Braided tensor categories and extensions of vertex operator algebras

Yi-Zhi Huang, Alexander Kirillov Jr. and James Lepowsky

Abstract

Let V be a vertex operator algebra satisfying suitable conditions such that in particular its module category has a natural vertex tensor category structure, and consequently, a natural braided tensor category structure. We prove that the notions of extension (i.e., enlargement) of V and of commutative associative algebra, with uniqueness of unit and with trivial twist, in the braided tensor category of V-modules are equivalent.

1 Introduction

In the constructions and applications of vertex operator algebras, such algebras obtained by adjoining suitable modules to a smaller vertex operator algebra play a significant role, as in [FLM], for example.

Given a vertex operator algebra V satisfying suitable conditions, a natural vertex tensor category structure, and as a consequence, a natural braided tensor category structure, on the category of V-modules was constructed by the first and the third authors in [HL1]–[HL7], [Hu1] and [Hu4]. Such braided tensor category structure on the category of V-modules (including even the construction of suitable tensor product modules) had been conjectured to exist in physics and mathematics starting from the work of Moore and Seiberg [MS]. Assuming the existence of such braided tensor category structure, it was proposed in [Ho], [FFFS] and [KO] that certain extensions of a vertex operator algebra V (by which we mean vertex operator algebras containing V as a subalgebra, in this paper) should be related to commutative associative algebras in the braided tensor category of V-modules. In [KO], Ostrik and the second author formulated a relation between the extensions of V and the commutative associative algebras in the category of V-modules. Assuming this relation, in [KO] they classified the extensions of the vertex operator algebras associated to standard (integrable highest weight) modules for the affine Lie algebra $\widehat{\mathfrak{sl}}_2$, relating these extensions to the representation theory of $U_q(\mathfrak{sl}_2)$ and obtaining a q-analogue of the McKay correspondence.

In the present paper, we establish close variants of this relation formulated in [KO] between the extensions of V and the commutative associative algebras in the category of V-modules. This requires the theory of vertex tensor categories. There are many classes of vertex operator algebras whose suitable module categories are vertex tensor categories (see the Appendix below), and for these module categories, the conclusions of the theorems in the present paper hold, under suitable additional conditions. These classes of vertex operator algebras include in particular the vertex operator algebras associated with positive definite even lattices; the vertex operator algebras associated with affine Lie algebras and positive integral levels; the "minimal series" of vertex operator algebras associated with the Virasoro algebra; Frenkel, Lepowsky and Meurman's moonshine module vertex operator algebra $V^\natural$; the fixed point vertex operator subalgebra of $V^\natural$ under the standard involution; and the "minimal series" of vertex operator superalgebras (suitably generalized vertex operator algebras) associated with the Neveu-Schwarz superalgebra and also the "unitary series" of vertex operator superalgebras associated with the $N = 2$ superconformal algebra. Since the construction of the necessary vertex tensor category structure is substantial, the proofs of these conclusions for these module categories are highly nontrivial.

More specifically, in contrast with quantum group theory, in which the associativity isomorphisms are trivial, the construction of the natural associativity isomorphisms for the vertex tensor category structures and the braided tensor category structures, and the proofs of their properties, form the hardest part of the work [HL1]–[HL7], [Hu1], [Hu4]. (See Section 2 below for a sketch of the necessary structures.) In particular, using strict tensor categories equivalent to the original ones would only make the necessary constructions and proofs more difficult, and this is true even in the monodromy-free case when the intertwining maps (induced by intertwining operators) are single-valued.

In our theorems, we always assume that the module category for the underlying vertex operator algebra has a natural vertex tensor category structure rather than just a braided tensor category structure, since our proofs are based on this structure. In our first theorem, we do not require that the braided tensor category be rigid. We obtain different variants of the relation formulated in [KO].

The first version of the present paper was written in 2002, and mathematically, the present paper is essentially the same as the 2002 version. The results have been generalized by Kong and the first author in [HK] to a relation between open-string vertex algebras containing V as a subalgebra and associative algebras in the category of V-modules, and by Kong in [K] to a relation between conformal full field algebras over $V^L \otimes V^R$, for suitable vertex operator algebras V^L and V^R equipped with nondegenerate invariant bilinear forms, and commutative Frobenius algebras with trivial twists in the category of $V^L \otimes V^R$-modules.

In the next section, we review the relevant aspects of the vertex tensor category structure ([HL1]–[HL7], [Hu1], [Hu4]). In Section 3 we formulate and prove our theorems, including the result that the notion of vertex operator algebra extension of a vertex operator algebra satisfying certain conditions is equivalent to the notion of commutative associative algebra, with uniqueness of unit and with trivial twist, in the braided tensor category of V-modules. In the Appendix, for the reader's convenience we recall general theorems stating that under suitable conditions, the module category for a vertex operator algebra has a natural vertex tensor category structure and in particular braided tensor category structure.

Acknowledgment The first and third author were partially supported by NSF grants DMS-0070800 and DMS-0401302.

2 Review of the vertex tensor category theory

Here we briefly review what we will need concerning vertex tensor category structure for the proofs of the main theorems of the present paper. This entails the vertex tensor category structure itself, including the resulting braided tensor category structure, on the category of modules for a vertex operator algebra V satisfying suitable conditions; this structure was constructed in [HL1]–[HL7], [Hu1], [Hu4], whose notation and terminology we shall be using. For suitable sufficient conditions, see the Appendix.

We take as our precise notions of vertex operator algebra (over $\mathbb{C}$) and of module (over $\mathbb{C}$) for a vertex operator algebra the definitions given in [HL3]. In particular, the central charge (or rank) of a vertex operator algebra is allowed to be in $\mathbb{C}$ and the weight-grading of a module is by $\mathbb{C}$ (that is, the (conformal) weights of homogeneous elements are complex numbers).

We shall use the notation Y for the vertex operator map of the vertex operator algebra V, $\mathbf{1}$ for the vacuum vector, ω for the conformal element (generating the Virasoro algebra), $L(n)$ ($n \in \mathbb{Z}$) for the Virasoro algebra operators, and $V_{(n)}$ ($n \in \mathbb{Z}$) for the $L(0)$-eigenspaces (weight spaces) of V, as well as the analogous relevant notation for V-modules. By definition, a vertex operator subalgebra of a vertex operator algebra has the same conformal element. A simple vertex operator algebra is one that is irreducible as a module for itself.

We shall need the $P(z)$-tensor product $W_1 \boxtimes_{P(z)} W_2$ of V-modules W_1 and W_2, for $z \in \mathbb{C} \setminus \{0\}$. There is a tensor product bifunctor associated to any sphere with three tubes, but in the present paper the $P(z)$-tensor product is enough. The V-module $W_1 \boxtimes_{P(z)} W_2$ is not based on the tensor product vector space $W_1 \otimes W_2$.

One important feature of the $P(z)$-tensor product $W_1 \boxtimes_{P(z)} W_2$ is a natural isomorphism between the space of module maps from $W_1 \boxtimes_{P(z)} W_2$ to a third module W_3 and the space of intertwining operators of type $\binom{W_3}{W_1\, W_2}$ (see [HL3], [HL5]). (This isomorphism is based on the choice of the branch of $\log z$ such that $0 \leq \arg z < 2\pi$; different choices give equivalent theories.) In particular, there exists a canonical intertwining operator $\mathcal{Y}(\cdot, x)$ (x a formal variable) of type $\binom{W_1 \boxtimes_{P(z)} W_2}{W_1\, W_2}$ corresponding to the identity module map from $W_1 \boxtimes_{P(z)} W_2$ to itself. For $w_1 \in W_1$ and $w_2 \in W_2$, one has a $P(z)$-tensor product element

$$w_1 \boxtimes_{P(z)} w_2 = \mathcal{Y}(w_1, z)w_2 \in \overline{W_1 \boxtimes_{P(z)} W_2}$$

of w_1 and w_2, with z replacing x according to the branch choice, giving a $P(z)$-intertwining map $\mathcal{Y}(\cdot, z)$ (as opposed to intertwining operator), where $\overline{W_1 \boxtimes_{P(z)} W_2}$ is the formal completion of $W_1 \boxtimes_{P(z)} W_2$. Moreover, the homogeneous components of $w_1 \boxtimes_{P(z)} w_2$ for all $w_1 \in W_1$ and $w_2 \in W_2$ span the module $W_1 \boxtimes_{P(z)} W_2$.

The existence of these tensor product elements is another important feature of the tensor category theory. Such elements provide a powerful method for proving the necessary theorems in the tensor category theory (see in particular [Hu1]): One first proves certain results about these tensor product elements. Since the homogeneous components of these elements span the tensor product modules, one obtains the desired results. (Worth mentioning here is the subtlety that the set of all tensor product elements has almost no intersection with the $P(z)$-tensor product module; generically, only very special elements like $\mathbf{1} \boxtimes_{P(z)} w$ lie in $V \boxtimes_{P(z)} W$, W a V-module.)

To prove the necessary results involving associativity and coherence, one also needs tensor product elements of more than two elements. Here we briefly describe the tensor product element of three elements (see [Hu1] for details). Let W_1, W_2, W_3 be V-modules and let $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$. Let z_1 and z_2 be nonzero complex numbers. Since in general $w_2 \boxtimes_{P(z_2)} w_3$ does not lie in $W_2 \boxtimes_{P(z_2)} W_3$, one cannot define $w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)$ simply as the $P(z_1)$-tensor product element of w_1 and $w_2 \boxtimes_{P(z_2)} w_3$. Such triple tensor product elements do not exist in general. But using the assumed convergence property for the product of intertwining operators, the series $\sum_{n\in\mathbb{R}} w_1 \boxtimes_{P(z_1)} \pi_n(w_2 \boxtimes_{P(z_2)} w_3)$ (π_n being the projection map from a graded space to the subspace of weight n) is weakly absolutely convergent when $|z_1| > |z_2| > 0$ and the sum, which one writes as $w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)$, lies in $\overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}$. Similarly, when $|z_2| > |z_1 - z_2| > 0$, one has $(w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3 \in \overline{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3}$. The homogeneous components of these triple tensor product elements also span the corresponding triple tensor product modules. More generally, one has

$$\mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3$$

and

$$\mathcal{Y}^1(\mathcal{Y}^2(w_1, z_1 - z_2)w_2, z_2)w_3$$

in the formal completions of suitable V-modules for suitable intertwining operators $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Y}^1$ and $\mathcal{Y}^2$, elements w_1, w_2 and w_3 of suitable V-modules, and complex numbers z_1 and z_2 satisfying $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively.

The $P(z)$-tensor product functors and elements are needed for the vertex tensor category structure and for the proof of the main theorems of the present paper. For the braided tensor category structure, one needs to specify one particular tensor product bifunctor. One can in fact take this tensor product for the braided tensor category to be any particular $P(z)$-tensor product, but one can and does choose the $P(1)$-tensor product $\boxtimes_{P(1)}$ (where $z = 1$), and one also denotes $\boxtimes_{P(1)}$ by $\boxtimes$.

The unit object of the vertex tensor category and of the braided tensor category is the vertex operator algebra V. The left unit isomorphism

$$l_W : V \boxtimes W \to W$$

is characterized by

$$l_W(\mathbf{1} \boxtimes w) = w$$

for $w \in W$. The right unit isomorphism

$$r_W : W \boxtimes V \to W$$

is characterized by

$$\overline{r_W}(w \boxtimes \mathbf{1}) = e^{L(-1)}w$$

for $w \in W$.

One needs natural parallel transport isomorphisms. Let W_1 and W_2 be V-modules. Given a path γ in $\mathbb{C} \setminus \{0\}$ from $z_1 \in \mathbb{C} \setminus \{0\}$ to $z_2 \in \mathbb{C} \setminus \{0\}$, the parallel transport isomorphism $\mathcal{T}_\gamma : W_1 \boxtimes_{P(z_1)} W_2 \to W_1 \boxtimes_{P(z_2)} W_2$ is defined as follows: Let $\mathcal{Y}$ be the intertwining operator corresponding to the intertwining map $\boxtimes_{P(z_2)}$ and let $l(z_1)$ be the value of the logarithm of z_1 determined uniquely by $\log z_2$ (satisfying $0 \leq \arg z_2 < 2\pi$) and the path γ. Then $\mathcal{T}_\gamma$ is characterized by

$$\overline{\mathcal{T}}_\gamma(w_1 \boxtimes_{P(z_1)} w_2) = \mathcal{Y}(w_1, e^{l(z_1)})w_2$$

for $w_1 \in W_1$ and $w_2 \in W_2$, where $\overline{\mathcal{T}}_\gamma$ is the natural extension of $\mathcal{T}_\gamma$ to the formal completion $\overline{W_1 \boxtimes_{P(z_1)} W_2}$ of $W_1 \boxtimes_{P(z_1)} W_2$. The parallel transport isomorphism depends only on the homotopy class of γ.

The braiding isomorphism (also called the commutativity isomorphism)

$$\mathcal{R}_{W_1 W_2} : W_1 \boxtimes W_2 \to W_2 \boxtimes W_1$$

is characterized as follows: Let γ_1^- be a path from -1 to 1 in the closed upper half plane with 0 deleted, with $\mathcal{T}_{\gamma_1^-}$ the corresponding parallel transport isomorphism. Then

$$\overline{\mathcal{R}}_{W_1 W_2}(w_1 \boxtimes w_2) = e^{L(-1)}\overline{\mathcal{T}}_{\gamma_1^-}(w_2 \boxtimes_{P(-1)} w_1) \tag{2.1}$$

for $w_1 \in W_1$, $w_2 \in W_2$. One can prove that

$$(\mathcal{R}_{W_2 W_1} \circ \mathcal{R}_{W_1 W_2})(w_1 \boxtimes w_2) = \mathcal{Y}(w_1, x)w_2|_{x^n = e^{2\pi n i},\, n \in \mathbb{R}},$$

where $w_1 \in W_1$, $w_2 \in W_2$ and $\mathcal{Y}$ is the intertwining operator corresponding to the identity map from $W_1 \boxtimes W_2$ to itself.

Given complex numbers z_1 and z_2 satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$ and V-modules W_1, W_2 and W_3, one is able to construct an associativity isomorphism

$$\mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)} : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \to (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$$

characterized by

$$\overline{\mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}}(w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)) = (w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3.$$

To obtain the associativity isomorphism

$$\mathcal{A}_{W_1,W_2,W_3} : W_1 \boxtimes (W_2 \boxtimes W_3) \to (W_1 \boxtimes W_2) \boxtimes W_3$$

for the braided tensor category structure, one needs some particular parallel transport isomorphisms. Let z_1 and z_2 be real numbers satisfying $z_1 > z_2 > z_1 - z_2 > 0$. Let γ_1 and γ_2 be paths in the real line with 0 deleted from 1 to z_1 and z_2, respectively, and γ_3 and γ_4 be paths in the real line with 0 deleted from z_2 and $z_1 - z_2$ to 1, respectively. Then the associativity isomorphism is given by

$$\mathcal{A}_{W_1,W_2,W_3} = \mathcal{T}_{\gamma_3} \circ (\mathcal{T}_{\gamma_4} \boxtimes_{P(z_2)} 1_{W_3}) \circ \mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)} \circ (1_{W_1} \boxtimes_{P(z_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1}. \quad (2.2)$$

The characterizations of the commutativity and associativity isomorphisms above make the proof of the coherence properties straightforward. Here we sketch the proof of the commutativity of the pentagon diagram. This commutativity states that the following identity for maps from $W_1 \boxtimes (W_2 \boxtimes (W_3 \boxtimes W_4))$ to $((W_1 \boxtimes W_2) \boxtimes W_3) \boxtimes W_4$ holds:

$$\begin{aligned} &\mathcal{A}_{W_1 \boxtimes W_2, W_3, W_4} \circ \mathcal{A}_{W_1, W_2, W_3 \boxtimes W_4} \\ &\quad = (\mathcal{A}_{W_1,W_2,W_3} \boxtimes 1_{W_4}) \circ \mathcal{A}_{W_1, W_2 \boxtimes W_3, W_4} \circ (1_{W_1} \boxtimes \mathcal{A}_{W_2,W_3,W_4}). \end{aligned} \quad (2.3)$$

By analogy with the construction of the tensor product elements $w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)$, one can also construct tensor product elements of four elements, including

$$w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} (w_3 \boxtimes_{P(z_3)} w_4)) \quad (2.4)$$

and

$$((w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2-z_3)} w_3) \boxtimes_{P(z_3)} w_4)) \tag{2.5}$$

for $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$, $w_4 \in W_4$, when

$$|z_1| > |z_2| > |z_3| > |z_1 - z_3| > |z_2 - z_3| > |z_1 - z_2| > 0.$$

Since $\overline{\mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}}$ maps $w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)$ to $(w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3$, the formal extensions of both

$$\mathcal{A}_{P(z_2),P(z_3)}^{P(z_2-z_3),P(z_3)} \circ \mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}$$

and

$$(\mathcal{A}_{P(z_1-z_3),P(z_2-z_3)}^{P(z_1-z_2),P(z_2-z_3)} \boxtimes_{P(z_3)} 1_{W_4}) \circ \mathcal{A}_{P(z_1),P(z_3)}^{P(z_1-z_3),P(z_3)} \circ (1_{W_1} \boxtimes_{P(z_1)} \mathcal{A}_{P(z_2),P(z_3)}^{P(z_2-z_3),P(z_3)})$$

map (2.4) to (2.5). Thus these two maps are the same:

$$\begin{aligned} &\mathcal{A}_{P(z_2),P(z_3)}^{P(z_2-z_3),P(z_3)} \circ \mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)} \\ &\quad = (\mathcal{A}_{P(z_1-z_3),P(z_2-z_3)}^{P(z_1-z_2),P(z_2-z_3)} \boxtimes_{P(z_3)} 1_{W_4}) \circ \mathcal{A}_{P(z_1),P(z_3)}^{P(z_1-z_3),P(z_3)} \\ &\qquad \circ (1_{W_1} \boxtimes_{P(z_1)} \mathcal{A}_{P(z_2),P(z_3)}^{P(z_2-z_3),P(z_3)}). \end{aligned} \tag{2.6}$$

Now one chooses z_1 and z_2 to be real numbers satisfying $z_1 > z_2 > z_1 - z_2 > 0$. From the characterization above of the associativity isomorphism for the braided tensor category structure and the fact that the parallel transport isomorphisms involved are associated to paths in the real line, the parallel transport isomorphisms all cancel, and (2.3) follows from (2.6).

For any V-module W, one also has a twist $\theta_W\colon W \to W$, given by $\theta_W(w) = e^{2\pi i L(0)} w$ for $w \in W$. It is immediate from the definition that if W is simple, then

$$\theta_W = e^{2\pi i \Delta_W},$$

where Δ_W is the conformal dimension of W, that is, the lowest eigenvalue of the weight operator $L(0)$ on W. The twist θ_W satisfies the following conditions:

$$\theta_V = 1_V, \tag{2.7}$$

$$\theta_{W_1 \boxtimes W_2} = \mathcal{R}_{W_2 W_1} \circ \mathcal{R}_{W_1 W_2} \circ (\theta_{W_1} \boxtimes \theta_{W_2}), \tag{2.8}$$

$$\theta_{W'} = (\theta_W)', \tag{2.9}$$

where W' is the V-module contragredient to W, and for $f: W \to W$ one denotes by $f': W' \to W'$ the adjoint operator (see [FHL]).

The first identity (2.7) is immediate from the definitions. For (2.8), note that for $w_1 \in W_1$ and $w_2 \in W_2$, $w_1 \boxtimes w_2 = \mathcal{Y}(w_1, 1)w_2$ and $\mathcal{R}_{W_2W_1}\mathcal{R}_{W_1W_2}(w_1 \boxtimes w_2) = \mathcal{Y}(w_1, e^{2\pi i})w_2$ where $\mathcal{Y}$ is the corresponding intertwining operator (see above). Thus

$$\overline{\mathcal{R}_{W_2W_1} \circ \mathcal{R}_{W_1W_2} \circ (\theta_{W_1} \boxtimes \theta_{W_2})}(w_1 \boxtimes w_2) = \mathcal{Y}(e^{2\pi i L(0)}w_1, e^{2\pi i})e^{2\pi i L(0)}w_2,$$

and (2.8) follows from basic property

$$e^{2\pi i L(0)}\mathcal{Y}(w_1, z)w_2 = \mathcal{Y}(e^{2\pi i L(0)}w_1, e^{2\pi i})e^{2\pi i L(0)}w_2.$$

The last identity (2.9) is immediate because $L(0)' = L(0)$.

If the tensor category of V-modules is rigid[1], then θ defines a structure of a balanced rigid category on this category.

3 The main theorems

Definition 3.1 Let $\mathcal{C}$ be a braided tensor category. A *commutative associative algebra A in $\mathcal{C}$* (or *$\mathcal{C}$-algebra* for short) is an object A of $\mathcal{C}$ together with morphisms

$$\mu: A \otimes A \to A$$

and

$$\iota_A: \mathbf{1}_{\mathcal{C}} \hookrightarrow A$$

such that the following conditions hold:

1. Associativity:

$$\mu \circ (\mu \otimes 1_A) \circ \mathcal{A} = \mu \circ (1_A \otimes \mu) : A \otimes (A \otimes A) \longrightarrow A,$$

   where $\mathcal{A}$ is the associativity isomorphism

$$\mathcal{A} : A \otimes (A \otimes A) \longrightarrow (A \otimes A) \otimes A.$$

[1]After the first version of the present paper was written, the rigidity of the tensor category of V-modules was proved by the first author when V satisfies certain stronger conditions. See the Appendix for more discussion and the reference.

2. Commutativity:
$$\mu \circ \mathcal{R} = \mu,$$
where $\mathcal{R}$ is the braiding isomorphism
$$\mathcal{R} : A \otimes A \longrightarrow A \otimes A.$$

3. Unit:
$$\mu \circ (\iota_A \otimes 1_A) \circ l_A^{-1} = 1_A,$$
where
$$l_A : \mathbf{1}_{\mathcal{C}} \otimes A \to A$$
is the left unit isomorphism.

Such an algebra A is called *haploid* if it satisfies the following additional condition:

4. Uniqueness of unit:
$$\dim \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, A) = 1.$$

(Since we are using the notation $\mathbf{1}$ for the vacuum vector of V, we use the notation $\mathbf{1}_{\mathcal{C}}$ for the unit object of $\mathcal{C}$.)

Recall that a balancing in a rigid braided tensor category is a natural isomorphism from the identity functor to the double-dual functor satisfying standard conditions. This is equivalent to defining a natural transformation, called a twist, from the identity functor on $\mathcal{C}$ to itself satisfying standard conditions. Note that in the case of module categories for vertex operator algebras, we have a twist θ_W on any module W even if the category is not rigid (see Section 2). Below we shall use this twist θ_W.

Theorem 3.2 *Let V be a vertex operator algebra such that its module category $\mathcal{C}$ has a natural vertex tensor category structure described in Section 2 (see the Appendix for the appropriate conditions on V). We assume in addition that V is simple; every V-module is completely reducible; for any irreducible V-module W the weights of elements of W are nonnegative real numbers and $W_{(0)} \neq 0$ if and only if W is equivalent to V viewed as a V-module; and $\dim V_{(0)} = 1$. Then the following two notions are equivalent:*

1. *An extension $V_e \supset V$ such that $\dim (V_e)_{(0)} = 1$, that is, a vertex operator algebra V_e such that V is a subalgebra of V_e and $\dim (V_e)_{(0)} = 1$.*

2. *A haploid $\mathcal{C}$-algebra V_e with $\theta_{V_e} = 1_{V_e}$.*

Proof. Let V_e be a vertex operator algebra such that V is a subalgebra of V_e. Then V_e is a V-module. Thus it is also an object of $\mathcal{C}$. Since V_e is a vertex operator algebra, we have the vertex operator map Y_e for V_e. This vertex operator map can be viewed as an intertwining operator for V of type $\binom{V_e}{V_e V_e}$. Let $\mu : V_e \boxtimes V_e \to V_e$ be the module map corresponding to the intertwining operator Y_e. Also, since V is a subalgebra of V_e, we have a morphism $\iota_{V_e} : V \to V_e$. We now verify that the triple (V_e, μ, ι_{V_e}) is indeed a haploid $\mathcal{C}$-algebra.

We first prove the associativity. From the construction, we have

$$\overline{\mu}(u \boxtimes v) = Y_e(u, 1)v$$

for $u, v \in V_e$, where $\overline{\mu} : \overline{V_e \boxtimes V_e} \to \overline{V}_e$ is the natural extension of μ. Let μ_z be the morphism from $V_e \boxtimes_{P(z)} V_e$ to V_e corresponding to the intertwining operator Y_e. Then we have $\mu = \mu_1$ and

$$\overline{\mu}_z(u \boxtimes v) = Y_e(u, z)v.$$

Thus for $u, v, w \in V_e$ and z_1, z_2 satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$,

$$\overline{\mu_{z_1} \circ (1_{V_e} \boxtimes \mu_{z_2})}(u \boxtimes_{P(z_1)} (v \boxtimes_{P(z_2)} w)) = Y_e(u, z_1)Y_e(v, z_2)w \quad (3.1)$$

$$\overline{\mu_{z_1} \circ (\mu_{z_1 - z_2} \boxtimes 1_{V_e})}((u \boxtimes_{P(z_1 - z_2)} v) \boxtimes_{P(z_2)} w) = Y_e(Y_e(u, z_1 - z_2)v, z_2)w. \quad (3.2)$$

By the associativity for Y_e,

$$Y_e(u, z_1)Y_e(v, z_2)w = Y_e(Y_e(u, z_1 - z_2)v, z_2)w. \quad (3.3)$$

We also have the associativity isomorphism

$$\mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)} : V_e \boxtimes_{P(z_1)} (V_e \boxtimes_{P(z_2)} V_e) \to (V_e \boxtimes_{P(z_1 - z_2)} V_e) \boxtimes_{P(z_2)} V_e,$$

characterized by

$$\overline{\mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}}(u \boxtimes_{P(z_1)} (v \boxtimes_{P(z_2)} w)) = (u \boxtimes_{P(z_1 - z_2)} v) \boxtimes_{P(z_2)} w \quad (3.4)$$

for $u, v, w \in V_e$. Combining (3.1)–(3.4), we obtain

$$(\mu_{z_1} \circ (1_{V_e} \boxtimes_{P(z_1)} \mu_{z_2})) = (\mu_{z_2} \circ (\mu_{z_1 - z_2} \boxtimes_{P(z_2)} 1_{V_e})) \circ \mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}. \quad (3.5)$$

From (3.5), we obtain

$$\begin{aligned}&(\mu_{z_1} \circ (1_{V_e} \boxtimes_{P(z_1)} \mu_{z_2})) \circ (1_{V_e} \boxtimes_{P(z_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1}\\ &\quad = (\mu_{z_2} \circ (\mu_{z_1 - z_2} \boxtimes_{P(z_2)} 1_{V_e})) \circ \mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)} \circ (1_{V_e} \boxtimes_{P(z_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1},\end{aligned} \tag{3.6}$$

where z_1, z_2 are real numbers satisfying $z_1 > z_2 > z_1 - z_2 > 0$, γ_1 and γ_2 are paths in the real line from 1 to z_1 and z_2, respectively, and $\mathcal{T}_{\gamma_1}$ and $\mathcal{T}_{\gamma_2}$ are the parallel transport isomorphisms associated to γ_1 and γ_2, respectively.

From the construction, we have

$$(\mu_{z_1} \circ (1_{V_e} \boxtimes_{P(z_1)} \mu_{z_2})) \circ (1_{V_e} \boxtimes_{P(z_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1} = \mu \circ (1_{V_e} \boxtimes \mu). \tag{3.7}$$

Similarly,

$$(\mu_{z_2} \circ (\mu_{z_1 - z_2} \boxtimes_{P(z_2)} 1_{V_e})) \circ (\mathcal{T}_{\gamma_3} \circ (\mathcal{T}_{\gamma_4} \boxtimes_{P(z_2)} 1_{V_e}))^{-1} = (\mu \circ (\mu \boxtimes 1_{V_e})), \tag{3.8}$$

where γ_3 and γ_4 are paths in the real line from z_2 and $z_1 - z_2$ to 1, respectively, and $\mathcal{T}_{\gamma_3}$ and $\mathcal{T}_{\gamma_4}$ are the parallel transport isomorphisms associated to γ_3 and γ_4, respectively. Combining (3.6)–(3.8) and (2.2), we obtain the associativity

$$\mu \circ (1_{V_e} \boxtimes \mu) = (\mu \circ (\mu \boxtimes 1_{V_e})) \circ \mathcal{A}.$$

Next we prove the commutativity. We recall that the braiding isomorphism $\mathcal{R}$ is characterized by

$$\overline{\mathcal{R}}(u \boxtimes v) = e^{L(-1)} \overline{\mathcal{T}}_{\gamma_1^-}(v \boxtimes_{P(-1)} u) \tag{3.9}$$

where $u, v \in V_e$, γ_1^- is a (clockwise) path from -1 to 1 in the closed upper half plane with 0 deleted, $\mathcal{T}_{\gamma_1^-}$ is the corresponding parallel transport isomorphism and $\overline{\mathcal{T}}_{\gamma_1^-}$ is the natural extension of $\mathcal{T}_{\gamma_1^-}$ to the formal completion $\overline{V_e \boxtimes V_e}$ of $V_e \boxtimes V_e$. Thus

$$\overline{\mu}(\overline{\mathcal{R}}(u \boxtimes v)) = \overline{\mu}(e^{L(-1)} \mathcal{T}_{\gamma_1^-}(v \boxtimes_{P(-1)} u)). \tag{3.10}$$

Let $\mathcal{Y}$ be the intertwining operator corresponding to the intertwining map $\boxtimes_{P(1)}$ of type $\binom{V_e \boxtimes V_e}{V_e\, V_e}$. Then

$$e^{L(-1)} \mathcal{T}_{\gamma_1^-}(v \boxtimes_{P(-1)} u) = e^{L(-1)} \mathcal{Y}(v, e^{\pi i}) u. \tag{3.11}$$

Since μ is a morphism, $\mu \circ \mathcal{Y} = Y_e$, and Y_e satisfies the skew-symmetry

$$Y_e(u,1)v = e^{L(-1)} Y_e(v,-1)u,$$

we have

$$\begin{aligned}\mu(e^{L(-1)}\mathcal{Y}(v,e^{\pi i})u) &= e^{L(-1)}Y_e(v,-1)u \\ &= Y_e(u,1)v \\ &= \overline{\mu}(u \boxtimes v). \end{aligned} \tag{3.12}$$

Combining (3.10)–(3.12), we obtain

$$\overline{\mu}(\overline{\mathcal{R}}(u \boxtimes v)) = \overline{\mu}(u \boxtimes v)$$

for $u, v \in V_e$, or equivalently, the commutativity

$$\mu \circ \mathcal{R} = \mu.$$

For the unit property, we note that the left unit isomorphism $l_{V_e} : V_e \to V \boxtimes V_e$ is defined by $l_{V_e}(u) = \mathbf{1} \boxtimes u$ and thus

$$\begin{aligned}(\mu \circ (\iota_{V_e} \boxtimes 1_{V_e}) \circ l_{V_e})(u) &= \mu((\iota_{V_e} \boxtimes 1_{V_e})(\mathbf{1} \boxtimes u)) \\ &= \mu(\mathbf{1} \boxtimes u) \\ &= Y_e(\mathbf{1},1)u \\ &= 1_{V_e}(u)\end{aligned}$$

for $u \in V_e$.

Finally, we prove the uniqueness of the unit. Let $f \in \mathrm{Hom}_{\mathcal{C}}(V, V_e)$. Since f preserves the grading and $\dim (V_e)_{(0)} = 1$, it is clear that f maps $\mathbf{1}$ to a scalar multiple of $\mathbf{1}$. Since as a module V is generated by $\mathbf{1}$, f is determined by the scalar. Conversely, given any scalar, we can construct an element of $\mathrm{Hom}_{\mathcal{C}}(V, V_e)$ such that it maps $\mathbf{1}$ to the scalar times $\mathbf{1}$. Thus $\dim \mathrm{Hom}_{\mathcal{C}}(V, V_e) = 1$.

Conversely, let (V_e, μ, ι_{V_e}) be a $\mathcal{C}$-algebra. In particular, V_e is a V-module. The module map $\mu : V_e \boxtimes V_e \to V_e$ corresponds to an intertwining operator Y_e of type $\binom{V_e}{V_e V_e}$ such that

$$\overline{\mu}(u \boxtimes v) = Y_e(u,1)v \tag{3.13}$$

for $u, v \in V_e$. Since we have an injective morphism $\iota : V \to V_e$, we can view the vacuum vector $\mathbf{1}$ and the conformal element ω of V as elements of V_e. We now show that $(V_e, Y_e, \mathbf{1}, \omega)$ is a vertex operator algebra satisfying Conditions 1, 2, 3, 4 and 5.

Since $\theta_{V_e} = 1_{V_e}$, V_e is $\mathbb{Z}$-graded. One immediate consequence is that $Y_e(u, x)v \in V_e((x))$ for $u, v \in V_e$. The skew symmetry for Y_e now follows immediately from $\mu \circ \mathcal{R} = \mu$ and the vacuum property follows immediately from the unit property $\mu \circ (\iota_{V_e} \boxtimes 1_{V_e}) \circ l_{V_e} = 1_{V_e}$. The creation property follows from the vacuum property and the skew-symmetry. The Virasoro algebra relations and the $L(0)$-grading property follow from the fact that V_e is a V-module. The $L(-1)$-derivative property follows from the fact that Y_e is an intertwining operator.

We now prove the associativity. As above, for any nonzero complex number z, let $\mu_z : V_e \boxtimes_{P(z)} V_e \to V_e$ be the module map corresponding to the intertwining operator Y_e. By definition, we have

$$\mu_z(u \boxtimes_{P(z)} v) = Y_e(u, z)v = (\mu \circ \mathcal{T}_\gamma)(u \boxtimes_{P(z)} v) \tag{3.14}$$

for $u, v \in V_e$, where z is a nonzero complex number and γ is a path from z to 1 in the complex plane with a cut along the positive real line. Also, (3.7)–(3.8) hold.

Compose both sides of the associativity

$$\mu \circ (1_{V_e} \boxtimes \mu) = (\mu \circ (\mu \boxtimes 1_{V_e})) \circ \mathcal{A}$$

for the $\mathcal{C}$-algebra V_e with

$$((1_{V_e} \boxtimes_{P(z_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1})^{-1},$$

where γ_1 and γ_2 are paths from 1 to z_1 and z_2, respectively, in the complex plane with a cut along the positive real line and z_1 and z_2 are complex numbers satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$. Then we obtain

$$\begin{aligned}
&\mu \circ (1_{V_e} \boxtimes \mu) \circ ((1_{V_e} \boxtimes_{P(z_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1})^{-1} \\
&\quad = (\mu \circ (\mu \boxtimes 1_{V_e})) \circ \mathcal{A} \circ ((1_{V_e} \boxtimes_{P(z_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1})^{-1}.
\end{aligned} \tag{3.15}$$

Using (3.7)–(2.2) and (3.15), we have

$$\begin{aligned}
&\mu_{z_1} \circ (1_{V_e} \boxtimes_{P(z_1)} \mu_{z_2}) \\
&\quad = \mu \circ (1_{V_e} \boxtimes \mu) \circ ((1_{V_e} \boxtimes_{P(z_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1})^{-1} \\
&\quad = (\mu \circ (\mu \boxtimes 1_{V_e})) \circ \mathcal{A} \circ ((1_{V_e} \boxtimes_{P(z_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1})^{-1} \\
&\quad = (\mu_{z_2} \circ (\mu_{z_1 - z_2} \boxtimes_{P(z_2)} 1_{V_e})) \circ (\mathcal{T}_{\gamma_3} \circ (\mathcal{T}_{\gamma_4} \boxtimes_{P(z_2)} 1_{V_e}))^{-1} \\
&\qquad\qquad \circ \mathcal{A} \circ ((1_{V_e} \boxtimes_{P(z_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1})^{-1} \\
&\quad = (\mu_{z_2} \circ (\mu_{z_1 - z_2} \boxtimes_{P(z_2)} 1_{V_e})) \circ \mathcal{A}_{P(z_1), P(z_2)}^{P(z_1 - z_2), P(z_2)}.
\end{aligned} \tag{3.16}$$

For the next step, we need the convergence of products and iterates of intertwining operators for V. Because of the convergence, $\overline{\mu_{z_1}} \circ (1_{V_e} \boxtimes_{P(z_1)} \overline{\mu_{z_2}})$ is well defined and is equal to $\overline{\mu_{z_1} \circ (1_{V_e} \boxtimes_{P(z_1)} \mu_{z_2})}$. Similarly, $\overline{\mu_{z_1}} \circ (\overline{\mu_{z_1-z_2}} \boxtimes_{P(z_2)} 1_{V_e})$ is well defined and is equal to $\overline{\mu_{z_1} \circ (\mu_{z_1-z_2} \boxtimes_{P(z_2)} 1_{V_e})}$. Thus (3.16) gives

$$\begin{aligned}
&\overline{\mu_{z_1}} \circ (1_{V_e} \boxtimes_{P(z_1)} \overline{\mu_{z_2}}) \\
&\qquad = \overline{\mu_{z_1}} \circ (\overline{\mu_{z_1-z_2}} \boxtimes_{P(z_2)} 1_{V_e}) \circ \overline{\mathcal{A}}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}.
\end{aligned} \tag{3.17}$$

Applying both sides of (3.17) to $u \boxtimes_{P(z_1)} (v \boxtimes_{P(z_2)} w)$ for $u, v, w \in V_e$, pairing the result with $v' \in V_e$ and using (3.14) and

$$\overline{\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}}(u \boxtimes_{P(z_1)} (v \boxtimes_{P(z_2)} w)) = (u \boxtimes_{P(z_1-z_2)} v) \boxtimes_{P(z_2)} w,$$

we obtain the associativity

$$\langle v', Y_e(u, z_1)Y_e(v, z_2)w\rangle = \langle v', Y_e(Y_e(u, z_1 - z_2)v, z_2)w\rangle$$

for $u, v, w \in V_e$, $v' \in V_e'$ and $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$.

Since associativity and skew-symmetry imply commutativity, and associativity and commutativity imply rationality (see [Hu2] and [Hu3]), we have proved that V_e is a vertex operator algebra.

From the uniqueness of unit, we have $\dim \mathrm{Hom}_{\mathcal{C}}(V, V_e) = 1$. Assume that there is an element of $(V_e)_{(0)}$ which is not proportional to $\mathbf{1}$. Then this element generates a V-submodule of the V-module V_e. Since every V-module is completely reducible, this V-submodule is completely reducible. Thus there exists an irreducible V-submodule of this V-submodule that is generated by an element of $(V_e)_{(0)}$ not proportional to $\mathbf{1}$. Since any irreducible V-module having a nonzero element of weight 0 must be equivalent to V, this irreducible V-submodule is equivalent to V. But this V-submodule is not equal to $\iota_{V_e}(V) \subset V_e$ since its generator of weight 0 is not proportional to $\mathbf{1}_e$. Thus $\dim \mathrm{Hom}_{\mathcal{C}}(V, V_e) > 1$, and this is a contradiction. Hence $\dim (V_e)_{(0)} = 1$.

∎

Remark 3.3 From the proof, we see that the additional assumptions in Theorem 3.2 (that is, the assumptions that V is simple, every V-module is completely reducible, for any irreducible V-module W the weights of elements of W are nonnegative real numbers and $W_{(0)} \neq 0$ if and only if W is equivalent to V viewed as a V-module, and $\dim V_{(0)} = 1$) are needed only in the proof concerning the uniqueness of unit. If we remove these additional assumptions, then the proof above shows that an extension $V_e \supset V$ is equivalent to a (not necessarily haploid) $\mathcal{C}$-algebra.

Theorem 3.4 *Let V, V_e, and $\mathcal{C}$ be as in Theorem 3.2. Then the category of modules for the vertex operator algebra V_e is isomorphic to the category $\mathrm{Rep}^0 V_e$ as defined in [KO], where V_e is considered as an algebra in $\mathcal{C}$.*

Proof. A V_e-module is clearly an object of $\mathrm{Rep}^0 V_e$. Conversely, the proof of Theorem 3.2 shows that the vertex operator map for an object of $\mathrm{Rep}^0 V_e$ satisfies the associativity and the skew-symmetry. Since for modules, the results that associativity and skew-symmetry imply commutativity and that associativity and commutativity imply rationality still hold (actually more general results hold for intertwining operators; see [Hu2] and [Hu3]), we see that this object is actually a V_e-module. This correspondence between the two categories is an isomorphism of categories. ∎

Theorem 3.5 *Let $V, \mathcal{C}$ and V_e be as in Theorem 3.2. Assume in addition that V_e is simple as a $\mathcal{C}$-algebra, that there are finitely many inequivalent irreducible V-modules, and that $\mathcal{C}$ is rigid. Then the fusion rules for V and for V_e are finite, every V_e-module is completely reducible, and V_e as a vertex operator algebra has finitely many inequivalent irreducible modules.*

Proof. Notice that under the assumptions, the category of V-modules has a natural structure of rigid balanced braided tensor category. In particular, we can use the results of Section 1 in [KO].

Since the tensor product of two V-modules as a V-module is a direct sum of irreducible V-modules, no irreducible V-module can have infinite multiplicity in the decomposition of this tensor product. Thus the fusion rules among irreducible V-modules are finite and the fusion rules among general V-modules are also finite. Since V is a subalgebra of V_e, V_e-modules are V-modules. Thus intertwining operators among V_e-modules are intertwining operators among these V_e-modules as V-modules. Since all the fusion rules for V-modules are finite, all the fusion rules for V_e are finite. By Theorem 3.3 in [KO], we also know that any V_e-module is completely reducible.

We now prove that there are only finitely many inequivalent irreducible V_e-modules. Assume that there are infinitely many. Since every V-module is completely reducible, any V_e-module as a V-module is equivalent to a direct sum of irreducible V-modules. Since there are only finitely many inequivalent irreducible V-modules, every irreducible V_e-module as a V-module is equivalent to a direct sum of copies of these irreducible V-modules. Take any irreducible V_e-module W. Then the multiplicity of each irreducible V-module in the decomposition of W as a V-module is finite. Since there are infinitely many irreducible V_e-modules, there must be an irreducible V_e-module W_1 such that the multiplicity of at least one irreducible V-module in the decomposition of W_1 as a V-module is larger than the corresponding multiplicity in the decomposition of W. Since W and W_1 are inequivalent as V-modules, they are also inequivalent as V_e-modules. We can continue this process to find infinitely many inequivalent irreducible V_e-modules W_i for $i \in \mathbb{Z}_+$ such that W_i as a V-module is equivalent to a V-submodule of W_{i+1}. In particular, W as a V-module is equivalent to a V-submodule of W_i for $i \in \mathbb{Z}_+$. For simplicity, we shall identify W with the equivalent V-submodules of W_i for $i \in \mathbb{Z}_+$. Then all these infinitely many irreducible V_e-modules are generated by W.

By the main result of [P] (see Theorem 1.10 in [KO]), the category of V_e-module is a braided tensor category with the tensor product bifunctor $\boxtimes^{V_e}$ defined as in the proof of Theorem 1.5 in [KO] (here we use the superscript V_e rather than the subscript to avoid confusion with the notation $\boxtimes_{P(z)}$ discussed in Section 2). For $u \in V_e$ and $w \in W$, $u \boxtimes w \in \overline{V_e \boxtimes W}$. By definition, $V_e \boxtimes^{V_e} W$ is a quotient of $V_e \boxtimes W$. In particular, $\overline{V_e \boxtimes^{V_e} W}$ is a quotient of $\overline{V_e \boxtimes W}$. We use $u \boxtimes^{V_e} w$ to denote the coset of $u \boxtimes w$ in $\overline{V_e \boxtimes^{V_e} W}$. For $i \in \mathbb{Z}_+$, there is a linear map from $V_e \boxtimes^{V_e} W$ to W_i such that its extension maps $u \boxtimes^{V_e} w$ to $Y_{W_i}(u, 1)w$. Clearly, this is a surjective V_e-module map from $V_e \boxtimes^{V_e} W$ to W_i for $i \in \mathbb{Z}_+$. In particular, W_i for $i \in \mathbb{Z}_+$ are irreducible quotients of $V_e \boxtimes^{V_e} W$ by V_e-submodules. Since every V_e-module is completely reducible, the V_e-module $V_e \boxtimes^{V_e} W$ is equivalent to a direct sum of irreducible V_e-modules. Since V has only finitely many irreducible modules and $V_e \boxtimes^{V_e} W$ is a V_e-module whose homogeneous subspaces are finite-dimensional, $V_e \boxtimes^{V_e} W$ must be a finite direct sum of irreducible V_e-modules. In fact, if it is an infinite direct sum, then as a V-module it is also an infinite direct sum of irreducible V-modules. Since there are only finitely many inequivalent irreducible V-modules, any such infinite direct sum must have infinite-dimensional homogeneous spaces, a contradiction. Thus any quotient V_e-module of $V_e \boxtimes^{V_e} W$ must also be equivalent to a finite direct sum of the direct summands of $V_e \boxtimes^{V_e} W$. In particular, each W_i for $i \in \mathbb{Z}_+$ must be equivalent to a direct summand of $V_e \boxtimes^{V_e} W$. Since there can only be finitely many such direct summands, we have a contradiction, so that there must be only finitely many inequivalent irreducible V_e-modules. ∎

Using Theorems 3.2–3.5, we obtain the following result:

Theorem 3.6 *Let V be a simple vertex operator algebra such whose module category $\mathcal{C}$ has a natural vertex tensor category structure as described in Section 2. We assume in addition the following: There are finitely many inequivalent irreducible V-modules; every V-module is completely reducible; for any irreducible V-module W, the weights of elements of W are nonnegative and $W_{(0)} \neq 0$ if and only if W is equivalent to V; $\dim V_{(0)} = 1$; and the braided tensor category structure on $\mathcal{C}$ is rigid. Then the following two notions are equivalent:*

1. *An extension $V_e \supset V$, where V_e is a vertex operator algebra such that V is a subalgebra of V_e, its module category has a natural braided tensor category structure as described in Section 2, and it also satisfies the additional assumptions above for V with V replaced by V_e, that is: There are finitely many inequivalent irreducible V_e-modules; every V_e-module is completely reducible; for any irreducible V_e-module W, the weights of elements of W are nonnegative and $W_{(0)} \neq 0$ if and only if W is equivalent to V_e; and $\dim(V_e)_{(0)} = 1$.*

2. *A haploid $\mathcal{C}$-algebra V_e with $\theta_{V_e} = 1_{V_e}$.*

In addition, the category of modules for such a vertex operator algebra V_e is isomorphic to the category $\mathrm{Rep}^0 V_e$ as defined in [KO], where V_e is the corresponding $\mathcal{C}$-algebra.

Proof. By Theorem 3.2, a $\mathcal{C}$-algebra V_e with $\theta_{V_e} = 1_{V_e}$ is equivalent to a vertex operator algebra V_e such that V is a subalgebra of V_e and $\dim(V_e)_{(0)} = 1$. By Theorem 3.4, the category of modules for such a vertex operator algebra V_e is isomorphic to the category $\mathrm{Rep}^0 V_e$. By the main result of [P] (see Theorem 1.10 in [KO]), the category $\mathrm{Rep}^0 V_e$ has a natural braided tensor category structure.

By Theorem 3.5, every V_e-module is completely reducible and V_e as a vertex operator algebra has finitely many irreducible modules. Since any

V_e-module is a V-module, for any irreducible V_e-module W, the weights of elements of W are nonnegative. If there is a V_e-module W such that $W_{(0)} \neq 0$, then W as a V-module must contain a V-submodule equivalent to V. Since W is irreducible as a V_e-module, W is generated as a V_e-module by this V-submodule. Since V_e is generated as a V_e-module by V and is also irreducible as a V_e-module, W is equivalent to V_e as a V_e-module. ∎

Remark 3.7 Let V and $\mathcal{C}$ be as in Theorem 3.6. Then by Lemma 1.20 and Theorem 1.15 in [KO], a $\mathcal{C}$-algebra V_e with $\theta_{V_e} = 1_{V_e}$ is rigid if and only if V_e as a V_e-module is irreducible. In this case, the braided tensor category of V_e-modules when V_e is viewed as a vertex operator algebra is rigid.

Remark 3.8 In the theorems given in this section, we have to assume that the category of V-modules has a natural structure of vertex tensor category structure, as described in the preceding section. The existence of only a braided tensor category structure (even if it is rigid and balanced) does not seem to be enough for these results to hold.

Appendix

Here we recall general theorems stating that under suitable conditions, the module category for a vertex operator algebra has a natural vertex tensor category structure and in particular braided tensor category structure. We also recall a general theorem stating that under suitable conditions, this braided tensor category is rigid and is in fact a modular tensor category.

We first state some conditions that we shall be considering here (where we are using notation and terminology from [HL3]–[HL7], [Hu1]):

1. Finite reductivity (called "rationality" in [HL3]): Every V-module is completely reducible, there exist only finitely many inequivalent irreducible modules, and the spaces of intertwining operators among triples of irreducible modules are all finite dimensional. (Note that this finite reductivity is different from the various notions of "rationality" in [Z] and other works. Also note that the first two conditions (every V-module is completely reducible and there exist only finitely many inequivalent irreducible modules) together with the finite dimensionality of the homogeneous subspaces of V-modules imply that every V-module is of finite length.)

2. The convergence and extension property for products: For any V-modules W_1, W_2, W_3, W_4 and W_5 and any intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of types $\binom{W_4}{W_1 W_5}$ and $\binom{W_5}{W_2 W_3}$, respectively, there exists an integer N (depending only on $\mathcal{Y}_1$ and $\mathcal{Y}_2$), and for any homogeneous elements $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ and any elements $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$, there exist $M \in \mathbb{N}$, $r_k, s_k \in \mathbb{R}$ ($k = 1, \ldots, M$), and analytic functions $f_k(z)$ on $|z| < 1$ ($k = 1, \ldots, M$) satisfying

$$\text{wt } w_{(1)} + \text{wt } w_{(2)} + s_k > N, \quad k = 1, \ldots, M,$$

such that

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_2)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_1=z_1,\, x_2=z_2}$$

is absolutely convergent when $|z_1| > |z_2| > 0$ and can be analytically extended to the multivalued analytic function

$$\sum_{k=1}^{M} z_2^{r_k}(z_1 - z_2)^{s_k} f_k\left(\frac{z_1 - z_2}{z_2}\right)$$

in the region $|z_2| > |z_1 - z_2| > 0$.

3. Convergence of products of intertwining operators: For any $n \geq 3$, any V-modules $W_0, \ldots, W_{n+1}$ and $\widetilde{W}_1, \ldots, \widetilde{W}_{n-1}$, any intertwining operators

$$\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_n$$

of types

$$\binom{W_0}{W_1 \widetilde{W}_1}, \binom{\widetilde{W}_1}{W_2 \widetilde{W}_2}, \ldots, \binom{\widetilde{W}_{n-1}}{W_n W_{n+1}},$$

respectively, and any $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1, \ldots, w_{(n+1)} \in W_{n+1}$, the series

$$\langle w'_{(0)}, \mathcal{Y}_1(w_{(1)}, x_1) \cdots \mathcal{Y}_n(w_{(n)}, x_n) w_{(n+1)}\rangle|_{x_j = z_j,\, j=1,\ldots,n} \tag{A.1}$$

is absolutely convergent in the region $|z_1| > \cdots > |z_n| > 0$ and its sum can be analytically extended to a multivalued analytic function on the region given by $z_i \neq 0$, $i = 1, \ldots, n$, $z_i \neq z_j$, $i \neq j$, such that for any set of possible singular points with either $z_i = 0$, $z_i = \infty$ or $z_i = z_j$

for $i \neq j$, this multivalued analytic function can be expanded near the singularity as a series having the same form as the expansion near the singular points of a solution of a system of differential equations with regular singular points.

4. Every finitely-generated lower-bounded generalized V-module is a module.

5. V is of positive energy ($V_{(0)} = \mathbb{C}\mathbf{1}$ and $V_{(n)} = 0$ for $n < 0$) and the V-module V' contragredient to V is equivalent to V.

6. Every grading-restricted generalized V-module is a (finite) direct sum of irreducible V-modules.

7. V is C_2-cofinite, that is, $\dim V/C_2(V) < \infty$ where $C_2(V)$ is the subspace of V spanned by the elements of the form $\mathrm{Res}_x x^{-2} Y(u, x)v$ for $u, v \in V$. (This condition was introduced by Zhu in [Z], where it was called "Condition C.")

The following theorem was proved in [HL3]–[HL7], [Hu1]:

Theorem A.1 *Let V be a vertex operator algebra satisfying Conditions 1, 2, 3 and 4 above. Then the category $\mathcal{C}$ of V-modules has a natural structure of vertex tensor category and, in particular, $\mathcal{C}$ has a natural structure of braided tensor category.*

In [Hu4], the first author proved that if all V-modules satisfy a condition called the C_1-cofiniteness condition, then the fusion rules are all finite, and if in addition all the grading-restricted generalized V-modules are completely reducible, then Conditions 2 and 3 above hold. In particular, one can replace Conditions 2 and 3 in the theorem by these conditions. In [Hu6], the first author proved that if V satisfies Conditions 5 and 7, then any finitely generated lower-bounded generalized V-module is grading restricted. Thus Conditions 5, 6 and 7 imply Condition 4. One consequence of these results is that Conditions 5, 6 and 7 imply Conditions 1, 2, 3 and 4. In particular, for a vertex operator algebra V satisfying Conditions 5, 6 and 7, the category of V-modules has a natural structure of braided tensor category.

The following theorem was proved in [Hu5]:

Theorem A.2 *Let V be a simple vertex operator algebra satisfying Conditions 5, 6 and 7 above. Then the braided tensor category $\mathcal{C}$ of V-modules is rigid and satisfies the nondegeneracy condition. In particular, $\mathcal{C}$ is a modular tensor category.*

Theorem A.1 has been generalized in a number of directions, including in particular non-semisimple module categories, by Zhang and the first and third authors. Since the proof of Theorem 3.2 requires only vertex tensor category structure on suitable module categories, the conclusion of Theorem 3.2 remains valid in this greater generality.

References

[FFFS] G. Felder, J. Fröhlich, J. Fuchs and C. Schweigert, Correlation functions and boundary conditions in rational conformal field theory and three-dimensional topology, *Compositio Math.* **131** (2002) 189–237.

[FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, *Memoirs Amer. Math. Soc.* **104**, 1993.

[FLM] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Appl. Math., Vol. 134, Academic Press, Boston, 1988.

[Ho] G. Höhn, Genera of vertex operator algebras and three dimensional topological quantum field theories, in: *Vertex Operator Algebras in Mathematics and Physics (Toronto, 2000)*, Fields Inst. Commun., Vol. 39, Amer. Math. Soc., Providence, 2003, 89–107.

[Hu1] Y.-Z. Huang, A theory of tensor products for module categories for a vertex operator algebra, IV, *J. Pure Appl. Alg.*, **100** (1995), 173–216.

[Hu2] Y.-Z. Huang, Virasoro vertex operator algebras, (nonmeromorphic) operator product expansion and the tensor product theory, *J. Alg.* **182** (1996), 201–234.

[Hu3] Y.-Z. Huang, Generalized rationality and a generalized Jacobi identity for intertwining operator algebras, *Selecta Math. (N.S.)* **6** (2000), 225–267.

[Hu4] Y.-Z. Huang, Differential equations and intertwining operators, *Comm. Contemp. Math.* **7** (2005), 375–400.

[Hu5] Y.-Z. Huang, Rigidity and modularity of vertex tensor categories, *Comm. Contemp. Math.* **10** (2008), 871–911.

[Hu6] Y.-Z. Huang, Cofiniteness conditions, projective covers and the logarithmic tensor product theory, *J. Pure Appl. Alg.* **213** (2009), 458–475.

[HK] Y.-Z. Huang and L. Kong, Open-string vertex algebras, tensor categories and operads, *Comm. Math. Phys.* **250** (2004), 433–471.

[HL1] Y.-Z. Huang and J. Lepowsky, Toward a theory of tensor products for representations of a vertex operator algebra, in: *Proc. 20th International Conference on Differential Geometric Methods in Theoretical Physics, New York, 1991*, ed. S. Catto and A. Rocha, World Scientific, Singapore, 1992, Vol. 1, 344–354.

[HL2] Y.-Z. Huang and J. Lepowsky, Tensor products of modules for a vertex operator algebras and vertex tensor categories, in: *Lie Theory and Geometry, in Honor of Bertram Kostant,* ed. R. Brylinski, J.-L. Brylinski, V. Guillemin and V. Kac, Birkhäuser, Boston, 1994, 349–383.

[HL3] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, I, *Selecta Mathematica (New Series)* **1** (1995), 699–756.

[HL4] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, II, *Selecta Mathematica (New Series)* **1** (1995), 757–786.

[HL5] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, III, *J. Pure Appl. Alg.* **100** (1995) 141–171.

[HL6] Y.-Z. Huang and J. Lepowsky, Intertwining operator algebras and vertex tensor categories for affine Lie algebras, *Duke Math. J.* **99** (1999), 113–134.

[HL7] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, V, to appear.

[KO] A. Kirillov Jr. and V. Ostrik, *On a q-analog of the McKay correspondence and the ADE classification of* $\widehat{\mathfrak{sl}}_2$ *conformal field theories*, *Advances in Math.* **171** (2002), 183–227.

[K] L. Kong, Full field algebras, operads and tensor categories, *Adv. in Math.* **213** (2007), 271–340.

[MS] G. Moore and N. Seiberg, Classical and quantum conformal field theory, *Comm. Math. Phys.* **123** (1989), 177–254.

[P] B. Pareigis, On braiding and dyslexia, *J. Alg.* **171** (1995), 413–425.

[Z] Y. Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* **9** (1996), 237–307.

Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019
E-mail address: yzhuang@math.rutgers.edu

Department of Mathematics, State University of New York at Stony Brook, Stony Brook, NY 11794
E-mail address: kirillov@math.sunysb.edu

Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019
E-mail address: lepowsky@math.rutgers.edu