Differential equations and conformal field theories

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Abstract

We discuss the recent results of the author on the existence of systems of differential equations for chiral genus-zero and genus-one correlation functions in conformal field theories.

1 Introduction

Two-dimensional conformal field theories form a particular class of nontopological quantum field theories which have now been formulated and studied rigorously using various methods from different branches of mathematics. In physics, these theories describe perturbative string theory and also critical phenomena in condensed matter physics. They are also used to describe such phenomena as disorder in condensed matter physics and to construct nonperturbative objects such as D-branes in string theory. In mathematics, they are closely related to infinite-dimensional Lie algebras, infinite-dimensional integrable systems, the Monster (the largest finite sporadic simple group), modular functions and modular forms, Riemann surfaces and algebraic curves, knot and three-manifold invariants, Calabi-Yau manifolds and mirror symmetry, and many other branches of mathematics. We also expect that many mathematical problems can be solved by constructing and studying the corresponding conformal field theories. Moreover, the study of such theories might provide hints to possible deep connections among these different branches of mathematics and will probably shed light on the construction and study of higher-dimensional nontopological quantum field theories.

Mathematically, a geometric formulation of conformal field theory was first given around 1987 by Segal [S1] [S2] [S3] and Kontsevich. In [S2] and [S3], Segal further introduced the important notions of modular functor and weakly conformal field theory which describe mathematically the subtle and deep chiral structures in conformal field theories. One urgent problem is to give a construction of (chiral) conformal field theories in this sense.

To construct conformal field theories in this sense and to study these conformal field theories, it is necessary to construct and study chiral correlation functions on Riemann surfaces. For chiral correlation functions on genus-zero Riemann surfaces (or simply called chiral genus-zero correlation functions) associated to lowest weight vectors in minimal models [BPZ] and in Wess-Zumino-Novikov-Witten models [W], Belavin-Polyakov-Zamolodchikov and Knizhnik-Zamolodchikov found in their seminal works [BPZ] and [KZ], respectively, that these functions actually satisfy certain systems of differential equations of regular singular points (now called the BPZ equations and the KZ equations, respectively). In the case of Wess-Zumino-Novikov-Witten models, it is also known from the works of Tsuchiya-Ueno-Yamada [TUY] and Bernard [B1] [B2] that chiral correlation functions on higher-genus Riemann surfaces (or simply called chiral higher-genus correlation functions) satisfy systems of differential equations of KZ type. These equations play fundamental roles in the construction and study of the minimal models and Wess-Zumino-Novikov-Witten models.

A natural question is whether for general conformal field theories satisfying natural conditions, there exist systems of differential equations of regular singular points satisfied by chiral genus-zero correlation functions. More generally, we are interested in whether there exist systems of differential equations for chiral higher-genus correlation functions. The existence of such equations will allow us to study chiral correlation functions using the theory of differential equations and to construct chiral conformal field theories using these correlation functions.

Recently, in [H8] and [H10], the author established the existence of such differential equations in the genus-zero and genus-one cases under suitable natural conditions and applied these equations to the construction of genus-zero and genus-one chiral theories. In the present paper, after a brief discussion of the notion of conformal field theories in the sense of Segal and Kontsevich, we give an overview of these differential equations. For details, see [H8] and [H10]. For a recent exposition on conformal field theories in the sense of Segal and Kontsevich and the author's program of constructing such theories from representations of vertex operator algebras, see [H9].

In the next section, we recall roughly what a conformal field theory is in the sense of Segal [S1] [S2] [S3] and Kontsevich and what a weakly conformal field theory is in the sense of Segal [S2] [S3]. We discuss systems of differential equations for chiral genus-zero and genus-one correlation functions in Sections 3 and 4, respectively.

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2 Conformal field theories

Consider the following geometric category: The objects are ordered finite sets of copies of S^1. The morphisms are conformal equivalence classes of Riemann surfaces whose boundary components are analytically parametrized by the copies of S^1 in their domains and codomains. The compositions of morphisms are given using the boundary parametrizations in the obvious way. This category has a symmetric monoidal category structure for which the monoidal structure is defined by disjoint unions of objects and morphisms.

Roughly speaking, a *conformal field theory* is a projective linear representation of this category, that is, a locally convex topological vector space H (called the *state space*) with a nondegenerate bilinear form and a projective functor from this category to the symmetric monoidal category with traces generated by H (that is, the category whose objects are tensor powers of H and morphisms are trace-class maps), satisfying some natural conditions.

Conformal field theories in general have holomorphic (or chiral) and antiholomorphic (or antichiral) parts. Both parts also satisfy an axiom system which defines *weakly conformal field theories*. Roughly speaking, weakly conformal field theories are representations of geometric categories obtained from holomorphic vector bundles over the moduli space of Riemann surfaces with parametrized boundaries.

Our strategy is to construct chiral or antichiral genus-zero and genus-one parts of conformal field theories first and then using these to construct the full conformal field theories. In the remaining part of this paper, we shall discuss only chiral genus-zero and genus-one theories.

3 Differential equations and chiral genus-zero correlation functions

We first explain the main ingredients of chiral or antichiral genus-zero theories. The chiral or antichiral parts of genus-zero theories have been shown to be essentially equivalent to algebras of intertwining operators among modules for suitable vertex operator algebras (see [H2]–[H5] and [H7]). So here we briefly describe vertex operator algebras, modules and intertwining operators.

A vertex operator algebra is a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ equipped with a *vertex operator map* $Y : V \otimes V \to V[[z, z^{-1}]]$, a *vacuum* $\mathbf{1} \in V$ and a *conformal element* $\omega \in V$, satisfying a number of axioms. One version of the main axiom is the following: For $u_1, u_2, v \in V$, $v' \in V' = \coprod_{n\in\mathbb{Z}} V^*_{(n)}$, the series

$$\langle v', Y(u_1, z_1)Y(u_2, z_2)v\rangle$$
$$\langle v', Y(u_2, z_2)Y(u_1, z_1)v\rangle$$
$$\langle v', Y(Y(u_1, z_1 - z_2)u_2, z_2)v\rangle$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$. Other axioms are:

$$\dim V_{(n)} < \infty,$$

$$V_{(n)} = 0$$

when n is sufficiently negative (these are called the *grading-restriction conditions*); for $u, v \in V$, $Y(u, z)v$ contain only finitely many negative power terms;

for $u \in V$,

$$Y(\mathbf{1}, z) = 1, \quad \lim_{z\to 0} Y(u, z)\mathbf{1} = u;$$

let $L(n) : V \to V$ be defined by $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$, then

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{c}{12}(m^3 - m)\delta_{m+b,0}$$

(c is called the *central charge* of V),

$$\frac{d}{dz}Y(u, z) = Y(L(-1)u, z) \qquad \text{for } \ u \in V$$

and

$$L(0)u = nu \qquad \text{for } \ u \in V_{(n)}$$

(n is called the *weight* of u and is denoted wt u). For $u \in V$, we write $Y(u, z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1}$ where $u_n \in \mathrm{End}\ V$.

A *V-module* is an $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{(n)}$ equipped with a vertex operator map $Y_W : V \otimes W \to W[[z, z^{-1}]]$ satisfying all those axioms for V which still make sense. Let W_1, W_2 and W_3 be V-modules. An *intertwining operators* of type $\binom{W_3}{W_1 W_2}$ is a linear map $\mathcal{Y} : W_1 \otimes W_2 \to W_3\{z\}$, where $W_3\{z\}$ is the space of all series in complex powers of z with coefficients in W_3, satisfying all those axioms for V which still make sense. That is, for $w_1 \in W_1$ and $w_2 \in W_2$, the real parts of the powers of z in nonzero terms in the series $\mathcal{Y}(w_1, z_2)w_2$ have a lower bound; for $u \in V$, $w_1 \in W_1$, $w_2 \in W_2$ and $w_3' \in W_3' = \coprod_{n\in\mathbb{C}} (W_3)^*_{(n)}$,

$$\langle w_3', Y_{W_3}(u, z_1)\mathcal{Y}(w_1, z_2)w_2\rangle$$
$$\langle w_3', \mathcal{Y}(w_1, z_2)Y_{W_2}(u, z_1)w_2\rangle$$
$$\langle w_3', \mathcal{Y}(Y_{W_1}(u, z_1 - z_2)w_1, z_2)w_2\rangle$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, to a common (multivalued) analytic function in z_1 and z_2 with the only possible singularities (branch points) at $z_1, z_2 = 0$ and $z_1 = z_2$; also

$$\frac{d}{dz}\mathcal{Y}(w_1, z) = Y(L(-1)w_1, z).$$

For more details on basic notions and properties in the theory of vertex operator algebras, see [FLM] and [FHL].

We need the following notions to state the result on differential equations in the genus-zero case: Let V be a vertex operator algebra and W a V-module. Let $C_1(W)$ be the subspace of W spanned by elements of the form $u_{-1}w$ for $u \in V_+ = \coprod_{n>0} V_{(n)}$ and $w \in W$. If $\dim W/C_1(W) < \infty$, we say that W is *C_1-cofinite* or W satisfies the *C_1-cofiniteness condition.*

For chiral genus-zero theories, the main objects we want to construct and study are chiral genus-zero correlation functions. Let W_i for $i = 0, \dots, n+1$ and $\widetilde{W}_i$ for $i = 1, \dots, n-1$ be V-modules and let $\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_{n-1}, \mathcal{Y}_n$ be intertwining operators of types $\binom{W_0'}{W_1 \widetilde{W}_1}$, $\binom{\widetilde{W}_1}{W_2 \widetilde{W}_2}, \dots, \binom{\widetilde{W}_{n-2}}{W_{n-1} \widetilde{W}_{n-1}}$, $\binom{\widetilde{W}_{n-1}}{W_n W_{n+1}}$, respectively.

Let $w_i \in W_i$ for $i = 0, \ldots, n+1$. Formally, chiral genus-zero correlation functions are given by series of the form

$$\langle w_0, \mathcal{Y}_1(w_1, z_1) \cdots \mathcal{Y}_n(w_n, z_n) w_{n+1} \rangle. \tag{3.1}$$

Theorem 3.1 *Let W_i for $i = 0, \ldots, n+1$ be V-modules satisfying the C_1-cofiniteness condition. Then for any $w_i \in W_i$ for $i = 0, \ldots, n+1$, there exist*

$$a_{k,\, l}(z_1, \ldots, z_n) \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}, (z_1 - z_2)^{-1}, (z_1 - z_3)^{-1}, \ldots, (z_{n-1} - z_n)^{-1}],$$

for $k = 1, \ldots, m$ and $l = 1, \ldots, n$, such that for any V-modules $\widetilde{W}_i$ for $i = 1, \ldots, n-1$, any intertwining operators $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_{n-1}, \mathcal{Y}_n$, of types $\binom{W_0'}{W_1 \widetilde{W}_1}$, $\binom{\widetilde{W}_1}{W_2 \widetilde{W}_2}, \ldots, \binom{\widetilde{W}_{n-2}}{W_{n-1} \widetilde{W}_{n-1}}$, $\binom{\widetilde{W}_{n-1}}{W_n W_{n+1}}$, respectively, the series (3.1) satisfy the expansions of the system of differential equations

$$\frac{\partial^m \varphi}{\partial z_l^m} + \sum_{k=1}^{m} a_{k,\, l}(z_1, \ldots, z_n) \frac{\partial^{m-k} \varphi}{\partial z_l^{m-k}} = 0, \quad l = 1, \ldots, n$$

in the region $|z_1| > \cdots |z_n| > 0$. Moreover, there exist $a_{k,\, l}(z_1, \ldots, z_n)$ for $k = 1, \ldots, m$ and $l = 1, \ldots, n$ such that the singular points of the corresponding system are regular.

Similar systems of differential equations have also been obtained by Nagatomo and Tsuchiya in [NT].

Using these equations and other results on vertex operator algebras, modules and intertwining operators, chiral genus-zero weakly conformal field theories in the sense of Segal have been constructed. In particular, the direct sum of a complete set of inequivalent irreducible modules for a suitable vertex operator algebra has a natural structure of an intertwining operator algebra. Thus for such a vertex operator algebra, (3.1) is absolutely convergent in the region $|z_1| > \cdots |z_n| > 0$ and associativity and commutativity for intertwining operators hold. For more details on intertwining operator algebras and chiral genus-zero weakly conformal field theories, see [H1]–[H9].

4 Differential equations and chiral genus-one correlation functions

The second logical step is to construct chiral genus-one theories, that is, to construct maps associated to genus-one surfaces and prove the axioms which make sense for genus-one surfaces. Assume we have a weakly conformal field theory in the sense of Segal. Then for given elements in the state space of the theory, these maps give certain functions on the moduli space of genus-one surfaces with punctures (the space of conformal equivalence classes of such surfaces). They can be viewed as (multivalued) functions of $z_1, \ldots, z_n \in \mathbb{C}$ and $\tau \in \mathbb{H}$ (the upper half plane). Here as usual, τ corresponds to a torus given by the parallelogram with vertices $0, 1, \tau$ and $1+\tau$ and $z_1, \dots, z_n$ correspond to points on the torus.

But functions of $z_1, \dots, z_n$ and τ are in general only functions on the Teichmüller space, not functions on the moduli space. To construct genus-one theories, we do need to construct mathematical objects (vector bundles and holomorphic sections) on the moduli space, not the Teichmüller space. The moduli space is the quotient of the Teichmüller space $\mathbb{H}$ by the modular group $SL(2, \mathbb{Z})$. So we have to construct $SL(2, \mathbb{Z})$-invariant spaces of functions of the form above. These functions in the $SL(2, \mathbb{Z})$-invariant spaces are called *chiral genus-one correlation functions*.

The first result in this step was obtained by Zhu [Z]. He constructed chiral genus-one correlation functions associated to elements of a suitable vertex operator algebra V. Using his method, Miyamoto [M] constructed chiral genus-one correlation functions associated to elements of V-modules among which at most one is not isomorphic to V. But Zhu's method cannot be generalized to construct chiral genus-one correlation functions associated to elements of V-modules among which at least two are not isomorphic to V, because he used a recurrence formula which cannot be generalized to this general case.

In [H10], the author solved completely the problem of constructing chiral genus-one correlation functions from chiral genus-zero correlation functions. As in the genus-zero case, one of the main tools is systems of differential equations.

To construct these functions from representaions of a vertex operator algebra, we need some conditions on the vertex operator algebra and its modules. We first need some concepts: Let V be a vertex operator algebra and W a V-module. Let $C_2(W)$ be the subspace of W spanned by elements of the form $u_{-2}w$ for $u, w \in W$. Then we say that W is *C_2-cofinite* or satisfies the *C_2-cofiniteness condition* if $\dim W/C_2(W) < \infty$. It is easy to see that if $V_{(n)} = 0$ when $n < 0$ and $V_{(0)} = \mathbb{C}\mathbf{1}$, a V-module W is C_2-cofinite implies W is C_1-cofinite.

We now assume that the vertex operator algebra V satisfies the condition that (3.1) is absolutely convergent in the region $|z_1| > \cdots |z_n| > 0$ and associativity and commutativity for intertwining operators hold. We shall use the notation $q_z = e^{2\pi i z}$ for $z \in \mathbb{C}$. Let W_i, $\widetilde{W}_i$ be V-modules, and $w_i \in W_i$ for $i = 1, \dots, n$. For any intertwining operators $\mathcal{Y}_i$, $i = 1, \dots, n$, of types $\binom{\widetilde{W}_{i-1}}{W_i \widetilde{W}_i}$, respectively, let

$$\begin{aligned} &F_{\mathcal{Y}_1, \dots, \mathcal{Y}_n}(w_1, \dots, w_n; z_1, \dots, z_n; \tau) \\ &\quad = \mathrm{Tr}_{\widetilde{W}_n} \mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1}) \cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n}) q_\tau^{L(0) - \frac{c}{24}}, \end{aligned} \tag{4.1}$$

where c is the central charge of V and for $z \in \mathbb{C} \setminus \{0\}$,

$$U(q_z) = e^{(\log |q_z| + i \arg q_z) L(0)} e^{-L^+(A)}, \quad 0 \leq \arg q_z < 2\pi,$$

$L^+(A) = \sum_{j \geq 1} A_j L(j)$ and A_j for $j \geq 1$ are determined by

$$\frac{1}{2\pi i} \log(1 + 2\pi i w) = \left(\exp\left(\sum_{j \in \mathbb{Z}_+} A_j w^{j+1} \frac{\partial}{\partial w} \right) \right) w$$

Let

$$\begin{aligned}
\wp_1(z;\tau) &= \frac{1}{z} + \sum_{(k,l)\neq(0,0)} \left(\frac{1}{z-(k\tau+l)} + \frac{1}{k\tau+l} + \frac{z}{(k\tau+l)^2} \right), \\
\wp_2(z;\tau) &= \frac{1}{z^2} + \sum_{(k,l)\neq(0,0)} \left(\frac{1}{(z-(k\tau+l))^2} - \frac{1}{(k\tau+l)^2} \right) = -\frac{\partial}{\partial z}\wp_1(z;\tau)
\end{aligned}$$

be the Weierstrass zeta function and the Weierstrass $\wp$-function, respectively, and let $\wp_m(z;\tau)$ for $m>2$ be the elliptic functions defined recursively by

$$\wp_{m+1}(z,\tau) = -\frac{1}{m}\frac{\partial}{\partial z}\wp_m(z;\tau).$$

We also need the Eisenstein series

$$\begin{aligned}
G_2(\tau) &= \frac{\pi^2}{3} + \sum_{m\in\mathbb{Z}\setminus\{0\}} \sum_{l\in\mathbb{Z}} \frac{1}{(m\tau+l)^2}, \\
G_{2k+2}(\tau) &= \sum_{(m,l)\neq(0,0)} \frac{1}{(m\tau+l)^{2k+2}}, \qquad\qquad k\geq 1.
\end{aligned}$$

See, for example, [K] and [L], for detailed discussions on these elliptic functions and the Eisenstein series. Let

$$R = \mathbb{C}[G_4(\tau), G_6(\tau), \wp_2(z_i-z_j;\tau), \wp_3(z_i-z_j;\tau)]_{i,j=1,\dots,n,\ i<j},$$

that is, the commutative associative algebra over $\mathbb{C}$ generated by the series $G_4(\tau)$, $G_6(\tau)$, $\wp_2(z_i-z_j;\tau)$ and $\wp_3(z_i-z_j;\tau)$ for $i,j=1,\dots,n$ satisfying $i<j$. For $m\geq 0$, let R_m be the subspace of R spanned by elements of the form

$$G_4^{k_1}(\tau)G_6^{k_2}(\tau)\wp_2^{k_3}(z_i-z_j;\tau)\wp_3^{k_4}(z_i-z_j;\tau)$$

for $k_1,k_2,k_3,k_4\geq 0$ satisfying $4k_1+6k_2+2k_3+3k_4=m$.

We introduce, for any $\alpha\in\mathbb{C}$, the notation

$$\mathcal{O}_j(\alpha) = 2\pi i\frac{\partial}{\partial\tau} + G_2(\tau)\alpha + G_2(\tau)\sum_{i=1}^{n} z_i\frac{\partial}{\partial z_i} - \sum_{i\neq j}\wp_1(z_i-z_j;\tau)\frac{\partial}{\partial z_i}$$

for $j=1,\dots,n$. We shall also use the notation $\prod_{j=1}^{m}\mathcal{O}(\alpha_j)$ to denote the ordered product $\mathcal{O}(\alpha_1)\cdots\mathcal{O}(\alpha_m)$.

Then we have the following result:

Theorem 4.1 *Let V be a vertex operator algebra satisfying the conditions stated above and let W_i for $i=1,\dots,n$ be V-modules satisfying the C_2-cofiniteness condition. Then for any homogeneous $w_i\in W_i$ ($i=1,\dots,n$), there exist*

$$a_{p,\,i}(z_1,\dots,z_n;\tau)\in R_p, \quad b_{p,\,i}(z_1,\dots,z_n;\tau)\in R_{2p}$$

for $p=1,\ldots,m$ *and* $i=1,\ldots,n$ *such that for any* V*-modules* $\widetilde{W}_i$ *(*$i=1,\ldots,n$*) and intertwining operators* $\mathcal{Y}_i$ *of types* $\binom{\widetilde{W}_{i-1}}{W_i \widetilde{W}_i}$ *(*$i=1,\ldots,n$, $\widetilde{W}_0=\widetilde{W}_n$*), respectively, the series (4.1) satisfies the expansion of the system of differential equations*

$$\frac{\partial^m \varphi}{\partial z_i^m}+\sum_{p=1}^m a_{p,\,i}(z_1,\ldots,z_n;\tau)\frac{\partial^{m-p}\varphi}{\partial z_i^{m-p}}=0, \tag{4.2}$$

$$\begin{aligned}&\prod_{k=1}^m \mathcal{O}_i\left(\sum_{i=1}^n \mathrm{wt}\ w_i+2(m-k)\right)\varphi\\ &\quad+\sum_{p=1}^m b_{p,\,i}(z_1,\ldots,z_n;\tau)\prod_{k=1}^{m-p}\mathcal{O}_i\left(\sum_{i=1}^n \mathrm{wt}\ w_i+2(m-p-k)\right)\varphi=0,\end{aligned} \tag{4.3}$$

$i=1,\ldots,n$, *in the regions* $1>|q_{z_1}|>\cdots>|q_{z_n}|>|q_\tau|>0$. *Moreover, for fixed* $\tau\in\mathbb{H}$, *the singular points of the (reduced) system (4.2) are regular.*

The elliptic functions and the Eisenstein series discussed above have the following modular transformation formulas (see, for example, [K]): For any

$$\left(\begin{array}{cc}\alpha & \beta\\ \gamma & \delta\end{array}\right)\in SL(2,\mathbb{Z}),$$

$$\begin{aligned}G_2\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)&=(\gamma\tau+\delta)^2G_2(\tau)-2\pi i\gamma(\gamma\tau+\delta),\\ G_{2k}\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)&=(\gamma\tau+\delta)^{2k}G_{2k}(\tau),\\ \wp_m\left(\frac{z}{\gamma\tau+\delta};\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)&=(\gamma\tau+\delta)^m\wp_m(z,\tau),\end{aligned}$$

for $k\geq 2$ and $m\geq 1$. Using these formulas, it is straightforward to verify the following modular invariance of the system (4.2)–(4.3):

Proposition 4.2 *Let* $\varphi(z_1,\ldots,z_n;\tau)$ *be a solution of the system (4.2)–(4.3). Then for any*

$$\left(\begin{array}{cc}\alpha & \beta\\ \gamma & \delta\end{array}\right)\in SL(2,\mathbb{Z}),$$

$$\left(\frac{1}{\gamma\tau+\delta}\right)^{\mathrm{wt}\ w_1+\cdots\mathrm{wt}\ w_n}\varphi\left(\frac{z_1}{c\tau+d},\ldots,\frac{z_n}{c\tau+d};\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)$$

is also a solution of the system (4.2)–(4.3).

Using these systems of differential equations, chiral genus-one correlation functions have been constructed as the analytic extensions of sums of series of the form (4.1) in the region $1>|q_{z_1}|>\cdots>|q_{z_n}|>|q_\tau|>0$. Together with other results in the representation theory of vertex operator algebras, it has been proved that for suitable vertex operator algebras, the vector space of these chiral genus-one correlation functions are invariant under a suitable action of the modular group $SL(2, \mathbb{Z})$. See [H10] for details.

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