinite, there exists a finite dimensional subspace X of V such that $X + C_{2n+2}(V) = V$. We need only show that $X + O_n(V) = V$.

By definition, $O_n(V)$ is spanned by elements of the form

$$\begin{aligned}&\mathrm{Res}_x x^{-2n-2} Y((1+x)^{L(0)+n}u, x)v \\ &\quad = u_{-2n-2}v + \sum_{k\in\mathbb{Z}_+} \binom{\mathrm{wt}\ u+n}{k} u_{k-2n-2}v\end{aligned}$$

for $u, v \in V$ and of the form $(L(-1) + L(0))u$ for $u \in V$. We use induction on the weight of elements of V. For any lowest weight vector $w \in V$, we have $w = \tilde{w} + \sum_{i=1}^{m} u^i_{-2n-2}v^i$ where $\tilde{w} \in X$ and $u^i, v^i \in V$ are homogeneous for $i = 1, \ldots, m$. Since $u^i_{-2n-2}v^i$ are also lowest weight vectors, $u^i_{k-2n-2}v^i = 0$ for $i = 1, \ldots, m$ and $k \in \mathbb{Z}_+$. Then we have

$$u^i_{-2n-2}v^i = \mathrm{Res}_x x^{-2n-2} Y((1+x)^{L(0)+n}u^i, x)v^i$$

for $i = 1, \ldots, m$. We obtain

$$w = \tilde{w} + \sum_{i=1}^{m} \mathrm{Res}_x x^{-2n-2} Y((1+x)^{L(0)+n}u^i, x)v^i \in X + O_n(V).$$

Assume that elements of weights less than l of V are contained in $X+O_n(V)$. Then for w in $V_{(l)}$, there exists homogeneous $\tilde{w} \in X$ and homogeneous $u^i, v^i \in V$ for $i = 1, \ldots, m$ such that $w = \tilde{w} + \sum_{i=1}^{m} u^i_{-2n-2}v^i$. Since the weights of $u^i_{k-2n-2}v^i$ for $i = 1, \ldots, m$ and $k \in \mathbb{Z}_+$ are less than l, by induction assumption,

$$u^i_{k-2n-2}v^i \in X + O_n(V)$$

for $i = 1, \ldots, m$ and $k \in \mathbb{Z}_+$. Thus

$$\begin{aligned} w &= \tilde{w} + \sum_{i=1}^{m} u^i_{-2n-2} v^i \\ &= \tilde{w} + \sum_{i=1}^{m} \mathrm{Res}_x x^{-2n-2} Y((1+x)^{L(0)+n} u^i, x) v^i \\ &\quad - \sum_{i=1}^{m} \sum_{k \in \mathbb{Z}_+} \binom{\mathrm{wt}\ u^i + n}{k} u^i_{k-2n-2} v^i \\ &\in X + O_n(V). \end{aligned}$$

By induction principle, $V = X + O_n(V)$, which implies that $A_n = V/O_n(V)$ is finite dimensional. ∎

Corollary 2.15 ([B]) *If V is C_2-cofinite and of positive energy, then for $n \in \mathbb{N}$, $A_n(V)$ is finite dimensional.*

Proof. This follow immediately from Proposition 2.8 and Proposition 2.14. ∎

Remark 2.16 Corollary 2.15 is in fact an easy special case of Corollary 5.5 in [B] when the weak V-module M there is equal to the vertex operator algebra V.

3 Projective covers of irreducible modules and the finite abelian category structure

Definition 3.1 Let $\mathcal{C}$ be a full subcategory of generalized V-modules. A *projective object* of $\mathcal{C}$ is an object W of $\mathcal{C}$ such that for any objects W_1 and W_2 of $\mathcal{C}$, any module map $p : W \to W_2$ and any surjective module map $q : W_1 \to W_2$, there exists a module map $\tilde{p} : W \to W_1$ such that $q \circ \tilde{p} = p$. Let W be an object of $\mathcal{C}$. A *projective cover of W in $\mathcal{C}$* is a projective object U of $\mathcal{C}$ and a surjective module map $p : U \to W$ such that for any projective object W_1 of $\mathcal{C}$ and any surjective module map $q : W_1 \to W$, there exists a surjective module map $\tilde{q} : W_1 \to U$ such that $p \circ \tilde{q} = q$.

In general, it is not clear whether an object of $\mathcal{C}$ has a projective cover in $\mathcal{C}$. But we have the following:

Proposition 3.2 *If $\mathcal{C}$ is closed under the operations of taking finite direct sums, quotients and generalized submodules and every object in $\mathcal{C}$ is completely reducible in $\mathcal{C}$, then any irreducible generalized V-module in $\mathcal{C}$ equipped with the identity map is a projective cover of the irreducible generalized V-module itself.*

Proof. Let W be an irreducible generalized V-module in $\mathcal{C}$ and $1_W : W \to W$ the identity map. We first show that W is projective. Let W_1 and W_2 be objects of $\mathcal{C}$, $p : W \to W_2$ a module map and $q : W_1 \to W_2$ a surjective module map. Since W_2 is completely reducible and W is irreducible, $p(W)$ is irreducible summand of W_2 and p is an isomorphism from W to $p(W)$. Since W_1 is also completely reducible and q is surjective, one of the irreducible summand of W_1 must be isomorphic to $p(W)$ under q. Let $\tilde{p} : W \to W_1$ be the composition of p and the inverse of the isomorphism from the irreducible summand of W_1 above to $p(W)$. By definition, we have $q \circ \tilde{p} = p$. So W is projective.

Now let W_1 be a projective object of $\mathcal{C}$ and $q : W_1 \to W$ a surjective module map. Let $\tilde{q} = q$. Then $1_W \circ \tilde{q} = q$ and so $(W, 1_W)$ is the projective cover of W. ∎

In this section, we shall construct projective covers of irreducible V-modules in the category of quasi-finite-dimensional generalized V-modules when V satisfies certain conditions. Our tools are the associative algebras $A_n(V)$, $A_n(V)$-modules and their relations with generalized V-modules. We first need to recall the constructions and results from [DLM1].

Let W be a weak V-module and let

$$\Omega_n(W) = \{w \in W \mid u_k w = 0 \text{ for homogeneous } u \in V, \text{wt } u - k - 1 < -n\}.$$

Theorem 3.3 ([DLM1]) *The map $v \mapsto v_{\text{wt } v-1}$ induces a structure of $A_n(V)$-module on $\Omega_n(W)$.* ∎

The space $\hat{V}$ of operators on V of the form u_n for $u \in V$ and $n \in \mathbb{Z}$, equipped with the Lie bracket for operators, is a Lie algebra by the commutator formula for vertex operators. With the grading given by the weights wt $u - n - 1$ of the operators u_n when u is homogeneous, $\hat{V}$ is in fact a

$\mathbb{Z}$-graded Lie algebra. We use $\hat{V}_{(n)}$ to denote the homogeneous subspace of weight n. Then $\hat{V}_{(0)}$ and $P_n(\hat{V}) = \coprod_{k=n+1}^{\infty} \hat{V}_{(-k)} \oplus \hat{V}_{(0)}$ are subalgebras of $\hat{V}$.

Proposition 3.4 ([DLM1]) *The map given by $v_{\mathrm{wt}\ v-1} \mapsto v + O_n(V)$ is a surjective homomorphism of Lie algebras from $\hat{V}_{(0)}$ to $A_n(V)$ equipped with the Lie bracket induced from the associative algebra structure.*

Let E be an $A_n(V)$-module. Then it is also a module for $A_n(V)$ when we view $A_n(V)$ as a Lie algebra. By the proposition above, E is also a $\hat{V}_{(0)}$-module. Let $\hat{V}_{(-k)}$ for $k < n$ act on E trivially. Then E becomes a $P_n(\hat{V})$-module. Let $U(\cdot)$ be the universal enveloping algebra functor from the category of Lie algebras to the category of associative algebras. Then $U(\hat{V}) \otimes_{U(P_n(\hat{V}))} E$ is a $\hat{V}$-module. If we let elements of $E \subset U(\hat{V}) \otimes_{U(P_n(\hat{V}))} E$ to have degree n, then the $\mathbb{Z}$-grading on $\hat{V}$ induces a $\mathbb{N}$-grading on

$$U(\hat{V}) \otimes_{U(P_n(\hat{V}))} E = \coprod_{m \in N} (U(\hat{V}) \otimes_{U(P_n(\hat{V}))} E)(m)$$

such that $U(\hat{V}) \otimes_{U(P_n(\hat{V}))} E$ is a graded $\hat{V}$-module. By the Poincaré-Birkhoff-Witt theorem, $(U(\hat{V}) \otimes_{U(P_n(\hat{V}))} E)(m) = U(\hat{V})_{m-n} E$ for $m \in \mathbb{Z}$, where $U(\hat{V})_{m-n}$ is the homogeneous subspace of $U(\hat{V})$ of degree $m - n$.

For $u \in V$, we define $Y_{M_n}(u, x) = \sum_{k \in \mathbb{Z}} u_k x^{-k-1}$. These operators give a vertex operator map

$$Y_{M_n} : V \otimes U(\hat{V}) \otimes_{U(P_n(\hat{V}))} E \to U(\hat{V}) \otimes_{U(P_n(\hat{V}))} E[[x, x^{-1}]$$

for $U(\hat{V}) \otimes_{U(P_n(\hat{V}))} E$. Let F be the subspace of $U(\hat{V}) \otimes_{U(P_n(\hat{V}))} E$ spanned by coefficients of

$$(x_2+x_0)^{\mathrm{wt}\ u+n} Y_{M_n}(u, x_0+x_2) Y_{M_n}(v, x_2) w - (x_2+x_0)^{\mathrm{wt}\ u+n} Y_{M_n}(Y(u, x_0)v, x_2) w$$

for $u, v \in V$ and $w \in E$ and let

$$M_n(E) = (U(\hat{V}) \otimes_{U(P_n(\hat{V}))} E)/U(\hat{V})F.$$

Theorem 3.5 ([DLM1]) *The vector space $M_n(E)$ equipped with vertex operator map induced from the one for $U(\hat{V}) \otimes_{U(P_n(\hat{V}))} E$ is an $\mathbb{N}$-gradable V-module with an $\mathbb{N}$-grading $M_n(E) = \coprod_{m \in \mathbb{N}} (M_n(E))(m)$ induced from the $\mathbb{N}$-grading of $U(\hat{V}) \otimes_{U(P_n(\hat{V}))} E$ such that $(M_n(E))(0) \neq 0$ and $(M_n(E))(n) = E$.*

The $\mathbb{N}$-gradable V-module $M_n(E)$ satisfies the following universal property: For any weak V-module W and $A_n(V)$-module map $\phi : E \to \Omega_n(W)$, there is a unique homomorphism $\bar{\phi} : M_n(E) \to W$ of weak V-modules such that $\bar{\phi}((M_n(E))(n)) = \phi(E)$.

Remark 3.6 In [DLM1], $M_n(E)$ is used to denote $U(\hat{V}) \otimes_{U(P_n(\hat{V}))} E$ and $\bar{M}_n(E)$ is used to denote what we denote by $M_n(E)$ in this paper. We use $M_n(E)$ instead of $\bar{M}_n(E)$ for simplicity. The reader should note the difference in notations.

This finishes our brief discussion of the material in [DLM1] needed in the present paper.

The constructions and results quoted above are for weak modules or $\mathbb{N}$-gradable V-modules. To apply them to our setting, we need the following:

Proposition 3.7 *If E is finite-dimensional, then $M_n(E)$ is a generalized V-module.*

Proof. By assumption, $(M_n(E))(n) = E$ is finite-dimensional. Since $L(0)$ preserve the homogeneous subspace $(M_n(E))(n)$ of $M_n(E)$, we can view $L(0)$ as an operator on the finite-dimensional vector space $(M_n(E))(n)$. Thus $(M_n(E))(n)$ can be decomposed into a direct sum of generalized eigenspaces of $L(0)$. This decomposition gives $(M_n(E))(n)$ a new grading. Since $M_n(E)$ is generated by $(M_n(E))(n)$, this new grading on $(M_n(E))(n)$ and the $\mathbb{Z}$-grading on V gives a new grading on $M_n(E)$ such that the homogeneous subspaces are generalized eigenspaces of $L(0)$. So $M_n(E)$ becomes a generalized V-module. ∎

Recall the C_1-cofiniteness condition for a vertex operator algebra[3] in the preceding section.

Proposition 3.8 *Let V be C_1-cofinite. If a lower-truncated generalized V-module W is generated by a finite-dimensional $A_n(V)$-submodule E of $\Omega_n(W)$ for some $n \in \mathbb{N}$, then W is quasi-finite dimensional.*

[3]Recall that by our convention, the C_1-cofiniteness condition for a vertex operator algebra means the C_1-cofiniteness condition in the sense of [Li] or the C_1^a-cofiniteness condition.

Proof. We need only discuss the case that W is generated by one homogeneous element $w \in \Omega_n(W)$ which generates a finite dimensional $A_n(V)$-submodule E of $\Omega_n(W)$. Since W is lower-truncated, it is an $\mathbb{N}$-gradable weak V-module. Since V is C_1-cofinite, we know from Theorem 3.10 in [KarL] that there are homogeneous $v^1, \dots, v^m \in V_+$ such that W is spanned by elements of the form $v^{i_1}_{p_1} \cdots v^{i_k}_{p_k} w$, for $k \in \mathbb{N}$, $1 \le i_1, \dots, i_k \le m$, $p_1, \dots, p_k \in \mathbb{Z}$ and $0 \le k_1 \le k_2 \le k$, satisfying

$$\mathrm{wt}\ v^{i_1}_{p_1} \ge \cdots \ge \mathrm{wt}\ v^{i_{k_1}}_{p_{k_1}} > 0,$$

$$0 > \mathrm{wt}\ v^{i_{k_1+1}}_{p_{k_1+1}} \ge \cdots \ge \mathrm{wt}\ v^{i_{k_2}}_{p_{k_2}}$$

and

$$\mathrm{wt}\ v^{i_{k_2+1}}_{p_{k_2+1}} = \cdots = \mathrm{wt}\ v^{i_k}_{p_k} = 0.$$

Since $v^{i_{k_2+1}}_{p_{k_2+1}} \cdots v^{i_k}_{p_k} w \in E$, we see that W is in fact spanned by elements of the form $v^{i_1}_{p_1} \cdots v^{i_{k_2}}_{p_{k_2}} \tilde{w}$, for $k_2 \in \mathbb{N}$, $1 \le i_1, \dots, i_{k_2} \le m$, $p_1, \dots, p_{k_2} \in \mathbb{Z}$, $0 \le k_1 \le k_2$ and $\tilde{w} \in E$, satisfying

$$\mathrm{wt}\ v^{i_1}_{p_1} \ge \cdots \ge \mathrm{wt}\ v^{i_{k_1}}_{p_{k_1}} > 0$$

and

$$0 > \mathrm{wt}\ v^{i_{k_1+1}}_{p_{k_1+1}} \ge \cdots \ge \mathrm{wt}\ v^{i_{k_2}}_{p_{k_2}}.$$

But since W is lower truncated, there exists $n \in \mathbb{N}$ such that for homogeneous $u^1, \dots, u^r \in V$ and $j_1, \dots, j_r \in \mathbb{Z}$ satisfying $\mathrm{wt}\ u^1_{j_1} \cdots u^r_{j_r} < -n$, $u^1_{j_1} \cdots u^r_{j_r} \tilde{w} = 0$ for any element $\tilde{w} \in E$. Thus when $n = 0$, elements of the form $v^{i_1}_{p_1} \cdots v^{i_{k_2}}_{p_{k_2}} \tilde{w}$ for $k_1 \in \mathbb{N}$, $1 \le i_1, \dots, i_{k_2} \le m$, $p_1, \dots, p_{k_2} \in \mathbb{Z}$, and $\tilde{w} \in E$, satisfying

$$\mathrm{wt}\ v^{i_1}_{p_1} \ge \cdots \ge \mathrm{wt}\ v^{i_{k_1}}_{p_{k_2}} > 0$$

span W, and when $n > 0$, elements of the form $v^{i_1}_{p_1} \cdots v^{i_{k_2}}_{p_{k_2}} \tilde{w}$ for $k_2 \in \mathbb{N}$, $1 \le i_1, \dots, i_{k_2} \le m$, $p_1, \dots, p_{k_2} \in \mathbb{Z}$, $0 \le k_1 \le k_2$ and $\tilde{w} \in E$, satisfying

$$\mathrm{wt}\ v^{i_1}_{p_1} \ge \cdots \ge \mathrm{wt}\ v^{i_{k_1}}_{p_{k_1}} > 0$$

and

$$0 > \mathrm{wt}\ v^{i_{k_1+1}}_{p_{k_1+1}} \ge \cdots \ge \mathrm{wt}\ v^{i_k}_{p_k} \ge -n$$

span W. From these inequalities and the fact that $v^{i_{k_1+1}}, \ldots, v^{i_{k_2}} \in V_+$, we see that when $n > 0$,

$$\begin{gathered} k_2 - k_1 \leq n, \\ p_{k_1+1}, \ldots, p_{k_2} > 0, \\ p_{k_1+1} \leq n + \mathrm{wt}\ v^{i_{k_1+1}} - 1 \leq n + \max(\mathrm{wt}\ v^1, \ldots, \mathrm{wt}\ v^m) - 1 \\ \cdots, \\ p_{k_2} \leq n + \mathrm{wt}\ v^{i_{k_2}} - 1 \leq n + \max(\mathrm{wt}\ v^1, \ldots, \mathrm{wt}\ v^m) - 1. \end{gathered}$$

If $\Re(\mathrm{wt}\ (v^{i_1}_{p_1} \cdots v^{i_{k_2}}_{p_{k_2}} \tilde{w})) \leq R$, then we have

$$\mathrm{wt}\ v^{i_1}_{p_1} + \cdots + \mathrm{wt}\ v^{i_{k_1}}_{p_{k_1}} + \mathrm{wt}\ v^{i_{k_1+1}}_{p_{k_1+1}} + \cdots + \mathrm{wt}\ v^{i_{k_2}}_{p_{k_2}} + \Re(\mathrm{wt}\ \tilde{w}) \leq R.$$

Since

$$\mathrm{wt}\ v^{i_{k_1+1}}_{p_{k_1+1}} + \cdots + \mathrm{wt}\ v^{i_{k_2}}_{p_{k_2}} \geq -n,$$

we obtain

$$0 < \mathrm{wt}\ v^{i_1}_{p_1} + \cdots + \mathrm{wt}\ v^{i_{k_1}}_{p_{k_1}} + \Re(\mathrm{wt}\ \tilde{w}) \leq R + n.$$

Combining with the equalities we have above, we also obtain

$$R + n - \Re(\mathrm{wt}\ \tilde{w}) \geq \mathrm{wt}\ v^{i_1}_{p_1} \geq \cdots \geq \mathrm{wt}\ v^{i_{k_1}}_{p_{k_1}} > 0.$$

Thus we have

$$\begin{gathered} k_1 \leq R + n - \Re(\mathrm{wt}\ \tilde{w}) = R + n - \Re(\mathrm{wt}\ w), \\ p_1, \ldots, p_{k_1} \geq -R - n + \Re(\mathrm{wt}\ \tilde{w}) = -R - n + \Re(\mathrm{wt}\ w), \\ p_1 < \mathrm{wt}\ v^{i_1} - 1 \leq \max(\mathrm{wt}\ v^1, \ldots, \mathrm{wt}\ v^m) - 1, \\ \cdots, \\ p_{k_1} < \mathrm{wt}\ v^{i_{k_1}} - 1 \leq \max(\mathrm{wt}\ v^1, \ldots, \mathrm{wt}\ v^m) - 1. \end{gathered}$$

All these inequalities for k_1, $k_2 - k_1$, $p_1, \ldots, p_k$ shows that there are only finitely many such numbers. Since E is finite-dimensional, we conclude that there are only finitely many elements of the form above which span the subspace $\coprod_{\Re(l) \leq R} W_{[l]}$ of W. Thus the generalized V-module W is quasi-finite dimensional. ∎

Since the positive energy property and the C_2-cofiniteness condition for V imply the C_1-cofiniteness condition for V, We have the following consequence:

Corollary 3.9 *Let V be of positive energy and C_2-cofinite. Then any finitely generated lower-truncated generalized V-module is quasi-finite dimensional.*

Proof. For any finitely generated lower-truncated generalized V-module W, there is $N \in \mathbb{N}$ such that the finitely many generators are contained in $\Omega_N(W)$. By Corollary 2.15, $A_n(V)$ is finite dimensional for $n \in \mathbb{N}$. In particular, $A_N(V)$ is finite dimensional. Thus the finitely many generators of W generate a finite-dimensional $A_N(V)$-submodule of $\Omega_N(W)$. Since V is C_1-cofinite, by Proposition 3.8, W is quasi-finite dimensional. ∎

Remark 3.10 This corollary can also be proved directly using a similar argument based on the spanning set for a weak V-module in [B], without using Theorem 3.10 in [KarL].

Corollary 3.11 *Let V be C_1-cofinite. If an $A_n(V)$-module E is finite dimensional, then $M_n(E)$ is quasi-finite dimensional.*

Proof. Since $M_n(E)$ is a lower-truncated generalized V-module generated by the finite-dimensional $A_n(V)$-submodule E of $\Omega_n(M_n(E))$, by Proposition 3.8, it is quasi-finite dimensional. ∎

This result together with Proposition 1.6 gives:

Theorem 3.12 *Let V be C_1-cofinite. For an irreducible $\mathbb{N}$-gradable weak V-module W, if $\Omega_0(W)$ is finite dimensional, then W is an ordinary V-module. If, in addition, $A_0(V)$ is semisimple, then every irreducible $\mathbb{N}$-gradable weak V-module is an ordinary V-module. In particular, if V is C_1-cofinite and $A_0(V)$ is semisimple, then every irreducible lower-truncated generalized V-module is an ordinary V-module.*

Proof. Since $\Omega_0(W)$ is finite dimensional, $M_0(\Omega_0(W))$ is a quasi-finite dimensional generalized V-module by Propositions 3.7 and 3.11. The identity map from $\Omega_0(W)$ to itself extends to a module map from $M_0(\Omega_0(W))$ to W. Since W is irreducible, this module map must be surjective. Thus W as the image of a quasi-finite dimensional generalized V-module must also be a quasi-finite dimensional generalized V-module. By Proposition 1.6, W must be an ordinary V-module.

If $A_0(V)$ is semisimple, every irreducible $A_0(V)$-module is finite dimensional. Let W be an irreducible $\mathbb{N}$-gradable weak V-module. Then $\Omega_0(W)$ is an irreducible $A_0(V)$-module and hence is finite dimensional. From what we have just proved, W must be an ordinary V-module. ∎

The following lemma is very useful:

Lemma 3.13 *Let W be a grading-restricted generalized V-module and W_1 the generalized V-submodule of W generated by a homogeneous element $w \in \Omega_0(W)$. Then there exists a generalized V-submodule W_2 of W_1 such that W_1/W_2 is an irreducible V-module and has the lowest weight* wt w.

Proof. Since $w \in \Omega_0(W)$, W_1 has the lowest weight wt w. Then $(W_1)_{[\mathrm{wt}\ w]}$ is an $A_0(V)$-module. Since W is grading restricted, $(W_1)_{[\mathrm{wt}\ w]}$ is finite dimensional. It is easy to see by induction that any finite-dimensional module for an associative algebra is always of finite length. In particular, $(W_1)_{[\mathrm{wt}\ w]}$ is of finite length. Then there exists a $A_0(V)$-submodule M of $(W_1)_{[\mathrm{wt}\ w]}$ such that $(W_1)_{[\mathrm{wt}\ w]}/M$ is irreducible. It is also clear that any generalized V-submodule of W_1 has a lowest weight. Let W_2 be the sum of all generalized V-submodules of W_1 whose lowest weight spaces either have weights with real parts larger than $\Re(\mathrm{wt}\ w)$ or are contained in M. Since W_2 is a sum of generalized V-submodule of W_1, it is also a generalized V-submodule of W_1. Since sums of elements of M and elements of weights with real parts larger than $\Re(\mathrm{wt}\ w)$ cannot be equal to w, W_2 cannot contain w. Thus W_2 is a proper generalized V-submodule of W_1.

Assume that W_3 is a proper generalized V-submodule of W_1 and contains W_2 as a generalized V-submodule. As a generalized submodule of W_1, W_3 also has a lowest weight. Since $M \subset W_2 \subset W_3$, the lowest weight space of W_3 must contain M. But the lowest weight space of W_3 must be M because otherwise it is an $A_0(V)$-module strictly larger than M but not containing w, contradictory to the fact that $(W_1)_{[\mathrm{wt}\ w]}/M$ is irreducible. We conclude that the lowest weight space of W_3 is in fact M. Thus, by definition of W_2, W_3 is contained in W_2. Since by assumption, W_3 contains W_2, we must have $W_3 = W_2$, proving that W_1/W_2 is irreducible. Since W is grading restricted, so is W_1/W_2. Thus W_1/W_2 is an irreducible V-module. Since $w \in W_1$ but $w \notin W_2$, the lowest weight of W_1/W_2 is wt w. ∎

Using Proposition 1.8 and Lemma 3.13, we obtain the following result:

Proposition 3.14 *Let S be the set of the lowest weights of irreducible V-modules. Assume that for any $N \in \mathbb{Z}$, the set $\{n \in S \mid \Re(n) \leq N\}$ is* *finite. Then a generalized V-module is grading-restricted if and only if it is quasi-finite dimensional. In particular, the conclusion holds if there are only finitely many inequivalent irreducible V-modules.*

Proof. In view of Proposition 1.8, we need only prove that a grading-restricted generalized V-module is quasi-finite dimensional.

Let W be a grading-restricted generalized V-module. We first show that there exists a subset S_0 of S such that $W = \coprod_{n\in S_0+\mathbb{N}} W_{[n]}$. Let w be a homogeneous element of W. We need only show that $\mathrm{wt}\ w \in S+\mathbb{N}$. The generalized V-submodule of W generated by w is also grading restricted. Note that this generalized V-submodule is graded by $\mathrm{wt}\ w+\mathbb{Z}$. Since it is lower truncated and graded by $\mathrm{wt}\ w+\mathbb{Z}$, it must have a lowest weight of the form $\mathrm{wt}\ -k$ for some $k\in\mathbb{N}$. By Lemma 3.13, this lowest weight $\mathrm{wt}\ -k$ must be the lowest weight of an irreducible V-module. So we obtain $\mathrm{wt}\ w-k\in S$ or $\mathrm{wt}\ w\in S+k\subset S+\mathbb{N}$.

Since for any $N\in\mathbb{Z}$, $\{n\in S_0 \mid \Re(n)\leq N\}\subset\{n\in S\mid \Re(n)\leq N\}$, by assumption, $\{n\in S_0\mid \Re(n)\leq N\}$ is also finite for any $N\in\mathbb{Z}$. For any $N\in\mathbb{Z}$, let K_N be a nonnegative integer satisfying $K_N\geq \max\{N-\Re(n)\mid n\in S_0\}$. Then we have

$$\coprod_{n\in S_0+\mathbb{N},\Re(n)\leq N} W_{[n]}\subset \coprod_{i=0}^{K_N}\coprod_{n\in S_0+i,\Re(n)\leq N} W_{[n]}.$$

Since for each n, $W_{[n]}$ is finite dimensional and for $i=0,\dots,K_N$, $\{n\in S_0+i\mid \Re(n)\leq N\}$ is finite, we see that

$$\coprod_{i=0}^{K_N}\coprod_{n\in S_0+i,\Re(n)\leq N} W_{[n]}$$

is finite dimensional. Thus W is quasi-finite dimensional. ∎

Proposition 3.15 *Assume that there exists $N\in\mathbb{Z}_+$ such that the real part of the lowest weight of any irreducible V-module is less than or equal to N. Then we have:*

1. *For any quasi-finite-dimensional generalized V-module W, $\Omega_0(W)$ is finite dimensional.*

2. *Any quasi-finite-dimensional generalized V-module is of finite length.*

Proof. Let w be a homogeneous element of $\Omega_0(W)$ and let W_1 be the generalized V-submodule of W generated by w. Then by Lemma 3.13, there exists a generalized V-submodule W_2 of W_1 such that W_1/W_2 is irreducible and the lowest weight of W_1/W_2 is wt w. By assumption, we have $\Re(\text{wt } w) \leq N$. So $\Omega_0(W) \subset \coprod_{\Re(n)\leq N} W_{[n]}$. Since W is quasi-finite dimensional, $\Omega_0(W)$ must be finite dimensional, proving the first conclusion.

We now prove the second conclusion. Assume that there is a quasi-finite-dimensional generalized V-module W which is not of finite length. We have just proved that $\Omega_0(W)$ is finite dimensional. It is easy to see by induction that any finite-dimensional module for an associative algebra must be of finite length. Since $\Omega_0(W)$ is a finite-dimensional $A_0(V)$-module, there exists a finite composition series

$$M_0 = \Omega_0(W) \supset M_1 \supset \cdots \supset M_k \supset M_{k+1} = 0$$

of Ω_0. Take a homogeneous element $w_1 \in M_0 \setminus M_1$. By Lemma 3.13, we see that wt w_1 is equal to the lowest weight of an irreducible generalized V-module.

Let U_i be the generalized V-module generated by M_i for $i = 0, \dots, k+1$. Since W is not of finite length, there exists i such that U_i/U_{i+1} is not of finite length. Since U_i/U_{i+1} is not of finite length, it is in particular not irreducible. So there exists a nonzero proper generalized V-submodule U of U_i/U_{i+1}. Since U_i/U_{i+1} is quasi-finite-dimensional, the nonzero proper generalized V-submodule U is also quasi-finite-dimensional. Then we can repeat the process of obtaining the homogeneous element w_1 above to obtain a nonzero homogeneous element $\tilde{w} \in \Omega_0(U)$.

Let $W_1 = W$ and let W_2 be the generalized V-submodule of U_i generated by the elements $w \in U_i$ such that $w + W_{i+1} \in \Omega_0(U)$. Then W_2 is a nonzero proper generalized V-submodule of $W_1 = W$. We know that M_i/M_{i+1} is an irreducible $A_0(V)$-submodule of $\Omega_0(U_i/U_{i+1})$ and it generates U_i/U_{i+1}. Thus $\Omega_0(U)$ cannot contain any element of M_i/M_{i+1}. In particular, w_1 cannot be in W_2. Let w_2 be a homogeneous element of W_1 such that $w_2 + W_{i+1} = \tilde{w}$. Since $\tilde{w} \in \Omega_0(U)$, by Lemma 3.13, wt $\tilde{w}$ is equal to the lowest weight of an irreducible V-module. Since wt $w_2 =$ wt $\tilde{w}$, wt w_2 is also the lowest weight of an irreducible V-module.

Repeating the process above, we obtain an infinite sequence $\{W_i\}_{i=1}^{\infty}$ of generalized V-submodules of W such that for $i \in \mathbb{Z}_+$, W_{i+1} is a nonzero proper V-submodule of W_i and a sequence $\{w_i\}_{i=1}^{\infty}$ of homogeneous elements of W such that $w_i \in W_i \backslash W_{i+1}$ and wt w_i is the lowest weights of an irreducible V-module.

The elements w_i, $i \in \mathbb{Z}_+$, are linearly independent. In fact, if they are not. there are $\lambda_j \in \mathbb{C}$ and w_{i_j} in the sequence above for $j = 1, \ldots, l$ such that λ_j are not all zero, $i_1 < \cdots < i_l$ and

$$\sum_{j=1}^{l} \lambda_j w_{i_j} = 0.$$

We can assume that $\lambda_1 \neq 0$. Thus w_{i_1} can be expressed as a linear combination of $w_{i_2}, \ldots, w_{i_l}$. Since $w_i \in W_i$ and $W_{i_j} \subset W_{i_2}$ for $j \geq 2$, we see that $w_{i_2}, \ldots w_{i_l} \in W_{i_2}$. So we see that w_{i_1} is a linear combination of elements of W_{i_2}. So w_{i_1} as a linear combination of elements of W_{i_2} must also be in W_{i_2}. But since $i_1 > i_2$, W_{i_2} is a proper submodule of W_{i_1}. By construction, $w_{i_1} \notin W_{i_1+1} \supset W_{i_2}$. Contradiction. So w_i, $i \in \mathbb{Z}_+$, are linearly independent.

On the other hand, since wt w_i are lowest weights of irreducible V-modules, the real parts of their weights must be less than or equal to N. Thus we have a linearly independent infinite subset of the finite-dimensional vector space $\coprod_{\Re(n) \leq N} W_{[n]}$. Contradiction. So W is of finite length. ∎

Corollary 3.16 *If there are only finitely many inequivalent irreducible V-modules, then every quasi-finite-dimensional generalized V-module, or equivalently, every grading-restricted generalized V-module, is of finite length. In particular, if $A_0(V)$ is finite dimensional, every quasi-finite-dimensional generalized V-module, or equivalently, every grading-restricted generalized V-module, is of finite length.*

Proof. In this case, a generalized V-module is quasi-finite dimensional if and only if it is grading restricted by Proposition 3.14, and the condition in Proposition 3.15 is clearly satisfied. Thus the conclusion is true. ∎

Corollary 3.17 *Assume that V is C_1-cofinite and that there exists $N \in \mathbb{Z}$ such that the real part of the lowest weight of any irreducible V-module is less than or equal to N. Let $n \in \mathbb{N}$ and let E be a finite-dimensional $A_n(V)$-module. Then $M_n(E)$ is quasi-finite dimensional and of finite length.*

Proof. Since V is C_1-cofinite and E is finite dimensional, by Proposition 3.11, $M_n(E)$ is quasi-finite dimensional. By Theorem 3.15, $M_n(E)$ is of finite length. ∎

Proposition 3.18 *Let E be an $A_n(V)$-module. Then any eigenspace or generalized eigenspace of the operator $\omega + O_n(V) \in A_n(V)$ on E is also an $A_n(V)$-module.*

Proof. This result follows immediately from the fact that $\omega + O_n(V)$ is in the center of $A_n(V)$. ∎

Corollary 3.19 *Let E be a finite-dimensional $A_n(V)$-module, $\lambda_1, \dots, \lambda_k$ be the generalized eigenvalues of the operator $\omega + O_n(V)$ on E and $E_{(\lambda_1)}, \dots, E_{(\lambda_k)}$ the generalized eigenspaces of eigenvalues $\lambda_1, \dots, \lambda_k$, respectively. Then $E_{(\lambda_i)}$ for $i = 1, \dots, k$ are $A_n(V)$-modules and $E = \coprod_{i=1}^{k} E_{(\lambda_i)}$.*

When $M = A_n(V)$, we have:

Proposition 3.20 *Assume that $A_n(V)$ is finite dimensional. Let $\lambda_1, \dots, \lambda_k$ be the generalized eigenvalues of the operator $\omega + O_n(V)$ on $A_n(V)$. Then the generalized eigenspaces $(A_n(V))_{(\lambda_i)}$ for $i = 1, \dots, k$ are projective $A_n(V)$-modules and $A_n(V) = \coprod_{i=1}^{k} (A_n(V))_{(\lambda_i)}$.*

Proof. Since $\omega + O_n(V)$ is in the center of $A_n(V)$, $(A_n(V))_{(\lambda_i)}$ is an $A_n(V)$-module. Since $(A_n(V))_{(\lambda_i)}$ is a direct summand of the free $A_n(V)$-module $A_n(V)$ itself, it is projective. ∎

If an $A_n(V)$-module E is a generalized eigenspace of the operator $\omega + O_n(V)$ with eigenvalue λ, we call E a *homogeneous $A_n(V)$-module of weight λ*.

Proposition 3.21 *Let E be a homogeneous $A_n(V)$-module of weight λ. Then $M_n(E) = \coprod_{m \in \mathbb{N}} (M_n(E))_{[\lambda - n + m]}$ and $(M_n(E))_{[\lambda]} = E$, where $(M_n(E))_{[\lambda - n + m]}$ is the generalized eigenspace of $L(0)$ with eigenvalue $\lambda - n + m$.*

Proof. The operator $L(0)$ acts on E as $\omega + O_n(V)$ and thus elements of E has weight λ. The conclusion of the proposition now follows from the construction of $M_n(E)$. ∎

Theorem 3.22 *Assume that V is C_1-cofinite and that there exists a positive integer N such that $|\Re(n_1) - \Re(n_2)| \leq N$ for the lowest weights n_1 and n_2 of any two irreducible V-modules. Let E be a finite-dimensional homogeneous projective $A_N(V)$-module whose weight is equal to the lowest weight of an irreducible V-module. Then $M_N(E)$ is projective in the category of finite length generalized V-modules.*

Proof. By Propositions 3.17, we know that $M_N(E)$ is of finite length.

Let W_1 and W_2 be finite length generalized V-modules, $f : M_N(E) \to W_2$ a module map and $g : W_1 \to W_2$ a surjective module map. Let the weight of E be n_E. Then n_E is the lowest weight of some irreducible V-module. By Proposition 3.21, E as a subspace of $M_N(E)$ is also of weight n_E. By assumption, the set of the real parts of the lowest weights of irreducible V-modules must be bounded and the real parts of weights of any finite length generalized V-module must be larger than or equal to the real part of the lowest weight of an irreducible V-module. Thus we see that E is in $\Omega_N(M_N(E))$. Then $f(E)$ is in $\Omega_N(W_2)$. Also $(W_1)_{(n_E)}$ and $(W_2)_{(n_E)}$ must be in $\Omega_N(W_1)$ and $\Omega_N(W_2)$, respectively. Thus $(W_1)_{(n_E)}$ and $(W_2)_{(n_E)}$ are $A_N(V)$-modules. Since f and g are module maps and g is surjective, we have $f(E) \subset (W_2)_{(n_E)}$ and $g((W_1)_{(n_E)}) = (W_2)_{(n_E)}$. So the restriction $\alpha = f|_E : E \to (W_2)_{(n_E)}$ and $\beta = g|_{(W_1)_{(n_E)}} : (W_1)_{(n_E)} \to (W_2)_{(n_E)}$ of f and g to E and $(W_1)_{(n_E)}$, respectively, are $A_N(V)$-module maps and β is surjective. Since E is projective, there exists an $A_N(V)$-module map $\tilde{\alpha} : E \to (W_1)_{(n_E)}$ such that $\beta \circ \tilde{\alpha} = \alpha$. By Theorem 3.5, there exists a unique V-module map $\tilde{f} : M_N(E) \to W_1$ extending $\tilde{\alpha}$. Since $g \circ \tilde{f} : M_N(E) \to W_2$ and $f : M_N(E) \to W_2$ are extensions of $\beta \circ \tilde{\alpha}$ and α, respectively, and $\beta \circ \tilde{\alpha} = \alpha$, by the uniqueness in Theorem 3.5, we have $g \circ \tilde{f} = f$, proving that $M_N(E)$ is projective. ∎

The following two theorems are the main results of this section:

Theorem 3.23 *Assume that V is C_1-cofinite and that there exists a positive integer N such that $|\Re(n_1) - \Re(n_2)| \leq N$ for the lowest weights n_1 and n_2 of any two irreducible V-modules and $A_N(V)$ is finite dimensional. Then any irreducible V-module W has a projective cover in the category of finite length generalized V-modules.*

Proof. Let W be an irreducible V-module with the lowest weight n_W. Then $W_{(n_W)}$ is a finite dimensional $A_N(V)$-module generated by an arbitrary element. Since $A_N(V)$ is finite dimensional, by Proposition 3.20, $A_N(V) =$

$\coprod_{i=1}^{k}(A_N(V))_{(\lambda_i)}$ and $(A_N(V))_{(\lambda_i)}$ for $i = 1, \dots, k$ are projective $A_N(V)$-modules. Since $W_{(n_W)}$ is an $A_N(V)$-module generated by one element, we have a surjective $A_N(V)$-module map from $A_N(V)$ to $W_{(n_W)}$. But the $A_N(V)$-module map must preserve the weights, so under the $A_N(V)$-module map, $(A_N(V))_{(\lambda_i)} = 0$ if $\lambda_i \neq n_W$. Thus we have a surjective $A_N(V)$-module map from $(A_N(V))_{(n_W)}$ to $W_{(n_W)}$.

Now we decompose the finite-dimensional $A_N(V)$-module $(A_N(V))_{(n_W)}$ into a direct sum of indecomposable $A_N(V)$-modules. Take an indecomposable $A_N(V)$-module E in the decomposition such that the image of E under the $A_N(V)$-module map from $(A_N(V))_{(n_W)}$ to $W_{(n_W)}$ is not 0. Since E is a direct summand of the projective $A_N(V)$-module $(A_N(V))_{(n_W)}$, E is also projective. Since W is an irreducible V-module, $W_{(n_W)}$ is an irreducible $A_N(V)$-module. Thus the image of E under the $A_N(V)$-module map from $(A_N(V))_{(n_W)}$ to $W_{(n_W)}$ must be equal to $W_{(n_W)}$. We denote the restriction to E of the $A_N(V)$-module map from $(A_N(V))_{(n_W)}$ to $W_{(n_W)}$ by α. Then α is surjective.

We first prove that (E, α) is a projective cover of $W_{(n_W)}$ in the category of $A_N(V)$-modules. Let E_1 be an $A_N(V)$-submodule of E such that $E_1 + \ker \alpha = E$. Let $e_1 : E_1 \to E$ be the embedding map. Then $\alpha_1 = \alpha \circ e_1$ where $\alpha_1 = \alpha|_{E_1}$ is the restriction of α to E_1. Since α is surjective, α_1 must also be surjective. Since E is projective and α_1 is surjective, there exists an $A_N(V)$-module map $\beta_1 : E \to E_1$ such that $\alpha_1 \circ \beta_1 = \alpha$. Also we have $\alpha_1 \circ \beta_1 \circ e_1 = \alpha \circ e_1 = \alpha_1$. Let $E_2 = \beta_1(E_1) \subset E_1$ and $\alpha_2 = \alpha_1|_{E_2} = \alpha|_{E_2} : E_2 \to W_{(n_W)}$. Then

$$\begin{aligned}
\alpha_2(E_2) &= (\alpha_1 \circ \beta_1)(E_1) \\
&= (\alpha_1 \circ \beta_1 \circ e_1)(E_1) \\
&= \alpha_1(E_1) \\
&= W_{(n_W)},
\end{aligned}$$

that is, α_2 is surjective. Since E is projective, we have an $A_N(V)$-module map $\beta_2 : E \to E_2$ such that $\alpha_2 \circ \beta_2 = \alpha$. Let $e_2 : E_2 \to E$ be the embedding from E_2 to E. Then we have $p \circ e_2 = \alpha_2$ and so we also have $\alpha_2 \circ \beta_2 \circ e_2 = \alpha \circ e_2 = \alpha_2$. Repeating this procedure, we obtain a sequence of $A_N(V)$-modules $E \supset E_1 \supset E_2 \supset \cdots$ and $A_N(V)$-module maps $\beta_i : E \to E_i$ and $\alpha_i : E_i \to E$ for $i \in \mathbb{Z}_+$ such that $E_{i+1} = \beta_i(E_i)$, $\alpha_i \circ \beta_i = \alpha$, $\alpha \circ e_i = \alpha_i$ and $\alpha_i \circ \beta_i \circ e_i = \alpha_i$, where $e_i : E_i \to E$ for $i \in \mathbb{Z}_+$ are the embedding maps from E_i to E. Since E is of finite length, there exists $l \in \mathbb{Z}_+$ such that $E_{l+1} = E_l$. Thus $E_l = E_{l+1} = \beta_l(E_l)$ and so $\beta_l \circ e_l : E_l \to E_l$ is surjective.

We now show that $\gamma = \beta_l \circ e_l$ must be an isomorphism. In fact, if not, then $\ker g \neq 0$. Let $K = \ker g$ and $\gamma^{-q}(K) = \gamma^{-1}(g^{-(q-1)}(K))$ for $q \in \mathbb{N}$. We have a sequence $K \subset \gamma^{-1}(K) \subset \gamma^{-2}(K) \subset \cdots$ of $A_N(V)$-submodules of E. Since E is of finite length, there must be $q \in \mathbb{N}$ such that $\gamma^{-(q+1)}(K) = \gamma^{-q}(K)$. Applying γ^{q+1} to both sides, we obtain $K = 0$, proving that γ is injective. Since γ is also surjective, it is an isomorphism.

Thus $(g^{-1} \circ \beta_l) \circ e_l = 1_{E_l}$, the identity map on E_l. This shows that $E = E_l \oplus \ker(g^{-1} \circ \beta_l)$. Since E is indecomposable and $E_l \neq 0$, we must have $\ker(g^{-1} \circ \beta_l) = 0$ and $E = E_l$. Thus $E_1 = E$, proving that (E, α) is a projective cover of $W_{(n_W)}$.

By Theorem 3.22, $M_N(E)$ is a projective finite length generalized V-module and by Theorem 3.5, there is a unique module map $p : M_N(E) \to W$ extending the $A_N(V)$-module map $\alpha : E \to W_{(n_W)}$ above. Since W is irreducible and $p \neq 0$, p must be surjective. If there is another projective finite length generalized V-module W_1 and a surjective module map $q : W_1 \to W$, there must be a module map $\tilde{q} : W_1 \to M_N(E)$ such that $p \circ \tilde{q} = q$. Since $(M_N(E))_{[n_W]} = E$, we see that $\tilde{q}((W_1)_{[n_W]}) \subset E$. Since q is surjective, $q((W_1)_{[n_W]}) = W_{(n_W)}$. Thus

$$\begin{aligned}
\alpha(\tilde{q}((W_1)_{[n_W]})) &= p(\tilde{q}((W_1)_{[n_W]})) \\
&= q((W_1)_{[n_W]}) \\
&= W_{(n_W)}.
\end{aligned}$$

This implies that $\tilde{q}((W_1)_{[n_W]}) + \ker \alpha = E$. Since (E, α) is a projective cover of $W_{(n_W)}$, we must have $\tilde{q}((W_1)_{[n_W]}) = E$. Since $M_N(E)$ is generated by E, the image of W_1 under $\tilde{q}$ is $M_N(E)$. So $\tilde{q}$ is surjective. Thus $(M_N(E), p)$ is a projective cover of W. ∎

Recall from [EO] that an abelian category over $\mathbb{C}$ is called *finite* if every object is of finite length, every space of morphisms is finite dimensional, there are only finitely many inequivalent simple objects and every simple object has a projective cover.

Theorem 3.24 *Assume that V is C_1-cofinite and that there exists a positive integer N such that $|\Re(n_1) - \Re(n_2)| \leq N$ for the lowest weights n_1 and n_2 of any two irreducible V-modules and $A_N(V)$ is finite dimensional. Then the category of grading-restricted generalized V-modules is a finite abelian category over $\mathbb{C}$.*

Proof. By Propositions 3.14 and 3.15, every object in the category is of finite length. By Theorem 3.23, every object in the category has a projective cover. Since $A_N(V)$ is finite dimensional, there are only finitely many inequivalent irreducible (simple) objects.

Let W_1 and W_2 be grading-restricted generalized V-modules. Then they are of finite length. By Proposition 1.7, they are both finitely generated. In particular, W_1 is generated by elements of weights whose real parts are less than or equal to some real number R. So module maps from W_1 to W_2 are determined uniquely by their restrictions to the subspace $\coprod_{\Re(n)\leq R}(W_1)_{[n]}$ of W_1. Since W_1 and W_2 are also quasi-finite dimensional and module maps preserve weights, these restrictions are maps between finite-dimensional vector spaces and in particular, the space of these restrictions is finite dimensional. Thus the space of module maps from W_1 to W_2 is finite dimensional. ∎

4 The tensor product bifunctors and braided tensor category structure

We consider a vertex operator algebra V satisfying the following conditions:

1. $V$is C_1-cofinite[4].
2. There exists a positive integer N such that $|\Re(n_1)-\Re(n_2)|\leq N$ for the lowest weights n_1 and n_2 of any two irreducible V-modules and such that $A_N(V)$ is finite dimensional.
3. Every irreducible V-module is $\mathbb{R}$-graded and C_1-cofinite[5].

Proposition 4.1 *If V is C_2-cofinite and of positive energy, then V satisfies Conditions 1–3 above.*

Proof. By Remark 2.7, V is C_1-cofinite. By Corollary 2.15, $A_n(V)$ are finite-dimensional for $n\in\mathbb{N}$. In particular, there are only finitely many inequivalent irreducible V-modules. So Condition 2 is satisfied. From Theorem 5.10

[4] Recall our convention that by a the vertex operator algebra V being C_1-cofinite, we mean that V is C_1-cofinite in the sense of [Li] or C_1^a-cofinite.

[5] Recall that by a V-module being C_1-cofinite, we mean that the V-module is C_1-cofinite in the sense of [H6] but not necessarily C_1^a-cofinite or C_1-cofinite in the sense of [Li].

in [Miy] and Proposition 5.3 in [ABD], we know that every irreducible V-modules is is $\mathbb{Q}$-graded and C_2-cofinite and thus is in particular C_1-cofinite. ∎

In this section, we shall assume that V satisfies Conditions 1–3 above. By Proposition 4.1, if V is C_2-cofinite and of positive energy, this assumption is satisfied. Thus the results in this section hold if V is C_2-cofinite and of positive energy.

Proposition 4.2 *For a vertex operator algebra V satisfying Conditions 1 and 2 above, every irreducible $\mathbb{N}$-gradable weak V-module is an irreducible V-module and there are only finitely many inequivalent irreducible V-modules. In particular, every lower-truncated irreducible generalized V-module is an irreducible V-module.*

Proof. Since $A_N(V)$ is finite dimensional, $A_0(V)$ as the image of a surjective homomorphism from $A_N(V)$ to $A_0(V)$ (see Proposition 2.4 in [DLM1]) must also be finite dimensional. Thus there are only finitely many inequivalent irreducible $A_0(V)$-modules. By Theorem 2.2.2 in [Zhu2], there are only finitely many inequivalent irreducible V-modules. For an irreducible $\mathbb{N}$-gradable weak V-module W, $\Omega_0(W)$ is an irreducible $A_0(V)$-module by Theorem 2.2.2 in [Zhu2] and thus must be finite dimensional. By Proposition 3.12, every irreducible $\mathbb{N}$-gradable weak V-module is an irreducible V-module. ∎

Proposition 4.3 *For a vertex operator algebra V satisfying Conditions 1 and 2 above, the category of grading-restricted generalized V-modules, the category of quasi-finite-dimensional generalized V-modules and the category of finite length generalized V-modules are the same.*

Proof. From Propositions 1.9 and 3.15, we see that the category of grading-restricted generalized V-modules is the same as the category of finite length generalized V-modules and from Propositions 1.8 and 3.14, we see that the category of grading-restricted generalized V-modules is the same as the category of quasi-finite-dimensional generalized V-modules. ∎

We use $\mathcal{C}$ to denote the category in the proposition above. In this section, we shall use the results obtained in the preceding sections to show that the category $\mathcal{C}$ satisfies all the assumptions to use the logarithmic tensor product theory in [HLZ1] and [HLZ2]. Thus by the results of this paper and the theory developed in [HLZ1] and [HLZ2], we shall see that $\mathcal{C}$ has a natural structure of braided tensor category.

Proposition 4.4 *Let W_1, W_2 and W_3 be objects of $\mathcal{C}$. Then the fusion rule $N_{W_1W_2}^{W_3}$ is finite.*

Proof. By the definition of the category $\mathcal{C}$, W_1, W_2 and W_3 are of finite length. Since every irreducible V-modules is C_1-cofinite, by Proposition 2.9, W_1, W_2 and W_3 are also C_1-cofinite. On the other hand, we also know that W_1, W_2 and W_3 are quasi-finite dimensional. Now the proof of Theorem 3.1 in [H6] still works when W_1, W_2 and W_3 are quasi-finite-dimensional generalized V-modules. So the fusion rule $N_{W_1W_2}^{W_3}$ is finite. ∎

Recall the definition of $W_1 \boxtimes_{P(z)} W_2$ for two generalized V-modules W_1 and W_2 in $\mathcal{C}$ in [HLZ1] and [HLZ2]. Note that $W_1 \boxtimes_{P(z)} W_2$ depends on our choice of $\mathcal{C}$.

Theorem 4.5 *Let W_1 and W_2 be objects in $\mathcal{C}$. Then $W_1 \boxtimes_{P(z)} W_2$ (defined using the category $\mathcal{C}$) is also in $\mathcal{C}$.*

Proof. By the definition of $W_1 \boxtimes_{P(z)} W_2$, it is a sum of finite length generalized V-submodules of $(W_1 \otimes W_2)^*$. We denote the set of these finite length generalized V-modules appearing in the sum and their finite sums (still finite length generalized V-modules) by S. We want to prove that $W_1 \boxtimes_{P(z)} W_2$ is also of finite length.

Assume that $W_1 \boxtimes_{P(z)} W_2$ is not of finite length. Then take any finite length generalized V-module M_1 in S. Since $W_1 \boxtimes_{P(z)} W_2$ is not of finite length, M_1 is not equal to $W_1 \boxtimes_{P(z)} W_2$. So we can find M_2 in S such that $M_1 \subset M_2$ but $M_1 \neq M_2$. For example, we can take any finite length generalized V-module in S which is not a generalized submodule of M_1 and then take M_2 to be the sum of M_1 and this finite length generalized V-module in S. Since $W_1 \boxtimes_{P(z)} W_2$ is not of finite length, this procedure can continue infinitely and we get a sequence $\{M_i\}_{i\in\mathbb{Z}_+}$ of finite length generalized V-modules in S such that $M_i \subset M_j$ when $i \leq j$ but $M_i \neq M_j$ when $i \neq j$. For every $i \in \mathbb{Z}_+$, since M_i is of finite length, M_i/M_{i+1} is also of finite length. Thus there exists a generalized V-submodule N_i of M_i such that $M_{i+1} \subset N_i$ and $(M_i/M_{i+1})/(N_i/M_{i+1})$ is an irreducible V-module, or equivalently, M_i/N_i is an irreducible V-module. (Note that by our assumption, every lower-truncated irreducible generalized V-module is an irreducible V-module.) Since there are only finitely many equivalence classes of irreducible V-modules, infinitely many of the irreducible V-modules M_i/N_i for $i \in \mathbb{Z}_+$ are isomorphic. Let $\{M_i/N_i\}_{i\in B}$ be an infinite set such that M_i/N_i for $i \in B \subset \mathbb{Z}_+$ are isomorphic to an irreducible V-module M.

By Theorem 3.23, there exists a projective cover (P, p) of M in the category $\mathcal{C}$. Since M_i/N_i are isomorphic to M, there exists surjective module map $\pi_i : M_i \to M$ whose kernel is N_i. Since P is projective, there exist module maps $p_i : P \to M_i$ such that $\pi_i \circ p_i = p$. If $p_i(P) \subset N_i$, then $p(P) = \pi_i(p_i(P)) = 0$ since the kernel of π_i is N_i. This is contradictory to the surjectivity of p. Thus $p_i(P)$ is not a generalized V-submodule of N_i.

The embedding $M_i \to W_1 \boxtimes_{P(z)} W_2$ give $P(z)$-intertwining maps J_i of types $\binom{M_i'}{W_1 W_2}$ as follows: For $m_i \in M_i$, $w_1 \in W_1$ and $w_2 \in W_2$,

$$\langle m_i, J_i(w_1 \otimes w_2)\rangle = m_i(w_1 \otimes w_2).$$

Let $I_i = p_i' \circ J_i$. Then I_i are $P(z)$-intertwining maps of types $\binom{P'}{W_1 W_2}$. We now show that these intertwining maps are linearly independent.

Assume that there exist $\lambda_i \in \mathbb{C}$, of which only finitely many are possibly not 0, such that

$$\sum_{i\in B} \lambda_i I_i = 0.$$

For $w \in P$, $w_1 \in W_1$ and $w_2 \in W_2$, we obtain

$$\begin{aligned} 0 &= \left\langle w, \left(\sum_{i\in B} \lambda_i I_i\right)(w_1 \otimes w_2)\right\rangle \\ &= \sum_{i\in B} \lambda_i \langle w, (p_i'(J_i(w_1 \otimes w_2))\rangle \\ &= \sum_{i\in B} \lambda_i \langle p_i(w), J_i(w_1 \otimes w_2)\rangle \\ &= \sum_{i\in B} \lambda_i (p_i(w))(w_1 \otimes w_2). \end{aligned}$$

Since w_1 and w_2 are arbitrary,

$$\sum_{i\in B} \lambda_i p_i(w) = 0.$$

Since $w \in P$ is also arbitrary,

$$\sum_{i \in B} \lambda_i p_i = 0.$$

If there exist $i \in B$ such that $\lambda_i \neq 0$. Then there exists $i_0 \in B$ which is the smallest in B such that $\lambda_{i_0} \neq 0$. We see that p_{i_0} can be written as a linear combination of p_i, $i \in B$ and $i > i_0$. We know that $p_i(P)$ is in M_i and $p_{i_0}(P)$ is in M_{i_0} but not in N_{i_0} which contains but is not equal to M_i for $i > i_0$. Contradiction. So $\lambda_i = 0$ for all $i \in B$, proving the linear independence of I_i.

Since I_i for $i \in B$ are linearly independent, the dimension of intertwining maps of type $\binom{P'}{W_1 W_2}$ is infinite and thus the fusion rule $N^P_{W_1 W_2} = \infty$. Since W_1, W_2 and P are in $\mathcal{C}$ and so must be quasi-finite dimensional, by Proposition 4.4, $N^P_{W_1 W_2} < \infty$. Contradiction. Thus $W_1 \boxtimes_{P(z)} W_2$ must be of finite length and thus is in the category $\mathcal{C}$. ∎

Corollary 4.6 *The category $\mathcal{C}$ is closed under $P(z)$-tensor products.* ∎

We now verify the other assumptions in [HLZ1] and [HLZ2].

Proposition 4.7 *For any object W in $\mathcal{C}$, the weights of W form a discrete set of rational numbers and there exists $K \in \mathbb{Z}_+$ such that $(L(0)-L(0)_s)^K = 0$ on W, where $L(0)_s$ is the semisimple part of $L(0)$.*

Proof. By Corollary 5.10 in [Miy], we know that weights of irreducible V-modules must be in $\mathbb{Q}$. Thus for each irreducible V-module, there exists $h \in \mathbb{Q}$ such that the weights of the irreducible V-module are given by $h + k$ for $k \in \mathbb{N}$. For any finite length generalized V-module W, there is a finite composition series $W = W_1 \supset \cdots \supset W_n \supset W_{n+1} = 0$ of generalized V-submodules of W such that W_i/W_{i+1} for $i = 1, \ldots, n$ are irreducible. Then W as a graded vector space is isomorphic to $\coprod_{i=1}^n (W_i/W_{i+1})$. Since W_i/W_{i+1} are irreducible, there are $h_i \in \mathbb{Q}$ such that the weights of W_i/W_{i+1} are $h_i + k$ for $k \in \mathbb{N}$. Thus the weights of W are $h_i + k$ for $k \in \mathbb{N}$, $i = 1, \ldots, n$ and clearly form a discrete subset of $\mathbb{Q}$.

For any weight m, $(L(0) - m)W_{[m]}$ is a subspace of $W_{[m]}$ invariant under $L(0)$. It cannot be equal to $W_{[m]}$ since there are eigenvectors of $L(0)$ in $W_{[m]}$. Since W/W_2 is an ordinary V-module, $(L(0) - m)(W/W_2)_{[m]} = 0$, that is, $(L(0)-m)W_{[m]} \subset (W_2)_{[m]}$. Similarly we have $(L(0)-m)^i(W_{[m]}) \subset (W_{i+1})_{[m]}$.

Since $W_{n+1} = 0$, we have $(L(0) - m)^n(W_{[m]}) = 0$. So we can take $K = n$ and then we have $(L(0) - L(0)_s)^K = 0$ on W. ∎

Together with Proposition 7.11 in [HLZ2], we see that Assumption 7.10 in [HLZ2] holds for our category $\mathcal{C}$:

Proposition 4.8 *For any objects W_1, W_2 and W_3 of $\mathcal{C}$, any logarithmic intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$ and any $w_1 \in W_1$ and $w_2 \in W_2$, the powers of x and $\log x$ occurring in $\mathcal{Y}(w_1, x)w_2$ form a unique expansion set of the form $D \times \{0, \dots, N\}$ where D is a discrete set of rational numbers depending on the discrete sets of the weights of W_1, W_2 and W_3 and N is dependent on the K's given in the preceding proposition for W_1, W_2 and W_3.* ∎

The definition of $\mathcal{C}$, Remark 1.4 and Proposition 1.5 give us the following:

Proposition 4.9 *The category $\mathcal{C}$ is a full subcategory of the category of grading-restricted generalized V-modules closed under the operations of taking contragredients, finite direct sums, generalized V-submodules and quotient generalized V-modules.* ∎

To formulate the next result, we need the following definitions (see [HLZ1] and [HLZ2] for the definitions of $P(z)$-intertwining map, the Virasoro operator $L'_{P(z_1-z_2)}(0)$ and the vertex operator map $Y'_{P(z)}$):

Definition 4.10 We say that the *convergence condition for intertwining maps in $\mathcal{C}$ holds* if for objects W_1, W_2, W_3, W_4 and M_1 of $\mathcal{C}$, $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > 0$, $P(z_1)$-intertwining map I_1 of type $\binom{W_4}{W_1 M_1}$ and $P(z_2)$-intertwining map I_2 of type $\binom{M_1}{W_2 W_3}$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$, $w'_{(4)} \in W'_4$, the series

$$\sum_{m \in \mathbb{C}} \langle w'_4, I_1(w_{(1)} \otimes \pi_m(I_2(w_{(2)} \otimes w_{(3)}))) \rangle$$

is absolutely convergent, where we use π_m to denote the projection from the algebraic completion $\overline{W}$ of a generalized V-module W to the homogeneous space of W of weight m.

Definition 4.11 Assume that the convergence condition for intertwining maps in $\mathcal{C}$ holds. We say that the *expansion condition for intertwining maps in $\mathcal{C}$ holds* if for objects W_1, W_2, W_3, W_4 and M_1 of $\mathcal{C}$, $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$, $P(z_1)$-intertwining map I_1 of type $\binom{W_4}{W_1 M_1}$ and $P(z_2)$-intertwining map I_2 of type $\binom{M_1}{W_2 W_3}$ and $w'_{(4)} \in W'_4$, the element of $(W_1 \otimes W_2 \otimes W_3)^*$ given by

$$w_{(1)} \otimes w_{(2)} \otimes w_{(3)} \mapsto \sum_{m \in \mathbb{C}} \langle w'_4, I_1(w_{(1)} \otimes \pi_m(I_2(w_{(2)} \otimes w_{(3)})))\rangle$$

satisfies the following $P^{(2)}(z_1 - z_2)$-local grading restriction condition for $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$:

(a) For any $w_{(3)} \in W_3$, the element $\mu^{(2)}_{\lambda, w_{(3)}} \in (W_1 \otimes W_2)^*$ given by

$$\mu^{(2)}_{\lambda, w_{(3)}}(w_{(1)} \otimes w_{(2)}) = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})$$

is the limit, in the locally convex topology defined by the pairing between $(W_1 \otimes W_2)^*$ and $W_1 \otimes W_2$, of an absolutely convergent series of generalized eigenvectors in $(W_1 \otimes W_2)^*$ with respect to the Virasoro operator $L'_{P(z_1 - z_2)}(0)$ on $(W_1 \otimes W_2)^*$.

(b) For any $w_{(3)} \in W_1$, let $W^{(2)}_{\lambda, w_{(3)}}$ be the smallest subspace of $(W_1 \otimes W_2)^*$ graded by eigenvalues of $L'_{P(z_1 - z_2)}(0)$, containing the terms in the series in (a) and stable under the component operators of the vertex operators $Y'_{P(z)}(v, x)$ on $(W_1 \otimes W_2)^*$ for $v \in V$. Then $W^{(2)}_{\lambda, w_{(3)}}$ has the properties

$$\dim(W^{(2)}_{\lambda, w_{(3)}})_{[n]} < \infty,$$
$$(W^{(2)}_{\lambda, w_{(2)}})_{[n+k]} = 0 \quad \text{for} \quad k \in \mathbb{Z} \text{ sufficiently negative}$$

for any $n \in \mathbb{C}$, where the subscripts denote the $\mathbb{C}$-grading by $L'_{P(z)}(0)$-eigenvalues.

The following result verifies the last assumption we need:

Theorem 4.12 *The convergence and the expansion conditions for intertwining maps in $\mathcal{C}$ hold (see [HLZ1] and Sections 7 and 9 in [HLZ2] for these* *conditions). For objects W_1, W_2, W_3, W_4, W_5, M_1 and M_2 of $\mathcal{C}$, logarithmic intertwining operators $\mathcal{Y}_1$, $\mathcal{Y}_2$ and $\mathcal{Y}_3$ of types $\binom{W_5}{W_1 M_1}$, $\binom{M_1}{W_2 M_2}$ and $\binom{M_2}{W_3 W_4}$, respectively, $z_1, z_2, z_3 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_3| > 0$ and $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$, $w_{(4)} \in W_4$ and $w'_{(5)} \in W'_5$, the series*

$$\sum_{m,n\in\mathbb{C}} \langle w'_{(5)}, \mathcal{Y}_1(w_{(1)}, z_1)\pi_m(\mathcal{Y}_2(w_{(2)}, z_2)\pi_n(\mathcal{Y}_3(w_{(3)}, z_3)w_{(4)}))\rangle_{W_5}$$

is absolutely convergent and can be analytically extended to a multivalued analytic function on the region given by $z_1, z_2, z_3 \neq 0$, $z_1 \neq z_2$, $z_1 \neq z_3$ and $z_2 \neq z_3$ with regular singular points at $z_1 = 0$, $z_2 = 0$, $z_3 = 0$, $z_1 = \infty$, $z_2 = \infty$, $z_3 = \infty$, $z_1 = z_2$, $z_1 = z_3$ or $z_2 = z_3$.

Proof. By Theorems 11.2 and 11.4 in [HLZ2] (see Remark 12.2 in [HLZ2]), we need only prove that every object of $\mathcal{C}$ satisfies the C_1-cofiniteness condition, every finitely-generated lower-truncated generalized V-module is in $\mathcal{C}$ and every object in $\mathcal{C}$ is quasi-finite dimensional. Our assumption in the beginning of this section gives the C_1-cofiniteness and our definition of the category gives the quasi-finite dimensionality. Let W be a finitely-generated lower-truncated generalized V-module. Let W_1 be the generalized V-submodule of W generated by $\Omega_N(W)$. If $W_1 \neq W$, we consider the quotient W/W_1. Then the difference between the the lowest weight of W/W_1 and the lowest weight of W must be larger than N; otherwise the homogeneous representatives of the lowest vectors of W/W_1 must be in $\Omega_N(W)$ which is impossible since $W_1 \neq W$. On the other hand, since $A_N(V)$ is finite dimensional, the $A_N(V)$-module generated by any nonzero lowest weight vector of W/W_1 is also finite-dimensional and is in particular grading restricted. Thus by Proposition 3.8, the generalized V-submodule of W/W_1 generated by a lowest weight vector is quasi-finite dimensional. By Lemma 3.13, we see that the lowest weight of W/W_1 is the lowest weight of an irreducible V-module. The same argument also shows that the lowest weight of W is the lowest weight of an irreducible V-module. But by assumption, the difference between the lowest weights of any two irreducible V-modules is less than or equal to N. So the difference between the lowest weight of W/W_1 and the lowest weight of W is less than or equal to N. Contradiction. So we must have $W_1 = W$. In particular, the finitely many generators of W must be in the generalized V-submodule generated by finitely many elements of $\Omega_N(W)$ and then this generalized V-submodule must be equal to W. Since

W is generated by finitely many elements of $\Omega_N(W)$, the $A_N(V)$-submodule generated by these elements must be finite-dimensional since $A_N(V)$ is finite dimensional. By Proposition 3.8, W is in $\mathcal{C}$. ∎

Using the results above and Theorem 12.13 in [HLZ2], we obtain:

Theorem 4.13 *The category $\mathcal{C}$, equipped with the tensor product functor $\boxtimes_{P(1)} = \boxtimes'_{P(1)}$, the unit object V, the braiding, associativity, the left and right unit isomorphisms given in Subsection 12.2 of [HLZ2], is a braided tensor category.* ∎

References

[A] T. Abe, A $\mathbb{Z}_2$-orbifold model of the symplectic fermionic vertex operator superalgebra, *Math. Z.* **255** (2007), 755–792.

[ABD] T. Abe, G. Buhl, C. Dong, Rationality, Regularity, and C_2-cofiniteness, *Trans. Amer. Math. Soc.* **356** (2004), 3391–3402.

[AM1] D. Adamović and A. Milas, Logarithmic intertwining operators and $\mathcal{W}(2, 2p-1)$-algebras, *J. Math. Phys.* **48**, 073503 (2007).

[AM2] D. Adamović and A. Milas, On the triplet vertex algebra $\mathcal{W}(p)$, *Adv. in Math.* **217** (2008), 2664-2699.

[AM3] D. Adamović and A. Milas, The N=1 triplet vertex operator superalgebras, to appear; arXiv:0712.0379.

[B] G. Buhl, A spanning set for VOA modules, *J. Algebra* **254** (2002), 125–151.

[CF] N. Carqueville and M. Flohr, Nonmeromorphic operator product expansion and C_2-cofiniteness for a family of $\mathcal{W}$-algebras, *J.Phys.* **A39** (2006), 951–966.

[DLM1] C. Dong, H. Li and G. Mason, Vertex operator algebras and associative algebras, *J. Algebra* **206** (1998), 67–96.

[DLM2] C. Dong, H. Li and G. Mason, Modular invariance of trace functions in orbifold theory, *Comm. Math. Phys.* **214** (2000), 1–56.

[EO] P. Etingof and V. Ostrik, Finite tensor categories, *Moscow Math. J.* **4** (2004), 627–654.

[FGST1] B. L. Feigin, A. M. Gaĭnutdinov, A. M. Semikhatov, and I. Yu Tipunin, I, The Kazhdan-Lusztig correspondence for the representation category of the triplet W-algebra in logarithmic conformal field theories (Russian), *Teoret. Mat. Fiz.* **148** (2006), no. 3, 398–427.

[FGST2] B. L. Feigin, A. M. Gaĭnutdinov, A. M. Semikhatov, and I. Yu Tipunin, Logarithmic extensions of minimal models: characters and modular transformations, *Nucl. Phys.* B **757** (2006), 303–343.

[FGST3] B. L. Feigin, A. M. Gaĭnutdinov, A. M. Semikhatov, and I. Yu Tipunin, Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center, *Comm. Math. Phys.* **265** (2006), 47–93.

[FHL] I.B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, preprint, 1989; Memoirs Amer. Math. Soc., Vol. 104, Number 494, American Math. Soc. Providence, 1993.

[Fl1] M. Flohr, On modular invariant partition functions of conformal field theories with logarithmic operators, *Int. J. Mod. Phys.* **A11** (1996), 4147–4172.

[Fl2] M. Flohr, On fusion rules in logarithmic conformal field theories, *Int. J. Mod. Phys.* **A12** (1996), 1943–1958.

[FG] M. Flohr and M. R. Gaberdiel, Logarithmic torus amplitudes, *J. Phys.* **A39** (2006), 1955–1968.

[FK] M. Flohr and H. Knuth, On Verlinde-Like formulas in $c_{p,1}$ logarithmic conformal field theories, to appear; arXiv:0705.0545.

[FGK] M. Flohr, C. Grabow and M. Koehn, Fermionic Expressions for the characters of $c(p, 1)$ logarithmic conformal field theories, *Nucl. Phys.* **B768** (2007), 263–276.

[FLM] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Appl. Math., Vol. 134, Academic Press, Boston, 1988.

[Fu] J. Fuchs, On nonsemisimple fusion rules and tensor categories, in: *Lie algebras, vertex operator algebras and their applications, Proceedings of a conference in honor of James Lepowsky and Robert Wilson, 2005*, ed. Y.-Z. Huang and K. Misra, Contemporary Mathematics, Vol. 442, Amer. Math. Soc., Providence, 2007.

[FHST] J. Fuchs, S. Hwang, A.M. Semikhatov and I. Yu. Tipunin, Nonsemisimple Fusion Algebras and the Verlinde Formula, *Comm. Math. Phys.* **247** (2004), no. 3, 713–742.

[G] V. Gurarie, Logarithmic operators in conformal field theory, *Nucl. Phys.* **B410** (1993), 535–549.

[GK1] M. R. Gaberdiel and H. G. Kausch, Indecomposable fusion products, *Nucl. Phys.* **B477** (1996), 298–318.

[GK2] M. R. Gaberdiel and H. G. Kausch, A rational logarithmic conformal field theory, *Phys. Lett.* **B386** (1996), 131–137.

[GN] M. R. Gaberdiel and A. Neitzke, Rationality, quasirationality and finite W-algebras, *Comm. Math.Phys.* **238** (2003) 305–331.

[H1] Y.-Z. Huang, A theory of tensor products for module categories for a vertex operator algebra, IV, *J. Pure Appl. Alg.* 100 (1995) 173–216.

[H2] Y.-Z. Huang, Intertwining operator algebras, genus-zero modular functors and genus-zero conformal field theories, in: *Operads: Proceedings of Renaissance Conferences*, ed. J.-L. Loday, J. Stasheff, and A. A. Voronov, Contemporary Math., Vol. 202, Amer. Math. Soc., Providence, 1997, 335–355.

[H3] Y.-Z. Huang, *Two-dimensional conformal geometry and vertex operator algebras*, Progress in Mathematics, Vol. 148, Birkhäuser, Boston, 1997.

[H4] Y.-Z. Huang, Genus-zero modular functors and intertwining operator algebras, *Internat. J. Math.* 9 (1998), 845–863.

[H5] Y.-Z. Huang, Riemann surfaces with boundaries and the theory of vertex operator algebras, in: *Vertex Operator Algebras in Mathematics and Physics*, ed. S. Berman, Y. Billig, Y.-Z. Huang and J. Lepowsky, Fields Institute Communications, Vol. 39, Amer. Math. Soc., Providence, 2003, 109–125.

[H6] Y.-Z. Huang, Differential equations and intertwining operators, *Comm. Contemp. Math.* **7** (2005), 375–400.

[H7] Y.-Z. Huang, Differential equations, duality and modular invariance, *Comm. Contemp. Math.***7** (2005), 649–706.

[H8] Y.-Z. Huang, Vertex operator algebras, the Verlinde conjecture and modular tensor categories, *Proc. Natl. Acad. Sci. USA* **102** (2005), 5352–5356.

[H9] Y.-Z. Huang, Vertex operator algebras, fusion rules and modular transformations, in: *Non-commutative Geometry and Representation Theory in Mathematical Physics*, ed. J. Fuchs, J. Mickelsson, G. Rozenblioum and A. Stolin, Contemporary Math. Vol. 391, Amer. Math. Soc., Providence, 2005, 135–148.

[H10] Y.-Z. Huang, Vertex operator algebras and the Verlinde conjecture, *Comm. Contemp. Math.* **10** (2008), 103-154.

[H11] Y.-Z. Huang, Rigidity and modularity of vertex tensor categories, *Comm. Contemp. Math.*, to appear; math.QA/0502533.

[HK1] Y.-Z. Huang and L. Kong, Full field algebras, *Comm. Math. Phys.***272** (2007), 345–396.

[HK2] Y.-Z. Huang and L. Kong, Modular invariance for conformal full field algebras, to appear; arXiv:math/0609570.

[HL1] Y.-Z. Huang and J. Lepowsky, Toward a theory of tensor products for representations of a vertex operator algebra, in: *Proc. 20th International Conference on Differential Geometric Methods in Theoretical Physics, New York, 1991*, ed. S. Catto and A. Rocha, World Scientific, Singapore, 1992, Vol. 1, 344–354.

[HL2] Y.-Z. Huang and J. Lepowsky, Tensor products of modules for a vertex operator algebras and vertex tensor categories, in: *Lie Theory and Geometry, in honor of Bertram Kostant,* ed. R. Brylinski, J.-L. Brylinski, V. Guillemin, V. Kac, Birkhäuser, Boston, 1994, 349–383.

[HL3] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, I, *Selecta Mathematica (New Series)* **1** (1995), 699–756.

[HL4] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, II, *Selecta Mathematica (New Series)* **1** (1995), 757–786.

[HL5] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, III, *J. Pure Appl. Alg.* **100** (1995) 141–171.

[HL7] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, V, to appear.

[HLZ1] Y.-Z. Huang, J. Lepowsky and L.Zhang, A logarithmic generalization of tensor product theory for modules for a vertex operator algebra, *Internat. J. Math.* **17** (2006), 975–1012.

[HLLZ] Y.-Z. Huang, J. Lepowsky, H. Li and L. Zhang, On the concepts of intertwining operator and tensor product module in vertex operator algebra theory, *J. Pure Appl. Algebra* **204** (2006), 507–535.

[HLZ2] Y.-Z. Huang, J. Lepowsky and L.Zhang, Logarithmic tensor product theory for generalized modules for a conformal vertex algebra, to appear; arXiv:0710.2687.

[K1] H. G. Kausch, Extended conformal algebras generated by multiplet of primary fields, *Phys. Lett.* **259** B (1991), 448–455.

[K2] H. G. Kausch, Symplectic fermions, *Nucl. Phys.* B **583** (2000), 513–541.

[KarL] M. Karel and H. Li, Certain generating subspaces for vertex operator algebras, *J. Alg.* **217** (1999), 393–421.

[KazL1] D. Kazhdan and G. Lusztig, Affine Lie algebras and quantum groups, *Duke Math. J., IMRN* **2** (1991), 21–29.

[KazL2] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, I, *J. Amer. Math. Soc.* **6** (1993), 905–947.

[KazL3] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, II, *J. Amer. Math. Soc.* **6** (1993), 949–1011.

[KazL4] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, III, *J. Amer. Math. Soc.* **7** (1994), 335–381.

[KazL5] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, IV, *J. Amer. Math. Soc.* **7** (1994), 383–453.

[Le] J. Lepowsky, From the representation theory of vertex operator algebras to modular tensor categories in conformal field theory, commentary on Y.-Z. Huang's PNAS article "Vertex operator algebras, the Verlinde conjecture and modular tensor categories", *Proc. Nat. Acad. Sci. USA* **102** (2005), 5304–5305.

[LL] J. Lepowsky and H. Li, *Introduction to Vertex Operator Algebras and Their Representations*, Progress in Math., Birkhäuser, Boston, 2003.

[Li] H. Li, Some finiteness properties of regular vertex operator algebras, *J. Alg.* **212** (1999), 495–514.

[Mil] A. Milas, Weak modules and logarithmic intertwining operators for vertex operator algebras, in *Recent Developments in Infinite-Dimensional Lie Algebras and Conformal Field Theory*, ed. S. Berman, P. Fendley, Y.-Z. Huang, K. Misra, and B. Parshall, Contemp. Math., Vol. 297, American Mathematical Society, Providence, RI, 2002, 201–225.

[Miy] M. Miyamoto, Modular invariance of vertex operator algebras satisfying C_2-cofiniteness, *Duke Math. J.* **122** (2004), 51-91.

[N] W. Nahm, Quasi-rational fusion products, *Int. J. Mod. Phys.* **B8** (1994), 3693–3702.

[Zha1] L. Zhang, Vertex operator algebras and Kazhdan-Lusztig's tensor category, Ph.D. thesis, Rutgers University, 2004.

[Zha2] L.Zhang, Vertex tensor category structure on a category of Kazhdan–Lusztig, *New York J. Math.*, to appear; math.QA/0701260.

[Zhu1] Y. Zhu, Vertex operators, elliptic functions and modular forms, Ph.D. thesis, Yale University, 1990.

[Zhu2] Y. Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* **9** (1996), 237–307.

Institut des Hautes Études Scientifiques, Le Bois-Marie, 35, Route De Chartres, F-91440 Bures-sur-Yvette, France
and
Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019 (permanent address)
E-mail address: yzhuang@math.rutgers.edu