A program to construct and study conformal field theories

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Operator Algebra Seminar at University of Tokyo

Outline

- Conformal field theories
 - A definition
 - Conjectures
 - Problems and a program
- 2 The major problems solved
 - The geometry of vertex operator algebras
 - Intertwining operators and vertex tensor categories
 - Modular invariance
 - Verlinde conjecture and Verlinde formula
 - Rigidity and modularity
 - Full rational conformal field theories
 - Open-closed conformal field theories
- 3 Unsolved problems
 - Higher-genus theories
 - Locally convex completions
 - Non-rational conformal field theories

Part 1 Conformal field theories

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A definition of conformal field theory

- 1987, Kontsevich and G. Segal: Definition of (two-dimensional) conformal field theory.
- 1988, G. Segal: Definitions of modular functor and weakly conformal field theory.
- A conformal field theory in the sense of Kontsevich-Segal is
 - a locally convex topological vector space H,
 - a nondegenerate hermitian form,
 - a projective functor from the category whose morphisms are Riemann surfaces with parametrized boundaries to the category of tensor powers of *H* and traceclass maps,

satisfying additional but natural conditions.

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Conjectures of Verlinde and Moore-Seiberg

- 1987, E. Verlinde:
 - Verlinde conjecture: For a rational conformal field theory, the modular transformaton *S* on the space of vacuum characters associated to $\tau \mapsto -1/\tau$ diagonaizes the matrices formed by fusion rules.
 - Verlinde formula for fusion rules:

$$N^{a_3}_{a_1a_2} = \sum_{a_4} rac{S^{a_4}_{a_1}S^{a_4}_{a_2}S^{a'_3}_{a_4}}{S^{a_4}_{e}}$$

- Obtained Moore-Seiberg polynomial equations using conjectures on operator product expansion and modular invariance for intertwining operators.
- Derived Verlinde conjecture and Verlinde fromula from these polynomial equations.
- Conjectured the existence of modular tensor category.

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Conjectures of Witten

• 1989, Witten:

- Obtained knot and three-manifold invariants from conjectures for rational conformal field theories.
- Conjectured that in the case of Wess-Zumino-Witten models, these invariants should be the same as those from the corresponding Chern-Simons theories.
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Unsolved problems

Problems and a program

The mathematical problems

- **Problem**: Give a constrution of conformal field theories satisfying the axioms of Kontsevich and Segal, or at least prove the existence of such conformal field theories. In particular, give a construction of the Wess-Zumino-Witten models and the minimal models, or at least prove the existence of these theories.
- **Problem**: Prove the conjectures of Verlinde, Moore-Seiberg and Witten.

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- **Problem**: Prove the conjectures of Verlinde, Moore-Seiberg and Witten.

- If there exists a conformal field theory satisfying the definition of Kontsevich and G. Segal, then the space of meromorphic fields form a vertex operator algebra.
- **Question**: Can we construct conformal field theories from vertex operator algebras?
- **Answer**: For rational conformal field theories, the answer is yes, except that we need also modules and intertwining operators for vertex operator algebras and that there are still some conjectures involving higher-genus Riemann surfaces to be proved. We believe that the answer is also yes for non-rational conformal field theories.
- In this talk, I will survey the main results and open problems in this long term program of 25 years.

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Part 2

The major problems solved

Conformal field theories

The major problems solved •••••••• Unsolved problems

The geometry of vertex operator algebras

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- 1986, Borcherds: Vertex algebras. Examples from even lattices. Insight on the moonshine module.
- 1988, Frenkel, Lepowsky and Meurman: Vertex operator algebras. Examples, including the moonshine module vertex operator algebra. Construction of modules and twisted modules.
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Vertex operator algebras

A vertex operator algebra consists the following data:

- A \mathbb{Z} -graded vector space $V = \prod_{n \in \mathbb{Z}} V_{(n)}$.
- A vertex operator map

$$egin{array}{rcl} Y_V: V\otimes V &
ightarrow V[[z,z^{-1}]], \ u\otimes v & \mapsto & Y(u,z)v. \end{array}$$

- A vacuum $\mathbf{1} \in V_{(0)}$.
- A conformal vector $\omega \in V_{(2)}$.

Vertex operator algebras

These data satisfy the following axioms:

- Grading-restriction property: dim $V_{(n)} < \infty$ for $n \in \mathbb{Z}$ and $V_{(n)} = 0$ when *n* is sufficiently negative.
- Lower-truncation property: For $u, v \in V$, Y(u, z)v contains only finitely many negative power terms.
- Axioms for the vacuum: For $u \in V$, $Y(\mathbf{1}, z)u = u$ and $\lim_{z\to 0} Y(u, z)\mathbf{1} = u$.
- Axioms for the conformal element: Let $L(n) : V \to V$ be defined by $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}$, then $[L(m), L(n)] = (m-n)L(m+n) + \frac{c}{12}(m^3 m)\delta_{m+b,0},$ $\frac{d}{dz}Y(u, z) = Y(L(-1)u, z) (L(-1)$ -derivative property) for $u \in V$ and L(0)u = nu for $u \in V_{(n)} (L(0)$ -grading property).

Vertex operator algebras

• Duality property: For $u_1, u_2, v \in V, v' \in V' = \coprod_{n \in \mathbb{Z}} V^*_{(n)}$, the series

$$\begin{array}{l} \langle v', Y(u_1, z_1) Y(u_2, z_2) v \rangle \\ \langle v', Y(u_2, z_2) Y(u_1, z_1) v \rangle \\ \langle v', Y(Y(u_1, z_1 - z_2) u_2, z_2) v \rangle \end{array}$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.

The early work of I. Frenkel and Tsukada

- 1986, I. Frenkel and Tsukada: Started a program to construct conformal field theories mathematically using path integrals. Obtained a geometric interpretation of meromorphic vertex operators and their basic properties.
- 1988, Tsukada: Constructed vertex operator algebras associated to positive-definite even lattices using path integrals.
- However, they did not solve the problem of giving a geometric formulation of the conformal element, the Virasoro algebra or, especially, the central charge for a vertex operator algebra. This geometric formulation is necessary if we want to construct conformal field theories in the sense of Kontsevich and G. Segal.

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The first major problem in the program

- 1988, Segal: The central charge of a conformal field theory should be interpreted as twice the power of the determinant line bundle over the moduli space of Riemann surfaces with parametrized boundaries.
- The first major problem to be solved in this program: From the works above, one can easily make a conjecture on what the geometric formulation of a vertex operator algebra (including the conformal element, the Virasoro algebra and the central charge) should be. Prove that the purely algebraic formulation of a vertex operator algebra is equivalent to this infinite-dimensional analytic and geometric formulation. This turned out to be a very difficult problem.
The geometry of vertex operator algebras

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The major problems solved

Unsolved problems

The geometry of vertex operator algebras

The solution

• 1991, H.: Solved this problem completely.

- Main hard part: Prove that certain formal series obtained from vertex operators and the Virasoro operators are expansions of certain analytic functions coming from genus-zero Riemann surfaces and the determinant line bundle. This was done by using a theorem of Fischer and Grauert in the deformation theory of complex manifolds and the holomorphicity of the sewing isomorphisms for the determinant lines.
- Geometric definition: A vertex operator algebra of central charge *c* is roughly speaking a meromorphic representation of the *c*/2-th power of the determinant line bundle over the moduli space of Riemann sphere with punctures and local coordinates vanishing at the punctures, equipped with the natural sewing operation.

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Intertwining operators

- The graded dimension (vacuum character) of a vertex operator algebra in general is not modular invariant and thus is not enough to construct a genus-one conformal field theory. For affine Lie algebras and the Virasoro algebra, one needs to use all modules to obtain a modular invariant vector space. The modular invariance requirement forces us to consider modules for the vertex operator algebra, not just the algebra itself.
- Consequently, we have to study "vertex operators" among different modules. These "vertex operators" were called chiral vertex operators by Moore and Seiberg and intertwining operators by Frenkel, Lepowsky and myself.

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The operator product expansion conjecture of Moore and Seibeig

1988, Moore and Seiberg: "Consider the operator product expansion:

$$\Phi_{\alpha,a}(z_1)\Phi_{\beta,b}(z_2) = \sum_{k} \sum_{c \in V_{kr}^{l}; d \in V_{kj}^{k}} F_{\rho k} \begin{bmatrix} i & j \\ l & r \end{bmatrix}_{ab}^{cd}$$
$$\times \sum_{K \in \mathcal{H}_{k}} \xi_{\rho,K,d}^{\alpha,\beta} \begin{bmatrix} i & j \\ l & r \end{bmatrix} (z_1, z_2, z_3)\Phi_{K,c}(z_3) \quad (3.9)$$

.... This expansion is an asymptotic expansion which is believed to have a finite radius of convergence. It is valid for $z_1 \sim z_2 \sim z_3$."

The second major problem

- There was no explanation or even discussion as to why this operator product expansion must hold. It was used by Moore and Seiberg as an additional hypothesis, **not a result**. Mathematically, it was clearly a conjecture.
- This operator product expansion is in fact equivalent to the associativity for intertwining operators:

 $\mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2) = \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)$

in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$.

• This was the second major problem to be solved. Since intertwining operators are in general multivalued and form vector spaces, the usual purely algebraic method used to study vertex operator algebras and modules does not work. It was necessary to develop a new method.

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The "intermediate modules" and the tensor product modules

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- 1995, H.: Proved the associativity for intertwining operators in the case of minimal models. A new construction of the moonshine module is obtained using this result.
- 1997, H. and Lepowsky: Proved the associativity for intertwining operators in the case of the Wess-Zumino-Witten models.
- 1999 and 2000, H. and Milas: Proved the associativity for intertwining operators in the case of N = 1 and N = 2 superconformal minimal models.

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- 2002, H.: Solved completely this second major problem by proving that the convergence and extension properties hold when the vertex operator algebra or its modules satisfy a certain purely algebraic cofiniteness condition in addition to certain other natural and purely algebraic conditions.
- Main idea: Derive differential equations with regular singular points and then use these differential equations to prove the convergence and extension properties.

- 2002, H.: Solved completely this second major problem by proving that the convergence and extension properties hold when the vertex operator algebra or its modules satisfy a certain purely algebraic cofiniteness condition in addition to certain other natural and purely algebraic conditions.
- Main idea: Derive differential equations with regular singular points and then use these differential equations to prove the convergence and extension properties.

Intertwining operators and vertex tensor categories

Braided tensor categories and vertex tensor categories

- The proof of the associativity for intertwining operators also gave immediately natural associativity isomorphisms for the tensor product bifunctors constructed by Lepowsky and me. The coherence for the associativity isomorphisms follows easily from a characterization of the associativity isomorphisms.
- The braiding isomorphism can be obtained easily from the skew-symmetry of intertwining operators. In particular, the category of modules has a natural structure of a braided tensor category.
- In fact, since the tensor product bifunctor constructed by Lepowsky and me depends on a sphere with punctures and local coordinates vanishing at the punctures, what we obtain is what we call a "vertex tensor category."

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Outline

- Conformal field theories
 - A definition
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Modular invariance

- Verlinde conjecture and Verlinde formula
- Rigidity and modularity
- Full rational conformal field theories
- Open-closed conformal field theories
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 - Higher-genus theories
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 - Non-rational conformal field theories

Modular invariance

The modular invariance conjecture of Moore and Seiberg

1988, Moore and Seiberg: "The final equation is obtained from the two-point function on the torus. The conformal blocks for the two-point function of $\beta_1 \in \mathcal{H}_{j_1}$, $\beta_2 \in \mathcal{H}_{j_2}$ are given by

$$\operatorname{Tr}_{i}\left[q^{L_{0}-\frac{c}{24}}\binom{i}{j_{1}\rho}_{z_{1}}(\beta_{1}\otimes\cdot)\binom{\rho}{j_{2}i}_{z_{2}}(\beta_{2}\otimes\cdot)\right]\cdot (dz_{1})^{\Delta_{\beta_{1}}}(dz_{2})^{\Delta_{\beta_{2}}}.$$
 (4.13)"

The third major problem

- By stating that the conformal blocks for the two-point function are given by the traces above, Moore and Seiberg in fact assumed the modular invariance of the space spanned by these traces. This modular invariance was used as an additional hypothesis, **not a result**.
- Mathematically, it was clearly a powerful conjecture. Many of the deep results in this program depend on the solution to this conjecture. This was the third major problem to be solved.
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- In 1990, in his important Ph.D. thesis work, Zhu proved a partial result on the modular invariance conjecture of Moore and Seiberg.
- This partial result stated that when the vertex operator algebra satisfies a positive energy condition, a complete reducibility condition and a condition now called C_2 -cofiniteness condition), the *q*-traces of products of *n* suitable modified vertex operators for irreducible modules can be analytically extended to meromorphic doubly-periodic functions on the plane with periods 1 and $\tau = (\log q)/2\pi i$ and span a vector space that is invariant under the action of the full modular group SL(2, \mathbb{Z}).



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The failure of Zhu's method

- 2000, Miyamoto: Generalized Zhu's partial result to the partial result for one intertwining operator and *n* vertex operators for modules, using Zhu's method.
- Unfortunately, the method developed by Zhu cannot be used or adapted to prove the (full) modular invariance conjecture of Moore and Seiberg mentioned above.
- The reason that Zhu's method cannot be used or adapted is the following: Zhu's method uses the commutator formula for vertex operators (acting on modules) to reduce the construction of genus-one *n*-point functions to the construction of genus-one one-point functions. But there is no commutator formula for general intertwining operators.

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Modular invariance

The new method

The new method consisted of the following steps:

- Prove that the q-traces of products of intertwining operators satisfy certain systems of modular invariant differential equations with regular singular points.
- Prove that the *q*-traces of products of intertwining operators are absolutely convergent and have genus-one associativity for intertwining operators (or genus-one operator product expansions) using the systems of differential equations and the associativity for intertwining operators.
- Reduce the proof of the modular invariance for genus-one *n*-point functions to the modular invariance for genus-one one-point functions (already proved by Zhu and Miyamoto) using the genus-one associativity for intertwining operators.
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The fourth major problem

- Moore and Seiberg showed that the Verlinde conjecture and Verlinde formula could indeed be derived from some basic conjectures—for example, the operator product expansion for chiral vertex operators and the full modular invariance conjecture on rational conformal field theories. But since Moore and Seiberg did not prove these conjectures, the Verlinde conjecture and Verlinde formula were still not proved in their paper.
- It turned out that the Verlinde conjecture and Verlinde formula were an important step in the program to **construct** rational conformal field theories from representations of vertex operator algebras. Thus, they should be viewed as the fourth major problem to be solved in this program.

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- This work in fact proved that the Moore-Seiberg polynomial equations hold for all such vertex operator algebras. Thus much stronger results were obtained. These stronger results played an important role in the proof of the rigidity and modularity conjecture which I will discuss next.

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The fifth major problem

- The rigidity of braided tensor category structure on the category of modules for a vertex operator algebra was an open problem for many years.
- Another closely related hard open problem (let's call it modularity) was the nondegenracy property and the identification of the *S*-matrix obtained from the ribbon tensor category structure with the action of the modular transformation associated to $\tau \mapsto -1/\tau$ on the space spanned by the graded dimension.
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Conformal field theories

The major problems solved

Unsolved problems

Rigidity and modularity

- 2005, H.: Proved the rigidity and modularity of the braided tensor category of modules for a vertex operator algebra satisfying the three conditions in the modular invariance theorem. In particular, combining this result with the work of Turaev, we obtain knot and three manifold invariants proposed first by Witten.
- The proof used a strong version of the Verlinde formula and thus logically used the modular invariance theorem. This was a surprise.
- For many years, there were widely circulated claims that the rigidity and modularity for the Wess-Zumino-Witten models had been proved, and that for general rational conformal field theories they could be proved in the same way. Such claims have been shown to be wrong and acknowledged as such in recent years.

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Unsolved problems

Rigidity and modularity

The case of Wess-Zumino-Witten models

- In the case of Wess-Zumino-Witten models, Finkelberg's tensor-category-equivalence theorem together with Kazhdan-Lusztig's rigidity theorem for negative levels had been thought to prove the rigidity for almost all (but not all) cases. But it turned out that Finkelberg's paper had a gap and it required either the Verlinde formula proved by Faltings, Teleman and me or, alternatively, the rigidity proved by me, to fill the gap and prove the equivalence theorem (again, for almost all, but not all, cases). Note that, as is mentioned above, my proof of the Verlinde formula or my proof of the rigidity needs the full modular invariance theorem.
- Finkelberg's tensor-category-equivalence theorem, after correction, still does not cover a few exceptional cases, including in particular, the *E*₈ leve 2 case.

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The sixth major problem

- The results discussed above are for chiral conformal field theories. Chiral conformal field theories are not enough. We need to put chiral and antichiral conformal field theories together in a suitable way to construct full conformal field theories. Here antichiral conformal field theories are just some chiral conformal field theories that will become the antichiral parts of full conformal field theories.
- Since the conformal fields in full conformal field theories must be single-valued but intertwining operators are in general multivalued, the construction of full field theories from chiral and antichiral conformal field theories are highly nontrivial.
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Conformal field theories

The major problems solved

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Full rational conformal field theories

- 2005 and 2006, Kong and H.: Solved this major problem for genus-zero and genus-one theories, respectively, in the so-called diagonal case, that is, the case that the state space of a full conformal field theory is the direct sum of the tensor products of irreducible modules for a vertex operator algebra and its contragredient modules.
- The main work is to construct a nondegenerate bilinear form on the space of intertwining operators satisfying natural properties. The difficult part is the proof of the nondegeneracy of the bilinear form. Recall that in the work of Witten on holomorphc factorization of Wess-Zumino-Witten models, the nondegeracy of the hermitian form on the space of conformal blocks was an assumption, **not a result**.

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- It was very surprising to us that the proof of the nodegeneracy of the bilinear form, a property for genus-zero chiral conformal field theories, needs the (full) modular invariance theorem, a genus-one property.
- The nondegeneracy of the bilinear form is in fact equivalent to the rigidity of the corresponding braided tensor category. This is another indication why the proof of the rigidity needed the (full) modular invariance theorem and was so difficult, even in the case of Wess-Zumino-Witten models.

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- The problems and solutions discussed above are all for closed conformal field theories. We also need to construct open-closed conformal field theories.
- One needs to construct open-string vertex operator agebras. Modules for open-string vertex algebras in fact correspond exactly to the important *D*-branes introduced and studied by string theorists.
- The connection between the open part and the closed part of an open-closed conformal field theory is given by what is called Cardy condition, studied first by Cardy.
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Part 3 Unsolved problems

Higher-genus theories

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- Conformal field theories
 - A definition
 - Conjectures
 - Problems and a program
- 2 The major problems solved
 - The geometry of vertex operator algebras
 - Intertwining operators and vertex tensor categories
 - Modular invariance
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Unsolved problems

Higher-genus theories

The major problem to be solved

- The major problem to be solved in the higher-genus case is a convergence problem similar to the convergence problem for products and iterates of intertwining operators in the genus-zero case and the convergence problem for traces of products and iterates of intertwining operators.
- To prove this convergence, one needs to prove some conjectures on certain types of functions on the infinite-dimensional Teichmüller spaces and moduli spaces of Riemann surfaces with parametrized boundaries.
- Recently Radnell, Schipper and Staubach have been making good progress in the study of these Teichmüller spaces and moduli spaces. I hope that they will soon be able to establish those conjectures as theorems on functions on these spaces.

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Locally convex completions

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Locally convex completions

- In 1998 and 2000, I constructed topological completions of vertex operator algebras and modules. This construction can be generalized easily to construct topological completions of the state spaces of the genus-zero chiral and full conformal field theories discussed above. But to obtain the full topological completions, we need first construct higher-genus theories.
- On the other hand, in the case that the state space has a natural inner product, there is another completion given by the corresponding norm.
- **Conjecture (H.)**: The topological completion obtained from higher-genus theory using the method giving topological completions of vertex operator algebras is the same as the topological completion obtained from the norm given by the inner product.

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Locally convex completions

- In 1998 and 2000, I constructed topological completions of vertex operator algebras and modules. This construction can be generalized easily to construct topological completions of the state spaces of the genus-zero chiral and full conformal field theories discussed above. But to obtain the full topological completions, we need first construct higher-genus theories.
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Locally convex completions

Connection with operator algebra approach

The right-hand side of the commutator formula for vertex operators

$$Y(u, z_1), Y(v, z_2)]$$

= $\operatorname{Res}_{z_0=0} z_2^{-1} e^{-z_0 \frac{\partial}{\partial z_1}} \delta\left(\frac{z_1}{z_2}\right) Y(Y(u, z_0), z_2)$

is given by a linear combination of the formal Laurent expansions of finitely many derivatives of the δ -funtion. Thus vertex operators with *z* in a proper interval of S^1 might give a Von Neumann algebra and when two proper intervals are disjoint, the two algebras commute with each other.

• **Question**: Is this indeed the correct way to look at the expected equivalence between the vertex operator algebra approach and the conformal nets approach?

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Outline

Conformal field theories

- A definition
- Conjectures
- Problems and a program

2 The major problems solved

- The geometry of vertex operator algebras
- Intertwining operators and vertex tensor categories
- Modular invariance
- Verlinde conjecture and Verlinde formula
- Rigidity and modularity
- Full rational conformal field theories
- Open-closed conformal field theories

3 Unsolved problems

- Higher-genus theories
- Locally convex completions
- Non-rational conformal field theories

- For non-rational conformal field theories, the main results we have now are on logarithmic conformal field theories.
- 2002, Miyamoto: A partial result on the modular invariance for logarithmic conformal field theories, using *q*-pseudo-traces instead of *q*-traces.
- 2003, H., Lepowsky and Zhang: Proved the logarithmic operator product expansion and constructed the vertex tensor category structure.
- **Conjecture (H.)**: Analytic extensions of suitably generalized *q*-pseudo-traces of products of logarithmic intertwining operators span a modular invariant vector space. Fiordalisi has made substantial progress.
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Calabi-Yau superconformal field theories

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Thanks!