On the applicability of logarithmic tensor category theory

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Abstract

We give results and observations which allow the application of the logarithmic tensor category theory of Lepowsky, Zhang and the author ([HLZ1]–[HLZ9]) to more general vertex (operator) algebras and their module categories than those studied in a paper by the author ([H3]).

1 Introduction

The logarithmic tensor category theory of Lepowsky, Zhang and the author ([HLZ1]–[HLZ9]) gave a construction of a braided tensor category structure (including a ribbon structure) on a module category $\mathcal{C}$ for a suitable vertex operator algebra or more general vertex algebra V. In this paper we broaden the applicability of this theory. After referring to the assumptions needed for invoking the theory in [HLZ1]–[HLZ9], we shall show that these assumptions can be verified in more general settings than before, and in addition, we can relax an assumption, thus yielding new families of logarithmic tensor categories.

Before we discuss the precise mathematics, we would like to mention that the present paper is essentially an addendum to [HLZ1]–[HLZ9] and is readable only if the reader consults these papers. The reader will need to consult the specific definitions, results and, especially, assumptions in [HLZ1]–[HLZ9] discussed in this paper.

The construction in [HLZ1]–[HLZ9] uses certain assumptions on V and $\mathcal{C}$: Assumptions 10.1 in [HLZ7] and Assumptions 12.1 and 12.2 in [HLZ9]. Parts 1 to 5 and most assumptions in Part 7 of Assumption 10.1 in [HLZ7], and Assumption 12.1 in [HLZ9] hold for most of the interesting examples and are relatively easy to verify. But there are examples of V and $\mathcal{C}$ for which the first half of Part 6 of Assumption 10.1 in [HLZ7] does not hold, that is, the weights of elements of generalized V-modules in $\mathcal{C}$ are not all real and we still would like to apply the theory of [HLZ1]–[HLZ9] to these examples. More importantly, the last part (the statement that $\mathcal{C}$ be closed under $P(z)$-tensor products for some $z \in \mathbb{C}^{\times}$) of Part 7 of Assumption 10.1 in [HLZ7], and Assumption 12.2 in [HLZ9], are related to some of the most important and deep properties of V and $\mathcal{C}$. When V and the objects of the category $\mathcal{C}$ have trivial A-gradings (for V) and $\tilde{A}$-grading (for the objects of $\mathcal{C}$), in other words, when the abelian groups A and $\tilde{A}$ are trivial, it was also proved in [HLZ8] that Assumption 12.2 follows from the condition that every finitely-generated lower bounded doubly-graded generalized V-module be an object of $\mathcal{C}$ together with the C_1-cofiniteness condition for objects of $\mathcal{C}$ in the sense of [H2].

In [H3], the author proved that V and the category $\mathcal{C}$ of grading-restricted generalized V-modules (that is, strongly graded generalized V-modules with trivial grading abelian group $\tilde{A}$ in the terminology used in [HLZ1]–[HLZ9]) satisfy all the assumptions in [HLZ7] and [HLZ9] if V is a vertex operator algebra (in particular, with trivial A) satisfying the following three conditions:

1. V is C_1^a-cofinite in the sense that $V/C_1^a(V)$ is finite dimensional, where $C_1^a(V)$ is the subspace of V spanned by the elements of the form u_nv for $u, v \in V_+ = \coprod_{n\in\mathbb{Z}_+} V_{(n)}$ and $L(-1)v$ for $v \in V$. (Here a in the superscript of C_1^a means "algebra" since this is the C_1-cofiniteness condition for V as a vertex operator algebra, not as a V-module.)

2. There exists a positive integer N such that $|\Re(n_1) - \Re(n_2)| \leq N$ for the lowest weights n_1 and n_2 of any two irreducible V-modules and such that $A_N(V)$ is finite dimensional.

3. Every irreducible V-module W is $\mathbb{R}$-graded and C_1-cofinite in the sense that $W/C_1(W)$ is finite dimensional, where $C_1(W)$ is the subspace of W spanned by the elements of the form u_nw for $u \in V_+ = \coprod_{n\in\mathbb{Z}_+} V_{(n)}$ and $w \in W$.

In [H3], the actions of $L(0)$ on objects of the category are in general not semisimple and thus intertwining operators among these objects in general are indeed logarithmic. If V is of positive energy (that is, $V_{(n)} = 0$ for $n < 0$ and $V_{(0)} = \mathbb{C}\mathbf{1}$) and C_2-cofinite (that is, $V/C_2(V)$ is finite dimensional where $C_2(V)$ is the subspace of V spanned by elements of the form $u_{-2}v$ for $u \in V_+$ and $v \in V$), then these three conditions hold. Since these three conditions, or the positive energy condition and C_2-cofiniteness condition, are relatively easy to verify and have indeed been verified for many interesting examples of vertex operator algebras and their module categories, the results in [H3] provide a practical method to apply the logarithmic tensor category theory in [HLZ1]–[HLZ9]. Up to now, the results in [H3] are still the best general results on the verifications of the main assumptions in [HLZ7] and [HLZ9].

An initial motivation for our embarking on the creation of the logarithmic tensor category [HLZ1]–[HLZ9] was to show that the braided tensor categories constructed by Kazhdan and Lusztig in [KL1]–[KL5] for suitable module categories for affine Lie algebras could be recast and understood as a special case of a new "logarithmic" generalization of the vertex tensor category theory constructed in [HL1]–[HL5] and [H1] for suitable module categories for a vertex operator algebra. In the new theory, the module categories would no longer be completely reducible and the actions of $L(0)$ on the modules would no longer be semisimple; in the Kazhdan-Lusztig work, the module categories have these properties. The theory, namely, [HLZ1]–[HLZ9], would be—and in fact is—very general and not limited to affine Lie algebras. In [Z], Zhang indeed placed the Kazhdan-Lusztig construction of braided tensor categories into the setting of vertex operator algebra theory, in order to apply an earlier version of [HLZ1]–[HLZ9] to construct these braided tensor categories. But Proposition 5.8 in [Z] is wrong. Fortunately, the mistake is minor and we correct it in the present paper.

Proposition 5.8 in [Z] is an attempt to verify the condition that every finitely-generated lower bounded doubly-graded generalized V-module is an object of the category considered (see the comments above about [HLZ8] and Assumption 12.2 in [HLZ9]). Also, in [Z], the objects in the category are not all graded by $\mathbb{R}$.

It is known that vertex operator algebras associated to, and module categories corresponding to, the braided tensor categories constructed by Kazhdan and Lusztig [KL1]–[KL5] do not satisfy the three conditions listed above (mainly the second one, but in some cases also the part of the third condition requiring that irreducible modules be $\mathbb{R}$-graded). Also, in recent years, new examples of interesting vertex (operator) algebras and their module categories have been constructed and studied. Some of these examples also do not satisfy all the three conditions listed above. It is therefore important to generalize the results in [H3], and even in [HLZ1]–[HLZ9], so that the logarithmic tensor category theory can be applied to these cases.

In this short paper, we give results and observations which can be used to apply the logarithmic tensor category theory in [HLZ1]–[HLZ9] to more general vertex (operator) algebras and module categories than those studied in [H3]. In particular, we correct the mistake in [Z] and make sure that the results in [Z] indeed hold even when the objects of $\mathcal{C}$ are not all graded by $\mathbb{R}$. This serves to complete the proof that the braided tensor categories of Kazhdan-Lusztig are indeed special cases of the logarithmic tensor category theory of [HLZ1]–[HLZ9].

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2 A unique expansion set result

Compared with the study of vertex (operator) algebras, modules and single (logarithmic) intertwining operators, the study of products and iterates of two or more (logarithmic) intertwining operators is much more difficult but is certainly also much richer and deeper. One needs to prove the convergence of the series obtained from these products and iterates and then to use analytic extension and expansion in different regions to obtained the desired results. The analytic extensions of these products and iterates are in general multivalued analytic functions for which many of the usual techniques, which work perfectly for the rational functions used in the study of vertex operator algebras and modules, do not work anymore. Among those techniques that do not work for intertwining operators is the Laurent expansion of a single-valued analytic function defined on an annulus. But we still need to prove that the multivalued analytic functions obtained from analytic extensions of the products and iterates of (logarithmic) intertwining operators can be expanded uniquely as series in powers of the variables and in nonnegative integral powers of the logarithms of the variables. In general, such expansions, even if they exist, might not be unique. The uniqueness is important for us to construct the (logarithmic) intertwining operators needed in our results. For this reason, a notion of unique expansion set was introduced in [HLZ6]:

We call a subset $\mathcal{S}$ of $\mathbb{C}\times\mathbb{C}$ a *unique expansion set* if the absolute convergence to 0 on some nonempty open subset of $\mathbb{C}^\times$ of any series

$$\sum_{(\alpha,\beta)\in\mathcal{S}} a_{\alpha,\beta} z^\alpha (\log z)^\beta, \quad a_{\alpha,\beta}\in\mathbb{C},$$

where $\log z = \log|z| + i\arg z$ and $0\le \arg z < 2\pi$, implies that $a_{\alpha,\beta}=0$ for all $(\alpha,\beta)\in\mathcal{S}$. Lemma 14.5 in [H1] can be restated as saying that the Cartesian product of a strictly increasing sequence of real numbers and $\{0\}$ is a unique expansion set. Proposition 7.8 in [HLZ6] states that for any $N\in\mathbb{N}$, $\mathbb{R}\times\{0,\dots,N\}$ is a unique expansion set. These results involve only real powers of the variable. It is known that $\mathbb{C}\times\{0\}$ is not a unique expansion set.

In this section, we give a generalization of Lemma 14.5 in [H1] to the case that the powers of the variable can be complex, with finitely many different imaginary parts. This generalization will allow us to apply the theory in [HLZ1]–[HLZ9] to suitable module categories for vertex (operator) algebras whose objects might have complex weights. In particular, this generalization justifies those statements in [Z] that also cover the case that the weights of elements of modules might be complex. It is still not known whether Proposition 7.8 in [HLZ6] (where the powers of the variables are real) can be generalized to the case that the powers of the variable contain finitely many different imaginary parts. But for existing examples, the result below, which generalizes Lemma 14.5 in [H1], is enough.

Proposition 2.1. *Let $\{n_i\}_{i\in\mathbb{Z}_+}$ be a sequence of strictly increasing real numbers (that is, $n_i\in\mathbb{R}$ for $i\in\mathbb{Z}_+$ and $n_1<n_2<n_3<\cdots$) and let $m_1,\dots,m_l$ be distinct real numbers. Let*

$$\mathbb{R}^{\{n_i\}_{i\in\mathbb{Z}_+}}_{m_1,\dots,m_l} = \{n_i + m_j\sqrt{-1}\mid i\in\mathbb{Z}_+,\ j=1,\dots,l\}.$$

Then for any $N\in\mathbb{N}$, $\mathbb{R}^{\{n_i\}_{i\in\mathbb{Z}_+}}_{m_1,\dots,m_l}\times\{0,\dots,N\}$ is a unique expansion set.

Proof. Let $a_{i,j,k}\in\mathbb{C}$ for $i\in\mathbb{Z}_+$, $j=1,\dots,l$ and $k=0,\dots,N$, and suppose that

$$\sum_{i\in\mathbb{Z}_+}\sum_{j=1}^{l}\sum_{k=0}^{N} a_{i,j,k} z^{n_i+m_j\sqrt{-1}}(\log z)^k = \sum_{i\in\mathbb{Z}_+}\sum_{j=1}^{l}\sum_{k=0}^{N} a_{i,j,k} e^{(n_i+m_j\sqrt{-1})\log z}(\log z)^k$$

is absolutely convergent to 0 for $z=z_1\in\mathbb{C}^\times$. (See Remark 2.2 below.) We want to prove that $a_{i,j,k}=0$ for $i\in\mathbb{Z}_+$, $j=1,\dots,l$ and $k=0,\dots,N$.

We may assume that $m_1>\cdots>m_l$. It is sufficient to prove that $a_{1,j,k}=0$ for $j=1,\dots,l$ and $k=0,\dots,N$, which we proceed to do.

For $z\in\mathbb{C}$ satisfying $|z|\le|z_1|$,

$$\sum_{i\in\mathbb{Z}_+}|a_{i,j,k}z^{n_i-n_1}| = \sum_{i\in\mathbb{Z}_+}|a_{i,j,k}||z|^{n_i-n_1} \le \sum_{i\in\mathbb{Z}_+}|a_{i,j,k}||z|_1^{n_i-n_1} = \sum_{i\in\mathbb{Z}_+}|a_{i,j,k}z_1^{n_i-n_1}|$$

is absolutely and uniformly convergent. Thus for $z \in \mathbb{C}$ satisfying $|z| < |z_1|$,

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}_+}\sum_{j=1}^{l}\sum_{k=0}^{N} a_{i,j,k} z^{(n_i-n_1)+(m_j-m_l)\sqrt{-1}}(\log z)^{k-N} \\
&\quad = \sum_{i\in\mathbb{Z}_+}\sum_{j=1}^{l}\sum_{k=0}^{N} a_{i,j,k} e^{((n_i-n_1)+(m_j-m_l)\sqrt{-1})\log z}(\log z)^{k-N}
\end{aligned} \tag{2.1}$$

is absolutely and uniformly convergent to 0.

Take $\log z = x + y\sqrt{-1}$ where $x \in \mathbb{R}$ and $y \in \mathbb{R}$ such that $e^x < |z_1|$ or equivalently $x < \log |z_1|$. Then $|z| < |z_1|$. Thus from (2.1), for such x and y, we have

$$\sum_{i\in\mathbb{Z}_+}\sum_{j=1}^{l}\sum_{k=0}^{N} a_{i,j,k} e^{(n_i-n_1)x} e^{(n_i-n_1)y\sqrt{-1}} e^{(m_j-m_l)\sqrt{-1}x} e^{-(m_j-m_l)y}(x+y\sqrt{-1})^{k-N} = 0. \tag{2.2}$$

Let x and y go to $-\infty$ and ∞, respectively, on both sides of (2.2). Since the series in the left-hand side of (2.2) is uniformly convergent, we can take the limit term by term. Thus we obtain $a_{1,l,N} = 0$.

Now assume that $a_{1,j,N} = 0$ for $j = p+1, \dots, l$. Then

$$\begin{aligned}
&\sum_{j=1}^{p}\sum_{k=0}^{N} a_{1,j,k} e^{(m_j-m_p)\sqrt{-1}x} e^{-(m_j-m_p)y}(x+y\sqrt{-1})^{k-N} \\
&\qquad + \sum_{i\in\mathbb{Z}_++1}\sum_{j=1}^{l}\sum_{k=0}^{N} a_{i,j,k} e^{(n_i-n_1)x} e^{(n_i-n_1)y\sqrt{-1}} e^{(m_j-m_p)\sqrt{-1}x} e^{-(m_j-m_p)y}(x+y\sqrt{-1})^{k-N} \\
&\quad = 0.
\end{aligned} \tag{2.3}$$

Let $y = \log |x|$ in (2.3) and then let x go to $-\infty$ on both sides of (2.3). Again by taking the limit on the left-hand side term by term, we obtain $a_{1,p,N} = 0$. Thus we have $a_{1,j,N} = 0$ for $j = 1, \dots, l$.

It follows that $a_{1,j,k} = 0$ for $j = 1, \dots, l$ and $k = 0, \dots, N-1$ as well. ∎

Remark 2.2. From the proof of Proposition 2.1, we see that we proved a stronger result. In fact, in the proof, we have assumed that the series is absolutely convergent only for a particular number $z = z_1 \in \mathbb{C}^\times$, not for z in a nonempty open subset of $\mathbb{C}^\times$. This assumption is weaker than the set to be a unique expansion set. Thus we have proved a result stronger than the statement of Proposition 2.1.

3 A sufficient condition for the expansion condition

In the proof of Theorem 4.12 in [H3], the author used the three conditions (in fact only the first two conditions) listed in the introduction to prove that every finitely-generated lower-truncated generalized V-module is in the category of grading-restricted generalized V-modules (where the grading abelian groups A and $\tilde{A}$ are trivial). The convergence and extension property for products and iterates of logarithmic intertwining operators also holds by Theorem 11.8 in [HLZ8]. These verify the two conditions needed in Theorem 11.4 in [HLZ8]. Then by Theorem 11.4 and Theorem 11.8 in [HLZ8], Assumption 12.2 in [HLZ9] holds.

When the first two conditions listed in the introduction do not hold, especially when the second condition does not hold, Condition 1 in Theorem 11.4 in [HLZ8] might not hold and thus we cannot use this theorem. But when Assumption 10.1 in [HLZ7] holds, Theorem 11.4 in [HLZ8] can be easily generalized by observing that the proof of Theorem 11.4 in [HLZ8] in fact proves the following stronger result (stronger in the sense that one assumption is weaker):

Theorem 3.1. *Suppose that Assumption 10.1 in [HLZ7] holds and the following two conditions are satisfied:*

1. *For any objects W_1 and W_2 of $\mathcal{C}$ and any $z \in \mathbb{C}^\times$, if the doubly-graded generalized V-module W_λ (or a doubly-graded V-module when $\mathcal{C}$ is in $\mathcal{M}_{sg}$) generated by a generalized eigenvector $\lambda \in (W_1 \otimes W_2)^*$ for $L_{P(z)}(0)$ satisfying the $P(z)$-compatibility condition is lower bounded, then W_λ is an object of $\mathcal{C}$.*

2. *The convergence and extension property for either products or iterates holds in $\mathcal{C}$ (or the convergence and extension property without logarithms for either products or iterates holds in $\mathcal{C}$, when $\mathcal{C}$ is in $\mathcal{M}_{sg}$).*

Then the convergence and expansion conditions for intertwining maps in $\mathcal{C}$ both hold.

Proof. Note that the only place in the proof of Theorem 11.4 in [HLZ8] where the condition that every finitely-generated lower bounded generalized V-module is in $\mathcal{C}$ (Condition 1 in that theorem) is used is in the last paragraph showing that $W_{\lambda_n^{(2)}(w'_{(4)}, w_{(3)})}$ is in $\mathcal{C}$. But to show that $W_{\lambda_n^{(2)}(w'_{(4)}, w_{(3)})}$ is in $\mathcal{C}$, Condition 1 in the statement of the present theorem is enough. Thus the theorem is proved. ∎

Remark 3.2. In Theorem 3.1, we assume in particular that the (generalized) weights are real numbers for any object in $\mathcal{C}$ (the first half of Part 6 of Assumption 10.1 in [HLZ7]). If we replace this part of the assumption by the assumption that every object in $\mathcal{C}$ is of finite length, that is, has a finite composition series in $\mathcal{C}$, then because of Proposition 2.1, the conclusion of Theorem 3.1 still holds.

Theorem 3.1 and Remark 3.2 can be used to construct the associativity isomorphism when Condition 1 in Theorem 11.4 in [HLZ8] is not satisfied (in particular, when V is a vertex operator algebra but Conditions 1 and 2 in the introduction are not satisfied or, as a special case, when V is a vertex operator algebra that is not C_2-cofinite), but Assumption 10.1 in [HLZ7] or the assumptions discussed in Remark 3.2 hold and the two conditions in Theorem 3.1 are satisfied.

4 A correction of a mistake in a paper of Lin Zhang

The tensor categories constructed by Kazhdan and Lusztig in [KL2] correspond to examples of vertex operator algebras and module categories that do not satisfy the second condition listed in the introduction (in particular, do not satisfy the C_2-cofiniteness condition). A construction of these tensor categories of Kazhdan-Lusztig using the logarithmic tensor category theory developed in [HLZ1]–[HLZ9] was given in [Z].

But it was noticed by Milas that one of the propositions in [Z] that is needed in applying Theorem 11.4 in [HLZ8] is wrong. Since this proposition in [Z] is wrong, we cannot use Theorem 11.4 in [HLZ8]. Instead, we use Theorem 3.1 and a result in [KL2] to give an almost trivial correction of the minor mistake in [Z].

To be more precise, the mistake is that Proposition 5.8 in [Z] is wrong. Proposition 5.8 in [Z], if correct, would verify Condition 1 in Theorem 11.4 in [HLZ8]. To correct this mistake, we need only verify Condition 1 in Theorem 3.1 above. We do this by using Theorem 7.9 in [KL2].

Proposition 4.1. *For any two objects W_1 and W_2 of $\mathcal{O}_\kappa$, if the generalized $V_{\hat{\mathfrak{g}}}(\ell, 0)$–module W_λ generated by a generalized eigenvector $\lambda \in (W_1 \otimes W_2)^*$ for $L_{P(z)}(0)$ satisfying the $P(z)$-compatibility condition is lower bounded, then W_λ is an object of $\mathcal{O}_\kappa$.*

Proof. Since W_λ is lower bounded, by definition, the elements of W_λ must be in $W_1 \circ_{P(z)} W_2$ (see [KL2] and [Z]). By Theorem 7.9 in [KL2], $W_1 \circ_{P(z)} W_2$ is in $\mathcal{O}_\kappa$. From [KL2] (also Theorem 5.1 in [Z]), $\mathcal{O}_\kappa$ consists of the $\hat{\mathfrak{g}}$-modules of level ℓ having a finite composition series all of whose irreducible subquotients are of the form $L(\ell, \mu)$ for various highest weights μ for the finite-dimensional Lie algebra $\mathfrak{g}$. Thus W_λ as a submodule of $W_1 \circ_{P(z)} W_2$ is also in $\mathcal{O}_\kappa$. ∎

Remark 4.2. In general, the objects in $\mathcal{O}_\kappa$ might have homogeneous subspaces of complex weights. In [Z], it was not justified that the results in [HLZ1]–[HLZ9] can indeed be applied in this case. What was missing is exactly a unique expansion set result that can be used in this case. Since any object in $\mathcal{O}_\kappa$ is of finite length, for a given object in $\mathcal{O}_\kappa$, there can only be finitely many different imaginary parts of the complex weights of the homogeneous subspaces of the object. Proposition 2.1 is exactly the unique expansion set result that we need in this case. Thus Proposition 2.1 fills this minor gap in [Z].

Together with [Z], Proposition 4.1 and Remark 4.2 serve to complete the proof that the braided tensor categories of Kazhdan-Lusztig are indeed special cases of the logarithmic tensor category theory of [HLZ1]–[HLZ9].

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