Lecture notes on representation theory and tensor categories

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1 Representations of associative algebras

Main references: [EGHLSVY], [FD] and [J].

Definition 1.1. An *associative algebra* is a ring together with a vector space structure over a field $\mathbb{F}$ such that the ring structure and the vector space structure are compatible. *Homomorphisms of associative algebras* are homomorphisms of the ring structures and the vector space structures. *Isomorphisms of associative algebras* are homomorphisms with inverses. If a subspace of an associave algebra is closed under the operations for the associative algebra, then it is called a *subalgebra of the associaive algebra*.

Examples:

1. Let M be a vector space over $\mathbb{F}$. The the set End M of all linear operators on M is an associative algebra.

2. Let M be a finite-dimensional vector space over $\mathbb{F}$ and T a linear operator on M. Consider the subspace $\mathbb{F}[T]$ of End M consisting of all polynomials in T. Then $\mathbb{F}[T]$ is an subalgebra of the associative algebra End M.

3. Let A be an associative algebra and $M_n(A)$ the set of matrices whose entries are elements of A. Then $M_n(A)$ has a natural structure of associative algebra. We call $M_n(A)$ the $n \times n$ matrix algebra over A.

4. Let G be a group. For a field $\mathbb{F}$, let $F[G]$ be the vector space over $\mathbb{F}$ with elements of G as a basis. Then $F[G]$ has a structure of associative algebra.

5. Let $\mathfrak{g}$ be a Lie algebra. Then the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is an associative algebra. (See Section 3 below for the definition of Lie algebra and the definition of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$.)

Definition 1.2. Let A be an associative algebra. A *(left) representation of* A is a vector space M and a homomorphism $\rho : A \to \text{End } M$ of associative algebras. The vector space M equipped with ρ is called a *(left) module for* A or an *(left)* A*-module* and is denoted (M, ρ) or simply M. A *homomorphism* from an A-module $(M_1, \rho 1)$ to another A-module (M_2, ρ_2) is a linear map $f : M_1 \to M_2$ such that $f \circ \rho_1(a) = \rho_2(a) \circ f$ for $a \in A$. An *isomorphism* or *equivalence* from an A-module to another A-module is a homomorphism with an inverse. If a subspace of an A-module (M, ρ) is invariant under the map $\rho(a)$ for all $a \in A$, then it is called an A*-submodule of* (M, ρ).

Example 1.3. Let A be an associative algebra. Then A is itself an A-module. Let I be a left ideal of A. Then A/I is also an A-module.

Let (M, ρ) be an A-module. We shall denote the space of all homomorphisms from M to itself by $\text{End}_A M$. Note that $\text{End}_A M$ is in general different from $\text{End } M = \text{End}_{\mathbb{F}} M$. For $a \in A$ and $x \in M$, for simplicity, we shall denote $(\rho(a))(x)$ by ax. Since ρ is a homomorphism of associative algebras, we have

$$a(bx) = (ab)x$$

for $a, b \in A$ and $x \in M$.

Example 1.4. For any element $x \in M$, the subspace

$$Ax = \{ax \in M \mid a \in A,\ x \in M\}$$

is clearly a submodule of M.

For any element $x \in M$, let

$$I_x = \{a \in A \mid ax = 0\}.$$

Then I_x is a left ideal of A.

Proposition 1.5. *The* A*-module* Ax *is isomorphic to the* A*-module* A/I_x*.*

The proof of this proposition is straightforward.

Definition 1.6. An *irreducible* A*-module* is an A-module such that its only A-submodule is 0 and itself.

Exercise 1.7. Show that M is an irreducible A-module if and only if M is isomorphic to A/I where I is a maximal left ideal of A.

Lemma 1.8 (Schur's Lemma). *Let* M *be an irreducible* A*-module. Then a homomorphism of* A *modules from* M *to* M *is either an isomorphism or* 0*.*

Proof. Let $f : M \to M$ be a homomorphism of A-modules. Then

$$\ker f = \{x \in M \mid f(x) = 0\}$$

is an A-submodule of M. Since M is irreducible, $\ker f$ is either 0 or M. If $\ker f = M$, then $f = 0$. If $\ker f = 0$, then f is injective. In particular, $f(M) \neq 0$. If $f(M)$ is not M, then it is an A-submodule of M which is neither 0 nor M. Contradiction. So $f(M) = M$, that is, f is also surjective. Thus f is an invertible linear map. It is easy to see that f^{-1} is also a homomorphism of A-modules. Hence f is an isomorphism of A-modules. ∎

Corollary 1.9. *Assume that the coefficient field $\mathbb{F}$ is algebraic closed. Let M be a finite-dimensional irreducible A-module. Then a homomorphism of A modules from M to M is proportional to the identity operator.*

Proof. Let f be such a homomorphism. Since M is finite dimensional and $\mathbb{F}$ is algebraic closed, f as a linear operator on M must have an eigenvalue $\lambda \in \mathbb{F}$ and an eigenspace M_λ. Then $f - \lambda 1_M$ (1_M being the identity operator on M) is also a homomorphism of A-modules from M to M. The kernel $\ker(f - \lambda 1_M)$ of $f - \lambda 1_M$ is exactly the eigenspace M_λ. So we see that M_λ is an A-submodule of M. Since M is irreducible, M_λ is either 0 or M. But as an eigenspace, M_λ is not 0. So $\ker(f - \lambda 1_M) = M_\lambda = M$. Thus $f = \lambda 1_M$. ∎

Corollary 1.10. *Let M be an irreducible A-module. Then* $\mathrm{End}_A\, M$ *is a division algebra, that is, an associative algebra whose nonzero elements are all invertible. In the case that the coefficient field $\mathbb{F}$ is algebraic closed,* $\mathrm{End}_A\, M = \mathbb{F}$.

Definition 1.11. Let M be an A-module and M_0 an A-submodule of M. Then there is a natural A-module structure on the quotient space M/M_0. The A-module M/M_0 is called the *quotient A-module* of M by M_0. An *A-module of finite length* is an A-module M and a finite sequence

$$M_0 = M \supset M_1 \supset \cdots \supset M_n \supset M_{i+1} = 0 \tag{1.1}$$

of A-submodules of M such that M_i/M_{i+1} for $i = 0, \ldots, n$ are irreducible A-modules. The (finite) sequence (1.1) is called a *finite composition series* of M and n is called the *length* of the composition series.

Exercise 1.12 (Half of Jordan-Hölder theorem). Prove that any two finite composition series of M must have the same length.

Definition 1.13. In view of the exercise above, we can define the *length* of an A-module of finite length to be the length of any finite composition series of M.

Definition 1.14. Let M^γ for $\gamma \in \Gamma$ be A-modules. Then there is a natural structure of A-module on the direct sum $\coprod_{\gamma\in\Gamma} M^\gamma$. A *complete reducible A-module* is an A-module isomorphic to a direct sum of irredubcible A-modules.

Exercise 1.15. If an A-moudle is completely reducible, then any A-submodule or any quotient of this A-module is also comnpletely reducible.

Theorem 1.16. *Let M be a complete reducible A-module of finite length. Then* $\mathrm{End}_A\, M$ *is isomorphic to a direct product of matrix algebras over division algebras. In the case that the coefficient field $\mathbb{F}$ is algebraic closed,* $\mathrm{End}_A\, M$ *is isomorphic to a direct product of matrix algebras over $\mathbb{F}$.*

Proof. Since M is a complete reducible A-module, there exist irreducible A-submodules M^γ for $\gamma \in \Gamma$ of M such that $M = \coprod_{\gamma\in\Gamma} M^\gamma$. Since M is of finite length, Γ must be a finite set. Among these finitely many irreducible A-modules, let $M^1, \cdots, M^n$ be inequivalent ones and for each $i = 1, \ldots, n$, let m_i be the number of irreducible A-modules in $\{M^\gamma \mid \gamma \in \Gamma\}$

isomorphic to M^i. Then M is isomorphic to $\coprod_{i=1}^n m_i M^i$ where $m_i M^i$ means the direct sum of m_i copies of M^i. We shall discuss $\coprod_{i=1}^n m_i M^i$ instead of M.

Let $f \in \mathrm{End}_A\, M$. For each $i = 1, \ldots, n$, we consider the map $f^i = \pi_i \circ f \circ e_i : m_i M^i \to m_i M^i$ where e_i is the embedding from $m_i M^i$ to $\coprod_{i=1}^n m_i M^i$ and π_i is the projection from $\coprod_{i=1}^n m_i M^i$ to $m_i M^i$. Since e_i and π_i are both homomorphisms of A-modules, f^i is also a homomorphism of A-modules. Consider $f^i_{kl} = p_k \circ f^i \circ \epsilon_l$ for $k, l = 1, \ldots, m_i$ where p_k is the projection from $m_i M^i$ to the k-th copy of M^i and ϵ_l is the embedding of M^i to the l-th copy of $m_i M^i$. Again, f^i_{kl} is a homomorphism of A-modules from M^i to M^i. Thus f^i_{kl} is an element of the division algebra $\mathrm{End}_A\, M^i$. From f^i_{kl} for $k, l = 1, \ldots, m_i$, $i = 1, \ldots, n$, we obtain n matrices (f^i_{kl}), $i = 1, \ldots, n$. These matrices give an element of the direct product of n matrix algebras over division algebras. In this way, we obtain a map from $\mathrm{End}_A\, M$ to the direct product of the n matrix algebras over the corresponding division algebras. It can be verified that this map is an isomorphism of associative algebras, proving the first part of the theorem.

In the case that $\mathbb{F}$ is algebraic closed, $\mathrm{End}_A\, M^i$ is equal to $\mathbb{F}$. Thus $\mathrm{End}_A\, M$ is isomorphic to a direct product of matrix algebras over $\mathbb{F}$. ∎

Exercise 1.17. Verify that the map given in the proof of the theorem above is indeed an isomorphism of associative algebras.

Definition 1.18. Let A be an associave algebra. Then the vector space A equipped with the *opposite multiplication* defined by $a \cdot^o b = ba$ for $a, b \in A$ is also an associative algebra. This associative algebra is called the *opposite algebra* of A and is denoted A^o.

Exercise 1.19. Let A be an associative algebra. Prove the following:

1. $(A^o)^o$ is isomorphic to A.

2. $\mathrm{End}_A\, A$ is isomorphic to the opposite algebra A^o of A.

3. $(M_n(A))^o$ is isomorphic to $M_n(A^o)$.

Definition 1.20. An associative algebra A is said to be *semisimple* if A as an A-module is completely reducible. An associative algebra A is said to be *simple* if A as an A-module is irreducible.

Theorem 1.21. *Let A be a semisimple associative algebra. Then we have:*

1. *There are only finitely many irreducible A-modules, all of which appear in the decomposition of the A module A as a direct sum of irreducible A-modules.*

2. *Every A-module is completely reducible.*

3. *A is isomorphic to a direct product of finitely many matrix algebras over division algebras.*

4. *In the case that the coefficient field $\mathbb{F}$ is algebraic closed and every irreducible A-module is finite-diemnsional, A is a direct product of fintely many matrix algebras over $\mathbb{F}$.*

Proof. Since as an A-module A is completely reducible, we have $A = \coprod_{\gamma\in\Gamma} M^\gamma$ where M^γ for $\gamma \in \Gamma$ are irreducible A-modules. On the other hand, since A contains the identity element e, there exists $\gamma_1, \dots, \gamma_n \in \Gamma$ and $e_1 \in M^{\gamma_1}, \dots, e_n \in M^{\gamma_n}$ such that $e = \sum_{i=1}^n e_i$. Thus we have

$$A = Ae = \coprod_{i=1}^n Ae_i = \coprod_{i=1}^n M^{\gamma_i}.$$

Let M be an irreducible A-module. Then for any nonzero $x \in M$, $M = Ax$. In particular, we have a homomorphism $p : A \to M$ of A-modules given by $p(a) = ax$. For $i = 1, \dots, n$, the embedding $e_i : M^{\gamma_i} \to A$ is also a homomorphism of A-modules. Then for $i = 1, \dots, n$, we have homomorphisms $p \circ e_i : M^{\gamma_i} \to M$ of A-modules. Since M^{γ_i} for $i = 1, \dots, n$ and M are all irreducible A-modules, $p \circ e_i$ must be either 0 or isomorphisms. If these maps are all 0, then $p = \sum_{i=1}^n p \circ e_i$ is 0. But that is not true because $p(e) = x \neq 0$. So at least one $p \circ e_i$ is not 0. Thus $p \circ e_i$ gives an isomorphism from M^{γ_i} to M, proving the first conclusion.

Let M be an A-module. Then M is the sum of Ax for $x \in M$. We know that Ax is isomorphic to A/I_x for some maximal ideal I_x which therefore is also an A-submodule of A and A is completely reducible. Thus as a quotient of a completely reducible A-module, Ax is also completely reducible. Hence M is a sum of irreducible A-modules. It is clear that a sum of irreducible A modules must be a direct sum of irreducible A-modules, proving the second conclusion.

We know that $\mathrm{End}_A\, A$ is isomorphic to a direct product of finitely many matrix algebras over division algebras. On the other hand, we also know that $\mathrm{End}_A\, A$ is isomorphic to A^o. So A^o is isomorphic to a direct product of finitely many matrix algebras over division algebras. Thus $(A^o)^o$ is isomorphic to a direct product of finitely many matrix algebras over the opposite algebras of the division algebras appearing in the decomposition of A^o. Since A is isomorphic to $(A^o)^o$, we obtain the third conclusion.

Note that the division algebras in the third conclusion are the algebra of all homomorphisms from an irreducible A module to itself. In the case that the coefficient field $\mathbb{F}$ is algebraic closed, these division algebras are just the field $\mathbb{F}$. So the last conclusion holds. ∎

Remark 1.22. Let A be a semisimple associative algebra. Then by the theorem above, A is isomorphic to $A_1 \oplus \cdots \oplus \cdots A_s$ where $A_1, \dots, A_s$ are simple associative algebras.

Definition 1.23. Let A be an associative algebra. A *right module for A* or an *right A-module* is a vector space M equipped with a homomorphism of associative algebras from A^o to End M. For any $a \in A$, we right the action of the image of a under the homomorphism from A^o to End M on $x \in M$ by xa. A *bimodule for A* or a *A-bimodule* is a vector space M with a left A-module structure and a right A-module structure such that the left and right A-module structure commute with each other. That is, for $a, b \in A$ and $x \in M$, $(ax)b = a(xb)$. Right A-submodules of a right A-module and A-bisubmodules of a A-bimodule are defined obviously. Homomorphisms and isomorphisms of right A-modules and A-bimodules are also defined obviously.

Example 1.24. Let A be an associative algebra. If M is a left A-module, then M is also a right A^o-module. The associative algebra A is an A-bimodule.

Definition 1.25. Let M_1 and M_2 be vector spaces. Then the *tensor product* $M_1 \otimes_{\mathbb{F}} M_2$ is the quotient space of the vector space spanned by elements of $M_1 \times M_2$ by the subspace spanned by elements of the forms $\lambda(x_1, x_2) - (\lambda x_1, x_2)$, $(\lambda x_1, x_2) - (x_1, \lambda x_2)$, $(x_1 + x_1', x_2) - (x_1, x_2) - (x_1', x_2)$ and $(x_1, x_2 + x') - (x_1, x_2) - (x_1, x_2')$ for $\lambda \in \mathbb{F}$, $x_1, x_1' \in M_1$ and $x_2, x_2' \in M_2$. The coset containing (x_1, x_2) is denoted by $x_1 \otimes x_2$. Let A be an associative algebra. If M_1 is a left A-module and M_2 is a right A-module, then $M_1 \otimes_{\mathbb{F}} M_2$ is a A-bimodule given by $a(x_1 \otimes x_2) = ax_1 \otimes x_2$ and $(x_1 \otimes x_2)a = x_1 \otimes x_2 a$ for $a \in A$, $x_1 \in M_1$ and $x_2 \in M_2$.

Definition 1.26. Let A be an associative algebra and M_1 and M_2 A-bimodules. A *product of* M_1 *and* M_2 is a A-bimodule M and a homomorphism $f : M_1 \otimes_{\mathbb{F}} M_2 \to M$ of A-bimodules such that $f(x_1 a \otimes x_2) = f(x_1 \otimes a x_2)$ for $a \in A$, $x_1 \in M_1$ and $x_2 \in M_2$. This product of M_1 and M_2 is denoted (M, f). A *tensor product of* M_1 *and* M_2 is a product (M, f) of M_1 and M_2 satisfying the following universal property: For any product (M', f') of M_1 and M_2, there exists a homomorphism $g : M \to M'$ of A-bimodules such that $f' = g \circ f$. From this universal property, any two tensor products of M_1 and M_2 are uniquely isomorphic. We shall denote the tensor product of M_1 and M_2 by $M_1 \otimes_A M_2$.

Let A be an associative algebra and M_1 and M_2 A-bimodules. Let $M_1 \otimes_A M_2$ be the quotient of $M_1 \otimes_{\mathbb{F}} M_2$ by the A-bisubmodule generated by elements of the form $x_1 a \otimes x_2 - x_1 \otimes a x_2$ for $a \in A$, $x_1 \in M_1$ and $x_2 \in M_2$. Let $\otimes_A : M_1 \otimes_{\mathbb{F}} M_2 \to M_1 \otimes_A M_2$ be the projection map. Then we have:

Proposition 1.27. *The pair* $(M_1 \otimes_A M_2, \otimes_A)$ *is a tensor product of* M_1 *and* M_2.

Proof. Let (M, f) be a product of M_1 and M_2. Since $f(x_1 a \otimes x_2) = f(x_1 \otimes a x_2)$ for $a \in A$, $x_1 \in M_1$ and $x_2 \in M_2$, the kernel of f contains the A-bisubmodule J generated by elements of the form $x_1 a \otimes x_2 - x_1 \otimes a x_2$ for $a \in A$, $x_1 \in M_1$ and $x_2 \in M_2$. Let

$$p : M_1 \otimes_A M_2 = (M_1 \otimes_{\mathbb{F}} M_2)/J \to (M_1 \otimes_{\mathbb{F}} M_2)/\ker f$$

be the projection and $\bar{f} : (M_1 \otimes_{\mathbb{F}} M_2)/\ker f \to M$ be the injective map induced from f. Then both p and $\bar{f}$ are homomorphisms of A-bimodules and $f = \bar{f} \circ p \circ \otimes_A$. Let $g = \bar{f} \circ p : M_1 \otimes_A M_2 = (M_1 \otimes_{\mathbb{F}} M_2)/J \to M$. Then g is also a homomorphism of A-bimodules and we have $g \circ \otimes_A = \bar{f} \circ p \circ \otimes_A = f$. ∎

For $x_1 \in M_1$ and $x_2 \in M_2$, we denote the image of $x_1 \otimes x_2$ under $\otimes_A$ by $x_1 \otimes_A x_2$.

2 Representations of finite groups

Main references: [EGHLSVY], [J].

Definition 2.1. Let G be a group. A *representation of* G is a homomorphism ρ of groups from G to the general linear group $GL(M)$ of invertile linear operators on a vector space M.

Let $\rho : G \to GL(M)$ be a map. Consider the group algebra $\mathbb{F}[G]$. For any element $a = \sum_{i=1}^{n} \lambda_i a_i \in \mathbb{F}[G]$ where $\lambda_i \in \mathbb{F}$ and $a_i \in G$ for $i = 1, \ldots, m$ and $x \in M$, we define

$$ax = \sum_{i=1}^{n} \lambda_i (\rho(a_i))(x)$$

and obtain a map from $\mathbb{F}[G]$ to End M. Conversely, if we have a map $\rho : \mathbb{F}[G] \to \text{End } M$ such that the image of G is in $GL(M)$, then the restriction of ρ to G is a map from G to $GL(M)$.

Proposition 2.2. *A map $\rho : G \to GL(M)$ is a representation of G if and only if the corresponding map from $\mathbb{F}[G]$ to End M is a representation of $\mathbb{F}[G]$.*

The proof is a straightforward verification.

From this proposition, we see that studying representations of a group is the same thing as studying the representations the corresponding group algebra.

Definition 2.3. A *subrepresentation* of a representation ρ of a group G is the representation of G corresponding to a submodule of the $\mathbb{F}[G]$-module corresponding to ρ. A representation of a group is called *irreducible*, *completely reducible* or *of finite length* if the corresponding representation of the group algebra is irreducible, completely reducible or of finite length. A representation of a group is *equivalent* to another representation of the group is the corresponding modules for the group algebra are isomorphic.

Definition 2.4. Let A be an associative algebra. Let M, M_1 and M_2 be A-modules. A sequence $0 \to M_1 \to M \to M_2 \to 0$ with homomorphisms of A-modules as arrows is called *exact* if for every A-module in the sequence, the image of the arrow before this A-module is equal to the kernel of the arrow after the A-module. An exact sequence $0 \to M_1 \to M \to M_2 \to 0$ is said to *split*, if M is isomorphic to $M_1 \oplus M_2$ as A-modules and the arrow from M_1 to M is the composition of the embedding of M_1 in $M_1 \oplus M_2$ and the isomorphism from $M_1 \oplus M_2$ to M and the arrow from M to M_2 is the composition of the isomorphism from M to $M_1 \oplus M_2$ and the projection from $M_1 \oplus M_2$ to M_2.

Exercise 2.5. Let A be an associative algebra. Prove that an A-module M is completely reducible if and only if an exact sequence $0 \to M_1 \to M \to M_2 \to 0$ of A-modules for any A-modules M_1 and M_2 splits. In particular, to prove that A is semisimple, it is enough to prove that an exact sequence $0 \to M_1 \to M \to M_2 \to 0$ of A-modules for any A-modules M, M_1 and M_2 splits.

Theorem 2.6 (Maschke's Theorem). *Let G be a finite group such that the characteristic of $\mathbb{F}$ does not divide the order $|G|$ of G. Then every representation of G is completely reducible. In particular, when the characteristic of $\mathbb{F}$ is 0, every representation of G is completely reducible.*

Proof. By the exercise above, we need only show that for any $\mathbb{F}[G]$-modules M, M_1 and M_2, an exact sequence $0 \to M_1 \to M \to M_2 \to 0$ splits.

Given $\mathbb{F}[G]$-modules M, M_1 and M_2 and an exact sequence $0 \to M_1 \to M \to M_2 \to 0$ of $\mathbb{F}[G]$-modules, the exactness of the sequence at M_1 allows us to embed M_1 as an A-submodule of M. We know that M as a vector space is linearly isomorphic to $M_1 \oplus M_2$. Let $l_0 : M \to M_1 \oplus M_2$ be a linear isomorphism. Let $p_0 : M \to M_1$ be the composition of this linear isomorphism l_0 and the projection from $M_1 \oplus M_2$ to M_1. By definition, we have

$p_0(x) = x$ for $x \in M_1$. Since the characteristic of $\mathbb{F}$ does not divide $|G|$, $\frac{1}{|G|}$ is a well-defined element of $\mathbb{F}$. Let

$$p = \frac{1}{|G|} \sum_{g \in G} \rho(g)^{-1} \circ p_0 \circ \rho(g).$$

Then p is also a linear map from M to M_1. For $g' \in G$,

$$\begin{aligned}
\rho(g')^{-1} \circ p \circ \rho(g') &= \frac{1}{|G|} \sum_{g \in G} \rho(g')^{-1} \circ \rho(g)^{-1} \circ p_0 \circ \rho(g) \circ \rho(g') \\
&= \frac{1}{|G|} \sum_{g \in G} \rho(gg')^{-1} \circ p_0 \circ \rho(gg') \\
&= \frac{1}{|G|} \sum_{g \in G} \rho(g)^{-1} \circ p_0 \circ \rho(g) \\
&= p.
\end{aligned}$$

Thus we have $p \circ \rho(g') = \rho(g') \circ p$ for $g' \in G$. In particular, p is a homomorphism of $\mathbb{F}[G]$-modules. Since M_1 is an $\mathbb{F}[G]$-module, $(\rho(g))(x) \in M_1$ for $g \in G$ and $x \in M_1$. Thus $(p_0 \circ \rho(g))(x) = p_0((\rho(g))(x)) = (\rho(g))(x)$ or equivalently $(\rho(g)^{-1} \circ p_0 \circ \rho(g))(x) = x$ for $g \in G$ and $x \in M_1$. Then we have

$$\begin{aligned}
p(x) &= \frac{1}{|G|} \sum_{g \in G} (\rho(g)^{-1} \circ p_0 \circ \rho(g))(x) \\
&= \frac{1}{|G|} \sum_{g \in G} x \\
&= x.
\end{aligned}$$

Let $q : M \to M_2$ be the arrow from M to M_2 in the exact sequence $0 \to M_1 \to M \to M_2 \to 0$. We define $l : M \to M_1 \oplus M_2$ by $l(x) = (p(x), q(x))$ for $x \in M$. Since both p and q are homomorphisms of $\mathbb{F}[G]$-modules, l is also a homomorphisms of $\mathbb{F}[G]$-modulesIf $l(x) = 0$, then $q(x) = 0$. Since the sequence is exact, x must be in M_1 (recall that we have embedded M_1 as an $\mathbb{F}[G]$-submodule of M. So $p(x) = x$. Then we have $(0, 0) = l(x) = (p(x), 0) = (x, 0)$ and we obtain $x = 0$. Thus l is injective. Let $(x_1, x_2) \in M_1 \oplus M_2$ where $x_1 \in M_1$ and $x_2 \in M_2$. The exactness of the sequence $0 \to M_1 \to M \to M_2 \to 0$ at M_2 says that q is surjective. So there exists $y \in M$ such that $q(y) = x_2$. We also have $x_1 = p(x_1)$. Let $x = x_1 - p(y) + y$. Then

$$\begin{aligned}
l(x) &= (p(x), q(x)) \\
&= (p(x_1) - p(p(y)) + p(y), q(x_1) - q(p(y)) + q(y)) \\
&= (x_1 - p(y) + p(y), q(y)) \\
&= (x_1, x_2).
\end{aligned}$$

Thus l is also surjective. Since l is both injective and surjective, it is an isomorphism of $\mathbb{F}[G]$-modules. Finally the embedding from M_1 to M is clearly the composition of the embedding of M_1 in $M_1 \oplus M_2$ and l. Also q is the composition of l and the projection from $M_1 \oplus M_2$ to M_2. Thus the exact sequence $0 \to M_1 \to M \to M_2 \to 0$ splits.

The second conclusion follows immediately. ∎

Using the theory of semisimple associative algebras, we have the following:

Theorem 2.7. *Let G be a finite group such that the characteristc of $\mathbb{F}$ does not divide $|G|$. Then there are only finitely many irreducible representations of G and every representation of G is completely reducible.*

Remark 2.8. For finite groups, we do not need the notions of right representations and bi-representations. The reason is that the opposite algebra $\mathbb{F}[G]^o$ is isomorphic to $\mathbb{F}[G]$. In fact, let $i : G \to G$ be defined by $i(g) = g^{-1}$ for $g \in G$. Then

$$i(g_1 g_2) = (g_1 g_2)^{-1} = g_2^{-1} g_1^{-1} = i(g_1 circ^o i(g_2)$$

for $g_1, g_2 \in G$. So i gives a homomorphism from $\mathbb{F}[G]$ to $\mathbb{F}[G]^o$. Since i is injective and surjective, it is in fact an isomorphism. Thus the notion of right $\mathbb{F}[G]$-module is the same as the notion of left $\mathbb{F}[G]$-module. Because of the same reason, we also do not need the notion of bi-$\mathbb{F}[G]$-module.

Let G be a group and $\rho : G \to GL(M)$ a representation of G. Let M^* be the dual space of M and $\rho^* : Gto\text{End } M^*$ the linear map defined by

$$\langle (\rho^*(g))(x^*), x\rangle = \langle x^*, (\rho(g)^{-1})(x)\rangle$$

for $g \in G$ $x \in M$ and $x^* \in M^*$, where $\langle \cdot, \cdot \rangle$ denotes the pairing between M^* and M.

Proposition 2.9. *The image of the map ρ^* is in $GL(M^*)$ and ρ^* is a representation of G.*

Proof. Let e be the identity of G. Then

$$\begin{aligned}
\langle (\rho^*(e))(x^*), x\rangle &= \langle x^*, (\rho(e)^{-1})(x)\rangle \\
&= \langle x^*, x\rangle \\
&= \langle 1_{M^*} x^*, x\rangle
\end{aligned}$$

for $x \in M$ and $x^* \in M^*$, where 1_{M^*} is the identity operator on M^*. Thus we have $(\rho^*(e)) = 1_{M^*}$.

For $g_1, g_2 \in G$, $x \in M$ and $x^* \in M^*$, we have

$$\begin{aligned}
\langle (\rho^*(g_1 g_2))(x^*), x\rangle &= \langle x^*, (\rho(g_2^{-1} g_1^{-1}))(x)\rangle \\
&= \langle x^*, (\rho(g_2)^{-1} \circ \rho(g_1)^{-1})(x)\rangle \\
&= \langle (\rho^*(g_1) \circ \rho^*(g_2))(x^*, x\rangle.
\end{aligned}$$

Thus we obtain

$$\rho^*(g_1 g_2) = \rho^*(g_1) \circ \rho^*(g_2).$$

In particular, for $g \in G$,

$$\rho^*(g) \circ \rho^*(g^{-1}) = \rho^*(e) = 1_{M^*}.$$

So for $g \in G$, $\rho^*(g)$ is invertible. Thus the image of ρ^* is in $GL(M^*)$. The formula we proved above now says that ρ^* is a representation of G. ∎

Definition 2.10. The representation ρ^* is called the *contragredient representation of* ρ.

Let $\rho_1 : G \to GL(M_1)$ and $\rho_2 : G \to GL(M_2)$ be representations of G. We now construct a *tensor product representation* $\rho_1 \otimes \rho_2$ of ρ_1 and ρ_2. Recall the tensor product vector space $M_1 \otimes M_2$. The vector space is spanned by elements of the form $x_1 \otimes x_2$ for $x_1 \in M_1$ and $x_2 \in M_2$. For $g \in G$, we define $(\rho_1 \otimes \rho_2)(g) \in \text{End}\,(M_1 \otimes M_2)$ by

$$((\rho_1 \otimes \rho_2)(g))(x_1 \otimes x_2) = (\rho_1(g))(x_1) \otimes (\rho_2(g))(x_2)$$

for $x_1 \in M_1$ and $x_2 \in M_2$. Since $M_1 \otimes M_2$ is spanned by elements of the form $x_1 \otimes x_2$ for $x_1 \in M_1$ and $x_2 \in M_2$ and $(\rho_1 \otimes \rho_2)(g)$ must be linear, the definition above gives us a unique element of End $(M_1 \otimes M_2)$.

Exercise 2.11. *Prove that for $g \in G$, $(\rho_1 \otimes \rho_2)(g)$ is in fact invertible and thus is in $GL(M_1 \otimes M_2)$. Prove that the linear map $\rho_1 \otimes \rho_2 : G \to GL(M_1 \otimes M_2)$ is a representation of G.*

Remark 2.12. Note that the underlying space of the tensor product representation is the tensor product of the vector spaces. If we consider the $\mathbb{F}[G]$-modules corresponding to these representations, then the corresponding tensor product modules are defined for two left $\mathbb{F}[G]$-modules, not two bi-$\mathbb{F}[G]$-modules. So the tensor product module constructed here is very different from the tensor product bimodule of two bimodules.

Definition 2.13. Let A be an associative algebra and let

$$\text{cent } A = \{a \in A \mid ab = ba, \text{ for all } b \in A\}.$$

Then cent A is a commutative associative algebra and is called the *center* of A.

Proposition 2.14. *Let $C_1, \dots, C_r$ be all the conjugation classes of G. Then $c_i = \sum_{g \in C_i} g$ for $i = 1, \dots, r$ form a basis of* cent $\mathbb{F}[G]$.

Proof. For $i = 1, \dots, r$ and $g' \in G$,

$$g'^{-1} c_i g' = \sum_{g \in C_i} g'^{-1} g g' = \sum_{g \in C_i} g = c_i.$$

So $c_i \in \text{cent } \mathbb{F}[G]$ for $i = 1, \dots, r$. Clearly, c_i for $i = 1, \dots, r$ are linearly independent.

Let $\sum_{g \in G} \gamma_g g$ be an element of $\mathbb{F}[G]$. Then for $g' \in G$,

$$g'^{-1} \left(\sum_{g \in G} \gamma_g g \right) g' = \sum_{g \in G} \gamma_g g'^{-1} g g' = \sum_{g \in G} \gamma_{g' g g'^{-1}} g.$$

So we see that $\sum_{g \in G} \gamma_g g$ is in cent $\mathbb{F}[G]$ if and only if $\gamma_{g' g g'^{-1}} = \gamma_g$ for all $g, g' \in G$. But this menas that $\gamma_{g' g g'^{-1}} = \gamma_g$ is in cent $\mathbb{F}[G]$ if and only if it is a linear combination of c_i for $i = 1, \dots, r$, proving that c_i for $i = 1, \dots, r$ indeed form a basis of cent $\mathbb{F}[G]$. ∎

Let A be a semisimple associative algebra. Then $A = A_1 \oplus \cdots \oplus A_s$ where $A_1, \dots, A_s$ are simple associative algebras which are isormorphic to matrix algebras over division algebras. Then it is clear that cent $A = \text{cent } A_1 \oplus \cdots \oplus \text{cent } A_s$. Since $A_1, \dots, A_s$ are isormorphic to matrix algebras over division algebras, cent $A_1, \dots, \text{cent } A_s$ are isormorphic to the center of the corresponding division algebras.

Proposition 2.15. *Let G be a finite group such that the characteristic of $\mathbb{F}$ does not divide $|G|$. Let $\mathbb{F}[G] = A_1 \oplus \cdots \oplus A_s$ where $A_1, \ldots, A_s$ are simple associative algebras which are isormorphic to matrix algebras over division algebras. Let r be the number on conjugation classes in G. Then $r \geq s$. In the case that $\mathbb{F}$ is algebraically closed, $r = s$.*

Proof. Since c_i for $i = 1, \ldots, r$ form a basis of cent $\mathbb{F}[G]$, the dimension of cent $\mathbb{F}[G]$ is r. On the other hand, since cent $\mathbb{F}[G] = \text{cent } A_1 \oplus \cdots \oplus \text{cent } A_s$,

$$r = \dim \text{cent } \mathbb{F}[G] = \sum_{i=1}^{s} \dim \text{cent } A_i \geq s.$$

In the case that $\mathbb{F}$ is algebraically closed, cent A_i for $i = 1, \ldots, s$ are equal to $\mathbb{F}$ and hence $\dim \text{cent } A_i = 1$ for $i = 1, \ldots, s$. Thus

$$r = \dim \text{cent } \mathbb{F}[G] = \sum_{i=1}^{s} \dim \text{cent } A_i = s.$$

∎

Definition 2.16. Let G be a finite graoup and $\rho : G \to GL(M)$ be a finite-dimensional representation of G. The *character* $\chi_\rho : G \to \mathbb{F}$ of ρ is the function defined by

$$\chi_\rho(g) = \text{Tr}\rho(g)$$

for $g \in G$. If $\mathbb{F} = \mathbb{C}$, χ_ρ is called a *complex character.*

Proposition 2.17. *Characters have the following properties:*

1. *Equivalent representations have the same character.*

2. *Any character is a class function, that is, a function on the set of conjugation classes.*

3. *If $\mathbb{F} = \mathbb{C}$, then the degree of ρ, that is, the dimension of M, is $\chi_\rho(e)$ where e is the identity of G.*

4. *Let $\rho : G \to GL(M)$ be a finite-dimensional representation of G and $\rho|_N : G \to GL(N)$ and $\rho|_{M/N} : G \to GL(M/N)$ be a subrepresentation and the corresponding quotient representation. Then*
$$\chi_\rho(g) = \chi_{\rho|_N}(g) + \chi_{\rho|_{m/N}}(g)$$
for $g \in G$.

5. *For finite-diemsnioanl representations ρ_1 and ρ_2 of G,*
$$\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1}\chi_{\rho_2}.$$

6. *The value of any complex character at $g \in G$ is a sum of m-th roots of unity where m is the exponent of G, that is, the least common multiple of the orders of the elements of G.*

7. Let ρ be a finite-dimensional complex representation of G. Then $\chi_{\rho^*} = \overline{\chi_\rho}$.

Proof. Parts 1 and 2 follows from the fact $\mathrm{Tr}AB = \mathrm{Tr}BA$, the definition of equivalence of representations and the definition of conjugate classes.

Since ρ is a homomorphism of groups, $\rho(e) = I_M$, the identity linear operator on M. Since $\mathbb{F} = \mathbb{C}$, $\dim M = \sum_{i=1}^{\dim M} 1$. Thus $\chi_\rho(e) = \mathrm{Tr}\rho(e) = \dim M = \deg \rho$, proving Part 3.

Chhose a basis $\{x_1, \ldots, x_n\}$ of N and then extend it to a basis $\{x_1, \ldots, x_m\}$ of M. Then $x_{n+1} + N, \ldots, x_m + N\}$ is a basis of M/N. For $g \in G$, let the matrices of the linear operators $\rho|_N(g)$ and $\rho|_{m/N}(g)$ under these bases be A and B. Then the matrix of the linear operator $\rho(g)$ under the basis above is

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

where C is a matrix of size $n \times (m - n)$ and 0 is the zero matrix of size $(m - n) \times n$. We know that $\chi_{\rho|_N}(g) = \mathrm{Tr}A$ and $\chi_{\rho|_{m/N}}(g) = \mathrm{Tr}B$. Thus Part 4 is true.

Let the underlying vector spaces for the representations ρ_1 and ρ_2 be M_1 and M_2. Let $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ be bases of M_1 and M_2, respectively. Then

$$\{x_1 \otimes y_1, \ldots, x_m \otimes y_1, \ldots, x_1 \otimes y_n, \ldots, x_m \otimes y_n\}$$

is a basis of $M_1 \otimes M_2$. For $g \in G$, let the matrices of the linear operators $\rho_1(g)$ and $\rho_2(g)$ under the bases above be $A = (a_{ij})$ and $B = (b_{kl})$. Then the matrix of the linear operator $\rho_1(g) \otimes \rho_2(g)$ under the basis above is $C = (c_{ik,jl})$ where $c_{ik,jl} = a_{ij} b_{kl}$ for $i, j = 1, \ldots, m$ and $k, l = 1, \ldots, n$. Then

$$\begin{aligned}
\chi_{\rho_1 \otimes \rho_2}(g) &= \mathrm{Tr}(\rho_1 \otimes \rho_2)(g) \\
&= \mathrm{Tr}C = \sum_{i=1}^{m} \sum_{k=1}^{n} c_{ik,ik} \\
&= \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ii} b_{kk} \\
&= \left(\sum_{i=1}^{m} a_{ii}\right) \left(\sum_{k=1}^{n} b_{kk}\right) \\
&= \chi_{\rho_1} \chi_{\rho_2}.
\end{aligned}$$

Since $g^m = e$, $\rho(g)^m = I_M$. So the minimal polynomial of $\rho(g)$ divides $x^m - 1$. This fact implies that the eigenvalues of $\rho(g)$ are roots of $x^m - 1$. Thus $\chi_\rho(g)$ is a sum of m-th roots of unity.

Let the eigenvalues of $\rho(g)$ be m-th roots $\omega_1, \ldots, \omega_n$ of unity. Then the eigenvalues of $\rho(g^{-1})$ are $\omega_1^{-1} = \overline{\omega_1}, \ldots, \omega_n^{-1} = \overline{\omega_n}$. From the definition of ρ^*, we see that the eigenvalues of ρ^* are equal to the eigenvalues of $\rho(g^{-1})$. Thus

$$\chi_{\rho^*}(g) = \sum_{i=1}^{n} \overline{\omega_i} = \overline{\sum_{i=1}^{n} \omega_i} = \overline{\chi_\rho(g)}.$$

∎

Theorem 2.18. *Let ρ be a complex representation of degree n of G. Then for any $g \in G$,*

$$|\chi_\rho(g)| \leq n = \deg \rho$$

and

$$|\chi_\rho(g)| = n$$

if and only if $\rho(g) = \omega I_M$ where ω is an m-th root of unity and I_M is the identity operator on M. In particular, if

$$\chi_\rho(g) = n,$$

then

$$\rho(g) = I_M.$$

Proof. From the proposition above, we know that there exist m-th roots $\omega_1, \ldots, \omega_n$ of unity such that

$$|\chi_\rho(g)| = \left|\sum_{i=1}^{n} \omega_i\right| \leq \sum_{i=1}^{n} |\omega_i| = \sum_{i=1}^{n} 1 = n.$$

Since $\omega_1, \ldots, \omega_n$ are roots of unity, $|\chi_\rho(g) = |\sum_{i=1}^{n} \omega_i|$ is n if and only if there exists an m-th root ω of unity such that $\omega_i = \omega$ for $i = 1, \ldots, n$. Then the sum $|\sum_{i=1}^{n} \omega_i|$ is n if and only if the matrix of $\rho(g)$ under a suitable basis of eigenvectors is ωI_n where I_n is the identity matrix of size $n \times n$. Thus $\rho(g) = \omega I_M$.

The last conclusion follows easily. ∎

Theorem 2.19 (Schur relations). *Let $\{\rho_1, \ldots, \rho_s\}$ be a set of representatives of the equivalence classes of irreducible complex representations of G and for each i let $\rho^{(i)}$ be a matrix representation given by ρ_i (that is, a homomorphism from G to the algebra of matrices by taking a basis of the representation space of ρ_i). Then if we write $\rho^{(i)} = (\rho_{kl}^{(i)})$, we have*

$$\begin{aligned} \sum_{g \in G} \rho_{kl}^{(j)}(g) \rho_{rt}^{(i)}(g^{-1}) &= 0 \text{ if } i \neq j, \\ \sum_{g \in G} \rho_{kl}^{(i)}(g) \rho_{rt}^{(i)}(g^{-1}) &= \delta_{kt}\delta_{lr}|G|/\deg \rho_i. \end{aligned}$$

Proof. Let M_i for $i = 1, \ldots, s$ be the underlying spaces of the irreducible representations ρ_i for $i = 1, \ldots, s$, respectively. Let f be a linear map from M_i to M_j. Let

$$\tilde{f} = \sum_{g \in G} \rho_j(g)^{-1} \circ f \circ \rho_i(g).$$

Then for $g' \in G$,

$$\begin{aligned} \rho_j(g')^{-1} \circ \tilde{f} \circ \rho_i(g') &= \sum_{g \in G} \rho_j(g')^{-1} \circ \rho_j(g)^{-1} \circ f \circ \rho_i(g) \circ \rho_i(g') \\ &= \sum_{g \in G} \rho_j(gg')^{-1} \circ f \circ \rho_i(gg') \\ &= \sum_{g \in G} \rho_j(g)^{-1} \circ f \circ \rho_i(g) \\ &= \tilde{f}, \end{aligned}$$

that is,

$$\rho_j(g') \circ \tilde{f} = \tilde{f} \circ \rho_i(g')$$

for $g' \in G$. Thus $\tilde{f}$ is a homomorphism of representations of G from ρ_i to ρ_j. Since ρ_i for $i = 1, \ldots, s$ are irreducible, when $i \neq j$, any homomorphism of representations of G from ρ_i to ρ_j is 0 and when $i = j$, any homomorphism of representations of G from ρ_i to ρ_i is proportional to the identity operator on M_i. Thus $\tilde{f} = 0$ when $i \neq j$ and $\tilde{f} = \lambda I_{M_i}$ when $i = j$.

For $i = 1, \ldots, s$, let $\{u_1^{(i)}, \ldots, u_{n_i}^{(i)}\}$ be a basis of M_i. Take f to be the linear maps f_{lt} from M_i to M_j defined by

$$f_{lr}(u_t^{(i)}) = \delta_{rt} u_l^{(j)},$$

for $r, t = 1, \ldots, n_i$ and $l = 1, \ldots, n_j$. In other words, f_{lr} is the unique linear map from M_i to M_j which maps $u_t^{(i)}$ to 0 when $r \neq t$ and to $u_l^{(j)}$ when $r = t$. When $i \neq j$, $\tilde{f}_{lr} = 0$ which gives

$$\sum_{g \in G} \rho_{kl}^{(j)}(g) \rho_{rt}^{(i)}(g^{-1}) = 0$$

for $r, t = 1, \ldots, n_i$ and $k, l = 1, \ldots, n_j$. When $i = j$, there exist $\lambda_{lr} \in \mathbb{C}$ for $r = 1, \ldots, n_i$, $l = 1, \ldots, n_j$ such that $\tilde{f}_{lr} = \lambda_{lr} I_{M_i}$ which gives

$$\sum_{g \in G} \rho_{kl}^{(i)}(g) \rho_{rt}^{(i)}(g^{-1}) = \delta_{kt} \lambda_{lr}.$$

Let $t = k$ and then sum over k. Since $\rho_{kl}^{(i)}(g)$ and $\rho_{rt}^{(i)}(g^{-1})$ are entries of matrices inverse to each other, we obtain

$$|G| \delta_{lr} = \sum_{g \in G} \delta_{lr} = \sum_{k=1}^{n_i} \lambda_{lr} = n_i \lambda_{lr} = (\deg \rho_i) \lambda_{lr}.$$

Thus

$$\lambda_{lr} = |G| \delta_{lr} / \deg \rho_i$$

and consequently

$$\sum_{g \in G} \rho_{kl}^{(i)}(g) \rho_{rt}^{(i)}(g^{-1}) = \delta_{kt} \delta_{lr} |G| / \deg \rho_i.$$

∎

Let $\mathbb{C}^G$ be the space of complex-valued functions on G. We define a positive definite hermitian form $(\cdot, \cdot)$ on $\mathbb{C}^G$ by

$$(\phi, \psi) = \frac{1}{|G|} \sum_{g \in G} \overline{\phi(g)} \psi(g)$$

for $\phi, \psi \in \mathbb{C}^G$. This hermitian form restricted to the subspace of complex class functions, that is, the subspace of complex functions on the set of conjugation classes of G, gives a positive definite hermitian form on this subspace. Then from Schur relation, we obtain:

Theorem 2.20. *Let $\{\rho_1, \ldots, \rho_s\}$ be a set of representatives of the equivalence classes of irreducible complex representations of G. Then $\{\chi_{\rho_1}, \ldots, \chi_{\rho_s}\}$ is an orthonormal basis of the space of complex class functions.*

Proof. The set $\{\chi_{\rho_1}, \ldots, \chi_{\rho_s}\}$ is orthonormal follows from Schur relation. In particular, this set is linearly independent. On the other hand, since the number of conjugation classes is s, the dimensional of complex class functions is s. Thus $\{\chi_{\rho_1}, \ldots, \chi_{\rho_s}\}$ a basis. ∎

Theorem 2.21. *Finite-dimensional representations of G are determined completely by their characters in the sense that their equivalence classes are uniquely determined by their characters.*

Proof. Let ρ be a finite-dimensional representation of G. Then there exist $m_1, \ldots, m_s \in \mathbb{N}$ ($\mathbb{N}$ is the set of nonegative intgers) such that ρ is equivalent to $m_1\rho_1 + \cdots + m_s\rho_s$. To determine the equivalence class of ρ, we need only determine $m_1, \ldots, m_s$. Since ρ is equivalent to $m_1\rho_1 + \cdots + m_s\rho_s$,

$$\chi_\rho = \sum_{i=1}^{s} m_i \chi_{\rho_i}.$$

Since $\{\chi_{\rho_1}, \ldots, \chi_{\rho_s}\}$ is an orthonormal basis of the space of complex class functions, we have

$$m_i = (\chi_{\rho_i}, \chi_\rho)$$

for $i = 1, \ldots, s$. Thus $m_1, \ldots, m_s$ are determined uniquely by the character χ_ρ of ρ and hence the equivalence class of ρ is determined uniquely by the character χ_ρ. ∎

3 Representations of Lie algebras

Main references: [H].

Definition 3.1. A *Lie algebra* is a vector space L over a field $\mathbb{F}$ equipped with a *bracket operation* $[\cdot, \cdot] : L \otimes L \to L$ satisfying the following conditions:

1. The *skew-symmetry*: For $x, y \in L$,

$$[x, y] = -y, x].$$

2. The *Jacobi identity*: For $x, y, z \in L$,

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

A *homomorphism* from a Lie algebra L_1 to another Lie algebra L_2 is a linear map f from L_1 to L_2 such that for $x, y \in L_1$, $f([x, y]_1) = [f(x), f(y)]_2$, where $[\cdot, \cdot]_1$ and $[\cdot, \cdot]_2$ are the bracket operations for L_1 and L_2, respectively. An *isomorphism* from a Lie algebra to another Lie algebra is an invertible homomorphism of Lie algebras.

Example 3.2. Let A be an associative algebra. We define a bracket operation $[\cdot,\cdot]$ by $[a,b] = ab - ba]$ for $a, b \in A$. Then A equipped with this bracket operation is a Lie algebra. In particular, for a vector space M, the space End M of all linear operators on M is an associative algebra. Then we have a Lie algebra structure on End M. We shall denote this Lie algebra by $\mathfrak{gl}(M)$.

Definition 3.3. Let L be a Lie algebra. A *reppresentation* of L is a vector space M and a homomorphism ρ of Lie algebra from L to $\mathfrak{gl}(M)$. The vector space M equipped with the representation ρ is called a *module for L* or an *L-module*. For an L-module, we shall denote $\rho(x)y$ for $x \in L$ and $y \in M$ by xy. A *homomorphism* of L-modules from an L-module M_1 to another L-module M_2 is a linear map from M_1 to M_2 such that $f(xy) = xf(y)$ for $x \in L$ and $y \in M_1$. An *isomorphism* from an L-module to another L-module is an invertible homomorphism of L-modules.

Definition 3.4. Let L be a Lie algebra. A *subalgebra* of a Lie algebra L is a subspace N of L such that the bracket operation $[\cdot,\cdot]$ for L maps $N \times N$ to N. An ideal of L is a subalgebra I of L such that $[x,y] \in I$ for $x \in I$ and $y \in L$. Let I be an ideal of L. Then L/I has a natural structure of a Lie algebra and is called the *quotient of L by I*. L is said to be *simple* if the only ideals of L are 0 and L and in addition, $[L,L] \neq 0$.

Definition 3.5. Let L be a Lie algebra. Let $L^{(1)} = [L,L]$, $L^{(2)} = [L^{(1)}, L^{(1)}]$, …, $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$, …. The Lie algebra L is said to be *solvable* if $L^{(i)} = 0$ for some i.

Proposition 3.6. *Let L be a Lie algebra.*

1. *If L is solvable, then all subalgebras and homomorphism images of L are sovable.*

2. *If I is a solvable ideal of L such that L/I is also solvable, then L is solvable.*

3. *If I and J are solvable ideals of L, then so is $I + J$.*

4. *There is a unique maximal solvable ideal of L.*

Proof. Part 1 follows immediately from the definitions.

Let I be a solvable ideal of L such that L/I is also solvable. Then there exists $m \in \mathbb{Z}_+$ such that $(L/I)^{(m)} = 0$, or equivalently, $L^{(m)} \subset I$ and there exists $n \in \mathbb{Z}_+$ such that $I^{(n)} = 0$. Since $L^{(m)} \subset I$ and $I^{(n)} = 0$, we have $L^{(m+n)} = (L^{(m)})^{(n)} \subset I^{(n)} = 0$, proving that L is solvable.

By the standard homomorphism theorem, we know that $(I + J)/J$ is isomorphic to $I/(I \cap J)$. Since $I/(I \cap J)$ is a homomorphism image of I, it is solvable. So $(I + J)/J$ is also solvable. But J is also aolvable. By Part 2, we see that $I + J$ is solvable.

Let S be a maximal solvable ideal (a solvable ideal such that any solvable ideal containing S must be equal to S). Let I be any solvable ideal. Then $S + I$ is also a solvable ideal. Since $S + I$ contains S, $S + I = S$ or equivalently, $I \subset S$. Thus such an S is unique. ∎

Definition 3.7. The unique maximal solvable ideal of L given in the proposition above is called the *radical* of L and is denoted Rad L. A Lie algebra is said to be *semisimple* if its radical is 0.

Definition 3.8. Let L be a vector space. The *tensor algebra generated by M* is the space

$$T(L) = \coprod_{n \in \mathbb{N}} L^{\otimes n},$$

where $\mathbb{N}$ is the set of nonnegative integers and $L^{\otimes n}$ is the tensor product of n copies of L (when $n = 0$, $L^{\otimes 0} = \mathbb{F}$), with the tensor product of elements as the multiplication. Let L be a Lie algebra. The the quotient of $T(L)$ by the two sided ideal I of $T(L)$ generated by elenets of the form $x \otimes y - y \otimes x - [x, y]$ for $x, y \in L$ is an associative algebra. This associative algebra is called *universal enveloping algebra of L* and is denoted by $U(L)$.

We shall use $x_1 \cdots x_n$ for $x_1, \ldots, x_n \in L$ to denote the element $x_1 \otimes \cdots \otimes x_n + I$ of $U(L)$. Then we see that $U(L)$ is spanned by elements of this form. In particular, elements of the form x for $x \in L$ form a subspace of $U(L)$ linearly isomorphic to L.

Proposition 3.9. *A vector space M is an L-module if and only if it is a $U(L)$-module.*

Proof. Let M be an L-module. For $x_1 \cdots x_n \in U(L)$ and $y \in M$, we define

$$(x_1 \cdots x_n)y = x_1(\cdots(x_n y)\cdots).$$

Since M is an L-module, it is easy to see that this is well defined, that is, if $x_1 \cdots x_n$ is equal to a linear combination of elements of the same form, then the action of this element on y defined above and by using the linear combination give the same result. It is also easy to see that this action gives a $U(L)$-module structure on M.

Conversely, given a $U(L)$-module M, since L can be viewed as a subspace of $U(L)$, we have an action of L on M. Using the definition of $U(L)$ and the meaning of $U(L)$-module, we see that M with this action of L is an L-module. ∎

Definition 3.10. Let L be a Lie algebra. Let $L^1 = [L, L]$, $L^2 = [L, L^1]$, ..., $L^i = [L, L^{i-1}]$, ... A Lie algebra is said to be *nilpotent* if $L^i = 0$ for some $i \in \mathbb{Z}_+$.

It is clear that $L^1 = L^{(1)}$ and $L^{(i)} \subset L^i$ for $i \geq 1$. So we have:

Proposition 3.11. *Let L be a Lie algebra. Then L is solvable if L or $[L, L]$ is nilpotent.* ∎

Theorem 3.12 (Cartan's criterion). *Let M be a finite-dimensional vector space and L be a subalgebra of the Lie algebra $\mathfrak{gl}(M)$. If $\mathrm{Tr} xy = 0$ for all $x \in [L, L]$ and $y \in L$, then L is solvable.*

The proof is omitted. See [H] for a proof.

Definition 3.13. Let L be a Lie algebra. A representation $\rho : L \to \mathfrak{gl}(M)$ is said to be *faithful* if $\ker \rho = 0$. In this case, the L-module M is also said to be *faithful*.

Definition 3.14. Let L be a finite-diemnsional Lie algebra and $\rho : L \to \mathfrak{gl}(M)$ a faithful representation of L. Define a bilinear form $\beta_\rho : L \otimes L \to \mathbb{F}$ by $\beta_\rho(x, y) = \mathrm{Tr}\rho()\rho(y)$ for $x, y \in L$. Let L be a finite-dimensional Lie algebra. The *Killing form* of L is the bilinear form $\kappa = \beta_{\mathrm{ad}}$ for the adjoint representation ad on L itself defined by $(\mathrm{ad}\ x)y = [x, y]$ for $x, y \in L$.

Exercise 3.15. Verify that the bilinear form β_ρ is associative, that is, $\beta_\rho([x,y],z) = \beta_\rho(x,[y,z])$ for $x, y, z \in L$.

Proposition 3.16. *Let L be a finite-diemnsional semisimple Lie algebra and $\rho : L \to \mathfrak{gl}(M)$ a finite-diemnsional faithful representation of L. Then β_ρ is nondegenerate, that is, $\beta_\rho(x,y) = 0$ for all $y \in L$ implies $x = 0$.*

Proof. Let $S = \{x \in L \mid \beta_\rho(x,y) = 0 \text{ forall } y \in L\}$ (S is called the *radical* of β_ρ). We need to show that $S = 0$.

Since $\rho(S)$ is a subalgebra of $\mathfrak{gl}(M)$, we can apply Cartan's criterion to $\rho(S)$. Since $\mathrm{Tr} xy = \beta_\rho(x,y) = 0$ for $x \in \rho(S)$ and $y \in \rho(L)$, we certainly have $\mathrm{Tr} xy = 0$ for $x \in [\rho(S), \rho(S)]$ and $y \in \rho(S)$. Thus $\rho(S)$ is solvable. Since ρ is faithful, S is isomorphic to $\rho(S)$ and is therefore also solvable. Since L is semisimple, $S = 0$. ∎

Since β_ρ is nondegenerate, it gives an isomorphism from L to the dual space L^* of L by $x \in L \mapsto \beta(x,\cdot)$. Let $\{x_1, \ldots, x_n\}$ be a basis of L and $\{x_1^*, \ldots, x_n^*$ the dual basis. By definition, we have

$$x_i^*(x_j) = \delta_{ij}$$

for $i, j = 1, \ldots, n$. Using the inverse of the isomorphism from L to L^*, The basis $\{x_1^*, \ldots, x_n^*$ corresponds to another basis $\{y_1, \ldots, y_n\}$ of L and satisfies

$$\beta_\rho(x_i, y_j) = \delta_{ij}$$

for $i, j = 1, \ldots, n$. We shall also call this basis the *dual basis of* $\{x_1, \ldots, x_n\}$ *with respect to the bilinear form* β_ρ or simply the dual basis of $\{x_1, \ldots, x_n\}$.

Definition 3.17. Let L be a finite-diemnsional semisimple Lie algebra and $\rho : L \to \mathfrak{gl}(M)$ a finite-diemnsional faithful representation of L. The *Casimir element* of ρ is

$$c_\rho = \sum_{i=1}^{n} \rho(x_i)\rho(y_i) \in \mathrm{End}\, M.$$

Exercise 3.18. Verify that the definition of the Casimir element above is independent of the choice of the basis $\{x_1, \ldots, x_n\}$.

Proposition 3.19. *Suppose that ρ is a faithful representation of L. Then the Casimir element commutes with $\rho(x)$ for $x \in L$.*

Proof. Let

$$[x, x_i] = \sum_{j=1}^{n} a_{ij} x_j$$

and

$$[x, y_i] = \sum_{j=1}^{n} b_{ij} y_j$$

for $i = 1, \ldots, n$. Then we have

$$\begin{aligned} a_{ik} &= \sum_{j=1}^{n} a_{ij}\delta_{jk} \\ &= \sum_{j=1}^{n} a_{ij}\beta_\rho(x_j, y_k) \\ &= \beta_\rho([x, x_i], y_k) \\ &= -\beta_\rho([x_i, x], y_k) \\ &= -\beta_\rho(x_i, [x, y_k]) \\ &= -\sum_{j=1}^{n} b_{kj}\beta_\rho(x_i, y_j) \\ &= -\sum_{j=1}^{n} b_{kj}\delta_{ij} \\ &= -b_{ki} \end{aligned}$$

for $i, k = 1, \ldots, n$. Thus

$$\begin{aligned} [\rho(x), c_\rho] &= \sum_{i=1}^{n}[\rho(x), \rho(x_i)\rho(y_i)] \\ &= \sum_{i=1}^{n} \rho(x)\rho(x_i)\rho(y_i) - \sum_{i=1}^{n} \rho(x_i)\rho(y_i)\rho(x) \\ &= \sum_{i=1}^{n}(\rho(x)\rho(x_i)\rho(y_i) - \rho(x_i)\rho(x)\rho(y_i)) + \sum_{i=1}^{n}(\rho(x_i)\rho(x)\rho(y_i)) - \rho(x_i)\rho(y_i)\rho(x) \\ &= \sum_{i=1}^{n}[\rho(x), \rho(x_i)]\rho(y_i) + \sum_{i=1}^{n} \rho(x_i)[\rho(x), \rho(y_i)] \\ &= \sum_{i=1}^{n} \rho([x, x_i]\rho(y_i) + \sum_{i=1}^{n} \rho(x_i)\rho([x, y_i] \\ &= \sum_{i,j=1}^{n} a_{ij}\rho(x_j)\rho(y_i) + \sum_{i,j=1}^{n} b_{ij}\rho(x_i)\rho(y_j) \\ &= \sum_{i,j=1}^{n} (a_{ij} + b_{ji})\rho(x_j)\rho(y_i) \\ & 0, \end{aligned}$$

proving that c_ρ commutes with $\rho(x)$ for $x \in L$. ∎

Let M_1 and M_2 be modules for a Lie algebra L. We now give a *tensor product module* of M_1 and M_2: Consider the tensor product vector space $M_1 \otimes M_2$. Define an action of L on

$M_1 \otimes M_2$ by

$$x(y_1 \otimes y_2) = xy_1 \otimes y_2 + x_1 \otimes xy_2$$

for $x \in L$, $x_1 \in M_1$ and $y_2 \in M_2$.

Exercise 3.20. Verify that $M_1 \otimes M_2$ with this action of L is indeed an L-module.

Let M_1 and M_2 be modules for a Lie algebra L. We next give an L-module structure on the vector space $\mathrm{Hom}(M_1, M_2)$ of all linear maps from M_1 to M_2: Define an action of L on $\mathrm{Hom}(M_1, M_2)$ by

$$(xf)(y_1) = xf(y_1) - f(xy_1)$$

for $x \in L$, $f \in \mathrm{Hom}(M_1, M_2)$ and $y_1 \in M_1$.

Exercise 3.21. Verify that $\mathrm{Hom}(M_1, M_2)$ with this action of L is indeed an L-module.

Let M be an L-module. We now consider then special case that $M_1 = M$ and $M_2 = \mathbb{F}$ with the trivial L-module structure (the action of elements of L on $\mathbb{F}$ is 0). Then $M^* = \mathrm{Hom}(M, \mathbb{F})$ and we obtain an L-module structure on M^*. This is called the *contragredient module of M*. We can also give the action of the action of L on M^* directly by

$$(xf)(y) = -f(xy)$$

for $x \in L$, $f \in M^*$ and $y \in M$.

In the rest of this section, we assume that F is algebraic closed of characteristic 0.

Theorem 3.22 (Weyl). *A finite-dimensional module for a finite-dimensional semisimple Lie algebra is completely reducible.*

Proof. We need only prove that an exact sequence

$$0 \to M_1 \to M \to M_2 \to 0$$

of finite-dimensional L-modules are completely reducible. Equivalently, we need only prove that for a finite-dimensional L-module M and a finite-dimensional L-submodule M_1, there exists a finite-dimensional L-module M_2 such that M is isomorphic to $M_1 \oplus M_2$.

If indeed we can find such M_2, then the projection p from M to M_1 is a homomorphism of L-modules. The projection p can be characterized as the linear map from M to M_1 such that $p|_{M_1} = I_{M_1}$ and $\ker p$ is isomorphic to M_2. So to prove the theorem, we need only to find a homomorphism of L-modules from M to M_1 such that its restriction to M_1 is the identity and its kernel is isomorphic to M_2.

To find such a homomorphism of L-modules from M to M_1, we consider $\mathrm{Hom}(M, M_1)$. We have given an L-module structure to this space. Such a homomorphism, if it exists, must belong to the subspace $\mathcal{M}$ of $\mathrm{Hom}(M, M_1)$ consisting of elements whose restriction to M_1 is proportional to the identity operator on M_1. On the other hand, we certainly do not want elements in this subspace whose restrictions to M_1 are 0. Let $\mathcal{M}_1$ be the space of all such elements. We claim that $\mathcal{M}$ is an L-submodule of $\mathrm{Hom}(M, M_1)$ and $\mathcal{M}_1$ is an L-submodule of $\mathcal{M}$. In fact, for $f \in \mathcal{M}$, there exists $\lambda \in \mathbb{F}$ such that $f|_{M_1} = \lambda I_{M_1}$. Then for $x \in L$ and $y \in M_1$, $(xf)(y) = xf(y) - f(xy) = \lambda xy - \lambda xy = 0$. Thus $(xf)|_{M_1} = 0$. The same proof also shows that $\mathcal{M}_1$ is an L-submodule of $\mathcal{M}$. Note that $\mathcal{M}/\mathcal{M}_1$ is one-dimensional because modulo elements of $\mathcal{M}_1$, elements of $\mathcal{M}$ are determined completely by its restrictions on M_1. If $\mathcal{M}$ can be decomposed as a direct sum of the L-submodule $\mathcal{M}_1$ and a one-dimensional L-submodule of $\mathcal{M}_1 \subset \mathrm{Hom}(M, M_1)$, then we can choose the homomorphism we are looking for to be a basis of this one-dimensional subspace of $\mathrm{Hom}(M, M_1)$.

We now prove that $\mathcal{M}$ can be decomposed as a direct sum of the L-submodule $\mathcal{M}_1$ and a one-dimensional L-submodule of $\mathcal{M}$. We have proved that $(xf)|_{M_1} = 0$ for $x \in L$. So $x\mathcal{M} \subset \mathcal{M}_1$ for $x \in L$. Thus L acts on the one-dimensional L-module $\mathcal{M}/\mathcal{M}_1$ trivially. In particular, the L-module $\mathcal{M}/\mathcal{M}_1$ is isomorphic to the trivial L-module $\mathbb{F}$.

We use induction on the dimension of $\mathcal{M}$. When the diemnsion of $\mathcal{M}$ is 1, $\mathcal{M}$ can certainly be decomposed as a direct sum of the L-submodule $\mathcal{M}_1 = 0$ and one-dimensional L-submodule $\mathcal{M}$ of $\mathcal{M}$. Now assume that when the dimension of $\mathcal{M}$ is less than k, the decompostion holds. We now consider the case that the dimension of $\mathcal{M}$ is k. If $\mathcal{M}_1$ is not irreducible, then there exists a nonzero proper L-submodule $\mathcal{M}_1'$ of $\mathcal{M}_1$. Then the dimension of $\mathcal{M}/\mathcal{M}_1'$ is less than k and $(\mathcal{M}/\mathcal{M}_1')/(\mathcal{M}_1/\mathcal{M}_1')$ is one-dimensional. By induction assumption, There is a one-dimensional L-submodule of $\mathcal{M}/\mathcal{M}_1'$ such that $\mathcal{M}/\mathcal{M}_1'$ is the direct sum of $\mathcal{M}_1/\mathcal{M}_1'$ and this one-dimensional L-submodule. But any L-submodule of $\mathcal{M}/\mathcal{M}_1'$ is of the form $\tilde{\mathcal{M}}/\mathcal{M}_1'$ where $\tilde{\mathcal{M}}$ is an L-submodule of $\mathcal{M}$. Now $\mathcal{M}_1'$ is an L-submodule of $\tilde{\mathcal{M}}$ such that $\tilde{\mathcal{M}}/\mathcal{M}_1'$ is one-dimensional. So we can use our induction assumption again to obtain a one-dimensional L-submodule X such that $\tilde{\mathcal{M}}$ is the direct sum of $\mathcal{M}_1'$ and X. We know that $\tilde{\mathcal{M}}/\mathcal{M}_1' \cap \mathcal{M}_1/\mathcal{M}_1' = 0$, $X \subset \tilde{\mathcal{M}}$ and $X \cap \mathcal{M}_1' = 0$. So $X \cap \mathcal{M}_1 = 0$. Thus

$$\dim \mathcal{M} = \dim \mathcal{M}_1 + 1 = \dim \mathcal{M}_1 + \dim X.$$

Since both $\mathcal{M}_1$ and X are L-submodules of $\mathcal{M}$ and their intersection is 0, their direct sum must be $\mathcal{M}$.

We still need prove the case that $\mathcal{M}_1$ is irreducible. If ρ is not faithful, then we consider the quotient $L/\ker\rho$. The representation ρ induces a faithful representation of $L/\ker\rho$. Since L is semisimple, Rad $L = 0$. The quotient as a homomorphism image of L is also semisimple. The complete reducibility of $\mathcal{M}$ as an L-module is equivalent to the complete reducibility of $L/\ker\rho$. Thus we can assume that ρ is faithful. Since the Casimir element c_ρ commutes with $\rho(x)$ for $x \in L$, c_ρ is in fact a homomorphism of L-modules from $\mathcal{M}$ to itself. In particualr, $c_\rho(\mathcal{M}_1) \subset \mathcal{M}_1$ and $\ker c_\rho$ is an L-submodule of $\mathcal{M}$. Since L acts on $\mathcal{M}/\mathcal{M}_1$ trivially, so does c_ρ. So $\mathrm{Tr} c_\rho = 0$ on $\mathcal{M}/\mathcal{M}_1$. But since $\mathcal{M}_1$ is irreducible, c_ρ acts as a scalar on $\mathcal{M}_1$. This scalar cannot be 0 since $\mathrm{Tr} c_\rho = \dim L$. Hence $\ker c_\rho$ must be a one-dimensional L-submodule of $\mathcal{M}$ such that $\ker c_\rho \cap \mathcal{M}_1 = 0$. Thus $\mathcal{M}$ is the direct sum of $\mathcal{M}_1$ and $\ker c_\rho$. ∎

Let $Z(L)$ be the center of L, that is

$$Z(L) = \{x \in L \mid [x, y] = 0 \text{ for } y \in L\}.$$

Then by definition,

$$Z(L) = \ker \mathrm{ad}_L$$

and $Z(L)$ is a solvable ideal of L.

Lemma 3.23. *A Lie algebra L is semisimple if and only if all abelian ideals of L are* 0.

Proof. Any abelian ideal of L is a solvable ideal of L and hence is in Rad L. Thus Rad $L = 0$ implies that all abelian ideals of L are 0.

Conversely, assume that all abelian ideals of L are 0. Since Rad L is a solvable ideal of L, there exists $n \in \mathbb{N}$ such that $(\text{Rad } L)^{(n)} = 0$ and $(\text{Rad } L)^{(n-1)} \neq 0$ if $n \neq 0$. If $n \neq 0$, then Rad $L)^{(n-1)}$ is a nonzero abelian ideal of L. Contradiction. So $n = 0$, that is, Rad $L = 0$. ∎

Theorem 3.24. *A Lie algebra L is semisimple if and only if its Killing form is nondegenerate.*

Proof. Assume that L is semisimple. Then Rad $L = 0$. Let S be the radical of the Killing form κ, that is,

$$S = \{x \in L \mid \kappa(x, y) = 0 \text{ forall } y \in L\}.$$

Then for $x \in S$ and $y \in L$ (in particular for $y \in [S, S]$), $\kappa(x, y) = 0$. By Cartan's criterion, $\text{ad}_L\ S$ is solvable. Since L is semisimple, $\ker \text{ad}_L = Z(L) \subset \text{Rad } L = 0$. So S is also solvable. Thus $S \subset \text{Rad } L = 0$, proving that κ is nondegenrate.

Conversely, assuming that the radical S of the Killing form κ is 0, we want to prove that L is semisimple. We prove that all abelian ideals of L are 0. Let I be an abelian ideal of L. For $x \in I$ and $y \in L$, $((\text{ad}_L\ x)(\text{ad}_L\ y))^2$ maps L to $[I, I]$. Since I is abelain, $[I, I] = 0$. Thus $((\text{ad}_L\ x)(\text{ad}_L\ y))^2 = 0$. Since the eigenvalues of any nilpotent operator are 0, we have

$$\kappa(x, y) = \text{Tr}(\text{ad}_L\ x)(\text{ad}_L\ y) = 0.$$

So $x \in S = 0$ and thus $x = 0$, proving that $I = 0$. ∎

Before we discuss construct representations of semisimple Lie algebras, we need the following result from linear algebra:

Theorem 3.25. *Let T be a linear operator on a finite-dimensional vector space M. Then there exist a unique diagonalizable (or semisimple) operator T_s and a unique nilpotent operator T_n on M such that $T = T_s + T_n$.*

Proof. Choose an ordered basis $\mathcal{B} = \{u_1, \ldots, u_n\}$ such that under this basis, the matrix $[T]_{\mathcal{B}}$ of T is a Jordan canonical form. Then $[T]_{\mathcal{B}} = S + N$ where S is a diagonal matrix whose diagonal entris are eigenvalues of T and N is a nilpotent Jordan canonical form whose eigenvalues are 0. Let T_s and T_n be the linear operators whose matrices under the basis $\mathcal{B}$ are S and N, respectively. Then we have $T = T_s + T_n$. Clearly, T_s and T_n are unique.

We can also obtain T_s and T_n and the decomposition $T = T_s + T_n$ using generalized eigenspaces of T as follows: Let $a_1, \ldots, a_k$ be distinct eigenvalues of T and $M_{a_1}, \ldots, M_{a_k}$ the corrsponding eigenspaces. Then $M = \oplus_{i=1}^k M_{a_i}$. Define $T_s : M \to M$ by $T_s(u) = a_i u$ for $u \in M_{a_i}$. Then T_s is certainly diagonalizable or semisimple. Let $T_n = T - T_s$. It is easy to see that T_n is nilpotent. By definition, we have $T = T_s + T_n$. ∎

Now we discuss representations of

$$\mathfrak{sl}(2,\mathbb{F}) = \{A \in M_{2\times 2} \mid \mathrm{Tr}A = 0\}$$

with the bracket operation defined by

$$[A, B] = AB - BA$$

for $A, B \in \mathfrak{sl}(2,\mathbb{F})$. The Lie algebra $\mathfrak{sl}(2,\mathbb{F})$ has a basis consisting the elements

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Their brackets or commutators are given by

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h.$$

Exercise 3.26. Prove that $\mathfrak{sl}(2,\mathbb{F})$ is semisimple.

Since $\mathfrak{sl}(2,\mathbb{F})$ is semisimple, we need only discuss finite-dimensional irreducible $\mathfrak{sl}(2,\mathbb{F})$-modules and how arbitrary finite-dimensional $\mathfrak{sl}(2,\mathbb{F})$-modules decompose into these finite-dimensional irreducible $\mathfrak{sl}(2,\mathbb{F})$-modules. We shall discuss only finite-dimensional irreducible $\mathfrak{sl}(2,\mathbb{F})$-modules below.

First we need:

Lemma 3.27. *Let $\rho : \mathfrak{sl}(2,\mathbb{F}) \to \mathfrak{gl}(M)$ be a representation of $\mathfrak{sl}(2,\mathbb{F})$. Then $\rho(h)$ is semisimple.*

Proof. Since any $\mathfrak{sl}(2,\mathbb{F})$-module is completely reducible, M is a direct sum of irreducible $\mathfrak{sl}(2,\mathbb{F})$-submodules of M. To prove that $\rho(h)$ is semisimple, it is enough to prove that the restriction of $\rho(h)$ to each of these irreducible $\mathfrak{sl}(2,\mathbb{F})$-submodules is semisimple. So we can assume that M is irreducible.

From the formulas for $[h, x]$ and $[h, y]$, we see that $\mathrm{ad}_{\mathfrak{sl}(2,\mathbb{F})}\, h$ is semisimple. Then since ρ is a homomorphism of Lie algebras, $\mathrm{ad}_{\rho(\mathfrak{sl}(2,\mathbb{F}))}\, \rho(h)$ is semisimple.

Since $\mathfrak{sl}(2,\mathbb{F})$ is semisimple (actually it is simple), we have $[\mathfrak{sl}(2,\mathbb{F}), \mathfrak{sl}(2,\mathbb{F})] = \mathfrak{sl}(2,\mathbb{F})$. Then we have $[\rho(\mathfrak{sl}(2,\mathbb{F})), \rho(\mathfrak{sl}(2,\mathbb{F}))] = \rho(\mathfrak{sl}(2,\mathbb{F}))$. Thus we have

$$\rho(\mathfrak{sl}(2,\mathbb{F})) = [\rho(\mathfrak{sl}(2,\mathbb{F})), \rho(\mathfrak{sl}(2,\mathbb{F}))] \subset [\mathfrak{gl}(M), \mathfrak{gl}(M)] = \mathfrak{sl}(M).$$

In particular, $\rho(h) \in \mathfrak{sl}(M)$. If we let $\rho(h) = \rho(h)_s + \rho(h)_n$ be the Jordan decomposition of the linear operator $\rho(h)$ on M, by definition, $\mathrm{Tr}\rho(h)_n = 0$, that is, $\rho(h)_n \in \mathfrak{sl}(M)$. Thus we also have $\rho(h)_s \in \mathfrak{sl}(M)$.

Let $\mathcal{B} = \{u_1, \ldots, u_n$ be a basis of M such that under this basis, the matrix $[\rho(h)]_{\mathcal{B}}$ of $\rho(h)$ is a Jordan canonical form. Then the matrix $[\rho(h)_s]_{\mathcal{B}}$ of $\rho(h)_s$ under $\mathcal{B}$ is a diagonal matrix diag $(a_1, \ldots, a_n)$ where $a_1, \ldots, a_n$ are eigenvalues of $\rho(h)$. Take a basis of $\mathfrak{gl}(M)$ to be the set of linear operators $T_{ij} \in \}\updownarrow(M)$ whose matrices under the basis $\mathcal{B}$ of M are E_{ij} for $i, j = 1, \ldots, n$ where E_{ij} is the matrix whose only nonzero entry is 1 at the i-th row and the j-th column. Then it is easy to verify by direct calculations that E_{ij} are generalized eigenvectors of the action of $[\rho(h)]_{\mathcal{B}}$ on the space $M_{n\times n}$ of $n \times n$ matrices by the bracket operation with eigenvalues $a_i - a_j$, that is,

$$\overbrace{[([\rho(h)]_{\mathcal{B}} - (a_i - a_j)I_n), \cdots, [([\rho(h)]_{\mathcal{B}} - (a_i - a_j)I_n)}^{k}, E_{ij}] \cdots] = 0$$

for sufficiently large k. Also, E_{ij} are eigenvectors of the action of $[\rho(h)_s]_{\mathcal{B}} = \text{diag }(a_1, \ldots, a_n)$ on the space $M_{n\times n}$ of $n \times n$ matrices by the bracket operation with eigenvalues $a_i - a_j$. Thus for the corresponding linear operators on M, we also have

$$(\text{ad}_{\mathfrak{gl}(M)}\ \rho(h))^k T_{ij} = \overbrace{[(\rho(h) - (a_i - a_j)I_M), \cdots, [(\rho(h) - (a_i - a_j)I_M)}^{k}, T_{ij}] \cdots] = 0$$

and T_{ij} are eigenvectors for $\text{ad}_{\mathfrak{gl}(M)}\ \rho(h)_s$ with eigenvalues $a_i - a_j$. Since $\text{ad}_{\mathfrak{gl}(M)}\ \rho(h)$ maps $\rho(\mathfrak{sl}(2,\mathbb{F}))$ to itself, $\rho(\mathfrak{sl}(2,\mathbb{F}))$ is also a direct sum of generalized eigenspace of the operator $\text{ad}_{\mathfrak{gl}(M)}\ \rho(h)$. In particular, $\text{ad}_{\mathfrak{gl}(M)}\ \rho(h)_s$ also maps $\rho(\mathfrak{sl}(2,\mathbb{F}))$ to itself. Moreover, the discussion above shows that $\text{ad}_{\mathfrak{gl}(M)}\ \rho(h)_s$ restricted to $\rho(\mathfrak{sl}(2,\mathbb{F}))$ is semisimple, $\text{ad}_{\mathfrak{gl}(M)}\ \rho(h) - \text{ad}_{\mathfrak{gl}(M)}\ \rho(h)_s$ restricted to $\rho(\mathfrak{sl}(2,\mathbb{F}))$ is nilpotent and $\text{ad}_{\mathfrak{gl}(M)}\ \rho(h)_s$ commutes with $\text{ad}_{\mathfrak{gl}(M)}\ \rho(h) - \text{ad}_{\mathfrak{gl}(M)}\ \rho(h)_s$. Thus $\text{ad}_{\mathfrak{gl}(M)}\ \rho(h)_s|_{\rho(\mathfrak{sl}(2,\mathbb{F})))}$ and $(\text{ad}_{\mathfrak{gl}(M)}\ \rho(h) - \text{ad}_{\mathfrak{gl}(M)}\ \rho(h)_s)|_{\rho(\mathfrak{sl}(2,\mathbb{F})))}$ are the semisimple and nilpotent parts, respectively, of $\text{ad}_{\rho(\mathfrak{sl}(2,\mathbb{F}))}\ \rho(h) = \text{ad}_{\mathfrak{gl}(M)}\ \rho(h)|_{\rho(\mathfrak{sl}(2,\mathbb{F})))}$. But we already showed that $\text{ad}_{\rho(\mathfrak{sl}(2,\mathbb{F}))}\ \rho(h)$ is semisimple. So

$$\text{ad}_{\rho(\mathfrak{sl}(2,\mathbb{F}))}\ \rho(h) = \text{ad}_{\mathfrak{gl}(M)}\ \rho(h)_s|_{\rho(\mathfrak{sl}(2,\mathbb{F})))}.$$

Since $\mathfrak{sl}(2,\mathbb{F})$ is semisimple, $\text{ad}_{\rho(\mathfrak{sl}(2,\mathbb{F}))}$ is faithful. Hence we have $\rho(h) = \rho(h)_s$, proving that $\rho(h)$ is semisimple. ∎

Let M be a finite-dimensional $\mathfrak{sl}(2,\mathbb{F})$-module. Then M is the direct sum of the eigenspaces M_{λ_i} of h with eigenvalues λ_i of h, respectively, for $i = 1, \ldots, k$. For $\lambda \neq \lambda_i$, we let $M_\lambda = 0$. Then we have

$$M = \coprod_{\lambda \in \mathbb{F}} M_\lambda.$$

Definition 3.28. The eigenvalues λ_i for $i = 1, \ldots, k$ are called *weights* of h or *weights* of the corresponding eigenvectors and the eigenspaces M_{λ_i} for $i = 1, \ldots, k$ are called *weight spaces* of h. An nonzero element $v \in M$ is called a *maximal vector* if $xv = 0$.

Theorem 3.29. *Let M be a finite-dimensional irreducible $\mathfrak{sl}(2,\mathbb{F})$-module. Let $m = \dim M - 1$. Then we have:*

1. *$M = \coprod_{i=0}^{m} M_{m-2i}$ and $\dim M_{m-2i} = 1$ for $i = 0, \ldots, m$.*

2. *Up to nonzero scalar multiples, M has a unique maximal vector in M_m.*

3. *Let $v_0 \in M_m$ be a maximal vector of M, $v_{-1} = 0$ and $v_i = \frac{1}{i!}y^i v_0$ for $i \in \mathbb{N}$. Then $v_i \neq 0$ and the action of $\mathfrak{sl}(2,\mathbb{F})$ on M is given by $hv_i = (m-2i)v_i$, $yv_i = (i+1)v_i$ and $xv_i = (m-i+1)v_i$ for $i \in \mathbb{N}$. In particular, up to isomorphisms, there exists at most one irreducible $\mathfrak{sl}(2,\mathbb{F})$-module of dimension $m+1$ for $m \in \mathbb{N}$.*

Proof. Since the action of h on M is semisimple, we have $M = \coprod_{\lambda \in \mathbb{F}} M_\lambda$. Since M is finite dimensional, there must be $\lambda \in \mathbb{F}$ such that $M_\lambda \neq 0$ but $M_{\lambda+2} = 0$. Take any nonzero element $v_0 \in M_\lambda$. Then

$$hxv_0 = xhv_0 + 2xv_0 = \lambda xv_0 + 2xv_0 = (\lambda + 2)xv_0$$

and thus $xv_0 \in M_{\lambda+2} = 0$. So v_0 is a maximal vector. Let $v_i = \frac{1}{i!}y^i v_0$ for $i \in \mathbb{N}$. Using the bracket formulas for x, y and h, we have $hv_i = (\lambda - 2i)v_i$, $yv_i = (i+1)v_{i+1}$ and $xv_i = (\lambda - i + 1)v_{i-1}$ for $i \in \mathbb{N}$.

Since M is finite dimensional, there must be $m \in \mathbb{N}$ such that $v_0, \ldots, v_m \neq 0$ but $v_{m+1} = 0$. Since $v_{m+1} = 0$, $v_i = 0$ for $i \geq m+1$. Since $v_0, \ldots, v_m$ are eigenvectors for h with distinct eigenvalues, they must be linearly independent. Also $v_0, \ldots, v_m$ span a vector space which is invariant under the action of x, y and h. So $v_0, \ldots, v_m$ span a submodule of M. Since M is irreducible, M must be equal to this submodule. So we see that M has a basis $\{v_0, \ldots, v_m\}$. Since $0 = xv_{m+1} = (\lambda - m - 1 + 1)v_m$ and $v_m \neq 0$, we obtain $\lambda = m$. Part 1 follows immediately.

Assume that there is another maximal vector u. Then $u = \alpha_0 v_0 + \cdots + \alpha_m v_m$ and

$$xu = \alpha_0 xv_0 + \cdots + \alpha_m xv_m = \alpha_1 mv_0 + \alpha_2(m-1)v_1 + \cdots + \alpha_m v_{m-1}.$$

Since u is a maximal vector,

$$\alpha_1 mv_0 + \alpha_2(m-1)v_1 + \cdots + \alpha_m v_{m-1} = xu = 0.$$

Thus we have $\alpha_1 = \cdots = \alpha_m$ and $u = \alpha_0 v_0$, proving Part 2.

Part 3 follows immediately. ∎

The theorem above gives the classification of irreducible $\mathfrak{sl}(2, \mathbb{F})$-modules. We still need to establish the existence. To establish the existence, we need the following Poincar'e-Birkhoff-Witt theorem in the case of finite-dimensional Lie algebras:

Theorem 3.30 (Poincaré-Birkhoff-Witt). *Let L be a finite-dimensional Lie algebra and $\{u_1, \ldots, u_n$ an ordered basis of L. Then elements of the form*

$$u_{i_1} \cdots u_{i_k}$$

for $k \in \mathbb{N}$ and $1 \leq i_1 \leq\leq i_k \leq n$ form a basis of $U(L)$ (when $k = 0$, the element is 1).

We omit the proof here. See [H].

We also need the following construction of "induced modules:"

Let L be a finite-dimensional Lie algebra and L_1 a subalgebra of L. Then $U(L_1)$ can be embedded into $U(L)$ as a subalgebra. Let M_1 be an L_1-module. Then $U(L) \otimes M_1$ is a $U(L)$-module. Let I be the $U(L)$-submodule of $U(L) \otimes M_1$ generated by elements of the form $ab \otimes c - a \otimes bc$ for $a \in U(L)$, $b \in U(L_1)$ and $c \in M_1$ where bc is the action of b on c. Then $(U(L) \otimes M_1)/I$ is also a $U(L)$-module and thus an L-module. This L-module is denoted by $\mathrm{Ind}_{U(L_1)}^{U(L)} M_1$ or $U(L) \otimes_{U(L_1)} M_1$ and is called an *induced module.*

Proposition 3.31. *Let L be a finite-dimensional Lie algebra and L_1 and L_2 are subalgebras o L such that $L = L_1 \oplus L_2$. Then the universal enveloping algebra $U(L)$ is linearly isomorphic to $U(L_1) \otimes U(L_2)$.*

Proof. We choose an ordered basis $\{u_1, \ldots, u_k\}$ of L_1 and an ordered basis $\{v_1, \ldots, v_l\}$ of L_2. Then $\{u_1, \ldots, u_k, v_1, \ldots, v_l\}$ is a basis of L. By the Poincar'e-Birkhoff-Witt theorem, elements of the form

$$u_{i_1} \cdots u_{i_p} v_{j_1} \cdots v_{j_q}$$

for $p, q \in \mathbb{N}$, $1 \leq i_1 \leq \cdots \leq i_p \leq k$ and $1 \leq j_1 \leq \cdots \leq j_q \leq l$ form a basis of $U(L)$. But also by the Poincar'e-Birkhoff-Witt theorem, the set of elements of the form

$$u_{i_1} \cdots u_{i_p} \otimes v_{j_1} \cdots v_{j_q}$$

for $p, q \in \mathbb{N}$, $1 \leq i_1 \leq \cdots \leq i_p \leq k$ and $1 \leq j_1 \leq \cdots \leq j_q \leq l$ form a basis of $U(L_1) \otimes U(L_2)$. It follows that $U(L)$ is linearly isomorphic to $U(L_1) \otimes U(L_2)$. ∎

Now come back to $\mathfrak{sl}(2, \mathbb{F})$-modules. Let $L_1 = \mathbb{F}x + \mathbb{F}h$ and $L_2 = \mathbb{F}y$. Then $\mathfrak{sl}(2, \mathbb{F}) = L_1 \oplus L_2$. Consider a one-dimensional vector space $\mathbb{F}v_0$ with a basis v_0. For $m \in \mathbb{N}$, we define an action of L_1 on on $\mathbb{F}v_0$ by $xv_0 = 0$ and $hv_0 = mv_0$. It is easy to see that this action gives an L_1-module structure to $\mathbb{F}v_0$. Now we have the induced module $U(\mathfrak{sl}(2, \mathbb{F})) \otimes_{U(L_1)} \mathbb{F}v_0$ for $\mathfrak{sl}(2, \mathbb{F})$. By the proposition above, this induced module is linearly isomorphic to the vector space $(U(L_2) \otimes U(L_1)) \otimes_{U(L_1)} \mathbb{F}v_0$ which in turn is linearly isomorphic to the vector space $U(L_2) \otimes \mathbb{F}v_0$. From the definition of L_2 we see that $U(L_2) \otimes \mathbb{F}v_0$ has a basis consiting of $\frac{1}{i!}y^i \otimes v_0$. From this basis, we see that the induced module $U(\mathfrak{sl}(2, \mathbb{F})) \otimes_{U(L_1)} \mathbb{F}v_0$ is infinite dimensional and is not what we are interested. What we are interested are finite dimensional and irreducible.

To obtain irreducible modules, we consider maximal submodules of $U(\mathfrak{sl}(2, \mathbb{F})) \otimes_{U(L_1)} \mathbb{F}v_0$. In fact, let J be the sum of all submodules of $U(\mathfrak{sl}(2, \mathbb{F})) \otimes_{U(L_1)} \mathbb{F}v_0$ which does not contain $1 \otimes v_0$. Then J is also a submodule. It is maximal because any submodule larger than this one must contain the element $1 \otimes v_0$ and thus is equal to $U(\mathfrak{sl}(2, \mathbb{F})) \otimes_{U(L_1)} \mathbb{F}v_0$. Thus we obtain an irreducible $\mathfrak{sl}(2, \mathbb{F})$-module $(U(\mathfrak{sl}(2, \mathbb{F})) \otimes_{U(L_1)} \mathbb{F}v_0)/J$.

Moreover, we have:

Theorem 3.32. *The dimension of the irreducible $\mathfrak{sl}(2, \mathbb{F})$-module $(U(\mathfrak{sl}(2, \mathbb{F})) \otimes_{U(L_1)} \mathbb{F}v_0)/J$ is $m + 1$.*

Proof. We first prove that in this irreducible $\mathfrak{sl}(2, \mathbb{F})$-module, $w = y^{m+1} \otimes v_0 = 0$. It is easy to see that

$$\begin{aligned} xw = xy^{m+1} \otimes v_0 &= y^{m+1}x \otimes v_0 + m(m+1)y^m \otimes v_0 - (m+1)y^m h \otimes v_0 \\ &= y^{m+1} \otimes xv_0 + m(m+1)y^m \otimes v_0 - (m+1)y^m \otimes hv_0 \\ &= (m+1)y^m \otimes v_0 - m(m+1)y^m \otimes v_0 \\ &= 0. \end{aligned}$$

Thus w is also a maximal vector. But an $\mathfrak{sl}(2, \mathbb{F})$-module cannot have more than one linearly independent maximal vector (see exercise below). Since the weight of w is not m, we must have $w = 0$, that is, $y^{m+1} \otimes v_0 = 0$.

We now have $y^i \otimes v_0 = 0$ for $i \geq m+1$. Since $U(\mathfrak{sl}(2,\mathbb{F})) \otimes_{U(L_1)} \mathbb{F}v_0$ is linearly isomorphic to the space $U(L_2) \otimes \mathbb{F}v_0$ which has a basis consisting of $\frac{1}{i!}y^i \otimes v_0$, we see that $(U(\mathfrak{sl}(2,\mathbb{F})) \otimes_{U(L_1)} \mathbb{F}v_0)/J$ is linearly spanned by elements of the form $\frac{1}{i!}y^i \otimes v_0$ for $i \leq m$. Thus $(U(\mathfrak{sl}(2,\mathbb{F})) \otimes_{U(L_1)} \mathbb{F}v_0)/J$ is finite dimensional. Since the weight of v_0 is m, by the theorem we proved before, $\dim(U(\mathfrak{sl}(2,\mathbb{F})) \otimes_{U(L_1)} \mathbb{F}v_0)/J = m+1$. ∎

Exercise 3.33. Prove that maximal vectors for the module $(U(\mathfrak{sl}(2,\mathbb{F})) \otimes_{U(L_1)} \mathbb{F}v_0)/J$ are unique up to a nonzero scalar.

Corollary 3.34. *There is a bijection between the set* $\mathbb{N}$ *of nonnegative integers and the set of equivalence classes of finite-dimensional irreducible* $\mathfrak{sl}(2,\mathbb{F})$*-modules.*

Now we quickly discuss the representation theory of general finite-dimensional semisimple Lie algebras. We shall describe only the main constructions and state the main results without giving any proofs.

Let L be a finite-dimensional semisimple Lie algebra. Then there must be a semisimple element of L. A *toral subalgebra* of L is a subalgebra of L consisting of semisimple elements.

Proposition 3.35. *A toral subalgebra of* L *is an abelian Lie algebra.*

Now we take a maximal toral subalgebra H of L, that is, a toral sublagebra of L such that any toral subalgebra containing H must be H. Since H is abelian and commuting operators have same eigenvectors, L is a direct sum of common eigenspaces of elements of H. For any eigenvector x of elements of H, there exists $\alpha \in H^*$ such that

$$[h,x] = \alpha(h)x.$$

Let Φ be the space of all nonzero such $\alpha \in H^*$ and let

$$L_\alpha = \{x \in L \mid [h,x] = \alpha(h)x \text{ for } h \in H\}.$$

Then we have

$$L = L_0 \oplus \coprod_{\alpha \in \Phi} L_\alpha.$$

$0 \in H^*$ is in Φ and $H \subset L_\alpha$. It can be proved that $L_0 = H$. Thus we have

$$L = H \oplus \coprod_{\alpha \in \Phi} L_\alpha.$$

It can be proved that one can find a basis Δ of the real vector space E spanned by elements of Φ such that $\Delta \subset \Phi$ and any element of Φ can be written as a linear combination of elements of Δ with either nonnegative coefficients or nonpositive coefficients. Elements of Φ are called *roots*. Elements of Δ are called *simple roots*. We fix a choice of $\Delta = \{\alpha_1, \dots, \alpha_l\}$.

Let M be an L-module. As in the case for $\mathfrak{sl}(2,\mathbb{F})$, the actions of elements of H on M must be semisimple. Since H is abelian, the actions of elements of H on M commute with each other. Thus

$$M = \coprod_{\lambda \in H^*} M_\lambda$$

where

$$M_\lambda = \{x \in M \mid hx = \lambda(h)x, \text{ for } h \in H\}$$

for $\lambda \in H^*$. When $M_\lambda \neq 0$, we say that λ is a *weight* of M and M_λ the *weight space* of weight λ. Let $\{\lambda_1, \ldots, \lambda_l\}$ be a basis of E determined by

$$\frac{2(\lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}$$

for $i, j = 1, \ldots, l$, where $(\cdot, \cdot)$ is the bilinear form on E induced from the Killing form on H. A weight λ is said to be *dominant* if it is a linear combination of $\lambda_1, \ldots, \lambda_l$ with nonnegative coefficients and is said to be *integral* if it is a linear combination of $\lambda_1, \ldots, \lambda_l$ with integral coefficients. Let Λ^+ be the set of dominant A weight is *dominant integral* if it is dominant and integral. Then we have the following result:

Theorem 3.36. *Let L be a finite-dimensional semisimple Lie algebra. Then there is a bijection from the set Λ^+ of dominant integral weights to the set of equivalence classes of finite-dimensional irreducible L-modules.*

The proof of this theorem is in spirit the same as the corresponding theorem above when $L = \mathfrak{sl}(2, \mathbb{F})$.

4 Basic concepts in category theory

Main references: [EGHLSVY], [J] and [M].

Definition 4.1. A *category* consists of the following data:

1. A collection of *objects*.

2. For two objects A and B, a set $\mathrm{Hom}(A, B)$ of *morphisms from A to B*.

3. For an object A, an *identity* $1_A \in \mathrm{Hom}(A, A)$.

4. For three objects A, B, C, a map

$$\begin{aligned} \circ : \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) &\to \mathrm{Hom}(A, C) \\ (f, g) &\mapsto f \circ g \end{aligned}$$

   called *composition* or *multiplication*.

These data must satisfy the following axioms:

1. The composition is associative, that is, for objects A, B, C, D and $f \in \mathrm{Hom}(C, D)$, $g \in \mathrm{Hom}(B, C)$, $h \in \mathrm{Hom}(A, B)$, we have $f \circ (g \circ h) = (f \circ g) \circ h$.

2. For an object A, the identity 1_A is the identity for the composition of morphisms when the morphisms involving A, that is, for an object B, $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, A)$, we have $1_A \circ g = g$ and $f \circ 1_A = f$.

We shall use $\mathcal{C}$, $\mathcal{D}$ and so on to denote categories. For a category $\mathcal{C}$, we use $\operatorname{Ob}\mathcal{C}$ to denote the collection of objects of $\mathcal{C}$.

Definition 4.2. Let $\mathcal{C}$ be a category. For any $A, B \in \operatorname{Ob}\mathcal{C}$, an element $f \in \operatorname{Hom}(A, B)$ is called an isomorphism if there exists $f^{-1} \in \operatorname{Hom}(B, A)$ such that $f \circ f^{-1} = 1_B$ and $f^{-1} \circ f = 1_A$.

Definition 4.3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant functor* (or a *contravariant functor*) from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data:

1. A map $\mathcal{F}$ from the collection $\operatorname{Ob}\mathcal{C}$ of objects of $\mathcal{C}$ to the collection $\operatorname{Ob}\mathcal{D}$ of objects of $\mathcal{D}$.

2. Given objects A and B of $\mathcal{C}$, a map, still denoted by $\mathcal{F}$, from $\operatorname{Hom}(A, B)$ to $\operatorname{Hom}(\mathcal{F}(A), \mathcal{F}(B))$ (or from $\operatorname{Hom}(A, B)$ to $\operatorname{Hom}(\mathcal{F}(B), \mathcal{F}(A))$ for a contravariant functor).

These data must satisfy the following axioms:

1. For objects A, B, C of $\mathcal{C}$ and morphisms $f \in \operatorname{Hom}(B, c)$, $g \in \operatorname{Hom}(A, B)$, we have
$$\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$$
(or
$$\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f)$$
for a contravariant functor).

2. For an object A of $\mathcal{C}$, $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$.

We shall denote the functor defined above by $\mathcal{F}$.

Definition 4.4. Let $\mathcal{F}$ and $\mathcal{G}$ be functors from $\mathcal{C}$ to $\mathcal{D}$. A *natural transformation* η *from* $\mathcal{F}$ *to* $\mathcal{G}$ consists of an element $\eta_A \in \operatorname{Hom}(\mathcal{F}(A), \mathcal{G}(A))$ for each object $A \in \operatorname{Ob}\mathcal{C}$ such that the following diagram is commutative for $A, B \in \operatorname{Ob}\mathcal{C}$ and $f, g \in \operatorname{Hom}(A, B)$:

$$\begin{array}{ccc}
\mathcal{F}(A) & \xrightarrow{\eta_A} & \mathcal{G}(A) \\
{\scriptstyle \mathcal{F}(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(g)} \\
\mathcal{F}(B) & \xrightarrow[\eta_B]{} & \mathcal{G}(B).
\end{array}$$

A *natural isomorphism* from $\mathcal{C}$ to $\mathcal{D}$ is a natural transformation η from $\mathcal{C}$ to $\mathcal{D}$ such that $\eta_A \in \operatorname{Hom}(\mathcal{F}(A), \mathcal{G}(A))$ for each object $A \in \operatorname{Ob}\mathcal{C}$ is an isomorphism.

Definition 4.5. Let $\mathcal{F}$ be a functor from a category $\mathcal{C}$ to a category $\mathcal{D}$ and $\mathcal{G}$ a functor from a category $\mathcal{D}$ to a category $\mathcal{E}$. The *composition* $\mathcal{G} \circ \mathcal{F}$ of $\mathcal{G}$ and $\mathcal{F}$ is a functor from $\mathcal{C}$ to $\mathcal{E}$ given by $(\mathcal{G} \circ \mathcal{F})(A) = \mathcal{G}(\mathcal{F}(A))$ for $A \in \operatorname{Ob}\mathcal{C}$ and $(\mathcal{G} \circ \mathcal{F})(f) = \mathcal{G}(\mathcal{F}(f))$ for $f \in \operatorname{Hom}(A, B)$ and $A, B \in \operatorname{Ob}\mathcal{C}$. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We say that $\mathcal{C}$ *is isomorphic to* $\mathcal{D}$ if there is a functor $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$ and a functor $\mathcal{F}^{-1}$ such that $\mathcal{F} \circ \mathcal{F}^{-1} = 1_{\mathcal{D}}$ and $\mathcal{F}^{-1} \circ \mathcal{F} = 1_{\mathcal{C}}$. We say that $\mathcal{C}$ *is equivalent to* $\mathcal{D}$ if there is a functor $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$ and a functor $\mathcal{G}$ such that $\mathcal{F} \circ \mathcal{G}$ is naturally isomorphic to $1_{\mathcal{D}}$ and $\mathcal{G} \circ \mathcal{F}$ is naturally isomorphic to $1_{\mathcal{C}}$.

Definition 4.6. Let A_j for $j \in I$ be objects of a category $\mathcal{C}$. A *product* of A_j for $j \in I$ is an object $\prod_{j\in I} A_j$ together with morphisms $p_j : \prod_{j\in I} A_j \to A_j$ satisfying the following universal property: For any object A of $\mathcal{C}$ and any morphism $f_j : A \to A_i$, there exists a unique morphism $f : A \to \prod_{j\in I} A_j$ such that such that $f_j = p_j \circ f$ for $i \in I$. A *coproduct* of A_j for $j \in J$ is an object $\coprod_{j\in I} A_j$ together with morphisms $i_j : A_j \to \prod_{j\in I} A_j$ satisfying the following universal property: For any object A of $\mathcal{C}$ and any morphism $f_j : A_j \to A$, there exists a unique morphism $f : \prod_{j\in I} A_i \to A$ such that $f_j = f \circ i_j$ for $i \in I$.

Exercise 4.7. Prove that products and coproducts of objects A_j for $j \in I$ in a category $\mathcal{C}$ are unique up to isomorphisms.

Definition 4.8. An *initial object* in a category $\mathcal{C}$ is an object I in $\mathcal{C}$ such that for any object X in $\mathcal{C}$, $\mathrm{Hom}(I, X)$ has one and only one element. An *terminal object* in a category $\mathcal{C}$ is an object T in $\mathcal{C}$ such that for any object X in $\mathcal{C}$, $\mathrm{Hom}(X, T)$ has one and only one element. A *zero object* in a category $\mathcal{C}$ is both an initial object and a terminal object.

Definition 4.9. Let $\mathcal{C}$ be a category containing a zero object 0. Let A and B be objects of $\mathcal{C}$ and let $f \in \mathrm{Hom}(A, B)$. A *kernel* of f is an object K and a morphism $k \in \mathrm{Hom}(K, A)$ satisfying $f \circ k = 0$ and the following universal property: For any object K' and morphism $k' \in \mathrm{Hom}(K', A)$ satisfying $f \circ k' = 0$, there exists a unique $g \in \mathrm{Hom}(K', K)$ such that $k' = k \circ g$. A *cokernel* of f is an object Q and a morphism $q \in \mathrm{Hom}(B, Q)$ satisfying $q \circ f = 0$ and the following universal property: For any object Q' and morphism $q' \in \mathrm{Hom}(B, Q')$ satisfying $q' \circ f = 0$, there exists a unique $u \in \mathrm{Hom}(Q, Q')$ such that $q' = u \circ q$.

Exercise 4.10. Prove that kernels and cokernels of of a morphism are unique up to isomorphisms.

Definition 4.11. Let $\mathcal{C}$ be a category containing a zero object 0. Let $A_1, \ldots, A_n$ be objects of $\mathcal{C}$. A *biproduct* of $A_1, \ldots, A_n$ is an object $A_1 \oplus \cdots \oplus A_n$ of $\mathcal{C}$ and $p_k : A_1 \oplus \cdots \oplus A_n \to A_k$ and $i_k : A_k \to A_1 \oplus \cdots \oplus A_n$ for $k = 1, \ldots, n$ such that $p_k \circ i_k = 1_{A_k}$ for $k = 1, \ldots, n$, $p_l \circ i_k = 0$ for $l \neq k$, $A_1 \oplus \cdots \oplus A_n$ equipped with p_k for $k = 1, \ldots, n$ is a product of $A_1, \ldots, A_n$ and $A_1 \oplus \cdots \oplus A_n$ equipped with i_k for $k = 1, \ldots, n$ is a coproduct of $A_1, \ldots, A_n$.

Definition 4.12. Let $\mathcal{C}$ be a category. Let A and B be objects of $\mathcal{C}$. A morphism $f \in \mathrm{Hom}(A, B)$ is said to be a *monomorphism* if for any object C and any $g_1, g_2 \in \mathrm{Hom}(C, A)$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$. A morphism $f \in \mathrm{Hom}(A, B)$ is said to be aN *epimorphism* if for any object C and any $g_1, g_2 \in \mathrm{Hom}(B, C)$, $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$.

Definition 4.13. An *abelian category* is a category $\mathcal{C}$ satisfying the following conditions:

1. For any objects A and B, $\mathrm{Hom}(A, B)$ is an abelian group and for any objects A, B and C, the map from $\mathrm{Hom}(B, A) \times \mathrm{Hom}(C, B)$ to $\mathrm{Hom}(C, A)$ given by the composition of morphisms is bilinear.

2. Every finite set of objects has a biproduct.

3. Every morphism has a kernel and cokernel.

4. Every monomorphism is a kernel of some morphism and every epimorphism is a cokernel of some morphism.

5 Monoidal categories and tensor categories

Main references: [EGNO], [M].

Definition 5.1. An *monoidal category* consists of the following data:

1. A category $\mathcal{C}$.
2. A bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *tensor product bifunctor*.
3. A natural isomorphism $\mathcal{A}$ from $\otimes \circ (1_{\mathcal{C}} \times \otimes)$ to $\otimes \circ (\otimes \times 1_{\mathcal{C}})$ called the *associativity isomorphism*.
4. An object $\mathbf{1}$ called the *unit object*.
5. A natural isomorphism l from $\mathbf{1} \otimes \cdot$ to $1_{\mathcal{C}}$ called the *left unit isomorphism* and a natural isomorphism r from $\cdot \otimes \mathbf{1}$ to $1_{\mathcal{C}}$ called the *right unit isomorphism*.

These data satisfy the following axioms:

1. The following *pentagon diagram* is commutative for objects A_1, A_2, A_3, A_4:

$$\begin{array}{ccccc} & & A_1 \otimes (A_2 \otimes (A_3 \otimes A_4)) & & \\ & \swarrow & & \searrow & \\ (A_1 \otimes A_2) \otimes (A_3 \otimes A_4) & & & & A_1 \otimes ((A_2 \otimes A_3) \otimes A_4) \\ \downarrow & & & & \downarrow \\ ((A_1 \otimes A_2) \otimes A_3) \otimes A_4 & & \longleftarrow & & (A_1 \otimes (A_2 \otimes A_3)) \otimes A_4 \end{array}$$

2. The following *triangle diagram* is commutative for objects A_1, A_2:

$$\begin{array}{ccc} (A_1 \otimes \mathbf{1}) \otimes A_2 & \longrightarrow & A_1 \otimes (\mathbf{1} \otimes A_2) \\ \downarrow & & \downarrow \\ A_1 \otimes A_2 & \underset{=}{\longrightarrow} & A_1 \otimes A_2. \end{array}$$

Definition 5.2. A *tensor category* is an abelian category equipped with a monoidal category structure such that the abelain category structure and the monoidal category structure are compatible in the sense that for objects A, B, C and D, the map $\otimes : \mathrm{Hom}(A, B) \times \mathrm{Hom}(C, D) \to \mathrm{Hom}(A \times C, B \otimes D)$ is bilinear.

Definition 5.3. Let $\mathcal{C}$ be a monoidal category. A *graph diagram in* $\mathcal{C}$ is a graph whose vertices are functors obtained from the tensor product bifunctor and the unit objects and the edges are natural isomorphisms obtained from the associativity isomorphisms, the left and the right unit isomorphisms. A graph diagram is *commutative* if the compositions of the isorphisms in any two paths with the same starting and ending vertices must be equal.

Theorem 5.4 (Mac Lane). *Let $\mathcal{C}$ be a monoidal category. Any graph diagram in $\mathcal{C}$ is commutative*

We omit the proof here; see [EGNO] and [M].

Definition 5.5. A *monoidal functor* from a monoidal categoory $\mathcal{C}$ to a monoidal category $\mathcal{D}$ is a triple $(\mathcal{F}, J, \varphi)$ where $\mathcal{F}$ is a functor from $\mathcal{C}$ to $\mathcal{D}$, J a natural transformation from the functor $\mathcal{F}(\cdot) \otimes_{\mathcal{D}} \mathcal{F}(\cdot)$ to the functor $\mathcal{F}(\cdot \otimes_{\mathcal{C}} \cdot)$ and φ an isomorphism from $\mathbf{1}_{\mathcal{D}}$ to $\mathcal{F}(\mathbf{1}_{\mathcal{C}})$ such that the diagram

$$\begin{array}{ccc} (\mathcal{F}(A_1) \otimes_{\mathcal{D}} \mathcal{F}(A_2)) \otimes_{\mathcal{D}} \mathcal{F}(A_3) & \longrightarrow & \mathcal{F}(A_1) \otimes_{\mathcal{D}} (\mathcal{F}(A_2) \otimes_{\mathcal{D}} \mathcal{F}(A_3)) \\ \downarrow & & \downarrow \\ (\mathcal{F}(A_1 \otimes_{\mathcal{C}} A_2) \otimes_{\mathcal{D}} \mathcal{F}(A_3) & & \mathcal{F}(A_1) \otimes_{\mathcal{D}} \mathcal{F}(A_2 \otimes_{\mathcal{C}} A_3) \\ \downarrow & & \downarrow \\ (\mathcal{F}((A_1 \otimes_{\mathcal{C}} A_2) \otimes_{\mathcal{C}} A_3) & \longrightarrow & \mathcal{F}(A_1 \otimes_{\mathcal{C}} (A_2 \otimes_{\mathcal{C}} A_3)) \end{array}$$

for objects A_1, A_2 and A_3 in $\mathcal{C}$ and the diagram

$$\begin{array}{ccc} \mathbf{1}_{\mathcal{D}} \otimes_{\mathcal{D}} \mathcal{F}(A) & \longrightarrow & \mathcal{F}(A) \\ \downarrow & & \uparrow \\ \mathcal{F}(\mathbf{1}_{\mathcal{C}} \otimes_{\mathcal{D}} \mathcal{F}(A) & \longrightarrow & \mathcal{F}(\mathbf{1}_{\mathcal{C}} \otimes_{\mathcal{C}} A) \end{array}$$

for an object A in $\mathcal{C}$ are commutative. A *monoidal eqivalence* from a monoidal categoory $\mathcal{C}$ to a monoidal category $\mathcal{D}$ is a monoidal functor $(\mathcal{F}, J, \varphi)$ from $\mathcal{C}$ to $\mathcal{D}$ such that $\mathcal{F}$ is an equivalence of categories and J is a natural isomorphism.

Definition 5.6. A monoidal category is *strict* if

$$\begin{aligned} \otimes \circ (1_{\mathcal{C}} \times \otimes) &= \otimes \circ (\otimes \times 1_{\mathcal{C}}), \\ \mathbf{1} \otimes \cdot &= 1_{\mathcal{C}} \\ \cdot \otimes \mathbf{1} &= 1_{\mathcal{C}} \end{aligned}$$

and the associativity, the left and the right unit isomorphisms are identities.

Theorem 5.7 (Mac Lane). *Any monoidal category is monoidal equivalent to a strict monoidal category.*

Exercise 5.8. Consider the category of bimodules for an associative algebra and the tensor product bifunctor we defined in the section on associative algebras. Show that there exists an associaitivity isomorphism such that the pentagon diagram is commutative.

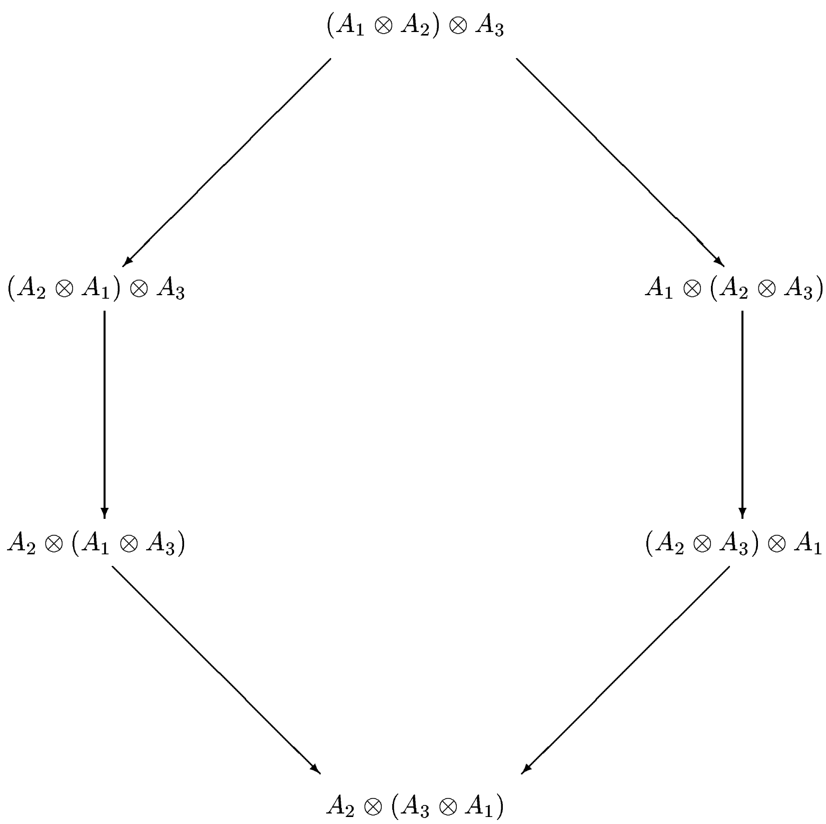
6 Symmetries and braidings

Main references: [T].

Definition 6.1. Let $\mathcal{C}$ be a monoidal category. A *symmetry* of $\mathcal{C}$ is a natural isomorphism C from $\otimes$ to $\otimes \circ \sigma_{12}$ (σ_{12} being the functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C} \times \mathcal{C}$ induced from the nontrivial element of S_2) such that for objects A_1, A_2, the morphism

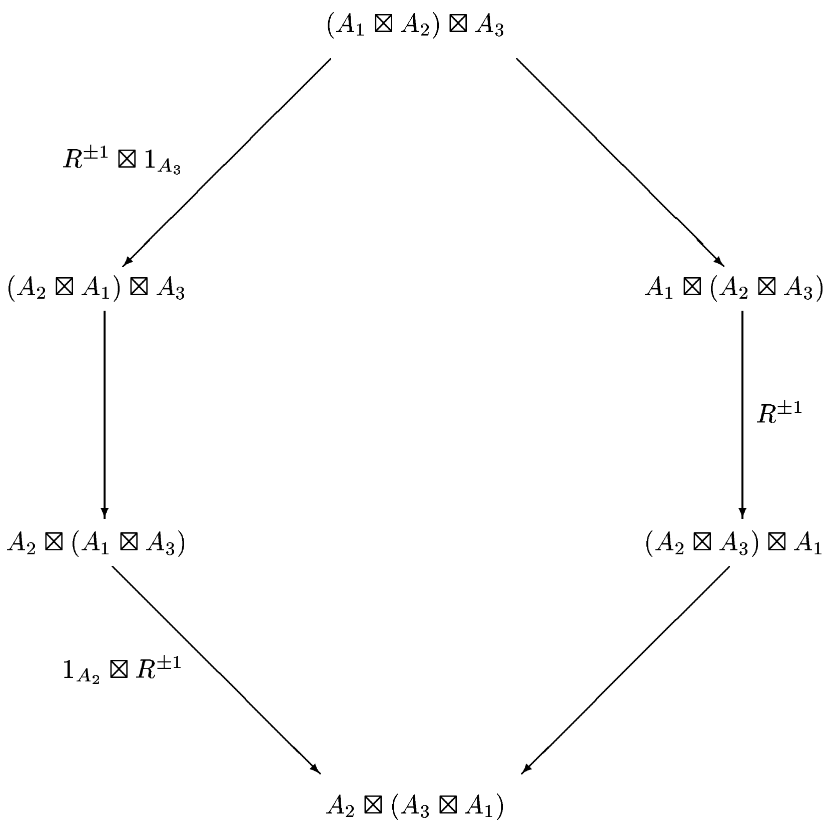
$$C_{A_2,A_1} \circ C_{A_1,A_2} : A_1 \otimes A_2 \to A_2 \otimes A_1 \to A_1 \otimes A_2$$

is equal to the identity $1_{A_1 \otimes A_2}$ and for objects A_1, A_2 and A_2, the *hexagon diagram*



is commutative. A *symmetric monoidal category* is a monoidal category with a symmetry. A *symmetric tensor category* is a tensor category with a symmetry.

Definition 6.2. Let $\mathcal{C}$ be a monoidal category. A *braiding* of $\mathcal{C}$ is a natural isomorphism R from $\otimes$ to $\otimes \circ \sigma_{12}$ such that for objects A_1, A_2 and A_2, the *hexagon diagrams*



is commutative. A *braided monoidal category* is a monoidal category with a braiding. A *braided tensor category* is a tensor category with a braiding.

7 Rigidity

Main references: [EGNO], [T].

Definition 7.1. Let $\mathcal{C}$ be a monoidal category. For an object A, a *right dual* of A is an object A^* and morphisms $\mathrm{ev}_A : A^* \otimes A \to \mathbf{1}$ and $\mathrm{coev}_A : \mathbf{1} \to A \otimes A^*$ such that the morphism obtained by composing the morphisms in

$$A \to \mathbf{1} \otimes A \to (A \otimes A^*) \otimes A \to A \otimes (A^* \otimes A) \to A \otimes \mathbf{1} \to A$$

is equal to the identity 1_A and the morphism obtained by composing the morphisms in

$$A^* \to A^* \otimes \mathbf{1} \to A^* \otimes (A \otimes A^*) \to (A^* \otimes A) \otimes A^* \to \mathbf{1} \otimes A^* \to A^*$$

is equal to the identity 1_{A^*}. A *left dual* of A is an object ${}^*A$ and morphisms $\mathrm{ev}'_A : A \otimes^* A \to \mathbf{1}$ and $\mathrm{coev}'_A : \mathbf{1} \to^* A \otimes A$ such that the morphism obtained by composing the morphisms in

$$A \to A \otimes \mathbf{1} \to A \otimes ({}^*A \otimes A) \to (A \otimes^* A) \otimes A \to \mathbf{1} \otimes A \to A$$

is equal to the identity 1_A and the morphism obtained by composing the morphisms in

$${}^*A \to \mathbf{1} \otimes^* A \to ({}^*A \otimes A) \otimes^* A \to^* A \otimes (A \otimes^* A) \to^* A \otimes \mathbf{1} \to^* A$$

is equal to the identity $1_{^*A}$.

Definition 7.2. A monoidal category $\mathcal{C}$ is said to be *rigid* if there are contravariant functors $^*\cdot : \mathcal{C} \to \mathcal{C}$ and $\cdot^* : \mathcal{C} \to \mathcal{C}$ such that for an object A, $^*A$ and A^* are left and right duals of A.

Exercise 7.3. Show that the category of finite-dimensional representations for a finite group and the category of finite-dimensional modules for a finite-dimensional Lie algebra are rigid symmetric tensor categories.

8 Ribbon categories and modular tensor categories

Main references: [T].

Definition 8.1. Let $\mathcal{C}$ be a braided onoidal category. A *twist* of $\mathcal{C}$ is a natural isomorphism $\theta : 1_{\mathcal{C}} \to 1_{\mathcal{C}}$ such that for objects A_1 and A_2,

$$\theta_{A_1 \otimes A_2} = R_{A_2, A_1} \circ R_{A_1, A_2} \circ (\theta_{A_1} \otimes \theta_{A_2}).$$

Definition 8.2. A *ribbon category* is a rigid braided monoidal category equipped with a twist.

Lemma 8.3. *In a ribbon category, the left dual and right dual can be taken to be the same.*

We omit the proof of this lemma.

Let $\mathcal{C}$ be a ribbon category and let $K = \mathrm{Hom}(\mathbf{1}, \mathbf{1})$. Then K is a monoid (a set with an associative product and an identity).

Lemma 8.4. *K is in fact commutative.*

In a ribbon category, we can define the "trace" of a morphism and the "dimension" of an object as follows:

Definition 8.5. Let $f \in \mathrm{Hom}(A, A)$ be a morphism in a ribbon category. The *trace* of f is defined to be

$$\mathrm{Tr}\ f = \mathrm{ev}_A \circ R_{A,A^*} \circ ((\theta_A \circ f) \otimes 1_{A^*}) \circ \mathrm{coev}_A \in K.$$

The dimension $\dim A$ of an object A is defined to be $\mathrm{Tr}\ 1_A$.

The trace of a morphism satisfies the properties that a trace should have.

Proposition 8.6. *Let $\mathcal{C}$ be a ribbon category. Then we have:*

1. *For $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, A)$, $\mathrm{Tr}\ fg = \mathrm{Tr}\ gf$.*
2. *For $f \in \mathrm{Hom}(A_1, A_2)$ and $g \in \mathrm{Hom}(A_3, A_4)$, $\mathrm{Tr}\ (f \otimes g) = (\mathrm{Tr}\ f)(\mathrm{Tr}\ g)$.*
3. *For $k \in K$, $\mathrm{Tr}\ k = k$.*

Example 8.7. The category of finite-dimensional representations of a finite group and the category of finite-dimensional modules for a finite-dimensional Lie algebra are ribbon categories whose braidings and twists are trivial.

Example 8.8. Let G be an mulplicative abelian group (an abelian group whose operation is written as a multiplication instead of an addition), K a commutative ring with identity and $c : G \times G \to K^*$ a bilinear form (K^* being the set of invertible elements of K), that is, for $g, g', h, h' \in G$, we have

$$\begin{aligned} c(gg', h) &= c(g, h)c(g', h), \\ c(g, hh') &= c(g, h)c(g, h'). \end{aligned}$$

We construct a ribbon category as follows: The objects of the category are elements of G. For any $g, h \in G$, $\mathrm{Hom}(g, h)$ is K if $g = h$ and 0 if $g \neq h$. The composition of two morphisms $g \to h \to\to f$ is the product of the two elements of K is $g = h = f$ and 0 otherwise. The tensor product of two objects $g, h \in G$ is their product gh. The tensor product $gg' \to hh'$ of two morphisms $g \to g'$ and $h \to h'$ is the product of the two elements in K if $g = h$ and $g' = h'$ and is 0 otherwise. The unit object is the identity of G. The associativity and left and right unit isomorphisms are the identity natural isomorphisms. For $g, h \in G$, the briading $gh \to hg = gh$ is defined to be $c(g, h)$. For $g \in G$, the twist $g \to g$ is defined to be $c(g, g)$. For $g \in G$, the (left and right) dual of g is g^{-1}. The morphisms ev_g, coev_g, ev'_g and coev'_g are the indentity of K. Then we have a ribbon category.

Exercise 8.9. Verify that the example above is indeed a ribbon category.

We now consider ribbon tensor categories, that is, rigid braided tensor categories with twists.

Let $\mathcal{C}$ be a ribbon tensor category. Then $K = \mathrm{Hom}(\mathbf{1}, \mathbf{1})$ acts on $\mathrm{Hom}(A, B)$ for any objects A and B by $kf = l_B \circ (k \otimes f) \circ l_A^{-1}$ for $k \in K$ and $f \in \mathrm{Hom}(A, B)$. This action gives $\mathrm{Hom}(A, B)$ a K-module structure.

Definition 8.10. An object A of a ribbon tensor category is said to be *irreducible* if $\mathrm{Hom}(A, A)$ is a free K-module of rank 1. A ribbon tensor category is said to be *semisimple* if the following conditions are satisfied:

1. For any simple objects A and B, $\mathrm{Hom}(A, B) = 0$ if A is not isomorphic to B.

2. Every object is a direct sum of finitely many irreducible objects.

Example 8.11. The unit object is an irreducible object.

Example 8.12. The ribbon tensor category of finite-dimensional representations over a field of a finite group such that the characteristic of the field does not divide the order of the group and the ribbon tensor category of finite-dimensional modules for a finite-dimensional semisimple Lie algebra are semisimple.

Definition 8.13. A *modular tensor category* is a semisimple ribbon tensor category $\mathcal{C}$, with finitely many equivalence classes of irreducible objects satisfying the following *nondegeneracy property*: Let $\{A_i\}_{i=1}^n$ be a set of representatives of the equivalence classes of irreducible objects of $\mathcal{C}$. Then the matrix (S_{ij}) where

$$S_{ij} = \mathrm{Tr}\, R_{A_j, A_i} \circ R_{A_i, A_j}$$

for $i, j = 1, \ldots, n$ is invertible.

Let I be the set of equivalence classes of irreduible objects in a modular tensor category. We shall use 0 to denote the equivalence class in I containing the unit object.

Proposition 8.14. *The dual object of an irreducible object is also irreducible.*

We omit gthe proof.

From this proposition, we see that there is a map $*: I \to I$ such that for any $i \in I$, i^* is the equivalence class in I such that objects in i^* are duals of objects in i.

We now choose one object A_i for each equivalence class $i \in I$. Then by definition, we have

$$S_{0,i} = S_{i,0} = \dim A_i$$

for $i \in I$.

Definition 8.15. Let $\mathcal{C}$ be a modular tensor category. Assuming that there exists $\mathcal{D} \in K$ such that

$$\mathcal{D}^2 = \sum_{i \in I} (\dim A_i)^2.$$

We call $\mathcal{D}$ the *rank* of $\mathcal{C}$.

If there is no such $\mathcal{D}$ in K, we can always enlarge K and the sets of morphisms such that in the new category, there exists such a $\mathcal{D}$.

For $i \in I$, the twist θ_{A_i} as an element of $\mathrm{Hom}(A_i, A_i)$ must be proportional to 1_{A_i}, that is, there exists $A_i \in K$ such that $\theta_i = A_i 1_{A_i}$. Since θ_{A_i} is an isomorphism, A_i is invertible. Let $\Delta = \sum_{i \in I} v_i^{-1} (\dim A_i)^2$, $T = (\delta_i^j v_i)$ and $J = (\delta_{i^*}^j)$. Then we have

$$\begin{aligned} (\mathcal{D}^{-1} S)^4 &= I, \\ (\mathcal{D}^{-1} T^{-1} S)^3 &= \Delta \mathcal{D}^{-1} (\mathcal{D}^{-1} S)^2. \end{aligned}$$

Let

$$\begin{aligned} s &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ t &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Then s and t are the generators of the modular group

$$SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}$$

satisfying the relations

$$s^4 = I,\ (ts)^3 = s^2.$$

Thus we see that $s \mapsto \mathcal{D}^{-1} S$ and $t \mapsto T^{-1}$ give a projective matrix representation of $SL(2, \mathbb{Z})$.

Since $\mathcal{C}$ is semisimple and I is the set of equivalence classes of irreducible objects in $\mathcal{C}$, we see that $A_i \otimes A_j$ for $i, j \in I$ must be isomorphic to $\oplus_{k \in I} N_{ij}^k A_k$, where N_{ij}^k are nonnegative integers giving the numbers of copies of A_k. These numbers N_{ij}^k afre called *fusion rules*.

Theorem 8.16 (Verlinde conjecture). *For $i, l, m \in I$, we have*

$$\sum_{j,k \in I} S_{mj}^{-1} N_{ij}^{k} S_{kl} = (\dim A_m)^{-1} S_{il} \delta_{lm}.$$

In fact, if we let

$$N_i = (N_{ij}^{k})$$

for $i \in I$, then the theorem above says that the matrix S diagonalizes N_i for $i \in I$ simultaneously.

Corollary 8.17 (Verlinde formula for fusion rules). *For $i, j, k \in I$, we have*

$$N_{ij}^{k} = \mathcal{D}^{-2} \sum_{l \in I} (\dim A_l)^{-1} S_{il} S_{jl} S_{k^* l}.$$

SWe omit the proofs of these results.

9 Construction of tensor categories

Main references: [EGNO], [T].

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