Differential equations and intertwining operators

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Abstract

We show that if every module W for a vertex operator algebra $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ satisfies the condition $\dim W/C_1(W) < \infty$, where $C_1(W)$ is the subspace of W spanned by elements of the form $u_{-1}w$ for $u \in V_+ = \coprod_{n>0} V_{(n)}$ and $w \in W$, then matrix elements of products and iterates of intertwining operators satisfy certain systems of differential equations. Moreover, for prescribed singular points, there exist such systems of differential equations such that the prescribed singular points are regular. The finiteness of the fusion rules is an immediate consequence of a result used to establish the existence of such systems. Using these systems of differential equations and some additional reductivity conditions, we prove that products of intertwining operators for V satisfy the convergence and extension property needed in the tensor product theory for V-modules. Consequently, when a vertex operator algebra V satisfies all the conditions mentioned above, we obtain a natural structure of vertex tensor category (consequently braided tensor category) on the category of V-modules and a natural structure of intertwining operator algebra on the direct sum of all (inequivalent) irreducible V-modules.

0 Introduction

In the present paper, we show that for a vertex operator algebra satisfying certain finiteness and reductivity conditions, matrix elements of products and iterates of intertwining operators satisfy certain systems of differential equations of regular singular points. Similar results are also obtained by Nagatomo and Tsuchiya in [NT]. Using these differential equations together with the results obtained by Lepowsky and the author in [HL1]–[HL4] [H1] and by the author in [H1] [H3]–[H7], we construct braided tensor categories and intertwining operator algebras.

In the study of conformal field theories associated to affine Lie algebras (the Wess-Zumino-Novikov-Witten models or WZNW models [W]) and to Virasoro algebras (the minimal models [BPZ]), the Knizhnik-Zamolodchikov equations [KZ] and the Belavin-Polyakov-Zamolodchikov equations [BPZ], respectively, play a fundamental role. Many important results for these and related theories, including the constructions of braided tensor category structures and the study of properties of correlation functions, are obtained using these equations (see, for example, [TK], [KazL], [V], [H3], [HL5] and [EFK]).

More generally, the tensor product theory for the category of modules for a vertex operator algebra was developed by Lepowsky and the author [HL1]–[HL4] [H1] and the theory of intertwining operator algebras was developed by the author [H1] [H3]–[H7]. These structures are essentially equivalent to chiral genus-zero weakly conformal field theories [S1] [S2] [H5] [H6]. In the construction of these structures from representations of vertex operator algebras, one of the most important steps is to prove the associativity of intertwining operators, or a weaker version which, in physicists' terminology, is called the (nonmeromorphic) operator product expansion of chiral vertex operators. It was proved in [H1] that if a vertex operator algebra V is rational in the sense of [HL1], every finitely-generated lower-truncated generalized V-module is a V-module and products of intertwining operators for V have a convergence and extension property (see Definition 3.2 for the precise description of the property), then the associativity of intertwining operators holds. Consequently the category of V-modules has a natural structure of vertex tensor category (and braided tensor category) and the direct sum of all (inequivalent) irreducible V-modules has a natural structure of intertwining operator algebra.

The results above reduce the construction of vertex tensor categories and intertwining operator algebras (in particular, the proof of the associativity of intertwining operators) to the proofs of the rationality of vertex operator algebras (in the sense of [HL1]), the condition on finitely-generated lower-truncated generalized V-modules and the convergence and extension property. Note that this rationality and the condition on finitely-generated lower-truncated generalized V-modules are both purely representation-theoretic properties. These and other related representation-theoretic properties have been discussed in a number of papers, including [Z2], [FZ], [H1], [DLM1]–

[DLM3], [L], [KarL], [GN], [B], [ABD] and [HKL].

The convergence and extension property, on the other hand, is very different from these purely representation-theoretic properties. Since this property is for all intertwining operators, it is impossible to prove it, even only the convergence part, by direct estimates. In the work [TK], Tsuchiya and Kanie used the Knizhnik-Zamolodchikov equations to show the convergence of the correlation function. For all the concrete examples (see [H2], [H3], [HL5], [HM1] and [HM2]), the convergence and extension property was proved using the particular differential equations of regular singular points associated to the examples, including the Knizhnik-Zamolodchikov equations and the Belavin-Polyakov-Zamolodchikov equations mentioned above. Also, since braided tensor categories and intertwining operator algebras give representations of the braid groups, from the solution to the Riemann-Hilbert problem, we know that there must be some differential equations such that the monodromies of the differential equations give these representations of the braid groups. In particular, it was expected that there should be differential equations satisfied by the products and iterates of intertwining operators. In [Ne], by using the fact that any finite-dimensional quotient space of a space of functions with an action of the derivative operators gives a system of differential equations, Neitzke observed that some cofiniteness conditions (including the one discussed in this paper) are related to systems of differential equations. Such differential equations can be used to define correlation functions. But the crucial problem of whether the products or iterates of intertwining operators or the formal series obtained from sewing three point correlation functions satisfy differential equations was not addressed and was unsolved.

In the present paper, we solve this problem by showing that if every module W for a vertex operator algebra $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ satisfies the condition $\dim W/C_1(W) < \infty$ where $C_1(W)$ is the subspace of W spanned by elements of the form $u_{-1}w$ for $u \in V_+ = \coprod_{n>0} V_{(n)}$ and $w \in W$, then matrix elements of products and iterates of intertwining operators satisfy certain systems of differential equations. Moreover, for any prescribed singular point, there exist such systems of differential equations such that this prescribed singular point is regular. The finiteness of the fusion rules is an immediate consequence of a result used to establish the existence of such systems. Together with some other reductivity and finiteness conditions, including the complete reducibility of generalized modules and the finiteness of the number of equivalence classes of modules, these systems of differential equations also give us the convergence and extension property. Consequently, if all the conditions mentioned above are satisfied, we obtain vertex tensor category structures (consequently braided tensor categories) on the category of V-modules and intertwining operator algebra structures on the direct sum of all (inequivalent) irreducible V-modules.

The results of the present paper have many applications. For example, in [H8], they have been used by the author in the construction of the chiral genus-one correlation functions and in the proof of the duality properties and modular invariance of these functions. In [Y], they have been used by Yamauchi in the study of module categories of simple current extensions of vertex operator algebras. They have also been also used to prove one of the Moore-Seiberg formulas in the author's proof of the Verlinde conjecture in [H9].

In [Z1] and [Z2], Zhu introduced the first cofiniteness condition (called the C_2-cofiniteness condition in this paper) in the theory of vertex operator algebras. The condition $\dim W/C_1(W) < \infty$, which we shall call the C_1-cofiniteness condition (or we say that W is C_1-cofinite), is a very minor restriction and is very easy to verify for familiar examples. Note that vertex operator algebras as modules for themselves always satisfy this condition. (A different but closely-related condition, called the C_1-finiteness condition, was discussed in [L].) This condition was first introduced by Nahm in [Na] in which it was called "quasi-rationality." In [AN], Abe and Nagatomo showed that if all modules for a vertex operator algebra satisfies the C_1-cofiniteness condition (called B_1 finiteness condition by these authors) and some additional conditions on the vertex operator algebras are satisfied, then the spaces of conformal blocks on the Riemann sphere are finite-dimensional. This finite-dimensionality is an immediate consequence of some results used to establish the existence of the systems of differential equations in the present paper; in particular, only the C_1-cofiniteness condition on modules is needed.

In [GN], Gaberdiel and Neitzke found a useful spanning set for a vertex operator algebra satisfying Zhu's C_2-cofiniteness condition and in [B], Buhl found useful generalizations of this spanning set for weak modules for such a vertex operator algebra. Using these spanning sets, Abe, Buhl and Dong [ABD] proved that for a vertex operator algebra V satisfying $V_{(n)} = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ and the C_2-cofiniteness condition, any irreducible weak V-module is also C_2-cofinite and the complete reducibility of every $\mathbb{N}$-gradable weak V-module implies that every weak V-module is a direct sum of V-modules. In Section 3, we also show that for such a vertex operator algebra, the complete reducibility of every V-module implies that every generalized V-module is a direct sum of V-modules. These results allow us to replace the conditions needed in the applications mentioned above by some stronger conditions which have been discussed extensively in a number of papers.

Note that intertwining operator algebras have a geometric formulation in terms of the sewing operation in the moduli space of spheres with punctures and local coordinates (see [H5] and [H6]). In fact, this formulation gives genus-zero modular functors and genus-zero weakly conformal field theories in the sense of Segal [S1] [S2], and, in particular, it gives the sewing property. It can be shown that the sewing property together with the generalized rationality property of intertwining operators (see [H7]) implies the factorization property for conformal blocks on genus-zero Riemann surfaces with punctures. In [NT], for a vertex operator algebra satisfying Zhu's C_2-cofiniteness condition, the condition that Zhu's algebra is semisimple and finite-dimensional and a condition stating that certain induced weak V-modules are irreducible, Nagatomo and Tsuchiya prove, among many other things, the factorization property of conformal blocks on genus-zero surfaces. Their factorization property is formulated and proved using the same method as in the work [TUY] by Tsuchiya, Ueno and Yamada on WZNW models. In particular, they prove the convergence of conformal blocks near the boundary points of the moduli space using certain holonomic systems of regular singularities near the singularities. The factorization theorem corresponds to the construction of the sewing operation for a modular functor in our formulation. On the other hand, the sewing property for the corresponding weakly conformal field theories (in particular, the associativity of intertwing operators and the operator product expansion of intertwining operators) are not discussed in [NT].

Though many results in the present paper and in [NT] are equivalent, the present work and the work [NT] develop into different directions. The present work proves and studies in details the properties of correlation functions (for example, associativity and commutativity of intertwining operators) while the work [NT] proves and studies in details the algebro-geometric properties of chiral genus-zero modular functors and conformal field theories (for example, the construction and study of sheaves of conformal blocks over the compactified moduli spaces).

We assume that the reader is familiar with the basic notions, notations and results in the theory of vertex operator algebras as presented in [FLM] and [FHL]. We also assume that the reader is familiar with the theory of intertwining operator algebras as developed and presented by the author in [H1], [H3], [H5], [H6] and [H7], based on the tensor product theory developed by Lepowsky and the author in [HL1]–[HL4] and [H1].

The present paper is organized as follows: The existence of systems of differential equations and the existence of systems with regular prescribed singular points are established in Section 1 and Section 2, respectively. In Section 3, we prove the finiteness of the fusion rules and the convergence and extension property. Consequently we obtain the vertex tensor category (and braided tensor category) structures and intertwining operator algebra structures. We also discuss conditions which imply some representation-theoretic conditions needed in the main application.

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1 Differential equations

Let $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ be a vertex operator algebra in the sense of [FLM] and [FHL]. For a V-module W, the vertex operator map is a linear map given by

$$\begin{aligned} Y : V &\to (\mathrm{End}\, W)[[x, x^{-1}]] \\ u &\mapsto Y(u, x) = \sum_{n\in\mathbb{Z}} u_n x^{-n-1}, \end{aligned}$$

where x is a formal variable and $u_n \in \mathrm{End}\, W$ for $n \in \mathbb{Z}$. In this paper, we shall use $x, x_1, x_2, ...$ to denote commuting formal variables and $z, z_1, z_2, ...$ to denote complex variables or complex numbers.

For any V-module W, let $C_1(W)$ be the subspace of W spanned by elements of the form $u_{-1}w$ for

$$u \in V_+ = \coprod_{n>0} V_{(n)}$$

and $w \in W$. If $\dim W/C_1(W) < \infty$, we say that W is C_1-*cofinite* or W satisfies the C_1-*cofiniteness condition.*

In this and the next sections, we shall consider only $\mathbb{R}$-graded V-modules whose subspace spanned by elements of weights less than or equal to any fixed $r \in \mathbb{R}$ is finite-dimensional. We shall call such a V-module a *discretely $\mathbb{R}$-graded V-module*. Clearly, finite direct sums of irreducible $\mathbb{R}$-graded V-modules are discretely $\mathbb{R}$-graded V-modules.

Let $R = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, (z_1 - z_2)^{-1}]$, W_i $(i = 0, 1, 2, 3)$ discretely $\mathbb{R}$-graded V-modules satisfying the C_1-cofiniteness condition and $T = R \otimes W_0 \otimes W_1 \otimes W_2 \otimes W_3$ which has a natural R-module structure. For simplicity, we shall omit one tensor symbol to write $f(z_1, z_2) \otimes w_0 \otimes w_1 \otimes w_2 \otimes w_3$ as $f(z_1, z_2) w_0 \otimes w_1 \otimes w_2 \otimes w_3$ in T. For a discretely $\mathbb{R}$-graded V-module W, as in [FHL], we use W' to denote the contragredient module of W. In particular, for $u \in V$ and $n \in \mathbb{Z}$, we have the operators u_n on W'. For $u \in V$ and $n \in \mathbb{Z}$, let $u_n^* : W \to W$ be the adjoint of $u_n : W' \to W'$. Note that since $\mathrm{wt}\ u_n = \mathrm{wt}\ u - n - 1$, we have $\mathrm{wt}\ u_n^* = -\mathrm{wt}\ u + n + 1$.

For $u \in V_+$ and $w_i \in W_i$, $i = 0, 1, 2, 3$, let J be the submodule of T generated by elements of the form (note the different meanings of the subscripts)

$$
\begin{aligned}
&\mathcal{A}(u, w_0, w_1, w_2, w_3)\\
&\quad = \sum_{k \geq 0} \binom{-1}{k} (-z_1)^k u^*_{-1-k} w_0 \otimes w_1 \otimes w_2 \otimes w_3 - w_0 \otimes u_{-1} w_1 \otimes w_2 \otimes w_3\\
&\qquad - \sum_{k \geq 0} \binom{-1}{k} (-(z_1 - z_2))^{-1-k} w_0 \otimes w_1 \otimes u_k w_2 \otimes w_3\\
&\qquad - \sum_{k \geq 0} \binom{-1}{k} (-z_1)^{-1-k} w_0 \otimes w_1 \otimes w_2 \otimes u_k w_3,
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{B}(u, w_0, w_1, w_2, w_3)\\
&\quad = \sum_{k \geq 0} \binom{-1}{k} (-z_2)^k u^*_{-1-k} w_0 \otimes w_1 \otimes w_2 \otimes w_3\\
&\qquad - \sum_{k \geq 0} \binom{-1}{k} (z_1 - z_2)^{-1-k} w_0 \otimes u_k w_1 \otimes w_2 \otimes w_3 - w_0 \otimes w_1 \otimes u_{-1} w_2 \otimes w_3\\
&\qquad - \sum_{k \geq 0} \binom{-1}{k} (-z_2)^{-1-k} w_0 \otimes w_1 \otimes w_2 \otimes u_k w_3,
\end{aligned}
$$

$$
\mathcal{C}(u, w_0, w_1, w_2, w_3)
$$

$$= u^*_{-1}w_0 \otimes w_1 \otimes w_2 \otimes w_3 - \sum_{k\geq 0}\binom{-1}{k} z_1^{-1-k} w_0 \otimes u_k w_1 \otimes w_2 \otimes w_3$$
$$-\sum_{k\geq 0}\binom{-1}{k} z_2^{-1-k} w_0 \otimes w_1 \otimes u_k w_2 \otimes w_3 - w_0 \otimes w_1 \otimes w_2 \otimes u_{-1}w_3$$

and

$$\begin{aligned}
&\mathcal{D}(u, w_0, w_1, w_2, w_3)\\
&\quad = u_{-1}w_0 \otimes w_1 \otimes w_2 \otimes w_3\\
&\quad\quad -\sum_{k\geq 0}\binom{-1}{k} z_1^{1+k} w_0 \otimes e^{z_1^{-1}L(1)}(-z_1^2)^{L(0)} u_k (-z_1^{-2})^{L(0)} e^{-z_1^{-1}L(1)} w_1 \otimes w_2 \otimes w_3\\
&\quad\quad -\sum_{k\geq 0}\binom{-1}{k} z_2^{1+k} w_0 \otimes w_1 \otimes e^{z_2^{-1}L(1)}(-z_2^2)^{L(0)} u_k (-z_2^{-2})^{L(0)} e^{-z_2^{-1}L(1)} w_2 \otimes w_3\\
&\quad\quad - w_0 \otimes w_1 \otimes w_2 \otimes u^*_{-1}w_3.
\end{aligned}$$

The gradings on W_i for $i = 0, 1, 2, 3$ induce a grading (also called *weight*) on $W_0 \otimes W_1 \otimes W_2 \otimes W_3$ and then also on T (here we define the weight of elements of R to be 0). Let $T_{(r)}$ be the homogeneous subspace of weight r for $r \in \mathbb{R}$. Then $T = \coprod_{r\in\mathbb{R}} T_{(r)}$, $T_{(r)}$ for $r \in \mathbb{R}$ are finitely-generated R-modules and $T_{(r)} = 0$ when r is sufficiently small. We also let $F_r(T) = \coprod_{s\leq r} T_{(s)}$ for $r \in \mathbb{R}$. Then $F_r(T)$, $r \in \mathbb{R}$, are finitely-generated R-modules, $F_r(T) \subset F_s(T)$ for $r \leq s$ and $\cup_{r\in\mathbb{R}} F_r(T) = T$. Let $F_r(J) = J \cap F_r(T)$ for $r \in \mathbb{R}$. Then $F_r(J)$ for $r \in \mathbb{R}$ are finitely-generated R-modules, $F_r(J) \subset F_s(J)$ for $r \leq s$ and $\cup_{r\in\mathbb{R}} F_r(J) = J$.

Proposition 1.1 *There exists $M \in \mathbb{Z}$ such that for any $r \in \mathbb{R}$, $F_r(T) \subset F_r(J) + F_M(T)$. In particular, $T = J + F_M(T)$.*

Proof. Since $\dim W_i/C_1(W_i) < \infty$ for $i = 0, 1, 2, 3$, there exists $M \in \mathbb{Z}$ such that

$$\begin{aligned}\coprod_{n>M} T_{(n)} \subset\ & R(C_1(W_0) \otimes W_1 \otimes W_2 \otimes W_3) + R(W_0 \otimes C_1(W_1) \otimes W_2 \otimes W_3)\\
& + R(W_0 \otimes W_1 \otimes C_1(W_2) \otimes W_3) + R(W_0 \otimes W_1 \otimes W_2 \otimes C_1(W_3)).
\end{aligned} \tag{1.1}$$

We use induction on $r \in \mathbb{R}$. If r is equal to M, $F_M(T) \subset F_M(J) + F_M(T)$. Now we assume that $F_r(T) \subset F_r(J) + F_M(T)$ for $r < s$ where $s > M$. We want to show that any homogeneous element of $T_{(s)}$ can be written as a sum of an element of $F_s(J)$ and an element of $F_M(T)$. Since $s > M$, by (1.1), any element of $T_{(s)}$ is an element of the right-hand side of (1.1). We shall discuss only the case that this element is in $R(W_0 \otimes C_1(W_1) \otimes W_2 \otimes W_3)$; the other cases are completely similar.

We need only discuss elements of the form $w_0 \otimes u_{-1}w_1 \otimes w_2 \otimes w_3$ where $w_i \in W_i$ for $i = 0, 1, 2, 3$ and $u \in V_+$. By assumption, the weight of $w_0 \otimes u_{-1}w_1 \otimes w_2 \otimes w_3$ is s and the weights of $u^*_{-1-k}w_0 \otimes w_1 \otimes w_2 \otimes w_3$, $w_0 \otimes w_1 \otimes w_2 \otimes u_k w_3$ and $w_0 \otimes w_1 \otimes u_k w_2 \otimes w_3$ for $k \geq 0$, are all less than the weight of $w_0 \otimes u_{-1}w_1 \otimes w_2 \otimes w_3$. So $\mathcal{A}(u, w_0, w_1, w_2, w_3) \in F_s(J)$. Thus we see that $w_0 \otimes u_{-1}w_1 \otimes w_2 \otimes w_3$ can be written as a sum of an element of $F_s(J)$ and elements of T of weights less than s. By the induction assumption, we know that $w_0 \otimes u_{-1}w_1 \otimes w_2 \otimes w_3$ can be written as a sum of an element of $F_s(J)$ and an element of $F_M(T)$.

Now we have

$$\begin{aligned} T &= \cup_{r\in\mathbb{R}} F_r(T) \\ &\subset \cup_{r\in\mathbb{R}} F_r(J) + F_M(T) \\ &= J + F_M(T). \end{aligned}$$

But we know that $J + F_M(T) \subset T$. Thus we have $T = J + F_M(T)$. ∎

We immediately obtain the following:

Corollary 1.2 *The quotient R-module T/J is finitely generated.*

Proof. Since $T = J + F_M(T)$ and $F_M(T)$ is finitely-generated, T/J is finitely-generated. ∎

For an element $\mathcal{W} \in T$, we shall use $[\mathcal{W}]$ to denote the equivalence class in T/J containing $\mathcal{W}$. We also have:

Corollary 1.3 *For any $w_i \in W_i$ ($i = 0, 1, 2, 3$), let M_1 and M_2 be the R-submodules of T/J generated by $[w_0 \otimes L(-1)^j w_1 \otimes w_2 \otimes w_3]$, $j \geq 0$, and by $[w_0 \otimes w_1 \otimes L(-1)^j w_2 \otimes w_3]$, $j \geq 0$, respectively. Then M_1 and M_2 are finitely generated. In particular, for any $w_i \in W_i$ ($i = 0, 1, 2, 3$), there exist $a_k(z_1, z_2), b_l(z_1, z_2) \in R$ for $k = 1, \ldots, m$ and $l = 1, \ldots, n$ such that*

$$[w_0 \otimes L(-1)^m w_1 \otimes w_2 \otimes w_3] + a_1(z_1, z_2)[w_0 \otimes L(-1)^{m-1} w_1 \otimes w_2 \otimes w_3]$$

$$+\cdots+a_m(z_1,z_2)[w_0\otimes w_1\otimes w_2\otimes w_3]=0, \tag{1.2}$$

$$[w_0\otimes w_1\otimes L(-1)^n w_2\otimes w_3]+b_1(z_1,z_2)[w_0\otimes w_1\otimes L(-1)^{n-1}w_2\otimes w_3]$$
$$+\cdots+b_n(z_1,z_2)[w_0\otimes w_1\otimes w_2\otimes w_3]=0. \tag{1.3}$$

Proof. Since R is a Noetherian ring, any R-submodule of the finitely-generated R-module T/J is also finitely generated. In particular, M_1 and M_2 are finitely generated. The second conclusion follows immediately. ∎

Now we establish the existence of systems of differential equations:

Theorem 1.4 *Let W_i for $i=0,1,2,3$ be discretely $\mathbb{R}$-graded V-modules satisfying the C_1-cofiniteness condition. Then for any $w_i\in W_i$ ($i=0,1,2,3$), there exist*

$$a_k(z_1,z_2), b_l(z_1,z_2)\in\mathbb{C}[z_1^{\pm1},z_2^{\pm1},(z_1-z_2)^{-1}]$$

for $k=1,\dots,m$ and $l=1,\dots,n$ such that for any discretely $\mathbb{R}$-graded V-modules W_4, W_5 and W_6, any intertwining operators $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Y}_3$, $\mathcal{Y}_4$, $\mathcal{Y}_5$ and $\mathcal{Y}_6$ of types $\binom{W_0'}{W_1W_4}$, $\binom{W_4}{W_2W_3}$, $\binom{W_5}{W_1W_2}$, $\binom{W_0'}{W_5W_3}$, $\binom{W_0'}{W_2W_6}$ and $\binom{W_6}{W_1W_3}$, respectively, the series

$$\langle w_0,\mathcal{Y}_1(w_1,z_1)\mathcal{Y}_2(w_2,z_2)w_3\rangle, \tag{1.4}$$

$$\langle w_0,\mathcal{Y}_4(\mathcal{Y}_3(w_1,z_1-z_2)w_2,z_2)w_3\rangle \tag{1.5}$$

and

$$\langle w_0,\mathcal{Y}_5(w_2,z_2)\mathcal{Y}_6(w_1,z_1)w_3\rangle, \tag{1.6}$$

satisfy the expansions of the system of differential equations

$$\frac{\partial^m\varphi}{\partial z_1^m}+a_1(z_1,z_2)\frac{\partial^{m-1}\varphi}{\partial z_1^{m-1}}+\cdots+a_m(z_1,z_2)\varphi = 0, \tag{1.7}$$

$$\frac{\partial^n\varphi}{\partial z_2^n}+b_1(z_1,z_2)\frac{\partial^{n-1}\varphi}{\partial z_2^{n-1}}+\cdots+b_n(z_1,z_2)\varphi = 0 \tag{1.8}$$

in the regions $|z_1|>|z_2|>0$, $|z_2|>|z_1-z_2|>0$ and $|z_2|>|z_1|>0$, respectively.

Proof. For simplicity, we assume that $w_i\in W_i$ for $i=0,1,2,3$ are homogeneous. Let $\Delta=\text{wt }w_0-\text{wt }w_1-\text{wt }w_2-\text{wt }w_3$. Let $\mathbb{C}(\{x\})$ be the space of all series of the form $\sum_{n\in\mathbb{R}}a_nx^n$, $a_n\in\mathbb{C}$ for $n\in\mathbb{R}$, such that $a_n=0$ when the real part of n is sufficiently negative. For any complex variable z, we can substitute $z^n = e^{n \log z}$ for x^n for $n \in \mathbb{R}$ to obtain a space $\mathbb{C}(\{z\})$ isomorphic to $\mathbb{C}(\{x\})$ (here and below we use the convention that $\log z$ is the value of the logarithm of z such that $0 \leq \arg z < 2\pi$). For formal variables x_1 and x_2, we consider the space $\mathbb{C}(\{z_2/z_1\})[x_1^{\pm 1}, x_2^{\pm 1}]$. Let I_{z_1,z_2} be the subspace of $\mathbb{C}(\{z_2/z_1\})[x_1^{\pm 1}, x_2^{\pm 1}]$ spanned by elements of the form

$$\left(\sum_{n \in \mathbb{R}} a_n (z_2/z_1)^n\right) x_1^k x_2^l - \left(\sum_{n \in \mathbb{R}} a_n (z_2/z_1)^{n+i}\right) x_{1+i}^k x_2^{l-i}$$

for $k, l, i \in \mathbb{Z}$. Let

$$\mathbb{C}(\{z_2/z_1\})[z_1^{\pm 1}, z_2^{\pm 1}] = \mathbb{C}(\{z_2/z_1\})[x_1^{\pm 1}, x_2^{\pm 1}]/I_{z_1,z_2}.$$

Then clearly $\mathbb{C}(\{z_2/z_1\})[z_1^{\pm 1}, z_2^{\pm 1}]$ is a $\mathbb{C}[[z_2/z_1]][z_1^{\pm 1}, z_2^{\pm 1}]$-module and can be identified with a subspace of $\mathbb{C}\{z_1, z_2\}$ which are the space of all series of the form $\sum_{m,n \in \mathbb{R}} b_{m,n} e^{m \log z_1} e^{n \log z_2}$, $b_{m,n} \in \mathbb{C}$ for $m, n \in \mathbb{R}$. Similarly we have $\mathbb{C}(\{(z_1 - z_2)/z_1\})[z_2^{\pm 1}, (z_1 - z_2)^{\pm 1}]$ and $\mathbb{C}(\{z_1/z_2\})[z_1^{\pm 1}, z_2^{\pm 1}]$, where in $\mathbb{C}(\{(z_1 - z_2)/z_1\})[z_2^{\pm 1}, (z_1 - z_2)^{\pm 1}]$, z_2 and $z_1 - z_2$ are viewed as independent complex variables. The spaces $\mathbb{C}(\{(z_1 - z_2)/z_1\})[z_2^{\pm 1}, (z_1 - z_2)^{\pm 1}]$ and $\mathbb{C}(\{z_1/z_2\})[z_1^{\pm 1}, z_2^{\pm 1}]$ are $\mathbb{C}[[(z_1 - z_2)/z_2]][z_2^{\pm 1}, (z_1 - z_2)^{\pm 1}]$- and $\mathbb{C}[[z_1/z_2]][z_1^{\pm 1}, z_2^{\pm 1}]$-modules, respectively, and can also be identified with a subspace of $\mathbb{C}\{z_1, z_2\}$. For any $r \in \mathbb{R}$, $z_1^r \mathbb{C}(\{z_2/z_1\})[z_1^{\pm 1}, z_2^{\pm 1}]$, $z_2^r \mathbb{C}(\{(z_1 - z_2)/z_1\})[z_2^{\pm 1}, (z_1 - z_2)^{\pm 1}]$ and $z_2^r \mathbb{C}(\{z_1/z_2\})[z_1^{\pm 1}, z_2^{\pm 1}]$ are also $\mathbb{C}[[z_2/z_1]][z_1^{\pm 1}, z_2^{\pm 1}]$-, $\mathbb{C}[[(z_1 - z_2)/z_2]][z_2^{\pm 1}, (z_1 - z_2)^{\pm 1}]$- and $\mathbb{C}[[z_1/z_2]][z_1^{\pm 1}, z_2^{\pm 1}]$-modules, respectively, and are also subspaces of $\mathbb{C}\{z_1, z_2\}$.

For any discretely $\mathbb{R}$-graded V-modules W_4, W_5 and W_6, any intertwining operators $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Y}_3$, $\mathcal{Y}_4$, $\mathcal{Y}_5$ and $\mathcal{Y}_6$ of types $\binom{W_0'}{W_1 W_4}$, $\binom{W_4}{W_2 W_3}$, $\binom{W_5}{W_1 W_2}$, $\binom{W_0'}{W_5 W_3}$, $\binom{W_0'}{W_2 W_6}$ and $\binom{W_6}{W_1 W_3}$, respectively, consider the maps

$$\begin{aligned}
\phi_{\mathcal{Y}_1, \mathcal{Y}_2} : T &\rightarrow z_1^{\Delta} \mathbb{C}(\{z_2/z_1\})[z_1^{\pm 1}, z_2^{\pm 1}], \\
\psi_{\mathcal{Y}_3, \mathcal{Y}_4} : T &\rightarrow z_2^{\Delta} \mathbb{C}(\{(z_1 - z_2)/z_1\})[z_2^{\pm 1}, (z_1 - z_2)^{\pm 1}], \\
\xi_{\mathcal{Y}_5, \mathcal{Y}_6} : T &\rightarrow z_2^{\Delta} \mathbb{C}(\{z_1/z_2\})[z_1^{\pm 1}, z_2^{\pm 1}],
\end{aligned}$$

defined by

$$\begin{aligned}
&\phi_{\mathcal{Y}_1, \mathcal{Y}_2}(f(z_1, z_2) w_0 \otimes w_1 \otimes w_2 \otimes w_3) \\
&\quad = \iota_{|z_1| > |z_2| > 0}(f(z_1, z_2)) \langle w_0, \mathcal{Y}_1(w_1, z_1) \mathcal{Y}_2(w_2, z_2) w_3 \rangle,
\end{aligned}$$

$$\psi(f(z_1, z_2)w_0 \otimes w_1 \otimes w_2 \otimes w_3)$$
$$= \iota_{|z_2|>|z_1-z_2|>0}(f(z_1, z_2))\langle w_0, \mathcal{Y}_4(\mathcal{Y}_3(w_1, z_1 - z_2)w_2, z_2)w_3\rangle,$$

$$\xi_{\mathcal{Y}_5,\mathcal{Y}_6}(f(z_1, z_2)w_0 \otimes w_1 \otimes w_2 \otimes w_3)$$
$$= \iota_{|z_2|>|z_1|>0}(f(z_1, z_2))\langle w_0, \mathcal{Y}_5(w_2, z_2)\mathcal{Y}_6(w_1, z_1)w_3\rangle,$$

respectively, where

$$\begin{aligned}
\iota_{|z_1|>|z_2|>0} : R &\to \mathbb{C}[[z_2/z_1]][z_1^{\pm 1}, z_2^{\pm 1}],\\
\iota_{|z_2|>|z_1-z_2|>0} : R &\to \mathbb{C}[[(z_1 - z_2)/z_2]][z_2^{\pm 1}, (z_1 - z_2)^{\pm 1}],\\
\iota_{|z_2|>|z_1|>0} : R &\to \mathbb{C}[[z_1/z_2]][z_1^{\pm 1}, z_2^{\pm 1}],
\end{aligned}$$

are the maps expanding elements of R as series in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1 - z_2| > 0$, $|z_2| > |z_1| > 0$, respectively.

Using the definition of J and the Jacobi identity defining intertwining operators, we have $\phi_{\mathcal{Y}_1,\mathcal{Y}_2}(J) = \psi_{\mathcal{Y}_3,\mathcal{Y}_4}(J) = \xi_{\mathcal{Y}_5,\mathcal{Y}_6}(J) = 0$. (In fact, we purposely choose $\mathcal{A}(u, w_0, w_1, w_2, w_3)$, $\mathcal{B}(u, w_0, w_1, w_2, w_3)$, $\mathcal{C}(u, w_0, w_1, w_2, w_3)$ and $\mathcal{D}(u, w_0, w_1, w_2, w_3)$ to be elements of the intersection of the right-hand side of (1.1) and the kernels of $\phi_{\mathcal{Y}_1,\mathcal{Y}_2}$, $\psi_{\mathcal{Y}_3,\mathcal{Y}_4}$ and $\xi_{\mathcal{Y}_5,\mathcal{Y}_6}$ for all $\mathcal{Y}_i$, $i = 1, \ldots, 6$.) Thus we have the induced maps

$$\begin{aligned}
\bar{\phi}_{\mathcal{Y}_1,\mathcal{Y}_2} : T/J &\to z_1^{\Delta}\mathbb{C}(\{z_2/z_1\})[z_1^{\pm 1}, z_2^{\pm 1}],\\
\bar{\psi}_{\mathcal{Y}_3,\mathcal{Y}_4} : T/J &\to z_2^{\Delta}\mathbb{C}(\{(z_1 - z_2)/z_1\})[z_2^{\pm 1}, (z_1 - z_2)^{\pm 1}],\\
\bar{\xi}_{\mathcal{Y}_5,\mathcal{Y}_6} : T/J &\to z_2^{\Delta}\mathbb{C}(\{z_1/z_2\})[z_1^{\pm 1}, z_2^{\pm 1}].
\end{aligned}$$

Applying $\bar{\phi}_{\mathcal{Y}_1,\mathcal{Y}_2}$, $\bar{\psi}_{\mathcal{Y}_3,\mathcal{Y}_4}$ and $\bar{\xi}_{\mathcal{Y}_5,\mathcal{Y}_6}$ to (1.2) and (1.3) and then use the $L(-1)$-derivative and $L(-1)$-bracket properties for intertwining operators, we see that (1.4), (1.5) and (1.6) indeed satisfy the expansions of the system in the regions $|z_1 > |z_2| > 0$, $|z_2| > |z_1 - z_2| > 0$ and $|z_2 > |z_1| > 0$, respectively. ∎

Remark 1.5 Note that in the theorems above, $a_k(z_1, z_2)$ for $k = 1, \ldots, m-1$ and $b_l(z_1, z_2)$ for $l = 1, \ldots, n-1$, and consequently the corresponding system, are independent of $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Y}_3$, $\mathcal{Y}_4$, $\mathcal{Y}_5$ and $\mathcal{Y}_6$.

The following result can be proved using the same method and so the proof is omitted:

Theorem 1.6 *Let W_i for $i = 0, \ldots, p+1$ be discretely $\mathbb{R}$-graded V-modules satisfying the C_1-cofiniteness condition. Then for any $w_i \in W_i$ for $i = 0, \ldots, p+1$, there exist*

$$a_{k_l,\, l}(z_1, \ldots, z_p) \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_p^{\pm 1}, (z_1 - z_2)^{-1}, (z_1 - z_3)^{-1}, \ldots, (z_{p-1} - z_p)^{-1}], \tag{1.9}$$

for $k_l = 1, \ldots, m_l$ and $l = 1, \ldots, p$, such that for any discretely $\mathbb{R}$-graded V-modules $\tilde{W}_q$ for $q = 1, \ldots, p-1$, any intertwining operators $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_{p-1}, \mathcal{Y}_p$, of types $\binom{W_0'}{W_1 \tilde{W}_1}$, $\binom{\tilde{W}_1}{W_2 \tilde{W}_2}, \ldots, \binom{\tilde{W}_{p-2}}{W_{p-1} \tilde{W}_{p-1}}$, $\binom{\tilde{W}_{p-1}}{W_p W_{p+1}}$, respectively, the series

$$\langle w_0, \mathcal{Y}_1(w_1, z_1) \cdots \mathcal{Y}_p(w_p, z_p) w_{p+1} \rangle \tag{1.10}$$

satisfy the expansions of the system of differential equations

$$\frac{\partial^{m_l} \varphi}{\partial z_l^{m_l}} + \sum_{k_l=1}^{m_l} a_{k_l,\, l}(z_1, \ldots, z_p) \frac{\partial^{m_l - k_l} \varphi}{\partial z_l^{m_l - k_l}} = 0, \quad l = 1, \ldots, p$$

in the region $|z_1| > \cdots |z_p| > 0$.

Remark 1.7 In this and the next section, we assume for simplicity that all the discretely $\mathbb{R}$-graded V-modules involved satisfy the C_1-cofiniteness condition. But the conclusions in Theorems 1.4 and 1.6 and in theorems in the next section still hold if for one of the discretely $\mathbb{R}$-graded V-modules, the C_1-cofiniteness condition is replaced by a much weaker condition that it is generated by lowest weight vectors (a lowest weight vector is a vector w such that $u_n w = 0$ when $\mathrm{wt}\ u - n - 1 < 0$). In fact, using the Jacobi identity, we can always reduce the proof of the general case to the case that the vector in this non-C_1-cofinite discretely $\mathbb{R}$-graded V-module is a lowest-weight vector. Then from the explicit expressions of $\mathcal{A}(u, w_0, w_1, w_2, w_3)$, $\mathcal{B}(u, w_0, w_1, w_2, w_3)$, $\mathcal{C}(u, w_0, w_1, w_2, w_3)$ and $\mathcal{D}(u, w_0, w_1, w_2, w_3)$, it is clear that every argument in this and the next section still works if we replace this non-C_1-cofinite discretely $\mathbb{R}$-graded V-module by the space of all its lowest-weight vectors.

2 The regularity of the singular points

We need certain filtrations on R and on the R-module T.

We discuss only the filtrations associated to the singular point $z_1 = z_2$. For $n \in \mathbb{Z}_+$, let $F_n^{(z_1=z_2)}(R)$ be the vector subspace of R spanned by elements of the form $f(z_1, z_2)(z_1 - z_2)^{-n}$ for $f(z_1, z_2) \in \mathbb{C}[z_1^{\pm}, z_2^{\pm}]$. Then with respect to this filtration, R is a filtered algebra, that is, $F_m^{(z_1=z_2)}(R) \subset F_n^{(z_1=z_2)}(R)$ for $m \leq n$, $R = \cup_{n\in\mathbb{Z}} F_n^{(z_1=z_2)}(R)$ and $F_m^{(z_1=z_2)}(R)F_n^{(z_1=z_2)}(R) \subset F_{m+n}^{(z_1=z_2)}(R)$ for any $m, n \in \mathbb{Z}_+$.

For convenience, we shall use σ to denote $\mathrm{wt}\ w_0 + \mathrm{wt}\ w_1 + \mathrm{wt}\ w_2 + \mathrm{wt}\ w_3$ for $w_i \in W_i$, $i = 0, 1, 2, 3$, when the dependence on w_0, w_1, w_2 and w_3 are clear. Let $F_r^{(z_1=z_2)}(T)$ for $r \in \mathbb{R}$ be the subspace of T spanned by elements of the form $f(z_1, z_2)(z_1 - z_2)^{-n} w_0 \otimes w_1 \otimes w_2 \otimes w_3$ where $f(z_1, z_2) \in \mathbb{C}[z_1^{\pm}, z_2^{\pm}]$, $n \in \mathbb{Z}_+$ and $w_i \in W_i$ $(i = 0, 1, 2, 3)$ satisfying $n + \sigma \leq r$. These subspaces give a filtration of T in the following sense: $F_r^{(z_1=z_2)}(T) \subset F_s^{(z_1=z_2)}(T)$ for $r \leq s$; $T = \cup_{r\in\mathbb{R}} F_r^{(z_1=z_2)}(T)$; $F_n^{(z_1=z_2)}(R)F_r^{(z_1=z_2)}(T) \subset F_{r+n}^{(z_1=z_2)}(T)$.

Let $F_r^{(z_1=z_2)}(J) = F_r^{(z_1=z_2)}(T) \cap J$ for $r \in \mathbb{R}$. We need the following refinement of Proposition 1.1:

Proposition 2.1 *For any $r \in \mathbb{R}$, $F_r^{(z_1=z_2)}(T) \subset F_r^{(z_1=z_2)}(J) + F_M(T)$.*

Proof. The proof is a refinement of the proof of Proposition 1.1. The only additional property we need is that the elements $\mathcal{A}(u, w_0, w_1, w_2, w_3)$, $\mathcal{B}(u, w_0, w_1, w_2, w_3)$, $\mathcal{C}(u, w_0, w_1, w_2, w_3)$ and $\mathcal{D}(u, w_0, w_1, w_2, w_3)$ are all in $F_{\mathrm{wt}\ u+\sigma}^{(z_1=z_2)}(J)$. This is clear. ∎

We also consider the ring $\mathbb{C}[z_1^{\pm}, z_2^{\pm}]$ and the $\mathbb{C}[z_1^{\pm}, z_2^{\pm}]$-module

$$T^{(z_1=z_2)} = \mathbb{C}[z_1^{\pm}, z_2^{\pm}] \otimes W_0 \otimes W_1 \otimes W_2 \otimes W_3.$$

Let $T_{(r)}^{(z_1=z_2)}$ for $r \in \mathbb{R}$ be the space of elements of $T^{(z_1=z_2)}$ of weight r. Then $T^{(z_1=z_2)} = \coprod_{r\in\mathbb{R}} T_{(r)}^{(z_1=z_2)}$.

Let $w_i \in W_i$ for $i = 0, 1, 2, 3$. Then by Proposition 2.1,

$$w_0 \otimes w_1 \otimes w_2 \otimes w_3 = \mathcal{W}_1 + \mathcal{W}_2$$

where $\mathcal{W}_1 \in F_\sigma^{(z_1=z_2)}(J)$ and $\mathcal{W}_2 \in F_M(T)$.

Lemma 2.2 *For any $s \in [0, 1)$, there exist $S \in \mathbb{R}$ such that $s + S \in \mathbb{Z}_+$ and for any $w_i \in W_i$, $i = 0, 1, 2, 3$, satisfying $\sigma \in s + \mathbb{Z}$, $(z_1 - z_2)^{\sigma+S}\mathcal{W}_2 \in T^{(z_1=z_2)}$.*

Proof. Let S be a real number such that $s+S\in\mathbb{Z}_+$ and such that for any $r\in\mathbb{R}$ satisfying $r\le -S$, $T_{(r)}=0$. By definition, elements of $F_r^{(z_1=z_2)}(T)$ for any $r\in\mathbb{R}$ are sums of elements of the form $f(z_1,z_2)(z_1-z_2)^{-n}\tilde{w}_0\otimes\tilde{w}_1\otimes\tilde{w}_2\otimes\tilde{w}_3$ where $f(z_1,z_2)\in\mathbb{C}[z_1^{\pm},z_2^{\pm}]$, $n\in\mathbb{Z}_+$ and $\tilde{w}_i\in W_i$ $(i=0,1,2,3)$ satisfying $n+\mathrm{wt}\ \tilde{w}_0+\mathrm{wt}\ \tilde{w}_1+\mathrm{wt}\ \tilde{w}_2+\mathrm{wt}\ \tilde{w}_3\le r$. Since $\mathrm{wt}\ \tilde{w}_0+\mathrm{wt}\ \tilde{w}_1+\mathrm{wt}\ \tilde{w}_2+\mathrm{wt}\ \tilde{w}_3>-S$, we obtain $r-n>-S$ or $r+S-n>0$. Thus $(z_1-z_2)^{r+S}F_r^{(z_1=z_2)}(T)\in T^{(z_1=z_2)}$.

By definition,

$$\mathcal{W}_2=w_0\otimes w_1\otimes w_2\otimes w_3-\mathcal{W}_1,$$

where

$$\mathcal{W}_1\in F_\sigma^{(z_1=z_2)}(J)\subset F_\sigma^{(z_1=z_2)}(T).$$

By the discussion above, $(z_1-z_2)^{\sigma+S}\mathcal{W}_1\in T^{(z_1=z_2)}$ and by definition,

$$w_0\otimes w_1\otimes w_2\otimes w_3\in T^{(z_1=z_2)}.$$

Thus $(z_1-z_2)^{\sigma+S}\mathcal{W}_2\in T^{(z_1=z_2)}$. ∎

Theorem 2.3 *Let W_i and $w_i\in W_i$ for $i=0,1,2,3$ be the same as in Theorem 1.4. For any possible singular point of the form ($z_1=0$, $z_2=0$, $z_1=\infty$, $z_2=\infty$, $z_1=z_2$), $z_1^{-1}(z_1-z_2)=0$, or $z_2^{-1}(z_1-z_2)=0$, there exist*

$$a_k(z_1,z_2),b_l(z_1,z_2)\in\mathbb{C}[z_1^{\pm1},z_2^{\pm1},(z_1-z_2)^{-1}]$$

for $k=1,\dots,m$ and $l=1,\dots,n$, such that this singular point of the system (1.7) and (1.8) satisfied by (1.4), (1.5) and (1.6) is regular.

Proof. We shall only prove the theorem for the singular points $z_1=z_2$ and $z_2^{-1}(z_1-z_2)=0$. The latter case is one of the most interesting case since it will give us the expansion of the solutions in the region $|z_2|>|z_1-z_2|>0$. The other cases are similar.

By Proposition 2.1,

$$w_0\otimes L(-1)^kw_1\otimes w_2\otimes w_3=\mathcal{W}_1^{(k)}+\mathcal{W}_2^{(k)}$$

for $k\ge 0$, where $\mathcal{W}_1^{(k)}\in F_{\sigma+k}^{(z_1=z_2)}(J)$ and $\mathcal{W}_2^{(k)}\in F_M(T)$.

By Lemma 2.2, there exists $S\in\mathbb{R}$ such that $\sigma+S\in\mathbb{Z}_+$ and

$$(z_1-z_2)^{\sigma+k+S}\mathcal{W}_2^{(k)}\in T^{(z_1=z_2)}$$

and thus

$$(z_1 - z_2)^{\sigma+k+S}\mathcal{W}_2^{(k)} \in \coprod_{r\leq M} T_{(r)}^{(z_1=z_2)}$$

for $k \geq 0$. Since $\mathbb{C}[z_1^{\pm}, z_2^{\pm}]$ is a Noetherian ring and $\coprod_{r\leq M} T_{(r)}^{(z_1=z_2)}$ is a finitely-generated $\mathbb{C}[z_1^{\pm}, z_2^{\pm}]$-module, the submodule of $\coprod_{r\leq M} T_{(r)}^{(z_1=z_2)}$ generated by $(z_1-z_2)^{\sigma+k+S}\mathcal{W}_2^{(k)}$ for $k \geq 0$ is also finitely generated. Let $(z_1-z_2)^{\sigma+k+S}\mathcal{W}_2^{(k)}$ for $k = 0, \dots, m-1$ be a set of generators of this submodule. Then there exist $c_k(z_1, z_2) \in \mathbb{C}[z_1^{\pm}, z_2^{\pm}]$ for $k = 0, \dots, m-1$ such that

$$(z_1 - z_2)^{\sigma+m+S}\mathcal{W}_2^{(m)} = -\sum_{k=0}^{m-1} c_k(z_1, z_2)(z_1 - z_2)^{\sigma+k+S}\mathcal{W}_2^{(k)}$$

or equivalently

$$\mathcal{W}_2^{(m)} + \sum_{k=0}^{m-1} c_k(z_1, z_2)(z_1 - z_2)^{k-m}\mathcal{W}_2^{(k)} = 0.$$

Thus

$$\begin{aligned} & w_0 \otimes L(-1)^m w_1 \otimes w_2 \otimes w_3 \\ &\quad + \sum_{k=0}^{m-1} c_k(z_1, z_2)(z_1 - z_2)^{k-m} w_0 \otimes L(-1)^k w_1 \otimes w_2 \otimes w_3 \\ &\quad\quad = \mathcal{W}_1^{(m)} + \sum_{k=0}^{m-1} c_k(z_1, z_2)(z_1 - z_2)^{k-m}\mathcal{W}_1^{(k)}. \end{aligned} \tag{2.1}$$

Since $\mathcal{W}_1^{(k)} \in F_{\sigma+k}^{(z_1=z_2)}(J) \subset J$, the right-hand side of (2.1) is in J. Thus we obtain

$$\begin{aligned} & [w_0 \otimes L(-1)^m w_1 \otimes w_2 \otimes w_3] \\ &\quad + \sum_{k=0}^{m-1} c_k(z_1, z_2)(z_1 - z_2)^{k-m}[w_0 \otimes L(-1)^k w_1 \otimes w_2 \otimes w_3] = 0. \end{aligned} \tag{2.2}$$

Similarly we can find $d_l(z_1, z_2) \in \mathbb{C}[z_1^{\pm}, z_2^{\pm}]$ for $l = 0, \dots, n$ such that

$$\begin{aligned} & [w_0 \otimes w_1 \otimes L(-1)^n w_2 \otimes w_3] \\ &\quad + \sum_{l=0}^{n-1} d_l(z_1, z_2)(z_1 - z_2)^{l-n}[w_0 \otimes w_1 \otimes L(-1)^l w_2 \otimes w_3] = 0. \end{aligned} \tag{2.3}$$

Now it is clear that the singular point $z_1 = z_2$ is regular.

To prove that the singular point $z_2^{-1}(z_1 - z_2)$ is also regular, we introduce new gradings on R and T. By asigning the degrees of z_1 and z_2 to be -1, we obtain a grading on R. Equipped with this grading, R is a $\mathbb{Z}$-graded ring. Together with the grading (by weights) on $W_0 \otimes W_1 \otimes W_2 \otimes W_4$, this grading on R gives another grading on T such that T is a graded module for R. Note that when u, w_0, w_1, w_2, w_3 are homogeneous, the elements $\mathcal{A}(u, w_0, w_1, w_2, w_3)$, $\mathcal{B}(u, w_0, w_1, w_2, w_3)$, $\mathcal{C}(u, w_0, w_1, w_2, w_3)$ and $\mathcal{D}(u, w_0, w_1, w_2, w_3)$ are all homogeneous with respect to this new grading. Thus this grading induces a grading on T/J such that T/J is also a graded R-module. Now note that for homogeneous w_0, w_1, w_2, w_3,

$$[w_0 \otimes L(-1)^l w_1 \otimes w_2 \otimes w_3], \quad [w_0 \otimes w_1 \otimes L(-1)^l w_2 \otimes w_3] \in T/J$$

for $l = 0, \dots, n$ are homogeneous of weights $\mathrm{wt}\ w_0 + \mathrm{wt}\ w_1 + \mathrm{wt}\ w_2 + \mathrm{wt}\ w_3 + l$, respectively. Thus by comparing the degrees of the terms in (2.2) and (2.3), we see that we can always choose $c_k(z_1, z_2)$ and $d_l(z_1, z_2)$ for $k = 1, \dots, m$, $l = 0, \dots, n$ to be elements of R of degree 0. Since $c_k(z_1, z_2)$ for $k = 0, \dots, m$ and $d_l(z_1, z_2)$ for $l = 0, \dots, n$ are of degree 0, they are actually functions of $z_3 = z_2^{-1}(z_1 - z_2)$. Using this fact and changing variables from z_1 and z_2 to z_3 and z_2, we see immediately that the singular point $z_3 = 0$ is regular. ∎

Remark 2.4 The regularity of the singular points proved in the theorem above is enough for our applications. In fact the sets of the singular points discussed in the theorem above all consist of only one point and thus the proof of the regularity is the same as the proof of the regularity of singular points of ordinary differential equations. Using the standard method in the theory of differential equations, we can actually prove the regularity of the sets of singular points consisting of two points such as $z_1 = z_2$, $z_2 = \infty$. Though it is not needed in the present paper and in our theory, we sketch a proof here. Let $\hat{\varphi}$ be the vector whose components are

$$\left((z_1 - z_2)\frac{\partial}{\partial z_1}\right)^i \left((z_1 - z_2)\frac{\partial}{\partial z_2}\right)^j \varphi$$

for $i = 0, \dots, m-1$ and $j = 0, \dots, n-1$. We also change the variables to $z_3 = z_2^{-1}(z_1 - z_2)$ and $z_4 = z_2^{-1}$. Then the system (1.7) and (1.8) now becomes

$$z_3 \frac{\partial}{\partial z_3} \hat{\varphi} = A(z_3)\hat{\varphi}, \tag{2.4}$$

$$z_4 \frac{\partial}{\partial z_4} \hat{\varphi} = \frac{-(1+z_3)A(z_3) - B(z_3)}{z_3} \hat{\varphi} \tag{2.5}$$

Using the theory of ordinary differential equations of regular singular points (see, for example, Appendix B of [K]), we can solve (2.4) first and then substitute the solutions into (2.5) to solve the system (1.7) and (1.8). From the explicit form of the solutions, we see immediately that the singular points $z_3 = 0$, $z_4 = 0$ of the system (2.4) and (2.5) are regular. So the the singular points $z_3 = 0$, $z_4 = 0$ of the system (1.7) and (1.8) are also regular.

In the literature, there is actually a stronger definition of regularity of singularities in the case of several variables. A set of singular points are called *regular* in this strong sense if the system is meromorphically equivalent to a compatible (or integrable) system with singularities of logarithmic type (see, for example, Chapter 6 of [BK] for definitions of singularities of logarithmic type and regular singularities). Let $G(z_3)$ be the $mn \times mn$ invertible matrix meromorphic in z_3 such that $G(z_3)\frac{-(1+z_3)A(z_3)-B(z_3)}{z_3}G(z_3)^{-1}$ is a Jordan canonical form. Use $G(z_3)$ as the meromorphic gauge transformation. Then it is easy to see that the system (2.4) and (2.5) becomes a system whose singular points $z_3 = 0$, $z_4 = 0$ are of logarithmic type. Thus the singular points $z_3 = 0$, $z_4 = 0$ of the system (2.4) and (2.5) are also regular in this strong sense.

The following result is proved in the same way and so the proof is omitted:

Theorem 2.5 *For any possible singular point of the system in Theorem 1.6 of the form $z_i = 0$ or $z_i = \infty$ or $z_i = z_j$ for $i \neq j$, there exist elements as in (1.9) such that this singular point of the system in Theorem 1.6 is regular.*

3 Applications

In this section, we prove the finiteness of the fusion rules and the convergence and extension property needed in the tensor product theory for the category of moddules for a vertex operator algebra developed by Lepowsky and the author in [HL1]–[HL4] and [H1] and in the theory of intertwtining operator algebras developed by the author in [H1] and [H3]–[H7]. We also discuss conditions which imply some representation-theoretic conditions needed in these theories. Using all these results, we obtain our main theorems on the constructions of vertex tensor category structures and intertwining operator algebras.

First we have:

Theorem 3.1 *Let V be a vertex operator algebra and W_1, W_2 and W_3 three discretely $\mathbb{R}$-graded V-modules. If W_1, W_2 and W_3' are C_1-cofinite, then the fusion rule among W_1, W_2 and W_3 is finite.*

Proof. Note that V is C_1-cofinite. Consider $W_3' \otimes V \otimes W_1 \otimes W_2$ and the corresponding R-module

$$T = R \otimes W_3' \otimes V \otimes W_1 \otimes W_2.$$

We fix $z_1^{(0)}, z_2^{(0)} \in \mathbb{C}$ satisfying $z_1^{(0)}, z_2^{(0)} \neq 0$ and $z_1^{(0)} \neq z_2^{(0)}$. Consider the evaluation maps from R to $\mathbb{C}$ and from T to $W_3' \otimes V \otimes W_1 \otimes W_2$ given by evaluating elements of R and T at $(z_1^{(0)}, z_2^{(0)})$. We use E to denote these maps. Since T/J is a finitely-generated R-module, $E(T)/E(J)$ is a finite-dimensional vector space. For any intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$, the map $\bar{\phi}_{Y_3, \mathcal{Y}}$ induces a map $E(\bar{\phi}_{Y_3, \mathcal{Y}})$ from $E(T)/E(J)$ to $\mathbb{C}$, where Y_3 is the vertex operator map defining the V-module structure on W_3. Thus we obtain a linear map from the space of intertwining operators of type $\binom{W_3}{W_1 W_2}$ to the space of linear maps from $E(T)/E(J)$ to $\mathbb{C}$. Since intertwining operators are actually determined by their values at one point (see [HL1]), we see that this linear map is injective. Thus the dimension of the space of intertwining operators of type $\binom{W_3}{W_1 W_2}$ is less than or equal to the dimension of the dual space of $E(T)/E(J)$. Since $E(T)/E(J)$ is finite dimensional, we see that the dimension of its dual space and therefore the dimension of the space of intertwining operators of type $\binom{W_3}{W_1 W_2}$ is finite. ∎

Remark 3.2 In fact, the conclusion of Theorem 3.1 still holds if for one of the discretely $\mathbb{R}$-graded V-modules W_1, W_2 and W_3, the C_1-cofiniteness condition is replaced by the condition that it is generated by lowest weight vectors. The reason is the same as in Remark 1.7. In particular, the fusion rule is finite when W_1 and W_2 are C_1-cofinite and W_3 is completely reducible.

Remark 3.3 Theorem 3.1 has been known to physicists for some time and originally goes back to the work of Nahm [Na]. In [L], Li proved that for irreducible V-modules satisfying a slightly weaker cofiniteness condition, the fusion rule is finite. In particular, in the case that W_1, W_2 and W_3 are irreducible, Theorem 3.1 is a consequence. In [AN], Abe and Nagatomo proved that the dimensions of the spaces of conformal blocks on the Riemann sphere are finite if the modules involved are C_1-cofinite (called B_1 finite in [AN]) and the vertex operator algebra satisfies some additional conditions. In particular, under these conditions, the fusion rules are finite.

To formulate the next result, we need:

Definition 3.4 Let V be a vertex operator algebra. We say that products of intertwining operators for V have the *convergence and extension property* if for any $\mathbb{C}$-graded V-modules W_i ($i = 0, 1, 2, 3$) and any intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of types $\binom{W_0'}{W_1 W_4}$ and $\binom{W_4}{W_2 W_3}$, respectively, there exists an integer N (depending only on $\mathcal{Y}_1$ and $\mathcal{Y}_2$), and for any $w_i \in W_i$ ($i = 0, 1, 2, 3$), there exist $r_i, s_i \in \mathbb{R}$ and analytic functions $f_i(z)$ on $|z| < 1$ for $i = 1, \dots, j$ satisfying

$$\mathrm{wt}\ w_{(1)} + \mathrm{wt}\ w_{(2)} + s_i > N, \quad i = 1, \dots, j, \tag{3.1}$$

such that (1.4) is absolutely convergent when $|z_1| > |z_2| > 0$ and can be analytically extended to the multivalued analytic function

$$\sum_{i=1}^{j} z_2^{r_i}(z_1 - z_2)^{s_i} f_i\left(\frac{z_1 - z_2}{z_2}\right) \tag{3.2}$$

when $|z_2| > |z_1 - z_2| > 0$.

This property is introduced and needed in [H1] in order to construct the associativity isomorphism between two iterated tensor products of three modules for a suitable vertex operator algebra. Also recall that a $\mathbb{C}$-graded generalized module for a vertex operator algebra is a $\mathbb{C}$-graded vector space equipped with a vertex operator map satisfying all the axioms for modules except the two grading-restriction axioms.

We have:

Theorem 3.5 *Let V be a vertex operator algebra satisfying the following conditions:*

1. *Every $\mathbb{C}$-graded generalized V-module is a direct sum of $\mathbb{C}$-graded irreducible V-modules.*

2. *There are only finitely many inequivalent $\mathbb{C}$-graded irreducible V-modules and they are all $\mathbb{R}$-graded.*

3. *Every $\mathbb{R}$-graded irreducible V-module satisfies the C_1-cofiniteness condition.*

Then every $\mathbb{C}$-graded V-module is a discretely $\mathbb{R}$-graded V-module and products of intertwining operators for V have the convergence and extension property. In addition, for any discretely $\mathbb{R}$-graded V-modules (or equivalently, $\mathbb{C}$-graded V-modules), intertwining operators and elements of the V-modules as in Theorem 1.6, (1.10) is absolutely convergent when $|z_1| > \cdots |z_p| > 0$ and can be analytically extended to the region given by $z_i \neq z_j$ $(i \neq j)$ $z_i = 0$ $(i = 1, \dots, n)$.

Proof. It is clear that every $\mathbb{C}$-graded V-module is a discretely $\mathbb{R}$-graded V-module. In the remaining part of this section, if every $\mathbb{C}$-graded V-module is a discretely $\mathbb{R}$-graded V-module, we shall call discretely $\mathbb{R}$-graded V-modules simply V-modules.

For any $w_i \in W_i$, $i = 0, 1, 2, 3$, using Theorems 1.4 and 2.3, the theory of differential equations of regular singular points (see, for example, Appendix B of [K]), it is easy to show that (1.4) is absolutely convergent when $|z_1| > |z_2| > 0$. In fact, the expansion coefficients of (1.4) as a series in powers of z_2 are series in powers of z_1 of the form $\langle w_0, \mathcal{Y}_1(w_1, z_1)w_4\rangle$ for $w_4 \in W_4$. These series in powers of z_1 have only finitely many terms and thus are convergent for any $z_1 \neq 0$. So for fixed $z_1 \neq 0$, (1.4) is a series in powers of z_2. Now fix $z_1 \neq 0$. By Theorem 2.3 the equation (1.8) as an ordinary differential equation with the variable z_2 on the region $|z_1| > |z_2| > 0$ has a regular singular point at $z_2 = 0$. Since (1.4) satisfies (1.8), it must be convergent as a series in powers of z_2. Since the coefficients of (1.8) are analytic in z_1, the sum of (1.4) is also analytic in z_1. So the sum is in fact analytic in both z_1 and z_2 and its expansion as a double series in powers of z_1 and z_2 must be absolutely convergent.

To prove the remaining part of the convergence and extension property, we need the regularity of the system (1.7) and (1.8) at the the singular point $z_2^{-1}(z_1 - z_2) = 0$. Using the $L(0)$-conjugation property for intertwining operators, we see that (1.4) is equal to

$$z_2^{\mathrm{wt}\ w_0-\mathrm{wt}\ w_2-\mathrm{wt}\ w_2-\mathrm{wt}\ w_2}\langle w_0, \mathcal{Y}_1(w_1, 1+z_2^{-1}(z_1-z_2))\mathcal{Y}_2(w_2, 1)w_3\rangle.$$

By Theorems 1.4, (1.4) satisfies (1.7) and by changing the variables to $z_3 = z_2^{-1}(z_1 - z_2)$ and z_2,

$$\langle w_0, \mathcal{Y}_1(w_1, 1+z_3)\mathcal{Y}_2(w_2, 1)w_3\rangle$$

satisfies an ordinary differential equation with the variable z_3. By Theorem 2.3, we see that the singular point $z_3 = 0$ of this ordinatry differential equation is regular. By the theory of (ordinary) differential equations of regular singular points (see, for example, Appendix B of [K]) and the fact that there are no logarithmic terms in the two intertwining operators in (1.4), we see that there exist $r_{i,k}, s_{i,k} \in \mathbb{R}$ and analytic functions $f_{i,k}(z)$ on $|z| < 1$ for $i = 1, \dots, j$, $k = 1, \dots, K$, such that (1.4) can be analytically extended to the multivalued analytic function

$$\sum_{k=1}^{K}\sum_{i=1}^{j} z_2^{r_{i,k}}(z_1-z_2)^{s_{i,k}} f_{i,k}\left(\frac{z_1-z_2}{z_2}\right)\left(\log\left(\frac{z_1-z_2}{z_2}\right)\right)^k \tag{3.3}$$

in the region $|z_2| > |z_1 - z_2| > 0$. Since (1.4) for general elements $w_i \in W_i$, $i = 0, 1, 2, 3$, are R-linear combinations of (1.4) for those $w_i \in W_i$ $(i = 0, 1, 2, 3)$ satisfying $\mathrm{wt}\ w_0 + \mathrm{wt}\ w_1 + \mathrm{wt}\ w_2 + \mathrm{wt}\ w_3 \leq M$, we see that K can be taken to be independent of $w_i \in W_i$, $i = 0, 1, 2, 3$. We now show that $K = 0$.

We see that (3.3) can be written as

$$\sum_{k=1}^{K} g_k(w_0, w_1, w_2, w_3; z_1, z_2)\left(\log\left(1-\frac{z_2}{z_1}\right)\right)^k, \tag{3.4}$$

where $g_k(w_0, w_1, w_2, w_3; z_1, z_2)$ for $k = 1, \dots, K$ are linear combinations of products of $z_2^{r_{i,k}}$, $(z_1 - z_2)^{s_{i,k}}$, $f_{i,k}\left(\frac{z_1-z_2}{z_2}\right)$ and $\left(\log\left(\frac{z_1}{z_2}\right)\right)^k$ for $i = 1, \dots, j$ and $k = 1, \dots, K$. Since (3.3) and $\left(\log\left(1-\frac{z_2}{z_1}\right)\right)^k$ for $k = 1, \dots, K$ can be analytically extended to the region $|z_1| > |z_2| > 0$, $g_k(w_0, w_1, w_2, w_3; z_1, z_2)$ for $k = 1, \dots, K$ can also be extended to this region. Consequently (1.4) is equal to (3.4).

Assume that $K \neq 0$. We can view $g_K(w_0, w_1, w_2, w_3; z_1, z_2)$ as the value at $w_1 \otimes w_2 \otimes w_3$ of the image of w_0 under a linear map from W_0 to $(W_1 \otimes$

$W_2 \otimes W_3)^*$. Then by the properties of (1.4) and the linear independence of $\left(\log\left(1-\frac{z_2}{z_1}\right)\right)^k$, $k = 1, \dots, K$, the image of w_0 under this linear map from W_0 to $(W_1 \otimes W_2 \otimes W_3)^*$ satisfy the $P(z_1, z_2)$-compatibility condition, the $P(z_1, z_2)$-local grading-restriction condition and the $P(z_2)$-local grading-restriction condition (see [H1]). Using Conditions 1–3, Theorem 3.1 Theorem 14.10 in [H1] and the construction of the associativity isomorphisms in [H1], we see that this image is in fact in $\Psi^{(1)}_{P(z_1,z_2)}(W_1 \boxdot_{P(z_1)} (W_1 \boxtimes W_3))$ where

$$\Psi^{(1)}_{P(z_1,z_2)} : W_1 \boxdot_{P(z_1)} (W_1 \boxtimes W_3) \to (W_1 \otimes W_2 \otimes W_3)^*$$

is defined by

$$(\Psi^{(1)}_{P(z_1,z_2)}(\nu))(w_1 \otimes w_2 \otimes w_3) = \langle \nu, w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)\rangle$$

for $\nu \in W_1 \boxdot_{P(z_1)} (W_1 \boxtimes W_3)$, $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$. In addition, by the properties of (1.4) and the linear independence of $\left(\log\left(1-\frac{z_2}{z_1}\right)\right)^k$, $k = 1, \dots, K$, $g_K(w_0, w_1, w_2, w_3; z_1, z_2)$ also satisfy the $L(-1)$-derivative properties

$$\begin{aligned}
\frac{\partial}{\partial z_1} g_K(w_0, w_1, w_2, w_3; z_1, z_2) &= g_K(w_0, L(-1)w_1, w_2, w_3; z_1, z_2), && (3.5)\\
\frac{\partial}{\partial z_2} g_K(w_0, w_1, w_2, w_3; z_1, z_2) &= g_K(w_0, w_1, L(-1)w_2, w_3; z_1, z_2). && (3.6)
\end{aligned}$$

Thus we see that $g_K(w_0, w_1, w_2, w_3; z_1, z_2)$ can also be written in the form of (1.4) with possibly different $\mathcal{Y}_1$ and $\mathcal{Y}_2$ in the region $|z_1| > |z_2| > 0$.

By the properties of (1.4) and the linear independence of $\left(\log\left(1-\frac{z_2}{z_1}\right)\right)^k$, $k = 1, \dots, K$, $g_{K-1}(w_0, w_1, w_2, w_3; z_1, z_2)$ also satisfies all the properties satisfied by (1.4) except for the $L(-1)$-derivative property. Thus for any fixed complex numbers z_1 and z_2 satisfying $|z_1| > |z_2| > 0$, using Conditions 1–3, Theorem 3.1, Theorem 14.10 in [H1], the construction of the associativity isomorphisms in [H1] and the same argument as the one above for $g_K(w_0, w_1, w_2, w_3; z_1, z_2)$, we see that $g_{K-1}(w_0, w_1, w_2, w_3; z_1, z_2)$ can also be written in the form of (1.4) with possibly different $\mathcal{Y}_1$ and $\mathcal{Y}_2$. Using Conditions 1–2 and Theorem 3.1, we can always find finite basis of spaces of intertwining operators among irreducible V-modules such that the products of these basis intertwining operators give us a finite basis of the space of linear functional on $W_0 \otimes W_1 \otimes W_3 \otimes W_4$ given by products of intertwining operators of the form (1.4) (evaluated at the given numbers z_1 and z_2). We denote the value of these basis elements of linear functionals at $w_0 \otimes w_1 \otimes w_2 \otimes w_3$ for $w_i \in W_i$, $i = 0, 1, 2, 3$, by $e_l(w_0, w_1, w_2, w_3; z_1, z_2)$, $l = 1, \dots, p$. Then we have

$$g_{K-1}(w_0, w_1, w_2, w_3; z_1, z_2) = \sum_{l=1}^{p} c_l(z_1, z_2) e_l(w_0, w_1, w_2, w_3; z_1, z_2), \qquad (3.7)$$

where we have written down the dependence of the coefficients $c_l(z_1, z_2)$, $l = 1, \dots, p$, on z_1 and z_2 explicitly. Since $g_{K-1}(w_0, w_1, w_2, w_3; z_1, z_2)$ and $e_l(w_0, w_1, w_2, w_3; z_1, z_2)$, $l = 1, \dots, p$, are all analytic in z_1 and z_2, we see that $c_l(z_1, z_2)$, $l = 1, \dots, p$, are also analytic in z_1 and z_2. The functions $e_l(w_0, w_1, w_2, w_3; z_1, z_2)$ for $l = 1, \dots, p$ also satisfy the $L(-1)$-derivative properties

$$\begin{aligned} \frac{\partial}{\partial z_1} e_l(w_0, w_1, w_2, w_3; z_1, z_2) &= e_l(w_0, L(-1)w_1, w_2, w_3; z_1, z_2), & (3.8)\\ \frac{\partial}{\partial z_2} e_l(w_0, w_1, w_2, w_3; z_1, z_2) &= e_l(w_0, w_1, L(-1)w_2, w_3; z_1, z_2). & (3.9) \end{aligned}$$

From the $L(-1)$-derivative property for intertwining operators and the linear independence of $\left(\log\left(1 - \frac{z_2}{z_1}\right)\right)^k$, $k = 1, \dots, K$, we obtain

$$\begin{aligned} &\frac{\partial}{\partial z_1} g_{K-1}(w_0, w_1, w_2, w_3; z_1, z_2)\\ &\quad + K \frac{\partial}{\partial z_1} \log\left(1 - \frac{z_2}{z_1}\right) g_K(w_0, w_1, w_2, w_3; z_1, z_2)\\ &\qquad = g_{K-1}(w_0, L(-1)w_1, w_2, w_3; z_1, z_2), & (3.10)\\ &\frac{\partial}{\partial z_2} g_{K-1}(w_0, w_1, w_2, w_3; z_1, z_2)\\ &\quad + K \frac{\partial}{\partial z_2} \log\left(1 - \frac{z_2}{z_1}\right) g_K(w_0, w_1, w_2, w_3; z_1, z_2)\\ &\qquad = g_{K-1}(w_0, w_1, L(-1)w_2, w_3; z_1, z_2). & (3.11) \end{aligned}$$

From (3.5), (3.6), (3.10) and (3.11), we obtain

$$\frac{\partial^2}{\partial z_1^2} g_{K-1}(w_0, w_1, w_2, w_3; z_1, z_2) - 2 \frac{\partial}{\partial z_1} g_{K-1}(w_0, L(-1)w_1, w_2, w_3; z_1, z_2)$$

$$
\begin{aligned}
&+g_{K-1}(w_0, L(-1)^2 w_1, w_2, w_3; z_1, z_2)\\
&+\left(\frac{\partial}{\partial z_1}\log\left(1-\frac{z_2}{z_1}\right)\right)^{-1}\frac{\partial^2}{\partial z_1^2}\log\left(1-\frac{z_2}{z_1}\right)\cdot\\
&\qquad\cdot g_{K-1}(w_0, L(-1)w_1, w_2, w_3; z_1, z_2)\\
&-\left(\frac{\partial}{\partial z_1}\log\left(1-\frac{z_2}{z_1}\right)\right)^{-1}\frac{\partial^2}{\partial z_1^2}\log\left(1-\frac{z_2}{z_1}\right)\cdot\\
&\qquad\cdot\frac{\partial}{\partial z_1}g_{K-1}(w_0, w_1, w_2, w_3; z_1, z_2)\\
&=0, \qquad (3.12)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial z_2^2}g_{K-1}(w_0, w_1, w_2, w_3; z_1, z_2) - 2\frac{\partial}{\partial z_2}g_{K-1}(w_0, w_1, L(-1)w_2, w_3; z_1, z_2)&\\
+g_{N-1}(w_0, w_1, L(-1)^2 w_2, w_3; z_1, z_2)&\\
+\left(\frac{\partial}{\partial z_2}\log\left(1-\frac{z_2}{z_1}\right)\right)^{-1}\frac{\partial^2}{\partial z_2^2}\log\left(1-\frac{z_2}{z_1}\right)\cdot&\\
\cdot g_{K-1}(w_0, w_1, L(-1)w_2, w_3; z_1, z_2)&\\
-\left(\frac{\partial}{\partial z_1}\log\left(1-\frac{z_2}{z_1}\right)\right)^{-1}\frac{\partial^2}{\partial z_1^2}\log\left(1-\frac{z_2}{z_1}\right)\cdot&\\
\cdot\frac{\partial}{\partial z_2}g_{K-1}(w_0, w_1, w_2, w_3; z_1, z_2)&\\
=0. \qquad (3.13)&
\end{aligned}
$$

Substituting (3.7) into (3.12) and (3.13) and then using the $L(-1)$-derivative properties (3.8) and (3.9) and the linear independence of $e_l(w_0, w_1, w_2, w_3; z_1, z_2)$ for $l = 1, \dots, p$, we obtain

$$
\begin{aligned}
\frac{\partial^2}{\partial z_1^2}c_l(z_1, z_2) &= \left(\frac{\partial}{\partial z_1}\log\left(1-\frac{z_2}{z_1}\right)\right)^{-1}\frac{\partial^2}{\partial z_1^2}\log\left(1-\frac{z_2}{z_1}\right)\frac{\partial}{\partial z_1}c_l(z_1, z_2),\\
\frac{\partial^2}{\partial z_2^2}c_l(z_1, z_2) &= \left(\frac{\partial}{\partial z_2}\log\left(1-\frac{z_2}{z_1}\right)\right)^{-1}\frac{\partial^2}{\partial z_2^2}\log\left(1-\frac{z_2}{z_1}\right)\frac{\partial}{\partial z_2}c_l(z_1, z_2)
\end{aligned}
$$

for $l = 1, \dots, p$. The general solution of this system of equations is

$$
c_l(z_1, z_2) = \lambda_1^{(l)}\log\left(1-\frac{z_2}{z_1}\right) + \lambda_2^{(l)}, \quad l = 1, \dots, p,
$$

where $\lambda_1^{(l)}$ and $\lambda_2^{(l)}$, $l = 1, \dots, p$, are constants. Since $g_{K-1}(w_0, w_1, w_2, w_3; z_1, z_2)$ cannot contain terms proportional to $\log\left(1-\frac{z_1}{z_2}\right)$, we have $\lambda_1^{(l)} = 0$ for

$l = 1, \dots, p$. So $c_l(z_1, z_2)$ for $l = 1, \dots, p$ are constants. Thus we have

$$\frac{\partial}{\partial z_1} g_{K-1}(w_0, w_1, w_2, w_3; z_1, z_2) = g_{K-1}(w_0, L(-1)w_1, w_2, w_3; z_1, z_2)$$

which contradicts to (3.10). So $K = 0$.

We now prove that there exists N such that (3.1) holds.

For any $w_0 \in W_0$, $w_3 \in W_3$ and $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$, let $\mu^{z_1,z_2}_{w_0,w_3} \in (W_1 \otimes W_2)^*$ be defined by

$$\mu^{z_1,z_2}_{w_0,w_3}(w_1 \otimes w_2) = \langle w_0, \mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(w_2, x_2)w_3\rangle|_{x_1^r = e^{r \log z_1}, x_2^r = e^{r \log z_2}, r \in \mathbb{R}}$$

for $w_1 \in W_1$ and $w_2 \in W_2$. A straightforward calculation shows that $\mu^{z_1,z_2}_{w_0,w_3}$ satisfies the $P(z_1 - z_2)$-compatibility condition (cf. [H1]).

We have proved that

$$\langle w_0, \mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(w_2, x_2)w_3\rangle = \sum_{i=1}^{j} z_2^{r_i}(z_1 - z_2)^{s_i} f_i\left(\frac{z_1 - z_2}{z_2}\right). \qquad (3.14)$$

Expanding f_i, $i = 1, \dots, j$, we can write the right-hand side of (3.14) as

$$\sum_{i=1}^{j} \sum_{m \in \mathbb{N}} C_{i,m}(w_0, w_1, w_2, w_3) z_2^{r_i - m}(z_1 - z_2)^{s_i + m}.$$

For $i = 1, \dots, j$, $m \in \mathbb{N}$, $w_0 \in W_0$ and $w_3 \in W_3$, let $\beta_{i,m}(w_0, w_3) \in (W_1 \otimes W_3)^*$ be defined by

$$(\beta_{i,m}(w_0, w_3))(w_1 \otimes w_2) = C_{i,m}(w_0, w_1, w_2, w_3)$$

for $w_1 \in W_1$ and $w_2 \in W_2$. Then we have

$$\mu^{z_1,z_2}_{w_0,w_3} = \sum_{i=1}^{j} \sum_{m \in \mathbb{N}} \beta_{i,m}(w_0, w_3) e^{(r_i - m)\log z_2} e^{(s_i + m)\log(z_1 - z_2)}.$$

Since $\mu^{z_1,z_2}_{w_0,w_3}$ satisfies the $P(z_1 - z_2)$-compatibility condition, $\beta_{i,m}(w_0, w_3)$ for $i = 1, \dots, j$ and $m \in \mathbb{N}$ also satisfy the $P(z_1 - z_2)$-compatibility condition.

A straightforward calculation shows that $\beta_{i,m}(w_0, w_3)$ for $i = 1, \dots, j$ and $m \in \mathbb{N}$ are eigenvectors of $L'_{P(z_1 - z_2)}(0)$ with eigenvalues $\mathrm{wt}\ w_1 + \mathrm{wt}\ w_2 + s_i + m$ (cf. [H1]). For fixed $i = 1, \dots, j$ and $m \in \mathbb{N}$, we consider the $\mathbb{C}$-graded generalized V-module generated by $\beta_{i,m}(w_0, w_3)$. By Condition 1, this $\mathbb{C}$-graded generalized V-module is a direct sum of irreducible V-modules. The generator $\beta_{i,m}(w_0, w_3)$ must belong to a finite sum of these irreducible V-modules and thus this $\mathbb{C}$-graded generalized V-module is in fact a module. So $\beta_{i,m}(w_0, w_3) \in W_1 \boxtimes_{P(z_1-z_2)} W_2$ for $i = 1, \dots, j$ and $m \in \mathbb{N}$. Let N be an integer such that $(W_1 \boxtimes_{P(z_1-z_2)} W_2)_{(r)} = 0$ when $r \leq N$. Since the weights of $\beta_{i,m}(w_0, w_3)$ are $\mathrm{wt}\ w_1 + \mathrm{wt}\ w_2 + s_i + m$ for $i = 1, \dots, j$ and $m \in \mathbb{N}$, we must have $\mathrm{wt}\ w_1 + \mathrm{wt}\ w_2 + s_i > N$.

The absolute convergence of (1.10) is proved using induction together with the same argument as in the proof of the absolute convergence of (1.4) except that instead of the systems (1.6) and Theorem 2.3, the systems in Theorem 1.6 and Theorem 2.5 are used. The analytic extension is obtained immediately using the system (1.6). ∎

Remark 3.6 The proof of Theorem 3.5 is in fact the main work in this paper. It uses the results obtained in Sections 1 and 2. Note that the proof of the convergence and the proof that the matrix elements of a product of intertwining operators can be analytically extended to a multivalued function of a certain form uses only the regularity of a singular point of ordinary differential equations induced from the system (1.7) and (1.8). There is no need to use the regularity of the sets of singular points of the two-variables system (1.7) and (1.8) as proved and discussed in Remark 2.4. The remaining part of the proof in this section is to show that there are no logarithm terms. This remaining part of the proof uses the version of the associativity theorem proved in [H1] stated in terms of the $P(z_1, z_2)$-compatibility condition, $P(z_1, z_2)$-local grading-restriction condition and the $P(z_2)$-local grading-restriction condition. Note what we want to prove here is the convergence and extension property. So the other more widely known version of the associativity theorem in [H1] stated in terms of convergence and extension property cannot be used here.

The following result is the main application we are interested in the present paper:

Theorem 3.7 *Let V be a vertex operator algebra satisfying the three conditions in Theorem 3.5. Then the direct sum of all (inequivalent) irreducible V-modules has a natural structure of intertwining operator algebra and the category of V-modules has a natural structure of vertex tensor category. In* *particular, the category of V-modules has a natural structure of braided tensor category.*

Proof. Combining Theorems 3.5, Theorem 3.1 and the results in [HL1]–[HL5], [H1], [H3]–[H5] and [H7], we obtain Theorem 3.7 immediately. ∎

Remark 3.8 Note that by the results obtained in [Z2], [FZ], [H1], [DLM1]–[DLM2], [L], [GN], [B], [ABD] and [HKL], the conclusion of the theorem above holds if various other useful conditions hold. Here we discuss one example. Let V be a vertex operator algebra. Let $C_2(V)$ be the subspace of V spanned by elements of the form $u_{-2}v$ for $u, v \in V$. Recall that V is said to be *C_2-cofinite* or satisfy the *C_2-cofiniteness condition* if $\dim V/C_2(V) < \infty$. We assume that $V_{(n)} = 0$ for $n < 0$ and $V_{(0)} = \mathbb{C}\mathbf{1}$, every $\mathbb{N}$-gradable weak V-module is completely reducible and V satisfies the C_2-cofiniteness condition. By the results of Abe, Buhl and Dong in [ABD], Condition 1 is a consequence of these three conditions and Condition 3 is a consequence of the C_2-cofiniteness condition for V. By a result of Anderson-Moore [AM] and Dong-Li-Mason [DLM3], in this case, every irreducible $\mathbb{C}$-graded V-module is in fact $\mathbb{Q}$-graded. Thus in this case, the conclusion of Theorem 3.7 holds.

In the example discussed in the remark above, we still need the condition that every $\mathbb{N}$-gradable weak V-module is completely reducible. Here we have a much stronger result:

Theorem 3.9 *Let V be a vertex operator algebra satisfying the following conditions:*

1. *For $n < 0$, $V_{(n)} = 0$ and $V_{(0)} = \mathbb{C}\mathbf{1}$.*
2. *Every $\mathbb{C}$-graded V-module is completely reducible.*
3. *V is C_2-cofinite.*

Then every $\mathbb{C}$-graded V-module is C_1-cofinite and every $\mathbb{C}$-graded generalized V-module is a direct sum of irreducible $\mathbb{C}$-graded V-modules. In particular, if V satisfies these three conditions and every (irreducible) $\mathbb{C}$-graded V-module is $\mathbb{R}$-graded, then the conclusions of Theorems 3.5 and 3.7 hold.

Proof. Since V satisfies Condition 3, Zhu's algebra $A(V)$ is finite-dimensional (see [DLM3]). Thus there are only finitely many irreducible $A(V)$-modules. Since there is a bijection between the set of equivalence classes of irreducible $A(V)$-modules and the set of equivalence classes of irreducible $\mathbb{C}$-graded V-modules (see [Z2]), we see that there are only finitely many inequivalent irreducible $\mathbb{C}$-graded V-modules.

Since V satisfies Conditions 1 and 3, by Corollary 3.18 in [KarL], every finitely-generated lower-truncated $\mathbb{C}$-graded generalized V-module is a $\mathbb{C}$-graded V-module.

The next part of the proof uses the results of Abe, Buhl and Dong [ABD] and, for a large part, is similar to the proof of some results in [ABD].

By Condition 2, to prove that every $\mathbb{C}$-graded V-module is C_1-cofinite, we need only consider irreducible $\mathbb{C}$-graded V-modules. By a result in [ABD], any irreducible $\mathbb{C}$-graded V-module is C_2-cofinite. Since C_2-cofinite $\mathbb{C}$-graded V-module is C_1-cofinite when V satisfies Condition 1, every $\mathbb{C}$-graded V-module is C_1-cofinite.

In [ABD], it was proved that if V satisfies Conditions 1 and 3, then any weak V-module W has a nonzero lowest-weight vector, that is, a vector $w \in W$ such that $v_n w = 0$ when $\mathrm{wt}\ v_n < 0$. In particular, any $\mathbb{C}$-graded generalized V-module has a nonzero lowest-weight vector. Clearly the $\mathbb{C}$-graded generalized V-submodule generated by w is a finitely-generated lower-truncated $\mathbb{C}$-graded generalized V-module. Since we have proved that such $\mathbb{C}$-graded generalized V-modules are $\mathbb{C}$-graded V-modules, the $\mathbb{C}$-graded generalized V-submodule generated by w is a $\mathbb{C}$-graded V-module. So we see that any nonzero $\mathbb{C}$-graded generalized V-module contains a non-zero V-submodule. By Condition 2, we see that any nonzero $\mathbb{C}$-graded generalized V-module in fact contains a nonzero irreducible $\mathbb{C}$-graded V-submodule.

Now a standard argument allows us to show that every $\mathbb{C}$-graded generalized V-module is a direct sum of irreducible $\mathbb{C}$-graded V-modules. In fact, given any $\mathbb{C}$-graded generalized V-module W, let $\tilde{W}$ be the sum of all irreducible $\mathbb{C}$-graded V-modules contained in W. Since we have proved that there are only finitely many irreducible $\mathbb{C}$-graded V-modules, $\tilde{W}$ is a lower-truncated $\mathbb{C}$-graded generalized V-module. If $\tilde{W} \neq W$, $W/\tilde{W}$ is a nonzero $\mathbb{C}$-graded generalized V-module and thus contains a nonzero irreducible $\mathbb{C}$-graded V-submodule $W_0/\tilde{W}$. Since both $W_0/\tilde{W}$ and $\tilde{W}$ are lower-truncated $\mathbb{C}$-graded generalized V-modules, W_0 is also such a $\mathbb{C}$-graded generalized V-module. It is clear that Condition 2 and the fact that every finitely-generated lower-truncated $\mathbb{C}$-graded generalized V-module is a $\mathbb{C}$-graded V-module are equivalent to the fact that every lower-truncated $\mathbb{C}$-graded generalized V-module is a direct sum of irreducible $\mathbb{C}$-graded V-modules. So W_0 is a direct sum of irreducible $\mathbb{C}$-graded V-modules. Since W_0 contains $\tilde{W}$ and $W_0/\tilde{W}$ is not 0, W_0 as a direct sum of irreducible $\mathbb{C}$-graded V-module contains more irreducible $\mathbb{C}$-graded V-submodules of W than $\tilde{W}$. Contradiction.

The last conclusion follows from the first two conclusions we have just proved and Theorems 3.5 and 3.7. ∎

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