First and second cohomologies of grading-restricted vertex algebras

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Abstract

Let V be a grading-restricted vertex algebra and W a V-module. We show that for any $m \in \mathbb{Z}_+$, the first cohomology $H^1_m(V, W)$ of V with coefficients in W introduced by the author is linearly isomorphic to the space of derivations from V to W. In particular, $H^1_m(V, W)$ for $m \in \mathbb{N}$ are equal (and can be denoted using the same notation $H^1(V, W)$). We also show that the second cohomology $H^2_{\frac{1}{2}}(V, W)$ of V with coefficients in W introduced by the author corresponds bijectively to the set of equivalence classes of square-zero extensions of V by W. In the case that $W = V$, we show that the second cohomology $H^2_{\frac{1}{2}}(V, V)$ corresponds bijectively to the set of equivalence classes of first order deformations of V.

1 Introduction

The present paper is a sequel to the paper [H]. We discuss the first and second cohomologies of grading-restricted vertex algebras introduced by the author in that paper.

Let V be a grading-restricted vertex algebra and W a V-module. Recall from [H] that for each $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, we have an n-th cohomology $H^n_m(V, W)$ of V with coefficients in W. For each $n \in \mathbb{N}$, We also have an n-th cohomology $H^n_\infty(V, W)$ of V with coefficients in W which is isomorphic to the inverse limit of the inverse system $\{H^n_m(V, W)\}_{m\in\mathbb{Z}_+}$. We also have an additional second cohomology $H^2_{\frac{1}{2}}(V, W)$ of V with coefficients in W. In the present paper, we discuss only $H^1_m(V, W)$ for $m \in \mathbb{Z}_+$ and $H^2_{\frac{1}{2}}(V, W)$.

Let V be a grading-restricted vertex algebra and W a V-module. A grading-preserving linear map $f : V \to W$ is called a *derivation* if

$$\begin{aligned} f(Y_V(u,z)v) &= Y_{WV}^W(f(u),z)v + Y_W(u,z)f(v) \\ &= e^{zL(-1)}Y_W(v,-z)f(u) + Y_W(u,z)f(v) \end{aligned}$$

for $u, v \in V$. We use Der (V, W) to denote the space of all such derivations. We have the following result for the first cohomologies of V with coefficients in W:

Theorem 1.1. *Let V be a grading-restricted vertex algebra and W a V-module. Then $H_m^1(V,W)$ is linearly isomorphic to the space of derivations from V to W for any $m \in \mathbb{Z}_+$, that is, $H_m^1(V,W)$ is linearly isomorphic to* Der (V, W) *for any $m \in \mathbb{Z}_+$.*

In particular, $H_m^1(V,W)$ for $m \in \mathbb{N}$ are isomorphic (and can be denoted using the same notation $H^1(V,W)$).

Definition 1.2. Let V be a grading-restricted vertex algebra. A *square-zero ideal of* V is an ideal W of V such that for any $u, v \in W$, $Y_V(u,x)v = 0$.

Definition 1.3. Let V be a grading-restricted vertex algebra and W a $\mathbb{Z}$-graded V-module. A *square-zero extension* (Λ, f, g) *of* V *by* W is a grading-restricted vertex algebra Λ together with a surjective homomorphism $f : \Lambda \to V$ of grading-restricted vertex algebras such that $\ker f$ is a square-zero ideal of Λ (and therefore a V-module) and an injective homomorphism g of V-modules from W to Λ such that $g(W) = \ker f$. Two square-zero extensions (Λ_1, f_1, g_1) and (Λ_2, f_2, g_2) of V by W are *equivalent* if there exists an isomorphism of grading-restricted vertex algebras $h : \Lambda_1 \to \Lambda_2$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \xrightarrow[g_1]{} & \Lambda_1 & \xrightarrow[f_1]{} & V & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle 1_W} & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle 1_V} & & \\ 0 & \longrightarrow & W & \xrightarrow[g_2]{} & \Lambda_2 & \xrightarrow[f_2]{} & V & \longrightarrow & 0, \end{array}$$

is commutative.

The notion of square-zero extension of V by W is an analogue of the notion of square-zero extension of an associative algebra by a bimodule. (see, for example, Section 9.3 of [W]).

We have the following result for the second cohomology $H^2_{\frac{1}{2}}(V, W)$ of V with coefficients in W:

Theorem 1.4. *Let V be a grading-restricted vertex algebra and W a V-module. Then the set of the equivalence classes of square-zero extensions of V by W corresponds bijectively to $H^2_{\frac{1}{2}}(V, W)$.*

Definition 1.5. Let t be a complex variable. A *family of grading-restricted vertex algebras up to the first order in t* is a $\mathbb{Z}$-graded vector space V, a family $Y_t : V \otimes V \to V((x))$ for $t \in \mathbb{C}$ of linear maps of the form $Y_t = Y_0 + t\Psi$ where Y_0 and Ψ are linear maps from $V \otimes V$ to $V((x))$ independent of t, and an element $\mathbf{1} \in V$ satisfying all the axioms for grading-restricted vertex algebras up to the first order in t. More precisely, the triple $(V, Y_t, \mathbf{1})$ satisfies the grading restriction condition, lower-truncation condition for vertex operators, $L(0)$-bracket formula and the following conditions:

1. *Identity property up to the first order in t*: $Y_t(\mathbf{1}, x) = 1_V + O(t^2)$.

2. *Creation property up to the first order in t*: For $u \in V$, $Y_t(u, x)\mathbf{1} \in V[[x]]$ and $\lim_{x \to 0} Y_t(u, x)\mathbf{1} = v + O(t^2)$.

3. *Duality up to the first order in t*: For $v_1, v_2, v_3 \in V$ and $v' \in V'$, the coefficients of t^0 and t^1 terms of
$$\langle v', Y_t(v_1, z_1)Y_t(v_2, z_2)v_3\rangle$$
$$\langle v', Y_t(v_2, z_2)Y_t(v_1, z_1)v_3\rangle$$
$$\langle v', Y_t(Y_t(v_1, z_1 - z_2)v_2, z_2)v_3\rangle$$
are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, to common rational functions in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.

Definition 1.6. Let $(V, Y_V, \mathbf{1})$ be a grading-restricted vertex algebra. A *first order deformation of V* is a family $Y_t : V \otimes V \to V((x))$ for $t \in \mathbb{C}$ of linear maps of the form $Y_t = Y_V + t\Psi$ where

$$\begin{aligned} \Psi : V \otimes V &\to V((x)) \\ v_1 \otimes v_2 &\to \Psi(v_1, x)v_2 \end{aligned}$$

is a linear map such that $(V, Y_t, \mathbf{1})$ for $t \in \mathbb{C}$ is a family of grading-restricted vertex algebras up to the first order in t. Two first order deformations $Y_t^{(1)}$

and $Y_t^{(2)}$, $t \in \mathbb{C}$, of $(V, Y_V, \mathbf{1})$ are *equivalent* if there exists a family $f_t : V \to V$, $t \in \mathbb{C}$ of linear maps of the form $f_t = 1_V + tg$ where $g : V \to V$ is a linear map preserving the gradings of V such that

$$f_t(Y_t^{(1)}(v_1, x)v_2) - Y_t^{(2)}(f_t(v_1), x)f_t(v_2) \in t^2 V((x)) \tag{1.1}$$

for $v_1, v_2 \in V$.

We have:

Theorem 1.7. *The set of equivalence classes of first order deformations of a grading-restricted vertex algebra is in bijection with the set of equivalence classes of square-zero extensions of V by V.*

From Theorems 1.4 and 1.7, we obtain immediately the following result for the second cohomology $H^2_{\frac{1}{2}}(V, V)$ of V with coefficients in V:

Theorem 1.8. *Let V be a grading-restricted vertex algebra. Then the set of the equivalence classes of first order deformations of V correspond bijectively to $H^2_{\frac{1}{2}}(V, V)$.*

We prove Theorems 1.1, 1.4 and 1.7 in Sections 2, 3 and 4, respectively.

Acknowledgments The author is grateful for partial support from NSF grant PHY-0901237.

2 First cohomologies and spaces of derivations

We prove Theorem 1.1 in the present section. First, we need the following:

Lemma 2.1. *Let $f : V \to W$ be a derivation. Then $f(\mathbf{1}) = 0$.*

Proof. By definition,

$$\begin{aligned} f(\mathbf{1}) &= f(Y_V(\mathbf{1}, z)\mathbf{1}) \\ &= \lim_{z \to 0} f(Y_V(\mathbf{1}, z)\mathbf{1}) \\ &= \lim_{z \to 0} e^{zL(-1)} Y_W(\mathbf{1}, -z) f(\mathbf{1}) + \lim_{z \to 0} Y_W(\mathbf{1}, z) f(\mathbf{1}) \\ &= 2f(\mathbf{1}). \end{aligned}$$

So $f(\mathbf{1}) = 0$. ∎

Let $\Phi : V \to \widetilde{W}_{z_1}$ be an element of $C^1_m(V, W)$ satisfying $\delta^1_m \Phi = 0$. Since Φ satisfies the $L(0)$-conjugation property, for $v \in V_{(n)}$ and $z \in \mathbb{C}^\times$,

$$\begin{aligned} z^{L(0)}(\Phi(v))(0) &= (\Phi(z^{L(0)}v))(0) \\ &= z^n(\Phi(v))(0). \end{aligned}$$

Thus $(\Phi(v))(0) \in W_{(n)}$. So $(\Phi(v))(0)$ is a grading-preserving linear map from V to W.

Since $\delta^1_m \Phi = 0$,

$$\begin{aligned} &R(\langle w', Y_W(v_1, z_1)(\Phi(v_2))(z_2)\rangle) - R(\langle w', (\Phi(Y_V(v_1, z_1 - z_2)v_2))(z_2)\rangle) \\ &\quad + R(\langle w', Y_W(v_2, z_2)(\Phi(v_1))(z_1)\rangle) \\ &= 0 \end{aligned}$$

for $v_1, v_2 \in V$ and $w' \in W'$. By $L(-1)$-derivative property for Φ and the vertex operator map Y_W,

$$R(\langle w', Y_W(v_2, z_2)(\Phi(v_1))(z_1)\rangle) = R(\langle w', e^{z_1 L(-1)} Y_W(v_2, -z_1 + z_2)(\Phi(v_1))(0)\rangle).$$

Thus we have

$$\begin{aligned} &R(\langle w', Y_W(v_1, z_1)(\Phi(v_2))(z_2)\rangle) - R(\langle w', (\Phi(Y_V(v_1, z_1 - z_2)v_2))(z_2)\rangle) \\ &\quad + R(\langle w', e^{z_1 L(-1)} Y_W(v_2, -z_1 + z_2)(\Phi(v_1))(0)\rangle) \\ &= 0. \end{aligned}$$

Let $z_2 = 0$. We obtain

$$\begin{aligned} &R(\langle w', Y_W(v_1, z_1)(\Phi(v_2))(0)\rangle) - R(\langle w', (\Phi(Y_V(v_1, z_1)v_2))(0)\rangle) \\ &\quad + R(\langle w', e^{z_1 L(-1)} Y_W(v_2, -z_1)(\Phi(v_1))(0)\rangle) \\ &= 0. \end{aligned}$$

Since w' is arbitrary, we obtain

$$\begin{aligned} &(\Phi(Y_V(v_1, z_1)v_2))(0) \\ &\quad = e^{z_1 L(-1)} Y_W(v_2, -z_1)(\Phi(v_1))(0) + Y_W(v_1, z_1)(\Phi(v_2))(0) \\ &\quad = Y^W_{WV}((\Phi(v_1))(0), z_1)(\Phi(v_2))(0) + Y_W(v_1, z_1)(\Phi(v_2))(0) \end{aligned}$$

for $v_1, v_2 \in V$. This means that $(\Phi(\cdot))(0) : V \to W$ is a derivation from V to W. Note that $\delta^0_m(C^0_m(V, W)) = 0$. So we obtain a linear map from $H^1(V, W)$ to the space of derivations from V to W.

Conversely, given any derivation f from V to W, let $\Phi_f : V \to \widetilde{W}_{z_1}$ be given by

$$(\Phi_f(v))(z_1) = f(Y_V(v, z_1)\mathbf{1}) = Y^W_{WV}(f(v), z_1)\mathbf{1}$$

for $v \in V$, where we have used Lemma 2.1. By Theorem 5.6.2 in [FHL], the map from V to $\widetilde{W}_{z_1}$ given by $v \mapsto Y^W_{WV}((\Phi(v))(0), z_1)\mathbf{1}$ is composable with m vertex operators for any $m \in \mathbb{N}$. Thus $\Phi_f \in C^1_m(V, W)$ for any $m \in \mathbb{N}$. For $v_1, v_2 \in V$ and $w' \in W'$,

$$\begin{aligned}
&((\delta^1_m \Phi_f)(v_1 \otimes v_2))(z_1, z_2)\\
&\quad = R(\langle w', Y_W(v_1, z_1)Y^W_{WV}(f(v_2), z_2)\mathbf{1}\rangle)\\
&\qquad -R(\langle w', Y^W_{WV}(f(Y_V(v_1, z_1 - z_2)v_2), z_2)\mathbf{1})\rangle)\\
&\qquad +R(\langle w', Y_W(v_2, z_2)Y^W_{WV}(f(v_1), z_1)\mathbf{1}\rangle)\\
&\quad = R(\langle w', Y_W(v_1, z_1)Y^W_{WV}(f(v_2), z_2)\mathbf{1}\rangle)\\
&\qquad -R(\langle w', Y^W_{WV}(Y^W_{WV}(f(v_1), z_1 - z_2)v_2), z_2)\mathbf{1})\rangle)\\
&\qquad -R(\langle w', Y^W_{WV}(Y_W(v_1, z_1 - z_2)f(v_2), z_2)\mathbf{1})\rangle)\\
&\qquad +R(\langle w', Y_W(v_2, z_2)Y^W_{WV}(f(v_1), z_1)\mathbf{1}\rangle)\\
&\quad = R(\langle w', Y_W(v_1, z_1)Y^W_{WV}(f(v_2), z_2)\mathbf{1}\rangle)\\
&\qquad -R(\langle w', e^{z_2 L_W(-1)}Y^W_{WV}(f(v_1), z_1 - z_2)v_2\rangle)\\
&\qquad -R(\langle w', e^{z_2 L_W(-1)}Y_W(v_1, z_1 - z_2)f(v_2)\rangle)\\
&\qquad +R(\langle w', Y_W(v_2, z_2)Y^W_{WV}(f(v_1), z_1)\mathbf{1}\rangle)\\
&\quad = R(\langle w', Y_W(v_1, z_1)Y^W_{WV}(f(v_2), z_2)\mathbf{1}\rangle)\\
&\qquad -R(\langle w', Y^W_{WV}(f(v_1), z_1)e^{z_2 L_V(-1)}v_2\rangle)\\
&\qquad -R(\langle w', Y_W(v_1, z_1)e^{z_2 L_W(-1)}f(v_2)\rangle)\\
&\qquad +R(\langle w', Y_W(v_2, z_2)Y^W_{WV}(f(v_1), z_1)\mathbf{1}\rangle)\\
&\quad = R(\langle w', Y_W(v_1, z_1)Y^W_{WV}(f(v_2), z_2)\mathbf{1}\rangle)\\
&\qquad -R(\langle w', Y^W_{WV}(f(v_1), z_1)Y_W(v_2, z_2)\mathbf{1}\rangle)\\
&\qquad -R(\langle w', Y_W(v_1, z_1)Y^W_{WV}(f(v_2), z_2)\mathbf{1}\rangle)\\
&\qquad +R(\langle w', Y_W(v_2, z_2)Y^W_{WV}(f(v_1), z_1)\mathbf{1}\rangle)\\
&\quad = -R(\langle w', Y^W_{WV}(f(v_1), z_1)Y_V(v_2, z_2)\mathbf{1}\rangle)\\
&\qquad +R(\langle w', Y_W(v_2, z_2)Y^W_{WV}(f(v_1), z_1)\mathbf{1}\rangle).
\end{aligned}$$

(2.1)

From Theorem 5.6.2 in [FHL], we know that the right-hand side of (2.1) is 0. So we obtain a linear map $f \mapsto \Phi_f$ from the space Der (V, W) to $H^1_m(V, W) = C^1_m(V, W)$.

Clearly these two maps are inverse to each other and thus Der (V, W) and $H^1_m(V, W)$ are isomorphic. ∎

3 Second cohomologies and square-zero extensions

In this section, we prove Theorem 1.4.

Let (Λ, f, g) be a square-zero extension of V by W. Then there is an injective linear map $\Gamma : V \to \Lambda$ such that the linear map $h : V \oplus W \to \Lambda$ given by $h(v, w) = \Gamma(v) + g(w)$ is a linear isomorphism. By definition, the restriction of h to W is the isomorphism g from W to $\ker f$. Then the grading-restricted vertex algebra structure and the V-module structure on Λ give a grading-restricted vertex algebra structure and a V-module structure on $V \oplus W$ such that the embedding $i_2 : W \to V \oplus W$ and the projection $p_1 : V \oplus W \to V$ are homomorphisms of grading-restricted vertex algebras. Moreover, $\ker p_1$ is a square-zero ideal of $V \oplus W$, i_2 is an injective homomorphism such that $i_2(W) = \ker p_1$ and the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \xrightarrow{i_2} & V \oplus W & \xrightarrow{p_1} & V & \longrightarrow & 0 \\ & & {\scriptstyle 1_W}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle 1_V} & & \\ 0 & \longrightarrow & W & \xrightarrow[g]{} & \Lambda & \xrightarrow[f]{} & V & \longrightarrow & 0 \end{array} \tag{3.1}$$

of V-modules is commutative. So we obtain a square-zero extension $(V \oplus W, p_1, i_2)$ equivalent to (Λ, f, g). We need only consider square-zero extension of V by W of the particular form $(V \oplus W, p_1, i_2)$. Note that the difference between two such square-zero extensions are in the vertex operator maps. So we use $(V \oplus W, Y_{V \oplus W}, p_1, i_2)$ to denote such a square-zero extension.

We now write down the vertex operator map for $V \oplus W$ explicitly. Since $(V \oplus W, Y_{V \oplus W}, p_1, i_2)$ is a square-zero extension of V, there exists $\Psi(u, x)v \in$

$W((x))$ for $u, v \in V$ such that

$$\begin{array}{rcl} Y_{V\oplus W}((v_1, 0), x)(v_2, 0) & = & (Y_V(v_1, x)v_2, \Psi(v_1, x)v_2), \\ Y_{V\oplus W}((v_1, 0), x)(0, w) & = & (0, Y_V(v_1, x)w_2), \\ Y_{V\oplus W}((0, w_1), x)(v_2, 0) & = & (0, Y_{WV}^W(w, x)v_2), \\ Y_{V\oplus W}((0, w_1), x)(0, w_2) & = & 0 \end{array}$$

for $v_1, v_2 \in V$ and $w_1, w_2 \in W$. Thus we have

$$\begin{aligned} & Y_{V\oplus W}((v_1, w_1), x)(v_2, w_2) \\ & \quad = (Y_V(v_1, x)v_2, Y_W(v_1, x)w_2 + Y_{WV}^W(w_1, x)v_2 + \Psi(v_1, x)v_2) \end{aligned} \tag{3.2}$$

for $v_1, v_2 \in V$ and $w_1, w_2 \in W$.

The vacuum of $V \oplus W$ is $(\mathbf{1}, 0)$. Since

$$\begin{array}{rcl} Y_{V\oplus W}((v, w), x)(\mathbf{1}, 0) & = & e^{xL_{V\oplus W}(-1)}(v, w) \\ & = & (e^{xL_V(-1)}v, e^{xL_W(-1)}w) \\ & = & (Y_V(v, x)\mathbf{1}, Y_{WV}^W(w, x)\mathbf{1}) \end{array}$$

for $v \in W$ and $w \in W$, we have

$$\Psi(v, x)\mathbf{1} = 0 \tag{3.3}$$

for $v \in V$.

We identify $(V \oplus W)'$ with $V' \oplus W'$. For $v_1, v_2 \in V$ and $w' \in W'$,

$$\begin{aligned} & \langle (0, w'), Y_{V\oplus W}((v_1, 0), z_1)Y_{V\oplus W}((v_2, 0), z_2)(\mathbf{1}, 0)\rangle \\ & \quad = \langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1} + Y_W(v_1, z_1)\Psi(v_2, z_2)\mathbf{1}\rangle \\ & \quad = \langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}\rangle, \\ & \langle (0, w'), Y_{V\oplus W}((v_2, 0), z_2)Y_{V\oplus W}((v_1, 0), z_1)(\mathbf{1}, 0)\rangle \\ & \quad = \langle w', \Psi(v_2, z_2)Y_V(v_1, z_1)\mathbf{1} + Y_W(v_2, z_2)\Psi(v_1, z_1)\mathbf{1}\rangle \\ & \quad = \langle w', \Psi(v_2, z_2)Y_V(v_1, z_1)\mathbf{1}\rangle, \\ & \langle (0, w'), Y_{V\oplus W}(Y_{V\oplus W}((v_1, 0), z_1 - z_2)(v_2, 0), z_2)(\mathbf{1}, 0)\rangle \\ & \quad = \langle w', Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)\mathbf{1} + \Psi(Y_V(v_1, z_1 - z_2)v_2, z_2)\mathbf{1}\rangle \\ & \quad = \langle w', Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)\mathbf{1}\rangle \end{aligned}$$

are absolutely convergent in the region $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to one rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$. Using our notation in [H], we denote this rational function by

$$R(\langle w', \Psi(v_1, z_1) Y_V(v_2, z_2)\mathbf{1}\rangle)$$

or

$$R(\langle w', \Psi(v_2, z_2) Y_V(v_1, z_1)\mathbf{1}\rangle)$$

or

$$R(\langle w', Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)\mathbf{1}\rangle).$$

Then we obtain an element, denoted by

$$E(\Psi(v_1, z_1) Y_V(v_2, z_2)\mathbf{1})$$

or

$$E(\Psi(v_2, z_2) Y_V(v_1, z_1)\mathbf{1})$$

or

$$E(Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)\mathbf{1}),$$

of $\widetilde{W}_{z_1,z_2}$ given by

$$\langle w', E(\Psi(v_1, z_1) Y_V(v_2, z_2)\mathbf{1})\rangle = R(\langle w', \Psi(v_1, z_1) Y_V(v_2, z_2)\mathbf{1}\rangle)$$

or

$$\langle w', E(\Psi(v_2, z_2) Y_V(v_1, z_1)\mathbf{1})\rangle = R(\langle w', \Psi(v_2, z_2) Y_V(v_1, z_1)\mathbf{1}\rangle)$$

or

$$\langle w', E(Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)\mathbf{1})\rangle = R(\langle w', Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)\mathbf{1}\rangle).$$

By definition, we have

$$\begin{aligned} E(\Psi(v_1, z_1) Y_V(v_2, z_2)\mathbf{1}) &= E(\Psi(v_2, z_2) Y_V(v_1, z_1)\mathbf{1}) \\ &= E(Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)\mathbf{1}) \end{aligned}$$

for $v_1, v_2 \in V$.

Let

$$\Phi : V \otimes V \to \widetilde{W}_{z_1,z_2}$$

be the linear map given by

$$\begin{aligned}(\Phi(v_1\otimes v_2))(z_1,z_2) &= E(\Psi(v_1,z_1)Y_V(v_2,z_2)\mathbf{1})\\ &= E(\Psi(v_2,z_2)Y_V(v_1,z_1)\mathbf{1})\\ &= E(Y_{WV}^W(\Psi(v_1,z_1-z_2)v_2,z_2)\mathbf{1}) \qquad (3.4)\end{aligned}$$

for $v_1,v_2\in V$ and $(z_1,z_2)\in F_2\mathbb{C}$. We first prove that $\Phi\in\widehat{C}^2_{\frac{1}{2}}(V,W)$.

By the $L(-1)$-derivative property and the $L(0)$-bracket formula for $V\oplus W$, we have

$$\begin{aligned}\frac{d}{dx}Y_{V\oplus W}((v,0),x) &= Y_{V\oplus W}((L_V(-1)v,0),x) \qquad (3.5)\\ &= [L_{V+W}(-1),Y_{V\oplus W}((v,0),x)], \qquad (3.6)\\ [L_{V+W}(0),Y_{V\oplus W}((v,0),x)] &= Y_{V\oplus W}((L_V(0)v,0),x)+x\frac{d}{dx}Y_{V\oplus W}((v,0),x) \qquad (3.7)\end{aligned}$$

for $v\in V$. By (3.5), (3.6), (3.7), (3.2) and the $L(-1)$-derivative property and the $L(0)$-bracket formula for V, we obtain

$$\begin{aligned}\frac{d}{dx}\Psi(v,x) &= \Psi(L_V(-1)v,x) \qquad (3.8)\\ &= L_W(-1)\Psi(v,x)-\Psi(v,x)L_V(-1), \qquad (3.9)\\ L_W(0)\Psi(v,x)-\Psi(v,x)L_V(0) &= \Psi(L_V(0)v,x)+x\frac{d}{dx}\Psi(v,x) \qquad (3.10)\end{aligned}$$

for $v\in V$. From (3.10), we obtain

$$z^{L_W(0)}\Psi(v,x)=\Psi(z^{L_V(0)}v,zx)z^{L_V(0)} \qquad (3.11)$$

for $v\in V$.

For $v_1,v_2\in V$ and $w'\in W'$, by (3.8) and the $L(-1)$-derivative property for V, we obtain

$$\begin{aligned}&\frac{\partial}{\partial z_1}\langle w',(\Phi(v_1\otimes v_2))(z_1,z_2)\rangle\\ &\quad=\frac{\partial}{\partial z_1}\langle w',E(\Psi(v_1,z_1)Y_V(v_2,z_2)\mathbf{1})\rangle\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial z_1} R(\langle w', \Psi(v_1, z_1) Y_V(v_2, z_2)\mathbf{1}\rangle) \\
&= R\left(\left\langle w', \frac{\partial}{\partial z_1}\Psi(v_1, z_1) Y_V(v_2, z_2)\mathbf{1}\right\rangle\right) \\
&= R(\langle w', \Psi(L_V(-1)v_1, z_1) Y_V(v_2, z_2)\mathbf{1}\rangle) \\
&= \langle w', E(\Psi(L_V(-1)v_1, z_1) Y_V(v_2, z_2)\mathbf{1})\rangle \\
&= \langle w', (\Phi(L_V(-1)v_1 \otimes v_2))(z_1, z_2)\rangle
\end{aligned} \tag{3.12}
$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial z_2}\langle w', (\Phi(v_1 \otimes v_2))(z_1, z_2)\rangle \\
&\quad= \frac{\partial}{\partial z_2}\langle w', E(\Psi(v_1, z_1) Y_V(v_2, z_2)\mathbf{1})\rangle \\
&\quad= \frac{\partial}{\partial z_2} R(\langle w', \Psi(v_1, z_1) Y_V(v_2, z_2)\mathbf{1}\rangle) \\
&\quad= R\left(\left\langle w', \Psi(v_1, z_1)\frac{\partial}{\partial z_2} Y_V(v_2, z_2)\mathbf{1}\right\rangle\right) \\
&\quad= R(\langle w', \Psi(v_1, z_1) Y_V(L_V(-1)v_2, z_2)\mathbf{1}\rangle) \\
&\quad= \langle w', E(\Psi(v_1, z_1) Y_V(L_V(-1)v_2, z_2)\mathbf{1})\rangle \\
&\quad= \langle w', (\Phi(v_1 \otimes L_V(-1)v_2))(z_1, z_2)\rangle.
\end{aligned} \tag{3.13}
$$

Using (3.8), (3.9) and the $L(-1)$-derivative property for V, we obtain

$$
\begin{aligned}
&\left(\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_2}\right)\langle w', (\Phi(v_1 \otimes v_2))(z_1, z_2)\rangle \\
&\quad= \left(\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_2}\right)\langle w', E(\Psi(v_1, z_1) Y_V(v_2, z_2)\mathbf{1})\rangle \\
&\quad= \left(\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_2}\right) R(\langle w', \Psi(v_1, z_1) Y_V(v_2, z_2)\mathbf{1})\rangle) \\
&\quad= R\left(\left\langle w', \frac{\partial}{\partial z_1}\Psi(v_1, z_1) Y_V(v_2, z_2)\mathbf{1}\right\rangle\right) \\
&\qquad+ R\left(\left\langle w', \Psi(v_1, z_1)\frac{\partial}{\partial z_2} Y_V(v_2, z_2)\mathbf{1}\right\rangle\right) \\
&\quad= R(\langle w', \Psi(L_V(-1)v_1, z_1) Y_V(v_2, z_2)\mathbf{1}\rangle) \\
&\qquad+ R(\langle w', \Psi(v_1, z_1) Y_V(L_V(-1)v_2, z_2)\mathbf{1}\rangle)
\end{aligned}
$$

$$
\begin{aligned}
&= R(\langle w', L_W(-1)\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}\rangle)\\
&= R(\langle L_{W'}(1)w', \Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}\rangle)\\
&= \langle L_{W'}(1)w', E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1})\rangle\\
&= \langle w', L_W(-1)E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1})\rangle\\
&= \langle w', L_W(-1)(\Phi(v_1 \otimes v_2))(z_1, z_2)\rangle
\end{aligned} \tag{3.14}
$$

for $v_1, v_2 \in V$ and $w' \in W'$, From (3.12), (3.13) and (3.14), we see that Φ satisfies the $L(-1)$-derivative property.

Also for $v_1, v_2 \in V$ and $w' \in W'$, by (3.11) and the $L(0)$-bracket formula for V, we have

$$
\begin{aligned}
&\langle w', z^{L_W(0)}(\Phi(v_1 \otimes v_2))(z_1, z_2)\rangle\\
&\quad = \langle w', z^{L_W(0)}E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1})\rangle\\
&\quad = \langle z^{L_{W'}(0)}w', E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1})\rangle\\
&\quad = R(\langle z^{L_{W'}(0)}w', \Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}\rangle)\\
&\quad = R(\langle w', z^{L_W(0)}\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}\rangle)\\
&\quad = R(\langle w', \Psi(z^{L_V(0)}v_1, zz_1)Y_V(z^{L_V(0)}v_2, zz_2)\mathbf{1}\rangle)\\
&\quad = \langle w', E(\Psi(z^{L_V(0)}v_1, zz_1)Y_V(z^{L_V(0)}v_2, zz_2)\mathbf{1})\rangle\\
&\quad = \langle w', (\Phi(z^{L_V(0)}v_1 \otimes z^{L_V(0)}v_2))(zz_1, zz_2)\rangle,
\end{aligned}
$$

that is, Φ satisfies the $L(0)$-conjugation property.

Since $V \oplus W$ is a grading-restricted vertex algebra, for $v_1, v_2, v_3 \in V$ and $w' \in W'$, the series

$$
\langle (0, w'), Y_{V\oplus W}((v_1, 0), z_1)Y_{V\oplus W}((v_2, 0), z_2)Y_{V\oplus W}((v_3, 0), z_3)(\mathbf{1}, 0)\rangle
$$

and

$$
\langle (0, w'), Y_{V\oplus W}(Y_{V\oplus W}((v_1, 0), z_1 - z_2)(v_2, 0), z_2)Y_{V\oplus W}((v_3, 0), z_3)(\mathbf{1}, 0)\rangle
$$

are absolutely convergent in the regions given by $|z_1| > |z_2| > |z_3| > 0$ and by $|z_2| > |z_1 - z_2|, |z_3| > 0$ and $|z_2 - z_3| > |z_1 - z_2|$, respectively, to a same rational function with the only possible poles at $z_1 = z_2$, $z_1 = z_3$, $z_2 = z_3$. But by (3.2) and (3.3), these series are equal to

$$
\langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle + \langle w', Y_W(v_1, z_1)\Psi(v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle
$$

and

$$\langle w', \Psi(Y_V(v_1, z_1 - z_2)v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle \\ +\langle w', Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle,$$

respectively, and are absolutely convergent to a same rational function which in our convention is equal to

$$R(\langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle + \langle w', Y_W(v_1, z_1)\Psi(v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle)$$

and

$$R(\langle w', \Psi(Y_V(v_1, z_1 - z_2)v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle \\ +\langle w', Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle).$$

In particular, we have proved that $\Phi \in \widehat{C}^2_{\frac{1}{2}}(V, W)$.

Since by (3.4),

$$\begin{aligned}(\Phi(v_1 \otimes v_2))(z_1, z_2) &= E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}) \\ &= E(\Psi(v_2, z_2)Y_V(v_1, z_1)\mathbf{1}) \\ &= (\Phi(v_2 \otimes v_1))(z_2, z_1) \\ &= (\sigma_{12}(\Phi(v_2 \otimes v_1)))(z_1, z_2)\end{aligned}$$

for $v_1, v_2 \in V$ and $(z_1, z_2) \in F_2\mathbb{C}$, that is,

$$\Phi(v_1 \otimes v_2) - \sigma_{12}(\Phi(v_2 \otimes v_1)) = 0$$

for $v_1, v_2 \in V$, We obtain

$$\begin{aligned}&\sum_{\sigma \in J_{2;1}} (-1)^{|\sigma|}\sigma(\Phi(v_1 \otimes v_2)) \\ &\quad = \Phi(v_1 \otimes v_2) - \sigma_{12}(\Phi(v_2 \otimes v_1)) \\ &\quad = 0\end{aligned}$$

for $v_1, v_2 \in V$. So $\Phi \in C^2(V, W)$.

Next we show that $\delta^2_{\frac{1}{2}}(\Phi) = 0$. For $v_1, v_2, v_3 \in V$, $w' \in W'$,

$$\langle w', ((\delta^2_{\frac{1}{2}}(\Phi))(v_1 \otimes v_2 \otimes v_3))(z_1, z_2, z_3)\rangle$$

$$
\begin{aligned}
&= R(\langle w', (E_W^{(1)}(v_1; \Phi(v_2 \otimes v_3)))(z_1, z_2, z_3)\rangle \\
&\qquad + \langle w', (\Phi(v_1 \otimes E^{(2)}(v_2 \otimes v_3; \mathbf{1})))(z_1, z_2, z_3)\rangle) \\
&\quad - R(\langle w', (\Phi(E^{(2)}(v_1 \otimes v_2; \mathbf{1}) \otimes v_3))(z_1, z_2, z_3)\rangle \\
&\qquad + \langle w', (E_{WV}^{W;(1)}(\Phi(v_1 \otimes v_2); v_3))(z_1, z_2, z_3)\rangle) \\
&= R(\langle w', Y_W(v_1, z_1)\Psi(v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle \\
&\qquad + \langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle) \\
&\quad - R(\langle w', \Psi(Y_V(v_1, z_1 - z_2)v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle \\
&\qquad + \langle w', Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle). \qquad (3.15)
\end{aligned}
$$

Since $V \oplus W$ is a grading-restricted vertex algebra, we have the associativity property

$$
\begin{aligned}
&R(\langle (0, w'), Y_{V\oplus W}((v_1, 0), z_1)Y_{V\oplus W}((v_2, 0), z_2)Y_{V\oplus W}((v_3, 0), z_3)(\mathbf{1}, 0)\rangle) \\
&\quad = R(\langle (0, w'), Y_{V\oplus W}(Y_{V\oplus W}((v_1, 0), z_1 - z_2)(v_2, 0), z_2) \cdot \\
&\qquad\qquad \cdot Y_{V\oplus W}((v_3, 0), z_3)(\mathbf{1}, 0)\rangle),
\end{aligned}
$$

which, by (3.2) and (3.3), is equivalent to

$$
\begin{aligned}
&R(\langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle \\
&\qquad + \langle w', Y_W(v_1, z_1)\Psi(v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle) \\
&\quad = R(\langle w', \Psi(Y_V(v_1, z_1 - z_2)v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle \\
&\qquad + \langle w', Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle),
\end{aligned}
$$

as we have noticed above. So the right-hand side of (3.15) is 0. Thus $\Phi + \delta_2^1 C_2^1(V, W)$ is an element of $H_{\frac{1}{2}}^2(V, W)$.

Conversely, given any element of $H_{\frac{1}{2}}^2(V, W)$, let $\Phi \in C_{\frac{1}{2}}^2(V, W)$ be a representative of this element. Then for any $v_1, v_2 \in V$, there exists N such that for $w' \in W'$, $\langle w', (\Phi(v_1 \otimes v_2))(z, 0)\rangle$ is a rational function of z with the only possible pole at $z = 0$ of order less than or equal to N. For $v_1, v_2 \in V$, let $\Psi(v_1, x)v_2 \in W((x))$ be given by

$$
\langle w', \Psi(v_1, x)v_2\rangle|_{x=z} = \langle w', (\Phi(v_1 \otimes v_2))(z, 0)\rangle.
$$

for $z \in \mathbb{C}^\times$. For $v_1, v_2 \in V$, define $Y_{V\oplus W}(v_1, x)v_2$ using (3.2). So we obtain a vertex operator map $Y_{V\oplus W}$. Reversing the proof above, we see that $V \oplus W$ equipped with the vertex operator map Y_{V+W} and the vacuum $(\mathbf{1}, 0)$ is a grading-restricted vertex algebra and together with the projection $p_1 : V \oplus W \to V$ and the embedding $i_2 : W \to V \oplus W$, $V \oplus W$ is a square-zero extension of V by W.

Next we prove that two elements of $\ker \delta^2_{\frac{1}{2}}$ obtained this way differ by an element of $\delta_1 C^1(V, W)$ if and only if the corresponding square-zero extensions of V by W are equivalent.

Let $\Phi_1, \Phi_2 \in \ker \delta^2_{\frac{1}{2}}$ be two such elements obtained from square-zero extensions $(V \oplus W, Y^{(1)}_{V\oplus W}, p_1, i_2)$ and $(V \oplus W, Y^{(2)}_{V\oplus W}, p_1, i_2)$. Assume that $\Phi_1 = \Phi_2 + \delta_1(\Gamma)$ where $\Gamma \in C^1(V, W)$. Since

$$\begin{aligned}
&\langle w', ((\delta_1(\Gamma))(v_1 \otimes v_2))(z_1, z_2)\rangle\\
&\quad = R(\langle w', Y_W(v_1, z_1)(\Gamma(v_2))(z_2)\rangle)\\
&\quad\quad -R(\langle w', (\Gamma(Y_V(v_1, z_1 - z_2)v_2))(z_2)\rangle)\\
&\quad\quad +R(\langle w', Y_W(v_2, z_2)(\Gamma(v_1))(z_1)\rangle),
\end{aligned}$$

we have

$$\begin{aligned}
&R(\langle w', \Psi_1(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}\rangle)\\
&\quad = \langle w', (\Phi_1(v_1 \otimes v_2))(z_1, z_2)\rangle\\
&\quad = \langle w', (\Phi_2(v_1 \otimes v_2))(z_1, z_2)\rangle\\
&\quad\quad +\langle w', (\delta_1(\Gamma))(z_1, z_2)\rangle\\
&\quad = R(\langle w', \Psi_2(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}\rangle\\
&\quad\quad +R(\langle w', Y_W(v_1, z_1)(\Gamma(v_2))(z_2)\rangle)\\
&\quad\quad -R(\langle w', (\Gamma(Y_V(v_1, z_1 - z_2)v_2))(z_2)\rangle)\\
&\quad\quad +R(\langle w', Y_W(v_2, z_2)(\Gamma(v_1))(z_1)\rangle)\\
&\quad = R(\langle w', \Psi_2(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}\rangle\\
&\quad\quad +R(\langle w', Y_W(v_1, z_1)(\Gamma(v_2))(z_2)\rangle)\\
&\quad\quad -R(\langle w', (\Gamma(Y_V(v_1, z_1 - z_2)v_2))(z_2)\rangle)\\
&\quad\quad +R(\langle w', e^{(z_1-z_2)L_W(-1)}Y_W(v_2, -z_1)(\Gamma(v_1))(z_2)\rangle). \qquad (3.16)
\end{aligned}$$

Let z_2 go to zero on both sides of (3.16). We obtain

$$\begin{aligned}
\langle w', \Psi_1(v_1, z_1)v_2\rangle &= \langle w', \Psi_2(v_1, z_1)v_2\rangle\\
&\quad +\langle w', Y_W(v_1, z_1)(\Gamma(v_2))(0)\rangle\\
&\quad -\langle w', (\Gamma(Y_V(v_1, z_1)v_2))(0)\rangle
\end{aligned}$$

$$\begin{aligned} & +\langle w', e^{z_1 L_W(-1)} Y_W(v_2, -z_1)(\Gamma(v_1))(0)\rangle \\ = & \ \langle w', \Psi_2(v_1, z_1)v_2\rangle \\ & +\langle w', Y_W(v_1, z_1)(\Gamma(v_2))(0)\rangle \\ & -\langle w', (\Gamma(Y_V(v_1, z_1)v_2))(0)\rangle \\ & +\langle w', Y_{WV}^W((\Gamma(v_1))(0), z_1)v_2\rangle. \end{aligned}$$

Then

$$\begin{aligned} \Psi_1(v_1, x)v_2 \ = \ & \Psi_2(v_1, x)v_2 + Y_W(v_1, x)(\Gamma(v_2))(0) \\ & -(\Gamma(Y_V(v_1, x)v_2))(0) + Y_{WV}^W((\Gamma(v_1))(0), x)v_2. \end{aligned} \tag{3.17}$$

For $v_1, v_2 \in V$ and $w_1, w_2 \in W$, by (3.2) and (3.17), we have

$$\begin{aligned} & Y^{(1)}_{V\oplus W}((v_1, w_1), x)(v_2, w_2) \\ & \quad = (Y_V(v_1, x)v_2, Y_W(v_1, x)w_2 + Y_{WV}^W(w_1, x)v_2 + \Psi_1(v_1, x)v_2) \\ & \quad = (Y_V(v_1, x)v_2, Y_W(v_1, x)w_2 + Y_{WV}^W(w_1, x)v_2 + \Psi_2(v_1, x)v_2) \\ & \qquad +(Y_V(v_1, x)v_2, Y_W(v_1, x)(\Gamma(v_2))(0)) \\ & \qquad -(Y_V(v_1, x)v_2, (\Gamma(Y_V(v_1, x)v_2))(0)) \\ & \qquad +(Y_V(v_1, x)v_2, Y_{WV}^W((\Gamma(v_1))(0), x)v_2) \\ & \quad = Y^{(2)}_{V\oplus W}((v_1, w_1 + (\Gamma(v_1))(0)), x)(v_2, w_2 + (\Gamma(v_2))(0)) \\ & \qquad -(Y_V(v_1, x)v_2, (\Gamma(Y_V(v_1, x)v_2))(0)). \end{aligned} \tag{3.18}$$

We now define a linear map $h : V \oplus W \to V \oplus W$ by

$$e(v, w) = (v, w + (\Gamma(v))(0))$$

for $v \in V$ and $w \in W$. Then h is a linear isomorphism and (3.18) can be rewritten as

$$h(Y^{(1)}_{V\oplus W}((v_1, w_1), x)(v_2, w_2)) = Y^{(2)}_{V\oplus W}(h(v_1, w_1), x)h(v_2, w_2). \tag{3.19}$$

for $v_1, v_2 \in V$ and $w_1, w_2 \in W$. Thus h is an isomorphism of grading-restricted vertex algebras from $(V\oplus W, Y^{(1)}_{V\oplus W}, (\mathbf{1}, 0))$ to $(V\oplus W, Y^{(2)}_{V\oplus W}, (\mathbf{1}, 0))$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \xrightarrow{i_2} & V\oplus W & \xrightarrow{p_1} & V & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_W} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle 1_V} & & \\ 0 & \longrightarrow & W & \xrightarrow{i_2} & V\oplus W & \xrightarrow{p_1} & V & \longrightarrow & 0 \end{array} \tag{3.20}$$

is commutative. Thus the two square-zero extensions of V by W are equivalent.

Conversely, let $(V \oplus W, Y^{(1)}_{V+W}, p_1, i_2)$ and $(V \oplus W, Y^{(2)}_{V+W}, p_1, i_2)$ be two equivalent square-zero extensions of V by W. So there exists an isomorphism $h : V \oplus W \to V \oplus W$ of grading-restricted vertex algebras such that (3.20) is commutative. We have the following lemma which is also needed in the next section:

Lemma 3.1. *There exists a linear map $g : V \to V$ such that*

$$h(v, w) = (v, w + g(v))$$

for $v \in V$ and $w \in W$.

Proof. Let $h(v, w) = (f(v, w), g(v, w))$ for $v \in V$ and $w \in W$. Then by (3.20), we have $f(v, w) = v$ and $g(0, w) = w$. Since h is linear, we have $g(v, w) = g(v, 0) + g(0, w) = w + g(v, 0)$. So $h(v, w) = (v, w + g(v, 0))$. Taking $g(v)$ to be $g(v, 0)$, we see that the conclusion holds. ∎

Let $(\Gamma(v))(z_1) = e^{z_1 L_W(-1)} g(v) \in \overline{W}$. Then $\Gamma : V \to \widetilde{W}_{z_1}$ is an element of $C^1_2(V, W)$. By definition, we have $g(v) = (\Gamma(v))(0)$ and $h(v, w) = (v, w + (\Gamma(v))(0))$ for $v \in V$ and $w \in W$. Let Φ_1 and Φ_2 be elements of $\ker \delta^2_{\frac{1}{2}}$ obtained from $(V \oplus W, Y^{(1)}_{V \oplus W}, p_1, i_2)$ and $(V \oplus W, Y^{(2)}_{V+W}, p_1, i_2)$, respectively, and Ψ_1 and Ψ_2 the corresponding maps from $V \otimes V$ to $W((x))$. Then since h is a homomorphism of grading-restricted vertex algebras, (3.19) holds for $v_1, v_2 \in V$ and $w_1, w_2 \in W$. Thus the two sides of (3.18) are equal for $v_1, v_2 \in V$ and $w_1, w_2 \in W$. So the two expressions in the middle of (3.18) are equal for $v_1, v_2 \in V$ and $w_1, w_2 \in W$. Thus we have (3.17) for $v_1, v_2 \in V$. Formula (3.17) implies that the two sides of (3.16) are equal for $v_1, v_2 \in V$. Thus the middle expressions in (3.16) are all equal for $v_1, v_2 \in V$. In particular, we obtain $\Phi_1 = \Phi_2 + \Gamma$. So Φ_1 and Φ_2 differ by an element of $\delta_1 C^1(V, W)$. ∎

4 Square-zero extensions and first order deformations

In this section, we prove Theorem 1.7.

Let $Y_t : V \otimes V \to V((x))$, $t \in U$, be a first order deformation of V. By definition, there exists

$$\begin{aligned} \Psi : V \otimes V &\to V((x)) \\ v_1 \otimes v_2 &\to \Psi(v_1, x)v_2 \end{aligned}$$

such that

$$Y_t(v_1, x)v_2 = Y_V(v_1, x)v_2 + t\Psi(v_1, x)v_2$$

for $v_1, v_2 \in V$ and $(V, Y_t, \mathbf{1})$ is a family of grading-restricted vertex algebras up to the first order in t.

The identity property for $(V, Y_t, \mathbf{1})$ up to the first order in t gives

$$Y_V(\mathbf{1}, x)v + t\Psi(\mathbf{1}, x)v = v + O(t^2)$$

for $v \in V$. So we obtain

$$\Psi(\mathbf{1}, x)v = 0 \tag{4.1}$$

for $v \in V$. The creation property for $(V, Y_t, \mathbf{1})$ up to the first order in t gives

$$\lim_{x\to 0}(Y_V(v, x) + t\Psi(v, x))\mathbf{1} = v + O(t^2)$$

for $v \in V$. Then we have

$$\lim_{x\to 0} \Psi(v, x)\mathbf{1} = 0 \tag{4.2}$$

for $v \in V$.

The duality property up to the first order in t can be written explicitly as follows: For $v_1, v_2, v_3 \in V$ and $v' \in V'$,

$$\langle v', (Y_V(v_1, z_1)\Psi(v_2, z_2) + \Psi(v_1, z_1)Y_V(v_2, z_2))v_3\rangle \tag{4.3}$$

$$\langle v', (Y_V(v_2, z_2)\Psi(v_1, z_1) + \Psi(v_2, z_2)Y_V(v_1, z_1))v_3\rangle \tag{4.4}$$

$$\langle v', (Y_V(\Psi(v_1, z_1 - z_2)v_2, z_2) + \Psi(Y_V(v_1, z_1 - z_2)v_2, z_2))v_3\rangle \tag{4.5}$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.

Let

$$\begin{aligned} Y_{V\oplus V} : (V \oplus V) \otimes (V \oplus V) &\to (V \oplus V)[[x, x^{-1}]] \\ (u_1, v_1) \otimes (u_2, v_2) &\mapsto Y_{V\oplus V}((u_1, v_1), x)(u_2, v_2) \end{aligned}$$

be given by

$$\begin{aligned}
&Y_{V\oplus V}((u_1, v_1), x)(u_2, v_2)\\
&\quad = (Y_V(u_1, x)u_2, Y_V(u_1, x)v_2 + Y_V(v_1, x)u_2 + \Psi(u_1, x)u_2) \qquad (4.6)
\end{aligned}$$

for $u_1, u_2, v_1, v_2 \in V$. By (4.6) and (4.1),

$$\begin{aligned}
Y_{V\oplus V}((\mathbf{1}, 0), x)(u, v) &= (Y_V(\mathbf{1}, x)u, Y_V(\mathbf{1}, x)v + Y_V(0, x)u + \Psi(\mathbf{1}, x)u)\\
&= (u, v)
\end{aligned}$$

for $u, v \in V$, that is, $(V \oplus V, Y_{V\oplus V}, (\mathbf{1}, 0))$ has the identity property. By (4.6) and (4.2),

$$\begin{aligned}
&\lim_{x\to 0} Y_{V\oplus V}((u, v), x)(\mathbf{1}, 0)\\
&\quad = (\lim_{x\to 0} Y_V(u, x)\mathbf{1}, \lim_{x\to 0} Y_V(u, x)0 + \lim_{x\to 0} Y_V(v, x)\mathbf{1} + \lim_{x\to 0} \Psi(u, x)\mathbf{1})\\
&\quad = (u, v)
\end{aligned}$$

for $u, v \in V$, that is, $(V \oplus V, Y_{V\oplus V}, (\mathbf{1}, 0))$ has the creation property.

By (4.6), we have

$$\begin{aligned}
&\langle (u', v'), Y_{V\oplus V}((u_1, v_1), z_1)Y_{V\oplus V}((u_2, v_2), z_2)(u_3, v_3)\rangle\\
&\quad = \langle (u', v'), Y_{V\oplus V}((u_1, v_1), z_1) \cdot\\
&\qquad\qquad \cdot(Y_V(u_2, z_2)u_3, Y_V(u_2, z_2)v_3 + Y_V(v_2, z_2)u_3 + \Psi(u_2, z_2)u_3)\rangle\\
&\quad = \langle u', Y_V(u_1, z_1)Y_V(u_2, z_2)u_3\rangle + \langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v_3\rangle\\
&\qquad +\langle v', Y_V(u_1, z_1)Y_V(v_2, z_2)u_3\rangle + \langle v', Y_V(u_1, z_1)\Psi(u_2, z_2)u_3\rangle\\
&\qquad +\langle v', Y_V(v_1, z_1)Y_V(u_2, z_2)u_3\rangle + \langle v', \Psi(u_1, z_1)Y_V(u_2, z_2)u_3\rangle. \qquad (4.7)
\end{aligned}$$

By the properties of V and the absolute convergence of (4.3), we see that the left-hand side of (4.7) is absolutely convergent when $|z_1| > |z_2| > 0$. Similarly, by (4.6), we have

$$\begin{aligned}
&\langle (u', v'), Y_{V\oplus V}((u_2, v_2), z_2)Y_{V\oplus V}((u_1, v_1), z_1)(u_3, v_3)\rangle\\
&\quad = \langle u', Y_V(u_2, z_2)Y_V(u_1, z_1)u_3\rangle + \langle v', Y_V(u_2, z_2)Y_V(u_1, z_1)v_3\rangle\\
&\qquad +\langle v', Y_V(u_2, z_2)Y_V(v_1, z_1)u_3\rangle + \langle v', Y_V(u_2, z_2)\Psi(u_1, z_1)u_3\rangle\\
&\qquad +\langle v', Y_V(v_2, z_2)Y_V(u_1, z_1)u_3\rangle + \langle v', \Psi(u_2, z_2)Y_V(u_1, z_1)u_3\rangle \qquad (4.8)
\end{aligned}$$

and the left-hand side of (4.8) is absolutely convergent when $|z_2| > |z_1| > 0$. Moreover, since (4.3) and (4.4) converges absolutely when $|z_1| > |z_2| > 0$

and when $|z_2| > |z_1| > 0$, respectively, to a common rational function with the only possible poles at $z_1, z_2, z_1 - z_2 = 0$, the left-hand side of (4.7) and left-hand side of (4.8) also converges absolutely when $|z_1| > |z_2| > 0$ and when $|z_2| > |z_1| > 0$, respectively, to a common rational function with the only possible pole at $z_1 - z_2 = 0$. By (4.6) again, we have

$$\begin{aligned}
&\langle (u', v'), Y_{V\oplus V}(Y_{V\oplus V}((u_1, v_1), z_1 - z_2)(u_2, v_2), z_2)(u_3, v_3)\rangle\\
&\quad = \langle (u', v'), Y_{V\oplus V}((Y_V(u_1, z_1 - z_2)u_2,\\
&\qquad\qquad Y_V(u_1, z_1 - z_2)v_2 + Y_V(v_1, z_1 - z_2)u_2\\
&\qquad\qquad\qquad +\Psi(u_1, z_1 - z_2)u_2), z_2)(u_3, v_3)\rangle\\
&\quad = \langle u', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)u_3\rangle + \langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v_3\rangle\\
&\qquad +\langle v', Y_V(Y_V(u_1, z_1 - z_2)v_2, z_2)u_3\rangle + \langle v', Y_V(Y_V(v_1, z_1 - z_2)u_2, z_2)u_3\rangle\\
&\qquad +\langle v', Y_V(\Psi(u_1, z_1 - z_2)u_2, z_2)u_3\rangle + \langle v', \Psi(Y_V(u_1, z_1 - z_2)u_2, z_2)u_3\rangle.
\end{aligned} \tag{4.9}$$

By the properties of V and the absolute convergence of (4.5) and (4.9), we see that the left-hand side of (4.9) is absolutely convergent when $|z_2| > |z_1 - z_2| > 0$. Moreover, since (4.3) and (4.5) converges absolutely when $|z_1| > |z_2| > 0$ and when $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function with the only possible poles at $z_1, z_2, z_1 - z_2 = 0$, the left-hand side of (4.7) and left-hand side of (4.9) also converges absolutely when $|z_1| > |z_2| > 0$ and when $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function with the only possible poles at $z_1, z_2, z_1 - z_2 = 0$. So $(V \oplus V, Y_{V\oplus V}, (\mathbf{1}, 0))$ has the duality property.

Note that the $L(-1)$-derivative property is in fact a consequence of the other axioms for vertex algebras. Thus $(V \oplus V, Y_{V\oplus V}, (\mathbf{1}, 0))$ is a grading-restricted vertex algebra.

By definition,

$$\begin{aligned}
&p_1(Y_{V\oplus V}((u_1, v_1), x)(u_2, v_2))\\
&\quad = p_1(Y_V(u_1, x)u_2, Y_V(u_1, x)v_2 + Y_V(v_1, x)u_2 + \Psi(u_1, x)u_2)\\
&\quad = Y_V(u_1, x)u_2\\
&\quad = Y_V(p_1(u_1, v_1), x)p_1(u_2, v_2)
\end{aligned}$$

for $u_1, u_2, v_1, v_2 \in V$. Also

$$\ker p_1 = 0 \oplus V$$

and

$$Y_{V\oplus V}((0, v_1), x)(0, v_2) = (0, 0)$$

for $v_1.v_2 \in V$. So p_1 is a surjective homomorphism of grading-restricted vertex algebras and $\ker p_1$ is a square-zero ideal of $V \oplus V$.

We use $Y_{V\oplus V}^{V}$ to denote the vertex operator map for $V \oplus V$ when $V \oplus V$ is viewed as a V-module. Then by definition,

$$\begin{aligned} i_2(Y_V(v_1, x)v_2) &= (0, Y_V(v_1, x)v_2) \\ &= Y_{V\oplus V}^{V}(v_1, x)(0, v_2) \\ &= Y_{V\oplus V}^{V}(v_1, x)i_2(v_2) \end{aligned}$$

for $v_1, v_2 \in V$. So i_2 is an injective homomorphism of V-modules. Clearly, we have $i_2(V) = \ker p_1$. Thus $(V \oplus V, Y_{V\oplus V}, p_1, i_2)$ is a square-zero extension of V by V.

Conversely, let $(V \oplus V, Y_{V\oplus V}, p_1, i_2)$ be a square-zero extension of V by V. Then there exists

$$\begin{aligned} \Psi : V \otimes V &\to V((x)) \\ v_1 \otimes v_2 &\to \Psi(v_1, x)v_2 \end{aligned}$$

such that

$$Y_{V\oplus V}((u_1, 0), x)(u_2, 0) = (Y_V(u_1, x)u_2, \Psi(u_1, x)u_2)$$

for $u_1, u_2 \in V$. The identity property and the creation property of the grading-restricted vertex algebra $(V \oplus V, Y_{V\oplus V}, (\mathbf{1}, 0))$ give (4.1) and (4.2). The duality property for $(V \oplus V, Y_{V\oplus V}, (\mathbf{1}, 0))$ gives (4.3), (4.4) and (4.5).

For $t \in \mathbb{C}$, define

$$Y_t(v_1, x)v_2 = Y_V(v_1, x)v_2 + t\Psi(v_1, x)v_2$$

for $v_1, v_2 \in V$. Then (4.1) and (4.2) imply that Y_t satisfies the identity property and the creation property up to the first order in t and (4.3), (4.4) and (4.5) imply that Y_t satisfies the duality property up to the first order in t. Thus $(V, Y_t, \mathbf{1})$ is a grading-restricted vertex algebras up to the first order in t, that is, Y_t is a first-order deformation of $(V, Y_V, \mathbf{1})$.

Now we prove that two first-order deformations of V are equivalent if and only if the corresponding square-zero extensions of V by V are equivalent.

Consider two equivalent first-order deformations of V given by $Y_t^{(1)} : V \otimes V \to V((x))$ and $Y_t^{(2)} : V \otimes V \to V((x))$ for $t \in \mathbb{C}$. Then there exist a family $f_t : V \to V$, $t \in \mathbb{C}$, of linear maps of the form $f_t = 1_V + tg$ where $g : V \to V$ is a linear map preserving the grading of V such that (1.1) holds for $v_1, v_2 \in V$. By definition, there exist linear maps

$$\begin{aligned} \Psi_1 : V \otimes V &\to V((x)) \\ v_1 \otimes v_2 &\to \Psi_1(v_1, x)v_2 \end{aligned}$$

and

$$\begin{aligned} \Psi_2 : V \otimes V &\to V((x)) \\ v_1 \otimes v_2 &\to \Psi_2(v_1, x)v_2 \end{aligned}$$

such that $Y_t^{(1)} = Y_V + t\Psi_1$ and $Y_t^{(2)} = Y_V + t\Psi_2$. By (1.1), we have

$$\begin{aligned} &\Psi_1(v_1, x)v_2 - \Psi_2(v_1, x)v_2 \\ &\quad = -g(Y_V(v_1, x)v_2) + Y_V(g(v_1), x)v_2 + Y_V(v_1, x)g(v_2) \end{aligned} \tag{4.10}$$

for $v_1, v_2 \in V$.

Let $(V \oplus V, Y_{V\oplus V}^{(1)}, p_1, i_2)$ and $(V \oplus V, Y_{V\oplus V}^{(2)}, p_1, i_2)$ be the square-zero extensions of V by V constructed from $Y_t^{(1)}$ and $Y_t^{(2)}$. Let $h : V \oplus V \to V \oplus V$ be defined by

$$h(v_1, v_2) = (v_1, v_2 + g(v_1))$$

for $v_1, v_2 \in V$. Clearly, h is a linear isomorphism. For $u_1, u_2, v_1, v_2 \in V$, by definition and (4.10),

$$\begin{aligned} &h(Y_{V\oplus V}^{(1)}((u_1, v_1), x)(u_2, v_2)) \\ &\quad = h(Y_V(u_1, x)u_2, Y_V(u_1, x)v_2 + Y_V(v_1, x)u_2 + \Psi_1(u_1, x)u_2) \\ &\quad = (Y_V(u_1, x)u_2, Y_V(u_1, x)v_2 + Y_V(v_1, x)u_2 \\ &\qquad\qquad +\Psi_1(u_1, x)u_2 + g(Y_V(u_1, x)u_2)) \\ &\quad = (Y_V(u_1, x)u_2, Y_V(u_1, x)v_2 + Y_V(v_1, x)u_2 \\ &\qquad\qquad +\Psi_2(u_1, x)u_2 + Y_V(g(u_1), x)u_2 + Y_V(u_1, x)g(u_2))) \\ &\quad = (Y_V(u_1, x)u_2, Y_V(u_1, x)(v_2 + g(u_2)) \\ &\qquad\qquad +Y_V((v_1 + g(u_1)), x)u_2 + \Psi_2(u_1, x)u_2) \\ &\quad = Y_{V\oplus V}^{(2)}(h(u_1, v_1), x)h(u_2, v_2). \end{aligned}$$

So h is in fact an isomorphism from the algebra $(V\oplus V, Y^{(1)}_{V\oplus V}, (\mathbf{1},0))$ to the algebra $(V\oplus V, Y^{(2)}_{V\oplus V}, (\mathbf{1},0))$. Now it is clear that the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V & \xrightarrow[i_2]{} & V\oplus V & \xrightarrow[p_1]{} & V & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle 1_W} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle 1_V} & & \\
0 & \longrightarrow & V & \xrightarrow[i_2]{} & V\oplus V & \xrightarrow[p_1]{} & V & \longrightarrow & 0,
\end{array}$$

So these two first order deformations are equivalent.

Conversely, let $(V\oplus V, Y^{(1)}_{V\oplus V}, p_1, i_2)$ and $(V\oplus V, Y^{(2)}_{V\oplus V}, p_1, i_2)$ be two equivalent square-zero extensions of V by V. Let $\Psi_1, \Psi_2 : V\otimes V \to V((x))$ be given by

$$\begin{aligned}
Y^{(1)}_{V\oplus V}((u_1,0),x)(u_2,0)) &= (Y_V(u_1,x)u_2, \Psi_1(u_1,x)u_2),\\
Y^{(2)}_{V\oplus V}((u_1,0),x)(u_2,0)) &= (Y_V(u_1,x)u_2, \Psi_2(u_1,x)u_2)
\end{aligned}$$

for $u_1, u_2 \in V$. Then $Y_t^{(1)}, Y_t^{(2)} : V\otimes V \to V((x))$ given by

$$\begin{aligned}
Y_t^{(1)}(v_1,x)v_2 = Y_V(v_1,x)v_2 + t\Psi_1(v_1,x)v_2,\\
Y_t^{(2)}(v_1,x)v_2 = Y_V(v_1,x)v_2 + t\Psi_2(v_1,x)v_2
\end{aligned}$$

for $v_1, v_2 \in V$ are first-order deformations of $(V, Y_V, \mathbf{1})$ by the proof above.

Let $h : V\oplus V \to V\oplus V$ be an equivalence from $(V\oplus V, Y^{(1)}_{V\oplus V}, p_1, i_2)$ to $(V\oplus V, Y^{(2)}_{V\oplus V}, p_1, i_2)$. Then by Lemma 3.1, there exists a linear map $g : V \to V$ such that

$$h(v_1, v_2) = (v_1, v_2 + g(v_1))$$

for $v_1, v_2 \in V$. Using the fact that h is an isomorphism of grading-restricted vertex algebras from $(V\oplus V, Y^{(1)}_{V\oplus V}, (\mathbf{1},0))$ to $(V\oplus V, Y^{(2)}_{V\oplus V}, (\mathbf{1},0))$, we obtain (4.10) which implies (1.1). Thus the two first-order deformations $Y_t^{(1)}$ and $Y_t^{(2)}$ are equivalent. ∎

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